Character formulas for unitary representations of cyclotomic rational Cherednik algebras

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Introduction

The purpose of this thesis is to complete the proofs of the character formulas from [**4**] for the diagonalizable and unitary irreducible representations in category $\mathcal{O}$ of the rational Cherednik algebra of type $W = G(\ell, 1, n)$. We establish two sorts of character formulas: firstly, for the diagonalizable irreducible representations we prove a combinatorial rule for computing the graded W-characters of these representations; secondly, for the unitary irreducible representations we are able to go farther, and deduce from the graded character formula a combinatorial formula for the dimensions of the Ext groups with standard modules (sometimes referred to as the *Kazhdan-Lusztig* character).

The main ideas of the proofs of these formulas have been published in [**4**], but that part of the proof having to do with the representation theory of a certain subalgebra of the Cherednik algebra was only sketched. The subalgebra is a certain W-analog of the degenerate affine Hecke algebra of the symmetric group, and the part of its representation theory that is relevant for this problem is the classification and description of those irreducible representations that are diagonalizable with respect to a large abelian subalgebra. In this thesis we complete the proofs in detail. Namely we show that the irreducible diagonalizable representations are indexed by certain combinatorial objects, the cyclotomic ℓ-skew shapes and that the representation indexed by a skew shape D has a basis of eigenvectors indexed by standard Young tableaux of shape D. Moreover we show that the restriction to W of the irreducible representation corresponding to D is described by Littlewood-Richardson numbers.

We now describe the objects and ideas involved somewhat more precisely, referring to the body of the thesis when appropriate.

The rational Cherednik algebra H_c is an algebra attached to a complex reflection group W depending on a set of parameters $c = (c_0, d_0, d_1, \ldots, d_{\ell-1})$ indexed by the conjugacy classes of reflections in W. An ℓ-partition λ of n is a sequence $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{\ell-1})$ of ℓ with n total boxes, we write $P_{\ell,n}$ for the set of all ℓ-partitions of n. The ℓ-partitions of n index the irreducible representations of $G(\ell, 1, n)$ and therefore also the standard and irreducible objects of category $\mathcal{O}_c$, we will write $\Delta_c(\lambda)$ and $L_c(\lambda)$ for the standard and irreducible objects of category $\mathcal{O}_c$ corresponding for the ℓ-partition λ.

For a box $b \in \lambda$, we define its charged content $\mathrm{ct}_c(b)$ by

$$\mathrm{ct}_c(b) = d_{\beta(b)} + \ell\,\mathrm{ct}(b) c_0$$

and write $\mathrm{ct}_c(\lambda)$ for the sum of the charged contents of the boxes of λ

$$\mathrm{ct}_c(\lambda)=\sum_{b\in\lambda}\mathrm{ct}_c(b)$$

For a skew ℓ-diagram D and ℓ-partitions λ,μ,ν we denote by c^D_λ and $c^\lambda_{\mu\nu}=c^{\lambda\setminus\mu}_\nu$ for a certain cyclotomic version of the Littlewood-Richardson numbers, which are simply products of certain classical Littlewood-Richardson numbers.

Also we define a certain set $\mathrm{Tab}_c(\lambda)$ of tableaux Q in λ satisfying the following properties:

(1) Q is a filling of the boxes of λ by non-negative integers such that $Q(b)\le Q(b')$ whenever $b\le b'$,

(2) If b is a box of λ and k is a positive integer such that

$$\mathrm{ct}_c(b)=d_{\beta(b)-k}+k$$

then $Q(b)<k$, and

(3) If b and b' are boxes of λ and k is a positive integer with $k=\beta(b)-\beta(b') \mod \ell$ and such that

$$\mathrm{ct}_c(b)-\mathrm{ct}_c(b')=k\pm\ell c_0$$

then

$$Q(b)\le Q(b')+k.$$

The set $\mathrm{Tab}_c(\lambda)$ is the indexing set for the irreducible $H_{\ell,n}$-modules of $L_c(\lambda)$. Given $Q\in\mathrm{Tab}_c(\lambda)$ we write $s_c(Q)$ for the skew diagram indexing its isotype as an $H_{\ell,n}$-module.

Now we are in conditions to state the main result of [**4**]

THEOREM 0.1. *Let λ be an ℓ-partition of n.*

1. If $L_c(\lambda)$ is $\mathfrak{t}$-diagonalizable, then

$$\mathrm{ch}(L_c(\lambda))=\sum_{\substack{Q\in\mathrm{Tab}_c(\lambda)\\ \mu\in P_{\ell,n}}} c^{s_c(Q)}_\mu[S^\mu]\,t^{|Q|}$$

2. If $L_c(\lambda)$ is unitary, then for each ℓ-partition μ of n

$$\dim_{\mathbf{C}}\left[\mathrm{Ext}^i_{\mathcal{O}_c}(\Delta_c(\mu),L_c(\lambda))\right]=\sum_{\substack{Q\in\mathrm{Tab}_c(\lambda)\\ \nu\in P_{\ell,n},\eta\in P_{\ell,n-i},\chi\in P_{\ell,i}\\ |Q|=\mathrm{ct}_c(\lambda)-\mathrm{ct}_c(\mu)-i}} c^{s_c(Q)}_\nu c^\nu_{\eta\chi} c^\mu_{\eta\chi^t}$$

Now we summarize the contents in this thesis. In Chapter 1 we define the rational Cherednik H_c algebra and state and prove the Poincaré-Birkhoff-Witt theorem (PBW), this result allows to describe the rational Cherednik algebra by generators and relations and to construct the standard modules. With standard modules defined, we introduce the category $\mathcal{O}_c$, simple and unitary representations and the characters of the rational Cherednik algebra. In Section 2 we define a highest weight category (in the sense of Cline, Parshall and Scott [**12**]) and observe that $\mathcal{O}_c$ satisfy its conditions.

In section 1 of Chapter 2, we describe in detail the combinatorial objects we will use, such as ℓ-partitions and skew shapes. In section 2 we define the groups of complex reflections $G(\ell,1,n)$ and in Section 3, the rational Cherednik algebra attached to them (which we will refer to as the *cyclotomic rational Cherednik algebra*).

In Chapter 3 we construct a subalgebra of H_c which is isomorphic to the cyclotomic degenerate affine Hecke algebra $H_{\ell,n}$. and we study in detail the class of diagonalizable $H_{\ell,n}$-modules with respect to a certain commutative subalgebra of $H_{\ell,n}$. In Chapter 4 we state and proof the main theorem using the classification of $H_{\ell,n}$-modules in the preceding Chapter.

We include an appendix with a catalog of unitary spectra and characters for the cyclotomic rational Cherednik algebra of type B, this is for $\ell = 2$ corresponding to the Weyl group of type B.

