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Automorphisms of double affine braid groups arising from topology

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The work of van der Lek

Linear Coxeter groups: Notation

Consider

- A (not necessarily finite) linear Coxeter group W acting on a finite dimensional real euclidean space $\mathfrak{h}$.
- A fundamental chamber C for W with walls $M_1, \ldots, M_\ell$.
- The orthogonal reflections s_j in the hyperplanes M_j (the *simple reflections*).
- m_{jk} which is the order of $s_j s_k$.
- The set $\mathcal{M}$ of reflection hyperplanes of all reflections in W (the *mirrors* of W).

Linear Artin groups

The *Artin group* of W, denoted A_W, is the group with the following presentation:

Generators: $T_1, \dots, T_\ell$.

The braid relations:

$$T_j T_k T_j \cdots = T_k T_j T_k \cdots$$

with $m_{jk} < \infty$ terms on each side, $j \neq k$.

In particular the map

$$A_W \to W, \quad T_j \mapsto s_j$$

extends to a surjective homomorphism, and by a theorem of Matsumoto (1964), there is a natural section (not homomorphism!)

$$T : W \to A_W, \quad s_j \mapsto T_j$$

The Tits cone

The *Tits cone* of W is

$$I = \bigcup_{w \in W} w(\overline{C}).$$

Denote by $\mathring{I}$ the interior of the Tits cone.

By a theorem of Vinberg (1971) it satisfies the following properties:

- I is a convex cone and $\mathring{I}$ is an open convex cone.
- $\mathring{I}$ consists of points in I with finite W-stabilizer.
- W acts properly discontinuously on $\mathring{I}$.

In particular, W is finite if and only if $I = \mathfrak{h}$.

The regular orbit space and braid groups

- Consider the *space of regular points*

$$Y = (\mathfrak{h} + i\mathring{I}) \setminus \bigcup_{M \in \mathcal{M}} M_{\mathbb{C}} \subseteq \mathfrak{h}_{\mathbb{C}}$$

where $(\cdot)_{\mathbb{C}} = - \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification functor.

The *regular orbit space* is the quotient $X = Y/W$.
Write $p : Y \to X$ for the canonical projection. This is a covering projection.

Choose a point $c \in C$, let $y_0 = ic$ and $x_0 = p(y_0)$.

The *braid group* of W is the fundamental group $\pi_1(X, x_0)$.

van der Lek's theorem: Linear case

If M is a wall of C, let $D_M(C)$ be the open halfspace of V determined by M that contains C.

Theorem 1 (van der Lek, 1983)

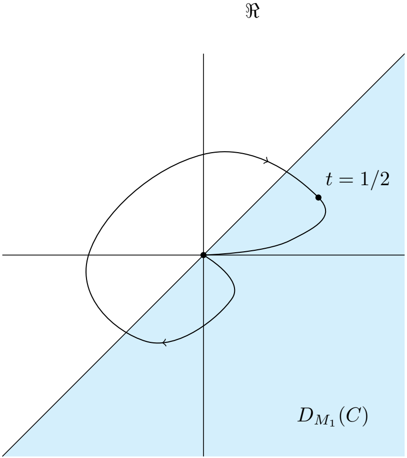
The braid group $\pi_1(X, x_0)$ is isomorphic to the Artin group A_W. Moreover the generators T_j, $j = 1, \ldots, \ell$ correspond under this isomorphism to path homotopy classes $[p \circ T^j]$ where

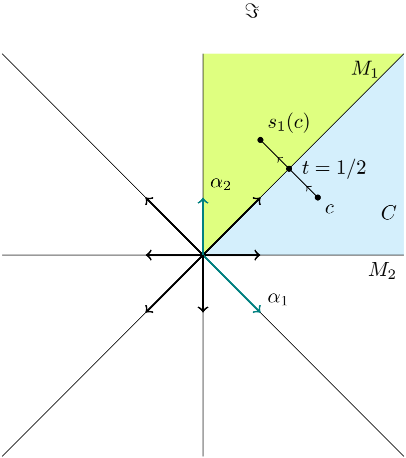
$$T^j : [0,1] \to Y, \quad t \mapsto \delta(t) + i((1-t)c + ts_j(c))$$

and $\delta : [0,1] \to \mathfrak{h}$ is any continuous function such that $\delta(0) = \delta(1) = 0$ and $\delta(1/2) \in D_{M_j}(C)$.