Contents

CHAPTER 1

Preliminaries

1. The rational Cherednik algebra

Let $\mathfrak{h}$ be a finite dimensional complex vector space and $\mathfrak{h}^*$ its dual space, with the natural pairing $\langle \cdot, \cdot \rangle$ between $\mathfrak{h}^*$ and $\mathfrak{h}$ defined by $\langle x, y \rangle = x(y)$. Let W be a finite subgroup of linear transformations of $\mathfrak{h}$ and R be the set of reflections in W, namely elements $w \in W$ such that $\operatorname{codim}(\operatorname{fix}_h(w)) = 1$, in other words the elements $w \in W$ whose fixed space is a hyperplane of $\mathfrak{h}$.

For each reflection $r \in R$ we fix a constant $c_r \in \mathbf{C}$ satisfying

$$c_{wrw^{-1}} = c_r \text{ for all } w \in W.$$

and $\alpha_r \in \mathfrak{h}^*$ such that

$$\operatorname{fix}_{\mathfrak{h}}(r) = \ker(\alpha_r).$$

Also there is a unique $\alpha_r^\vee \in \mathfrak{h}$ such that

$$r(x) = x - \langle x, \alpha_r^\vee \rangle \alpha_r, \quad \text{for } x \in \mathfrak{h}^*. \tag{1.1}$$

We define the partial derivation in the direction y as the function ∂_y characterized uniquely by the following properties

(1) $\partial_y(fg) = f\partial_y(g) + \partial_y(f)g \quad$ for $f, g \in \mathbf{C}[\mathfrak{h}]$ and
(2) $\partial_y(x) = x(y) \quad$ if $x \in \mathfrak{h}^*$.

For each $y \in \mathfrak{h}$ define the *Dunkl operator* D_y acting on $\mathbf{C}[\mathfrak{h}]$ by the formula

$$D_y(f) = \partial_y(f) - \sum_{r \in R} c_r \langle \alpha_r, y \rangle \frac{f - rf}{\alpha_r} \text{ for } f \in \mathbf{C}[\mathfrak{h}], \tag{1.2}$$

EXAMPLE 1. *(1) Let $W = \mathbf{Z}_2$ and $\mathfrak{h} = \mathbf{C}$. Up to scalars, there is only one Dunkl operator and is given by*

$$D = \partial_x - \frac{c(1-s)}{x}.$$

We write s for the unique reflection in W, whose action is given by $(sf)(x) = f(-x)$.

(2) Let $W = S_n$ and $\mathfrak{h} = \mathbf{C}^n$, write $\{y_1, \ldots, y_n\}$ for a basis of $\mathfrak{h}$ with dual basis $\{x_1, \ldots, x_n\}$. In this case the set of reflections R consists of the transpositions $s_{i,j}$ with $i < j$. And c reduces to one parameter. Then the Dunkl operator for $y_i \in \mathfrak{h}$ is given by

$$D_{y_i} f = \partial_{x_i}(f) - c\left(\sum_{j \neq i} \frac{f - s_{i,j} f}{x_i - x_j}\right),$$

for $f \in \mathbf{C}[\mathfrak{h}] \cong \mathbf{C}[x_1, \ldots, x_n]$.

Using the fact that α_r generates the ideal of functions vanishing on the fixed space of r we have that the quotient $\dfrac{f - rf}{\alpha_r}$ is actually an element of $\mathbf{C}[\mathfrak{h}]$ and hence Dunkl operators D_y are well defined.

One of Dunkl's important results was that Dunkl operators commute with each other, which is not at all obvious from the formula. To prove the commutativity we follow Etingof's idea [**3**] instead of the original proof of Dunkl. The partial derivation introduced above satisfy useful properties. .

REMARK 1. *Note that, for $w \in W$ and $y \in \mathfrak{h}$, $w\partial_y w^{-1}$ is a derivation of $\mathbf{C}[\mathfrak{h}]$ satisfying $w\partial_y w^{-1}(x) = \partial_{wy} x$. Thus by uniqueness of ∂_y we have $w\partial_y w^{-1} = \partial_{wy}$. Analogously we can check $\partial_{y_1} - \partial_{y_2} = \partial_{y_1 - y_2}$ and $\partial_{ty} = t\partial_y$.*

LEMMA 1. *The Dunkl operators satisfy the following properties*

(1) $wD_y w^{-1} = D_{wy}$ *for* $w \in W, y \in \mathfrak{h}$,
(2) $[D_y, f] = \partial_y(f) - \sum_{r\in R} c_r \langle \alpha_r, y\rangle \dfrac{f - rf}{\alpha_r} r$ *for* $f \in \mathbf{C}[\mathfrak{h}]$,
(3) $D_{y_1} - D_{y_2} = D_{y_1 - y_2}$ *and* $D_{ty} = tD_y$, $y_1, y_2, y \in \mathfrak{h}$, $t \in \mathbf{C}$.

PROOF. (1) For the first part notice that $\mathrm{fix}(wrw^{-1}) = \ker(w\alpha_r)$. As $\mathrm{fix}(wrw^{-1}) = \ker(\alpha_{wrw^{-1}})$ and the definition of the Dunkl operators does not depend on the choice of α_r, we have

$$\ker(w\alpha_r) = \ker(\alpha_{wrw^{-1}}).$$

We compute $wD_{w^{-1}y} w^{-1}$ acting on $f \in \mathbf{C}[\mathfrak{h}]$.

$$\begin{aligned}
wD_{w^{-1}y} w^{-1}(f) &= w\left(D_{w^{-1}y}(w^{-1}f)\right) \\
&= w\left(\partial_{w^{-1}y}(w^{-1}f) - \sum_{r\in R} c_r \langle \alpha_r, w^{-1}y\rangle \frac{w^{-1}f - rw^{-1}f}{\alpha_r}\right) \\
&= w\partial_{w^{-1}y} w^{-1} f - \sum_{r\in R} c_r w\alpha_r(y) \frac{f - wrw^{-1}f}{w\alpha_r} \\
&= \partial_y(f) - \sum_{r\in R} c_r \alpha_{wrw^{-1}}(y) \frac{f - (wrw^{-1})f}{\alpha_{wrw^{-1}}}
\end{aligned}$$

As $c_r = c_{wrw^{-1}}$ we can substitute r by wrw^{-1} in the sum obtaining

$$wD_{w^{-1}y} w^{-1} f = \partial_y f - \sum_{r\in R} c_r \langle \alpha_r, y\rangle \frac{f - rf}{\alpha_r} = D_y f.$$

Finally the assertion follows by using this result with wy instead of y.