Example in type B_2: Generators

T^1:

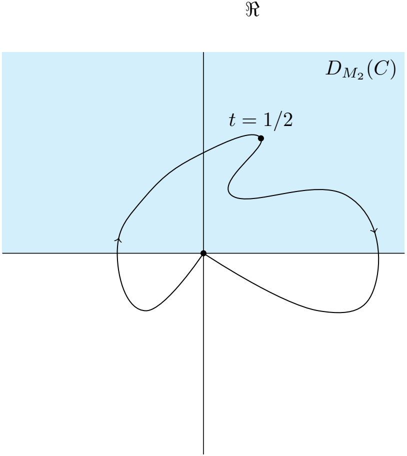


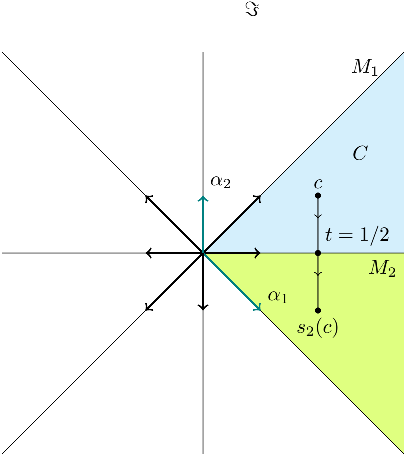


Thanks to Luigi Loayza Montesinos for these beautiful drawings!

Example in type B_2: Generators

T^2:





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Example in type B_2: The braid relation

The braid relation in this case is

$$T_1T_2T_1T_2 = T_2T_1T_2T_1$$

and, by lifting, the LHS and RHS correspond, respectively, to the following paths in Y:

$$T^1 * s_1T^2 * s_1s_2T^1 * s_1s_2s_1T^2$$

and

$$T^2 * s_2T^1 * s_2s_1T^2 * s_2s_1s_2T^1,$$

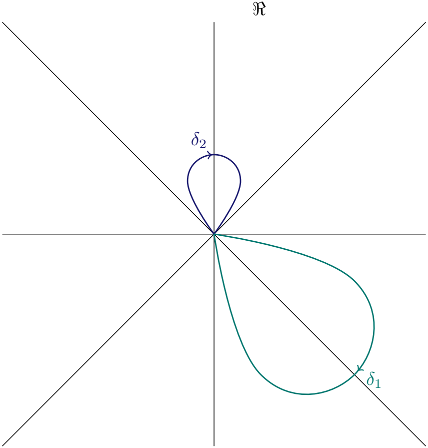
that is

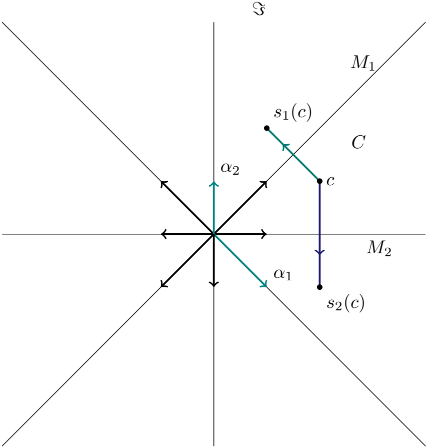
$$[p \circ (T^1 * s_1T^2 * s_1s_2T^1 * s_1s_2s_1T^2)] = T_1T_2T_1T_2$$