(2) The second part follows by direct computation of the Dunkl operator D_y acting on a product of two polynomial functions $f, g \in \mathbf{C}[\mathfrak{h}]$

$$\begin{aligned}
D_y(fg) &= \partial_y(fg) - \sum_{r\in R} \langle \alpha_r, y\rangle \frac{fg - rfg}{\alpha_r} \\
&= \partial_y(g)g + f\partial_y(g) - \sum_{r\in R} c_r \langle \alpha_r, y\rangle \frac{fg - rf\cdot rg}{\alpha_r}
\end{aligned}$$

$$= \partial_y(f)g + f\partial_y(g) - \sum_{r\in r}\langle\alpha_r, y\rangle\left(f\frac{g - rg}{\alpha_r} + \frac{f - rf}{\alpha_r}rg\right)$$
$$= \left(\partial_y(f) - \sum_{r\in R}\langle\alpha_r, y\rangle\frac{f - rf}{\alpha_r}r\right)g + fD_y(g)$$

This implies that $D_y f(g) - fD_y(g) = \left(\partial_y(f) - \sum_{r\in R}\langle\alpha_r, y\rangle\frac{f - rf}{\alpha_r}r\right)g$, and second part follows.

(3) The last part follows from Remark 1.

□

THEOREM 1.1 (Dunkl, C.). *The Dunkl operators D_y, $y\in\mathfrak{h}$ are pairwise commutative*

PROOF. Equivalently we will prove that

$$[D_{y_1}, D_{y_2}] = 0 \quad \text{for all } y_1, y_2 \in \mathfrak{h}.$$

Notice that for $x\in\mathfrak{h}^*$ we have $D_y(x) = x(y) - \sum_{r\in R} c_r\langle\alpha_r, y\rangle\langle x, \alpha_r^\vee\rangle$. From the definition (1.2) of Dunkl operators we know that the quantity $\langle\alpha_r, y\rangle\langle x, \alpha_r^\vee\rangle$ is indepedent of the choice of α_r and $\alpha_r^\vee$ satisfying such equation. Then, by Lemma 1

$$\begin{aligned}
[[D_{y_1}, x], D_{y_2}] &= [x(y_1) - \sum_{r\in R}\langle\alpha_r, y_1\rangle\langle x, \alpha_r^\vee\rangle r, D_{y_2}] \\
&= \sum_{r\in R} c_r\langle\alpha_r, y_1\rangle\langle x, \alpha_r^\vee\rangle(D_{y_2}r - rD_{y_2}) \\
&= \sum_{r\in R} c_r\langle\alpha_r, y_1\rangle\langle x, \alpha_r^\vee\rangle(D_{y_2}r - D_{ry_2}r) \\
&= \sum_{r\in R} c_r\langle\alpha_r, y_1\rangle\langle x, \alpha_r^\vee\rangle(D_{y_2} - D_{ry_2})r \\
&= \sum_{r\in R} c_r\langle\alpha_r, y_1\rangle\langle x, \alpha_r^\vee\rangle(D_{y_2 - ry_2})r \\
&= \sum_{r\in R} c_r\langle\alpha_r, y_1\rangle\langle x, \alpha_r^\vee\rangle D_{\langle\alpha_r, y_2\rangle\alpha_r^\vee}r \\
&= \sum_{r\in R} c_r\langle\alpha_r, y_1\rangle\langle x, \alpha_r^\vee\rangle\langle\alpha_r, y_2\rangle D_{\alpha_r^\vee}r.
\end{aligned}$$

Interchanging y_1 and y_2 gives $[[D_{y_2}, x], D_{y_1}] = \sum_{r\in R} c_r\langle\alpha_r, y_2\rangle\langle x, \alpha_r^\vee\rangle\langle\alpha_r, y_1\rangle D_{\alpha_r^\vee}r$. Using the Jacobi identity we have

$$[[D_{y_1}, D_{y_2}], x] = [[D_{y_1}, x], D_{y_2}] - [[D_{y_2}, x], D_{y_1}] = 0.$$

This implies that

$$[[D_{y_1}, D_{y_2}], f] = 0 \quad \text{for all } f \in \mathbf{C}[\mathfrak{h}],$$

hence

$$[D_{y_1}, D_{y_2}] = 0.$$

□

The Rational Cherednik algebra associated to $(W,\mathfrak{h})$ is the subalgebra $H_c = H_c(W,\mathfrak{h})$ of $\mathrm{End}(\mathbf{C}[\mathfrak{h}])$ generated by the group W, the ring $\mathbf{C}[\mathfrak{h}]$ (acting on itself by multiplication) and the commuting Dunkl operators $D_y,\ y\in\mathfrak{h}$, where the *deformation parameter c* is the tuple $(c_r)_{r\in R}$.

Notice that the rational Cherednik algebra has a similar behavior to what we would expect from the enveloping algebras of a Lie algebra. For instance, the rational Cherednik algebra has a version of the Poincaré-Birkhoff-Witt theorem (or PBW theorem), a result that in Lie algebras setup gives an explicit description of its universal enveloping algebra. The following version we give of PBW theorem for the rational Cherednik algebra will allow us to give a presentation of H_c given by generators and relations.

THEOREM 1.2 (PBW theorem for rational Cherednik Algebra). *The map*

$$\mathbf{C}[\mathfrak{h}]\otimes\mathbf{C}W\otimes\mathbf{C}[\mathfrak{h}^*]\xrightarrow{\sim} H_c$$

given by multiplication is an isomorphism of vector spaces.

This result is actually a consequence of the commutativity of the Dunkl operators, for a proof of this we refer to Etingof and Ma [**3**].

PROPOSITION 1. *The rational Cherednik algebra H_c may be presented as the algebra generated by the group algebra $\mathbf{C}W$, the commuting operators $y\in\mathfrak{h}$ and $x\in\mathfrak{h}^*$ subject to relations*

$$wxw^{-1}=w(x),\quad wyw^{-1}=w(y)$$

and

$$yx-xy=\langle x,y\rangle-\sum_{r\in R}c_r\langle\alpha_r,y\rangle\langle x,\alpha_r^\vee\rangle r \tag{1.3}$$

PROOF. Consider the following two maps

$$\begin{aligned} T(\mathfrak{h}^*\oplus\mathfrak{h}) &\to H_c \\ x &\mapsto x \text{ “multiplication by } x\text{"} \\ y &\mapsto D_y \end{aligned} \qquad \text{and} \qquad \begin{aligned} W &\to H_c \\ w &\mapsto w \text{ as operator in } H_c \end{aligned}$$

then the map

$$\begin{aligned} T(\mathfrak{h}^*\oplus\mathfrak{h})\rtimes W &\to H_c \\ f\otimes w &\mapsto fw \end{aligned}$$

is an algebra homomorphism. Let I be the ideal of $T(\mathfrak{h}^*\oplus\mathfrak{h})\rtimes W$ generated by

$$\begin{aligned} x_1x_2-x_2x_1, &\quad \forall x_1,x_2\in h^* \\ y_1y_2-y_2y_1, &\quad \forall y_1,y_2\in\mathfrak{h} \\ yx-xy-x(y)+\sum_{r\in R}c_r\langle\alpha_r,y\rangle\langle x,\alpha_r^\vee\rangle r, &\quad \forall x\in\mathfrak{h}^*, y\in\mathfrak{h} \end{aligned}$$