and

$$[p \circ (T^2 * s_2T^1 * s_2s_1T^2 * s_2s_1s_2T^1)] = T_2T_1T_2T_1.$$

T^1 vs. T^2





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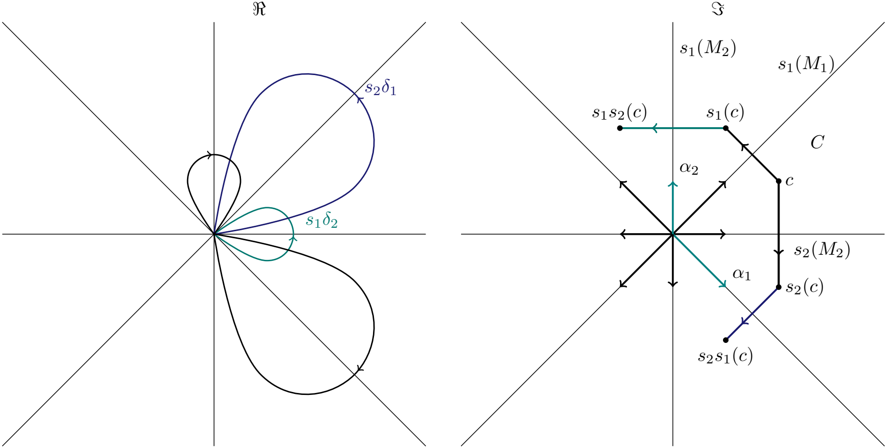
Example in type B_2: The braid relation

$T^1 * s_1 T^2$ vs. $T^2 * s_2 T^1$



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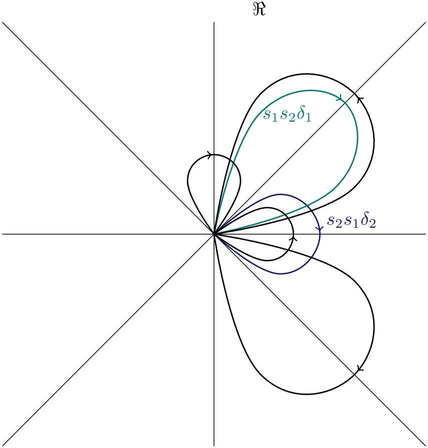
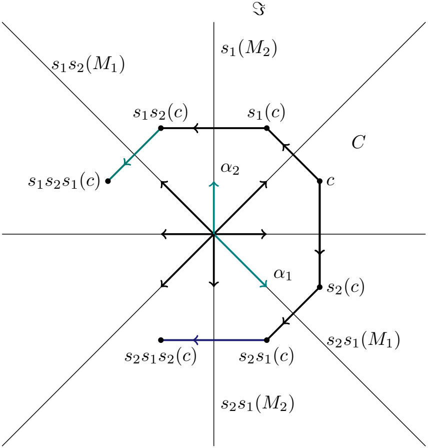
Example in type B_2: The braid relation

$$T^1 * s_1T^2 * s_1s_2T^1 \quad \text{vs.} \quad T^2 * s_2T^1 * s_2s_1T^2$$





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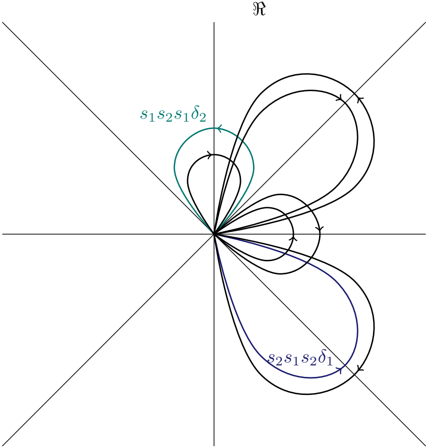
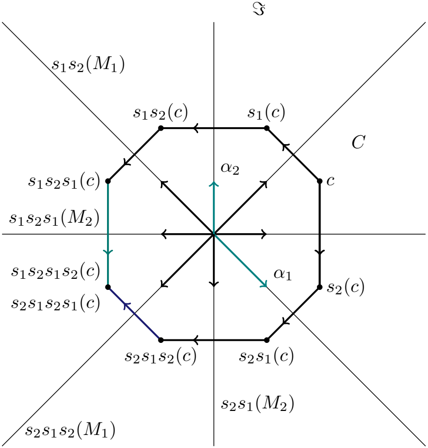
Example in type B_2: The braid relation