then, there exists a morphism

$$\left(T(\mathfrak{h}^*\oplus\mathfrak{h})\rtimes W\right)\big/ I\to H_c.$$

The natural map

$$\mathbf{C}[\mathfrak{h}] \otimes \mathbf{C}W \otimes \mathbf{C}[\mathfrak{h}^*] \twoheadrightarrow {T(\mathfrak{h}^* \oplus \mathfrak{h}) \rtimes W}\big/_{I}$$
$$f \otimes w \otimes g \mapsto \overline{f} \otimes \overline{w} \otimes \overline{g}$$

is a surjective vector spaces morphism, since we have enough relations in I to write each word in the right order. By PBW theorem the composition

$$\mathbf{C}[\mathfrak{h}] \otimes \mathbf{C}W \otimes \mathbf{C}[\mathfrak{h}^*] \twoheadrightarrow {T(\mathfrak{h}^* \oplus \mathfrak{h}) \rtimes W}\big/_{I} \to H_c$$

is an isomorphism, this implies that the second map is an isomorphism as well. □

EXAMPLE 2. *Using the previous proposition we can give explicit presentations for the corresponding Rational Cherednik algebras given in Example 1*

(1) $H_c(\mathbf{Z}_2, \mathbf{C})$ *is the algebra generated by variables* x, y, s *subject to*
- $s^2 = 1$
- $sx = -xs$
- $sy = -ys$
- $yx - xy = 1 - 2cs$

(2) $H_c(S_n, \mathbf{C}^n)$ *is the algebra generated by the symmetric group* S_n*, the commuting variables* $x_1, \dots, x_n$ *and commuting variables* $y_1, \dots, y_n$ *subject to relations*
- $wx_iw^{-1} = w_{w(i)}$ *and* $wy_iw^{-1} = y_{w(i)}$ *for* $w \in S_n$ *and* $1 \le i \le n$
- $y_ix_j - x_jy_i = cs_{i.j}$ *for* $1 \le i \neq j \le n$
- $y_ix_i - x_iy_i = 1 - c\sum_{j\neq i} s_{i,j}$.

1.1. Standard modules and the category $\mathcal{O}_c$. Let E be an irreducible representation of $\mathbf{C}W$. Define the *standard module* $\Delta_c(E)$ for H_c as the induced representation

$$\Delta_c(E) = \mathrm{Ind}_{\mathbf{C}[\mathfrak{h}^*]\rtimes W}^{H_c}(E),$$

where $\mathbf{C}[\mathfrak{h}^*] \rtimes W$ is the subalgebra of H_c generated by $\mathfrak{h}$ and W, and it acts on E by

$$ye = 0, \quad \forall y \in \mathfrak{h}, \forall e \in E.$$

By the PBW theorem, we have that

$$\Delta_c(E) \cong \mathbf{C}[\mathfrak{h}] \otimes E$$

as a $\mathbf{C}[\mathfrak{h}] \rtimes W$-module. These standard modules can be seen as analogs of the Verma modules for complex semisimple Lie algebras.

EXAMPLE 3. *The group* $\mathbf{Z}_2$ *has two irreducible (one-dimensional) representations, namely* $triv$ *and* sgn*, consequently there are two standard modules,* $\Delta_c(triv) \cong \mathbf{C}[x] \otimes triv$ *and* $\Delta_c(sgn) \cong \mathbf{C}[x] \otimes sgn$*. The action of* $\mathbf{C}[y]$ *on* $\Delta_c(E)$ *is given by*

$$y \cdot (x^i \otimes e) = \partial_x(x^i) \otimes e - c\frac{(x^i - (-x)^i)}{x} \otimes se \tag{1.4}$$

$$= \begin{cases} (i - c(1-(-1)^i))x^{i-1} \otimes e, & \text{if } E = triv \\ (i + c(1-(-1)^i))x^{i-1} \otimes e, & \text{if } E = sgn \end{cases} \tag{1.5}$$

Let $\mathcal{O}_c$ be the Serre subcategory of H_c-mod generated by the standard modules. In the following proposition we give an explicit characterization of the objects of the category $\mathcal{O}_c$.

PROPOSITION 2. *The category $\mathcal{O}_c$ consists of finitely generated H_c-modules M that are locally nilpotent for the action of each $y \in \mathfrak{h}$, in the sense that for each $m \in M$, there is some positive integer n with $y^n \cdot m = 0$ for all $y \in \mathfrak{h}$.*

For a proof of this proposition we refer to Ginzburg, Guay, Opdam an Rouquier [**5**]. We will introduce a parametrization that simplifies the expression of many numbers arising naturally in the study of the Cherednik algebra, in particular we will use it later to give an explicit presentation for the rational Cherednik algebra attached to the group $G(\ell, 1, n)$, which is the object of study of this work.

Let $\mathcal{A}$ be the set of hyperplanes H in $\mathfrak{h}$, where $H = \text{fix}(r)$ for some $r \in R$. For each $H \in \mathcal{A}$ we choose $\alpha_H \in \mathfrak{h}^*$ such that $H = \ker(\alpha_H)$. The subgroup $W_H = \{w \in W \mid w(v) = v \text{ if } v \in H\}$ is a cyclic subgroup and we write $W_H^\vee$ for its character group.

For $\chi \in W_H^\vee$ the corresponding primitive idempotent is given by

$$e_H, \chi = \frac{1}{n_H} \sum_{w \in W_H} \chi(w^{-1}) w \in \mathbf{C}W_H, \tag{1.6}$$

where n_H denotes the size of W_H. Define c_H, χ by

$$c_{H,\chi} n_H = \sum_{r \in W_H - \{1\}} c_r (1 - \chi(r)). \tag{1.7}$$

PROPOSITION 3. *With this reparametrization for each $y \in \mathfrak{h}$ the equation* (1.2) *for the Dunkl operator D_y becomes*

$$D_y(f) = \partial_y(f) - \sum_{H \in \mathcal{A}} \frac{\langle \alpha_H, y \rangle}{\alpha_H} \sum_{\chi \in W_H - \{1\}} c_{H,\chi} n_H e_{H,\chi} f, \quad f \in \mathbf{C}[\mathfrak{h}]. \tag{1.8}$$

PROOF. This is obtained by straightforward computation, using the following formulas.

$$\sum_{\chi \in W_H^\vee} e_{H,\chi} = 1 \tag{1.9}$$

which comes from the orthogonality relations in $W_H^\vee$, and

$$w = \sum_{\chi \in W_H^\vee} \chi(w) e_{H,\chi}, \quad \text{for } w \in W_H, \tag{1.10}$$

which follows from the relation arising since $w \in W$ and $e_{H,\chi}$, for $\chi \in W_H^\vee$ are basis of the group algebra $\mathbf{C}W_H$. Recalling that $\alpha_r \in \mathfrak{h}^*$ is chosen in such a way that $\ker(\alpha_r) = \text{fix}(r)$, then for $H \in \mathcal{A}$ with $\text{fix}(r) = H$ (or equivalently $r \in W_H$) we have $\lambda \alpha_H = \alpha_r$ for some scalar $\lambda \in \mathbf{C}^\times$ and therefore

$$\frac{\langle \alpha_r, y \rangle}{\alpha_r} = \frac{\langle \alpha_H, y \rangle}{\alpha_H} \qquad \text{for all } r \in W_H.$$