$$T^1 * s_1 T^2 * s_1 s_2 T^1 * s_1 s_2 s_1 T^2 \quad \text{vs.} \quad T^2 * s_2 T^1 * s_2 s_1 T^2 * s_2 s_1 s_2 T^1$$





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Root bases

A *root basis* in $\mathfrak{h}$ is a pair $(B, (\cdot)^\vee)$ where $B = \{\alpha_1, \ldots, \alpha_\ell\}$ is a finite subset of $\mathfrak{h}$ and

$$(\cdot)^\vee : B \to \mathfrak{h}^*, \quad \alpha \mapsto \alpha^\vee$$

is a map such that

- B (resp. $B^\vee$) is a linearly independent subset of $\mathfrak{h}$ (resp. $\mathfrak{h}^*$).
- $\alpha^\vee(\alpha) = 2$ for all $\alpha \in B$.
- $\beta^\vee(\alpha) \in \mathbb{Z}_{\leq 0}$ for all $\alpha \neq \beta$ in B.
- $\beta^\vee(\alpha) = 0 \Rightarrow \alpha^\vee(\beta) = 0$.

Let

$$s_i(x) = x - \alpha_i^\vee(x)\alpha_i$$

the reflection associated to α_i and W the group generated by $\{s_1, \ldots, s_\ell\}$. The *fundamental chamber* is

$$C = \{x \in \mathfrak{h} \mid \alpha_j^\vee(x) > 0, \quad j = 1, \ldots, \ell\}.$$

Generalized root systems

The set

$$R = W(B) = \{w(\alpha) \mid w \in W, \alpha \in B\}$$

is called a *generalized root system* and W its *Weyl group*.

- The set $R^\vee = W(B^\vee)$ is the *dual* of R.
- The elements of R are called *roots* and the elements of $R^\vee$ *co-roots*.
- The lattices

$$Q = \bigoplus_{j=1}^{\ell} \mathbb{Z}\alpha_j \quad \text{and} \quad Q^\vee = \bigoplus_{j=1}^{\ell} \mathbb{Z}\alpha_j^\vee.$$

  are called the *root lattice* and the *co-root lattice*, respectively.

Extended Weyl group

If $h \in \mathfrak{h}$ we write

$$t(h) : \mathfrak{h} \to \mathfrak{h}, \quad x \mapsto x + h$$

for the translation by the vector h.

Write $L = Q$.

The *extended Weyl group of* R is

$$\widetilde{W} = W \ltimes t(L).$$

Let $X^L = \{X^\lambda \mid \lambda \in L\}$ be a multiplicative abelian group isomorphic to L:

$$X^0 = 1, \quad X^{\lambda+\mu} = X^\lambda X^\mu, \quad (X^\lambda)^{-1} = X^{-\lambda}.$$

Write $X_j = X^{\alpha_j}$.

Extended Artin groups

There are several equivalent definitions of the *extended Artin group* $A_{\widetilde{W}}$ of $\widetilde{W}$. This is one of them:

It is the group with generators $T_1, \ldots, T_\ell$, $X_1, \ldots, X_\ell$ such that

- The subgroup generated by $T_1, \ldots, T_\ell$ is isomorphic to A_W;
- The subgroup generated by $X_1, \ldots, X_\ell$ is isomorphic to X^L;
- These generators also satisfy the *push-relations*:

$$T_j^{(n)} X^\lambda = X^{s_j(\lambda)} T_j^{(n+\langle \alpha_j^\vee, \lambda \rangle)}$$

where

$$T_j^{(2m)} := X_j^m T_j X_j^m$$

and

$$T_j^{(2m-1)} = X_j^m T_j^{-1} X_j^m.$$

Extended braid groups

The group $\widetilde{W}$ acts on $\mathfrak{h}_{\mathbb{C}}$, but the action of $t(L)$ is by translations only of the real part.