Then for each $y \in \mathfrak{h}$ and $f \in \mathbf{C}[\mathfrak{h}]$ we have

$$\begin{aligned}
D_y(f) &= \partial_y(f) - \sum_{r\in R} c_r \langle \alpha_r, y\rangle \frac{f - rf}{\alpha_r} \\
&= \partial_y(f) - \sum_{H\in\mathcal{A}} \sum_{r\in W_H - \{1\}} c_r \langle \alpha_r, y\rangle \frac{f - rf}{\alpha_r} \\
&= \partial_y(f) - \sum_{H\in\mathcal{A}} \frac{\langle \alpha_H, y\rangle}{\alpha_H} \left(\sum_{r\in W_H-\{1\}} c_r(1-r)\right) f
\end{aligned}$$

From equations (1.9) and (1.10) we have

$$\begin{aligned}
\sum_{r\in W_H-\{1\}} c_r(1-r) &= \sum_{r\in W_H-\{1\}} c_r \left(\sum_{\chi\in W_H^\vee} e_{H,\chi} - \sum_{\chi\in W_H^\vee} \chi(r) e_{H,\chi}\right) \\
&= \sum_{r\in W_H\{1\}} c_r \left(\sum_{\chi\in W_H^\vee-\{1\}} (1-\chi(r)) e_{H,\chi}\right) \\
&= \sum_{\chi\in W_H^\vee-\{1\}} \left(\sum_{r\in W_H-\{1\}} c_r(1-\chi(r))\right) e_{H,\chi} \\
&= \sum_{\chi\in W_H^\vee-\{1\}} c_{H,\chi} n_H e_{H,\chi}
\end{aligned}$$

which finishes the proof. □

PROPOSITION 4. *In terms of the parameters $c_{H,\chi}$ the equation* (1.3) *is*

$$yx - xy = \langle x, y\rangle - \sum_{H\in\mathcal{A}} \frac{\langle \alpha_H, y\rangle\langle x, \alpha_H^\vee\rangle}{\langle \alpha_H, \alpha_H^\vee\rangle} \sum_{\chi\in W_H^\vee} (c_{H,\chi\otimes\det^{-1}} - c_{H,\chi}) n_H e_{H,\chi} \tag{1.11}$$

PROOF. Note that $\frac{\langle \alpha_r, y\rangle\langle x, \alpha_r^\vee\rangle}{\langle \alpha_r, \alpha_r^\vee\rangle} = \frac{\langle \alpha_H, y\rangle\langle x, \alpha_H^\vee\rangle}{\langle \alpha_H, \alpha_H^\vee\rangle}$ for $r \in W_H$, then we have

$$\begin{aligned}
yx - xy &= \langle x, y\rangle - \sum_{r\in R} c_r \langle \alpha_r, y\rangle\langle x, \alpha_r^\vee\rangle r \\
&= \langle x, y\rangle - \sum_{H\in\mathcal{A}} \sum_{r\in W_H-\{1\}} c_r \langle \alpha_r, y\rangle\langle x, \alpha_r^\vee\rangle r \frac{\langle \alpha_r, \alpha_r^\vee\rangle}{\langle \alpha_r, \alpha_r^\vee\rangle} \\
&= \langle x, y\rangle - \sum_{H\in\mathcal{A}} \sum_{r\in W_H-\{1\}} c_r \frac{\langle \alpha_H, y\rangle\langle x, \alpha_H^\vee\rangle}{\langle \alpha_H, \alpha_H^\vee\rangle} r \langle \alpha_r, \alpha_r^\vee\rangle \\
&= \langle x, y\rangle - \sum_{H\in\mathcal{A}} \frac{\langle \alpha_H, y\rangle\langle x, \alpha_H^\vee\rangle}{\langle \alpha_H, \alpha_H^\vee\rangle} \sum_{r\in W_H-\{1\}} \langle \alpha_r, \alpha_r^\vee\rangle c_r r.
\end{aligned}$$

To finish the proof we use equation (1.10) and $\langle \alpha_r, \alpha_r^\vee\rangle = 1 - \det_{\mathfrak{h}} r^{-1}$, then

$$\sum_{r\in W_H-\{1\}} \langle \alpha_r, \alpha_r^\vee\rangle c_r r = \sum_{r\in W_H-\{1\}} \langle \alpha_r, \alpha_r^\vee\rangle c_r \sum_{\chi\in W_H^\vee} \chi(r) e_{H,\chi}$$

$$= \sum_{\chi \in W_H^\vee} \sum_{r \in W_H - \{1\}} (1 - \det r^{-1}) c_r \chi(r) e_{H,\chi}$$
$$= \sum_{\chi \in W_H^\vee} \left(\sum_{r \in W_H - \{1\}} c_r (1 - \chi \otimes \det{}^{-1} r) - \sum_{r \in W_H - \{1\}} c_r (1 - \chi(r)) \right) e_{H,\chi}$$
$$= \sum_{\chi \in W_H^\vee} (c_{H,\chi \otimes \det^{-1}} - c_{H,\chi}) n_n e_{H,\chi}$$

$\square$

1.2. The Fourier transform and the contravariant form.

PROPOSITION 5. *There exists a conjugate linear isomorphism $f : \mathfrak{h} \to \mathfrak{h}^*$ satisfying*

$$f(w(y)) = w(f(y)), \quad \forall w \in W, y \in \mathfrak{h}.$$

If W acts irreducibly on $\mathfrak{h}$ then such isomorphism is unique up to scalars.

PROOF. Since $W \subseteq GL(\mathfrak{h})$ is finite, then there exists a W-invariant positive definite hermitian form on $\mathfrak{h}$, we denote such form by $(\cdot,\cdot)$. Then the function f given by $f(y) = (\cdot, y)$ is the desired isomorphism. Uniqueness holds by Schur's lemma. $\square$

We fix an isomorphism as above, and we denote it by conjugation. By abuse we use the same notation for the inverse, then $\overline{y} \in \mathfrak{h}^*$ for $y \in \mathfrak{h}$, $\overline{x} \in \mathfrak{h}$ for $x \in \mathfrak{h}^*$ and $\overline{\overline{y}} = y$.

THEOREM 1.3. *Suppose c_r is real, in the sense that $c_r = \overline{c_{r^{-1}}}$, for all $r \in R$. Then there exists a unique conjugate linear antiautomorphism*

$$H_c \to H_c$$
$$h \mapsto \overline{h}$$

extending the maps

$$\mathfrak{h} \rightleftarrows \mathfrak{h}^*$$
$$y \mapsto \overline{y}$$
$$\overline{x} \leftarrow\!\shortmid x$$

as above, such that $\overline{w} = w^{-1}$, for all $w \in W$.

PROOF. Write $\overline{A}$ for he $\mathbf{C}$-algebra A with the conjugate $\mathbf{C}$-vector space structure

$$\alpha \cdot a = \overline{\alpha} a, \quad \forall a \in A \text{ and } \alpha \in \mathbf{C}.$$

The maps

$$\mathfrak{h} \rightleftarrows \mathfrak{h}^* \qquad \text{and} \qquad W \to W$$
$$y \mapsto \overline{y} \qquad\qquad w \mapsto \overline{w} = w^{-1}$$
$$\overline{x} \leftarrow\!\shortmid x$$

fit together to give

$$T(\mathfrak{h}^* \oplus \mathfrak{h}) \to \overline{T(\mathfrak{h}^* \oplus \mathfrak{h})}$$
$$x \mapsto \overline{x}$$
$$y \mapsto \overline{y}$$

which induces

$$\begin{array}{ccc} T(\mathfrak{h}^* \oplus \mathfrak{h}) \rtimes W & \longrightarrow & \overline{T(\mathfrak{h}^* \oplus \mathfrak{h}) \rtimes W} \\ \downarrow & & \downarrow \\ H_c & \dashrightarrow & \overline{H_c} \end{array}$$

Using the presentation of H_c given by Proposition 1, it remains to check that the following relations hold in $\overline{H_c}$

$$\overline{x_1}\,\overline{x_2} = \overline{x_2}\,\overline{x_1} \quad \forall x_1, x_2 \in h^*$$
$$\overline{y_1}\,\overline{y_2} = \overline{y_2}\,\overline{y_1}, \quad \forall y_1, y_2 \in \mathfrak{h}$$
$$\overline{y}\,\overline{x} - \overline{x}\,\overline{y} = \overline{y}(\overline{x}) + \sum_{r \in R} c_r \langle \alpha_r, \overline{x} \rangle \langle \overline{y}, \alpha_r^\vee \rangle r, \quad \forall x \in \mathfrak{h}^*, y \in \mathfrak{h}.$$

First and second relations are immediate since the morphism constructed is linear. To check the last relation we need to prove that $\overline{\langle x, y \rangle} = \langle \overline{y}, \overline{x} \rangle$, in fact, by the previous proposition we can consider $y_1 \in \mathfrak{h}$ to be the image of $x \in \mathfrak{h}^*$, thus $\overline{x} = y_1$ and $\overline{y_1} = x$. Then $x = (\cdot, y_1)$, and

$$\overline{\langle x, y \rangle} = \overline{x(y)} = \overline{(y, y_1)} = (y_1, y) = (\overline{x}, y) = \overline{y}(\overline{x}) = \langle \overline{y}, \overline{x} \rangle$$

This implies that $r^{-1}(x) = x - \langle x, \overline{\alpha_r} \rangle \overline{\alpha_r^\vee}$ for all $x \in \mathfrak{h}^*$, which finishes the proof. □

We abuse notation and also denote by conjugation to this antiautomorphism, which we call *the Fourier transform.*

EXAMPLE 4. *In $H_c(\mathbf{Z}_2, \mathbf{C})$ the Fourier transform is given by $\overline{x} = y$, $\overline{y} = x$, $\overline{s} = s$ and complex conjugation on scalars.*

Fix a positive definite W-equivariant hermitian form $(\cdot, \cdot)$ on E. Each standard module has a *contravariant form* $\langle \cdot, \cdot \rangle_c$ defined by the formula

$$\langle G_1 \otimes e_1, G_2 \otimes e_2 \rangle_c = (e_1, (\overline{G_1} \cdot G_2 \otimes e_2)(0)), \tag{1.12}$$

where we evaluate at zero by considering each element of $\mathbf{C}[\mathfrak{h}] \otimes E$ as a function on $\mathfrak{h}$ with values in E.

PROPOSITION 6. *The form is conjugate linear in the first variable, linear in the second variable and satisfies*

$$\langle G_1, G_2 \rangle_c = \overline{\langle G_2, G_1 \rangle_c}. \tag{1.13}$$

Moreover, for all $h \in H_c$ and $G_1, G_2 \in \Delta_c(E)$ we have

$$\langle h \cdot G_1, G_2 \rangle_c = \langle G_1, \overline{h} \cdot G_2 \rangle_c. \tag{1.14}$$

The property (1.13) is a “baby case" of a symmetry in the double affine Hecke algebra, wich is connected to the evaluation and norm formulas for Macdonald polynomials.

PROOF. To prove (1.13) note that

$$\overline{G_1}G_2 = \sum_{w\in W} c_w w \tag{1.15}$$

where $c_w = c_w^0 + \sum_{\deg fg>0} f(x)g(y)$. Hence $\overline{G_1}\cdot G_2 \otimes e_2(0) = \sum_{w\in W} c_w^0 w\otimes e_2$.
Using the Fourier transform in (1.15) we have that $\overline{G_2}G_1\otimes e_1(0) = \sum_{w\in W} \overline{c_w^0} w^{-1}\otimes e_1$. Then

$$\begin{aligned}
\langle G_1\otimes e_1, G_2\otimes e_2\rangle_c &\stackrel{def}{=} \left(e_1, (\overline{G_1}G_2\otimes e_2)(0)\right)\\
&= \left(e_1, \sum_{w\in W} c_w^0 w\otimes e_2\right)\\
&= \left(\sum_{w\in W} \overline{c_w^0} w^{-1}\otimes e_1, e_2\right)\\
&= \left((\overline{G_2}G_1\otimes e_1)(0), e_2\right)\\
&= \overline{(e_2, (\overline{G_2}G_1\otimes e_1)(0))}\\
&= \overline{\langle G_2\otimes e_2, G_1\otimes e_1\rangle_c}.
\end{aligned}$$

Property (1.14) can be checked on generators straightforward from the definition, which is enough to prove the result and finishes the proof. □

EXAMPLE 5. *The contravariant form is determined by the action of Dunkl operators in the standard modules, in* $H_c(\mathbf{Z}_2,\mathbf{C})$ *the contravariant form is given by* $\langle x^j\otimes e, x^i\otimes e\rangle_c = 0$ *if* $i\neq j$ *and*

$$\langle x^i\otimes e, x^i\otimes e\rangle_c = \begin{cases} \prod_{t=1}^i (t-(1-(-1)^t)c), & \text{if } E = triv\\ \prod_{t=1}^i (t+(1-(-1)^t)c), & \text{if } E = sgn\end{cases} \tag{1.16}$$

1.3. The grading element.

DEFINITION 1. *Let* $\{y_1,\dots,y_n\}$ *be a basis of* $\mathfrak{h}$ *and* $\{x_1,\dots,x_2\}$ *its corresponding dual basis. We define* the Euler element *as the operator*

$$\mathrm{eu} = \sum_{i=1}^n x_i\partial_{y_i}.$$

Using the definition of the Dunkl operator, the Euler element can be written as $\mathrm{eu} = \sum_{i=1}^n x_i D_{y_i} + \sum_{r\in R} c_r(1-r)$, in this way we have that $\mathrm{eu}\in H_c$. Note that since the Fourier transform exchanges multiplication with differentiation, in the same way this occur in a Weyl algebra, the Euler element is stable under de Fourier transform, i.e. $\overline{\mathrm{eu}} = \mathrm{eu}$.