- Let C be the fundamental chamber and choose $c \in C$.
- Consider as before the Tits cone I of W.
- Consider the space of regular points

$$\widetilde{Y} = (\mathfrak{h} + i\mathring{I}) \setminus \{\text{reflection hyperplanes of } \widetilde{W}\}.$$

- The *regular orbit space* is $\widetilde{X} = \widetilde{Y}/\widetilde{W}$. Let $\tilde{p} : \widetilde{Y} \to \widetilde{X}$ be the canonical covering projection.

  Let $\tilde{y}_0 = ic$ and $\tilde{x}_0 = \tilde{p}(\tilde{y}_0)$.

The *extended braid group* of $\widetilde{W}$ is the fundamental group $\pi_1(\widetilde{X}, \tilde{x}_0)$.

van der Lek's theorem: Extended case

Theorem 2 (van der Lek, 1983)

The extended braid group $\pi_1(\widetilde{X}, \tilde{x}_0)$ *is isomorphic to the extended Artin group* $A_{\widetilde{W}}$*. Moreover the generators* T_j *and* X_j *correspond under this isomorphism to path homotopy classes* $[p \circ T^j]$ *and* $[p \circ X^j]$ *where*

$$T^j(t) = \delta(t)\alpha_j + i((1-t)c + ts_j(c))$$

where $\delta : [0,1] \to [0,1/2)$ *is continuous, with* $\delta(0) = \delta(1) = 0$ *and* $\delta(1/2) > 0$*, and*

$$X^j(t) = t\alpha_j + ic,$$

respectively, $j = 1, \ldots, \ell$*.*

Affine Weyl groups

- Assume R is an irreducible root system in $\mathfrak{h}$.
- Let $\alpha_0^\vee$ be the longest coroot and α_0 be the corresponding root.
- Let $s_0 = t(\alpha_0)s_{\alpha_0}$ where s_{α_0} is the reflection corresponding to α_0.
- Let W_a be the affine Weyl group of R.

The action of W_a on $\mathfrak{h}$ and on $\mathfrak{h}^*$ is by affine transformations. But its action on the space of affine-linear functionals $\mathfrak{h}^* \to \mathbb{R}$ is linear, and this space is isomorphic to $\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{R}\delta$.
Thus we can consider W_a as a linear Coxeter group acting on $\widetilde{\mathfrak{h}}$.
Define $\hat{\mathfrak{h}} = \widetilde{\mathfrak{h}} \oplus \mathbb{R}\Lambda$ and

$$\psi : \hat{\mathfrak{h}} \to \mathbb{R}, \quad x + s\delta + t\Lambda \mapsto t.$$

We extend the inner product on $\mathfrak{h}$ by

$$\langle \mathfrak{h}, \delta \rangle = \langle \mathfrak{h}, \Lambda \rangle = 0 \quad \langle \delta, \delta \rangle = \langle \Lambda, \Lambda \rangle = 0, \quad \langle \Lambda, \delta \rangle = 1$$

In this case, the interior of the Tits cone is $\mathring{I} = \psi^{-1}((0, +\infty))$.

Complexifications

Let $\tau \in \mathbb{C}^\times$ and denote the complexification of ψ again by ψ.