PROPOSITION 7. *The Euler element satisfies the following properties*

(1) $[eu, x] = x$, *for all* $x\in\mathfrak{h}^*$
(2) $[\mathrm{eu}, y] = -y$, *for all* $y\in\mathfrak{h}$

(3) $[\mathrm{eu}, w] = 0$*, for all* $w \in W$*.*

PROOF. (1) We compute directly, using that $\mathrm{eu} = \sum_{i=1}^n x_i D_{y_i} + \sum_{r\in R} c_r(1-r)$

$$\begin{aligned}
[\mathrm{eu}, x] &= \left[\sum_{i=1}^n x_i D_{y_i} + \sum_{r\in R} c_r(1-r), x\right] \\
&= \left(\sum_{i=1}^n x_i D_{y_i} + \sum_{r\in R} c_r(1-r)\right) x - x\left(\sum_{i=1}^n x_i D_{y_i} + \sum_{r\in R} c_r(1-r)\right) \\
&= \sum_{i=1}^n x_i (D_{y_i} x - x D_{y_i}) + \sum_{r\in R} (xr - rx) \\
&= \sum_{i=1}^n x_i [D_{y_i}, x] - + \sum_{r\in R} (xr - rx) \\
&= \sum_{i=1}^n x_i \left(x(y_i) - \sum_{r\in R} c_r \langle \alpha_r, y_i\rangle \langle x, \alpha_r^\vee\rangle r\right) - + \sum_{r\in R} (xr - rx) \\
&= \sum_{i=1}^n x_i x(y_i) - \sum_{r\in R} c_r \langle x, \alpha_r^\vee\rangle \left(\sum_{i=1}^n x_i \alpha_r(y_i)\right) r + \sum_{r\in R} (xr - rx) \\
&= x + \sum_{r\in R} c_r((x - \langle x, \alpha_r^\vee\rangle \alpha_r) r - rx) \\
&= x + \sum_{r\in R} c_r(r(x) r - rx) \\
&= x + \sum_{r\in R} c_r(rx - rx) \\
&= x
\end{aligned}$$

(2) Using that $\overline{y} \in \mathfrak{h}^*$ and the Fourier transform used in part (a) we have

$$\begin{aligned}
\overline{[\mathrm{eu}, y]} &= \overline{y\mathrm{eu} - \mathrm{eu}y} = \overline{\mathrm{eu}y} - \overline{y\mathrm{eu}} = \overline{y}\,\overline{\mathrm{eu}} - \overline{\mathrm{eu}}\,\overline{y} = \overline{y}\,\mathrm{eu} - \mathrm{eu}\,\overline{y} \\
&= [\overline{y}, \mathrm{eu}] = -[\mathrm{eu}, \overline{y}] = -\overline{y}
\end{aligned}$$

(3) We prove that $w\,\mathrm{eu}\,w^{-1} = \mathrm{eu}$ which is equivalent to $[\mathrm{eu}, w] = 0$, in fact

$$\begin{aligned}
w\,\mathrm{eu}\,w^{-1} &= w\left(\sum_{i=1}^n x_i D_{y_i} + \sum_{r\in R} c_r(1-r)\right) w^{-1} \\
&= \sum_{i=1}^n w x_i D_{y_i} w^{-1} + \sum_{r\in R} c_r w(1-r) w^{-1} \\
&= \sum_{i=1}^n w x_i w^{-1} w D_{y_i} w^{-1} + \sum_{r\in R} c_r(1 - w r w^{-1}) \\
&= \sum_{i=1}^n w(x_i) D_{w(y_i)} + \sum_{r\in R} c_r(1 - w r w^{-1})
\end{aligned}$$

equality holds since for $w \in GL(\mathfrak{h})$ the set $\{w(y_i) : 1 \le i \le n\}$ is another basis of $\mathfrak{h}$ with dual basis $\{w(x_i) : 1 \le i \le n\}$ and that $c_r = c_{wrw^{-1}}$.

□

By Schur's lemma, the Euler element acts on E by a constant we denote by c_E. Since Dunkl operators act by 0 on E, we have that $\mathrm{eu} = \sum_{r\in R} c_r(1-r)$ on E. In terms of idempotents defined in subsection 1.1 $\mathrm{eu} = \sum_{H\in\mathcal{A}} \sum_{\chi \in W_H^\vee - \{1\}} c_{H,\chi} n_H e_{H,\chi}$ on E.

EXAMPLE 6. *For* $H_c(\mathbf{Z}_2, \mathbf{C})$ *we have that* $\mathrm{eu} = xd_x = xD + c(1-s)$ *and*

$$eu(1 \otimes e) = (xD + c(1-s))(1 \otimes e) = c \otimes e - c \otimes se$$

then $c_E = \begin{cases} 0, & \text{if } E = triv \\ 2c, & \text{if } E = sgn \end{cases}$

For an element in $\Delta_c(E)$ we have $\mathrm{eu}(x \otimes e) = (1 + c_E)(x \otimes e)$, i.e. the Euler element increases in 1 the grade. Inductively, for $f \in \mathbf{C}[\mathfrak{h}]$ homogeneous of degree d, we have

$$\mathrm{eu} f = (c_E + d) f.$$

This allow us to define the part of grade d on $\Delta_c(E)$ by

$$\Delta_c(E)_{c_E + d} = \left\{ m \in \Delta_c(E) \mid \mathrm{eu}\, m = dm \right\}$$

and then the grading on $\Delta_c(E)$ is given by

$$\Delta_c(E) = \bigoplus_{d \in \mathbf{Z}_{\geq 0}} \Delta_c(E)_{c_E + d}.$$

Analogously the Euler element decreases in 1 the degree of an element $y \in \mathfrak{h}$, with this and properties in Proposition 7 the eigenspace decomposition of H_c under the action of $[\mathrm{eu}, \cdot]$ gives a grading on H_c, where W is in degree 0, $\mathfrak{h}^*$ is in degree 1 and $\mathfrak{h}$ in degree -1. Hence each object M of $\mathcal{O}_c$ is $\mathbf{C}$-graded with finite dimensional weight spaces $M = \bigoplus_{d \in \mathbf{C}} M_d$, where

$$M_d = \{ m \in M \mid (\mathrm{eu} - d)^N \cdot m = 0 \text{ for } N \text{ sufficiently large} \}.$$

Moreover, $M_d = 0$ unless $d \in c_E + \mathbf{Z}_{\geq 0}$ for some irreducible representation E of W, since this is valid for standard modules and the category $\mathcal{O}_c$ is the Serre subcategory generated by them.