- Let $Y_a \subseteq \mathfrak{h}'_{\mathbb{C}}$ and X_a be the space of regular points considering W_a as a linear Coxeter group.
- Write
$$\mathfrak{h}_{\mathbb{C}}^{\tau} = \psi^{-1}(\tau).$$
- Define
$$Y_\tau = \mathfrak{h}_{\mathbb{C}}^{\tau} \setminus \{ \text{ reflection hyperplanes of } W_a\}$$
and
$$X_\tau = Y_\tau / W_a.$$
- If $\Im(\tau) > 0$ space Y_τ is a deformation rectract of Y_a and there is a deformation retraction invariant under W_a so that X_τ is also a deformation retract of X_a.
- In this case, $W_a = \tilde{W}$ and there is a homeomorphism $X_\tau \cong \widetilde{X}$ provided $\Im(\tau) > 0$.

van der Lek's theorem: Affine case

Theorem 3 (van der Lek, 1983)

The retraction $X_a \to \widetilde{X}$ defined above induces a commutative diagram of groups and isomorphisms

$$\begin{array}{ccc} \pi_1(X_a, *) & \longrightarrow & \pi_1(\widetilde{X}, *) \\ \downarrow & & \downarrow \\ A_{W_a} & \longrightarrow & A_{\widetilde{W}} \end{array}$$

Moreover, the isomorphism in the bottom row is given by

$$T_j \mapsto T_j, \quad j = 1, \ldots, \ell \quad \textit{and} \quad T_0 \mapsto X^{\alpha_0} T(s_{\alpha_0})$$

where $T : W \to A_W$ is the section given by Matsumoto's theorem.

The Heisenberg group

The Heisenberg group

- Consider two lattices L and M in a real euclidean vector space $\mathring{\mathfrak{h}}_{\mathbb{R}}$.
- Assume that the inner product on $\mathring{\mathfrak{h}}_{\mathbb{R}}$ induces a perfect pairing

$$\langle \cdot, \cdot \rangle : L \times M \to \frac{1}{e}\mathbb{Z}$$

  for some integer $e \geq 1$.
- Let

$$X^L = \{X^\lambda \mid \lambda \in L\} \quad \text{and} \quad Y^M = \{Y^\mu \mid \mu \in M\}$$

  be multiplicative abelian groups isomorphic to L and M, respectively.

The *Heisenberg group* N_L^M associated to this data is the group generated by X^L, Y^M and a central element q_0 subject to the relations

$$X^\lambda Y^\mu = Y^\mu X^\lambda q^{\langle \lambda, \mu \rangle}, \quad \lambda \in L, \mu \in M,$$

where $q = q_0^e$.

Automorphisms of the Heisenberg group

Consider the *theta subgroup* of the modular group

$$\Gamma_\vartheta = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; ab \equiv cd \equiv 0 \pmod 2 \right\} \leq \mathrm{SL}_2(\mathbb{Z}).$$

This group acts by automorphisms on N_L^L as follows. Let

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

and define $\varphi_A : N_L^L \to N_L^L$ by $\varphi_A(q_0) = q_0$,

$$\varphi_A(X^\lambda) = X^{a\lambda} Y^{b\lambda} q^{-ab/2} \quad \text{and} \quad \varphi_A(Y^\lambda) = X^{c\lambda} Y^{d\lambda} q^{-cd/2}.$$

Question.
Is it possible to construct these automorphisms using topological methods?

A representation of the Heisenberg group

Let $\mathring{\mathfrak{h}}$ be the complexification of $\mathring{\mathfrak{h}}_{\mathbb{R}}$ and extend the inner product by bilinearity. Add to $\mathring{\mathfrak{h}}$ a hyperbolic plane, that is, let

$$\mathfrak{h} = \mathring{\mathfrak{h}} \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda$$

where

$$\langle \mathring{\mathfrak{h}}, \delta \rangle = \langle \mathring{\mathfrak{h}}, \Lambda \rangle = 0,$$

$$\langle \delta, \delta \rangle = \langle \Lambda, \Lambda \rangle = 0 \quad \text{and} \quad \langle \delta, \Lambda \rangle = 1.$$

The Heisenberg group N_L^L acts on $\mathfrak{h}$ by the formulas

$$\begin{aligned} X^\lambda \cdot v &= v + 2\pi i \lambda, \\ Y^\lambda \cdot v &= v + \langle v, \delta \rangle \lambda - \left(\langle v, \lambda \rangle + \frac{1}{2} \langle \lambda, \lambda \rangle \langle v, \delta \rangle \right) \delta \\ q_0 \cdot v &= v + 2\pi i \delta. \end{aligned}$$

Heisenberg group as a fundamental group

The subspace

$$Y = \{v \in \mathfrak{h} \mid \Re(\langle v, \delta \rangle) > 0\}$$

is invariant under the action of N_L^L as are the subspaces

$$Y_\tau = \{v \in Y \mid \langle v, \delta \rangle = \tau\}.$$

Let

$$X_\tau = Y_\tau / N_L^L, \quad p : Y_\tau \to X_\tau$$

choose a regular point $y_0 \in Y_\tau$ and let $x_0 = p(y_0)$.
The monodromy map

$$\pi_1(X_\tau, x_0) \to N_L^L$$

is an isomorphism since Y_τ is a universal covering space of X_τ.

Relation to van der Lek's work

Note that the space Y is precisely the space of regular points for the action of N_L^L and that when $\Re(\tau) > 0$ the space Y_τ is a deformation retract of Y.
If we define the regular orbit space as

$$X = Y/N_L^L$$

we have that X_τ is a deformation retract of X and thus N_L^L is also the fundamental group of X.

Our goals

Our first goal is to construct topologically automorphisms of the Heisenberg group, in particular the automorphisms arising from the theta subgroup of the modular group.

Why?

If all of these constructions are "natural enough" we can extend these to the following more general construction:

Let W be a group of orthogonal transformations of $\mathring{\mathfrak{h}}_{\mathbb{R}}$ leaving L and M invariant. We define

$$W(L, M) = N_L^M \rtimes W.$$

When W is the Weyl group of a finite root system and L, M are certain choices of the (co-)root or (co-)weight lattices, this construction produces the so-called *double affine Weyl group.*

Double affine Artin/braid groups

Consider an affine Coxeter system $(W_a, \{s_0, \ldots, s_\ell\})$ acting by affine transformations on $\mathfrak{h}$ and linearly on $\mathfrak{h}'$ as before.
We can iterate the above constructions and extend W_a by a lattice M.
More precisely, we construct the extended Weyl group of W_a, that is

$$\widetilde{W_a}.$$

which we call the *double affine Weyl group*.

The *double affine Artin group* of W, denoted $\mathfrak{B}_W$, is the extended Artin group of the double affine Weyl group.

As before we can construct its space of regular points $\widetilde{Y_a}$ and the regular orbit space $\widetilde{X_a}$.

The *double affine braid group* is the fundamental group $\pi_1(\widetilde{X_a}, *)$.

A presentation

By definition $\mathfrak{B}_W$ is generated by A_{W_a}, (a multiplicative version Y^M of) a root lattice M and a central element q_0 subject to the relations

$$T_j Y^\mu = Y^{s_j(\mu)} T_j \quad \text{if } \langle \alpha_j^\vee, \mu \rangle = 0,$$
$$T_j Y^\mu T_j = Y^{s_j(\mu)} \quad \text{if } \langle \alpha_j^\vee, \mu \rangle = 1,$$

where $q = q_0^e$.

The goal

In 1992 Ivan Cherednik (followed later by Ion), using the ideas of van der Lek, proposed a connection between the double affine Hecke algebra and the fundamental group of a certain topological space.

Surprisingly for us, this topological point of view has not been exploited at all (as far as we know) and the topological techniques developed by van der Lek seem to be powerful enough to suggest the following

Main goal.
Produce **topologically** several involutions, automorphisms, homomorphisms of/between double affine braid groups.

Let the journey begin!