1.4. Simple and unitary representations.

PROPOSITION 8. *The standard module* $\Delta_c(E)$ *has a unique maximal submodule.*

PROOF. Let $M \subset \Delta_c(E)$ be a proper submodule, since $\Delta_c(E) = \bigoplus_{d \in \mathbf{Z}_{\geq 0}} \Delta(E)_{c_E + d}$, we have

$$M = \bigoplus_{d \in \mathbf{Z}_{\geq 0}} \left(M \cap \Delta_c(E)_{c_e + d} \right).$$

By hypothesis we have that $M \neq \Delta_c(E)$, which implies that $M \cap \Delta_c(E)_{c_E} = 0$. Then

$$M \subset \bigoplus_{d \in \mathbf{Z}_{> 0}} \Delta_c(E)_{c_E + d},$$

and therefore

$$\sum_{M \subsetneq \Delta_c(E)} M$$

is the unique maximal submodule of $\Delta_c(E)$. $\square$

Note that form the previous proposition we have that $Rad(\Delta_c(E)) = \sum_{M \subsetneq \Delta_c(E)} M$ and then $\mathrm{Rad}(\langle\cdot,\cdot\rangle_c) = \mathrm{Rad}(\Delta_c(E))$. From what has been worked so far we define $L_c(E)$ to be the unique irreducible quotient of $\Delta_c(E)$, i.e.

$$L_c(E) = \Delta_c(E)/Rad(\langle\cdot,\cdot\rangle_c) \tag{1.17}$$

The contravariant form descends to this quotient and it is non degenerate, and we say that $L_c(E)$ is a *unitary representation* if the contravariant form in $L_c(E)$ is positive definite.

COROLLARY 1. *Let L be an irreducible H_c-module in $\mathcal{O}_c$, then there exists $E \in \mathrm{Irr}\,\mathbf{C}W$ such that $L \cong L(E)$.*

PROOF. Let M be a H_c-module in $\mathcal{O}_c$, define

$$H^0(\mathfrak{h}, M) := \{m \in M \mid y \cdot m = 0, \forall y \in \mathfrak{h}\}$$

which is nonzero since M is locally nilpotent for the action of each $y \in \mathfrak{h}$. Then for an irreducible $\mathbf{C}W$-module E there exists a nonzero map $E \to H^0(\mathfrak{h}, M)$ and by Frobenius reciprocity we have a nonzero map $\Delta(E) \to M$. This map is surjective when $M = L$ is irreducible, therefore $L \cong L(E)$. □

By the previous proposition we can establish a very useful correspondence between H_c-modules and $\mathbf{C}W$-modules given by the map

$$\begin{aligned} Irr\mathbf{C}W &\to Irr\mathcal{O}_c \\ E &\mapsto L_c(E). \end{aligned}$$

EXAMPLE 7. *For $H_c(\mathbf{Z}_2, \mathbf{C})$ the contravariant form $\langle\cdot,\cdot\rangle_c$ is non-degenerate and $L_c(E) = \Delta_c(E)$, unless $c \in \frac{1}{2} + \mathbf{Z}$. In the other case, the irreducible module depends on the representation E as follows*

- $\underline{E = triv}$:
  If $c \in \frac{1}{2} + \mathbf{Z}_{\geq 0}$ then $\mathrm{Rad}\langle\cdot,\cdot\rangle_c$ is spanned by $\{x^i \otimes e \mid i \geq 2c\}$. Then $L_c(triv)$ is spanned by $\{x^i \otimes e \mid 0 \leq i \leq 2c-1\}$.
- $\underline{E = sgn}$:
  If $c \in \frac{1}{2} + \mathbf{Z}_{<0}$ then $\mathrm{Rad}\langle\cdot,\cdot\rangle_c$ is spanned by $\{x^i \otimes e \mid i \geq -2c\}$. Then $L_c(sgn)$ is spanned by $\{x^i \otimes e \mid 0 \leq i \leq -2c-1\}$.

1.5. Characters. For $E \in \mathrm{Irr}(\mathbf{C}W)$ we define two types of characters for the irreducible H_c-module $L_c(E)$. The *graded character* of $L_c(E)$ is given by

$$\mathrm{ch}(L_c(E)) = \sum_d [L_c(E)_d]\, t^d \tag{1.18}$$

where $[L_c(E)] \in K(\mathbf{C}W)$ is an object of the Grothendieck group of $\mathrm{Irr}(\mathbf{C}W)$ and t is a formal variable.

The *Kazhdan-Lusztig* character of $L_c(E)$ is given by

$$\mathrm{ch}(L_c(E)) = \sum_i \dim\left(\mathrm{Ext}^i(\Delta_c(F)_i, L_c(E))\right)[F]q^i$$

where q is a formal variable.

EXAMPLE 8. *In $H_c(\mathbf{Z}_2,\mathbf{C})$, we compute the graded character. In this case $K(\mathbf{C}W) = \{[triv],[sgn]\}$ then*

(1) $\underline{\mathrm{ch}(L_c(triv))}$:
- *If $c \notin \frac{1}{2}+\mathbf{Z}_{\geq 0}$ then*

$$L_c(triv)_d = \Delta_c(triv)_d = \{m \in \Delta_c(triv) \mid \mathrm{eu}\cdot m = dm\}$$

which is spanned by $\{x^d \otimes e\}$, where $e \in E$. Note that

$$s\cdot(x^d \otimes e) = (s(x^d)\otimes se) = (-1)^d x^d \otimes e = \begin{cases} x^d \otimes e & \text{if } d \text{ is even} \\ -x^d\otimes e & \text{if } d \text{ is odd}\end{cases}$$

then $[(triv)_d] = \begin{cases} [triv] & \text{if } d \text{ is even} \\ [sgn] & \text{if } d \text{ is odd.}\end{cases}$

- *If $c \in \frac{1}{2}+\mathbf{Z}_{\geq 0}$ we have*

$$L_c(triv)_d = \{m \in L_c(triv) \mid (eu-d)^N\cdot m = 0,\ \text{for some } N \text{ sufficiently large}\}.$$

In this case we have that $L_c(triv)_d$ is spanned by $\{x^d\otimes e\}$ if $d<2c$ and $\{0\}$ in other case. Then

$$[(triv)_d] = \begin{cases} [triv] & \text{if } d<2c \text{ is even} \\ [sgn] & \text{if } d \text{ is odd} \\ 0 & \text{if } d\geq 2c.\end{cases}$$

(2) $\underline{\mathrm{ch}(L_c(sgn))}$:
- *If $c \notin \frac{1}{2}-\mathbf{Z}_{\geq 0}$, since $c_{sgn} = 2c$ we have that $L_c(sgn)_d$ is spanned by $\{x^{d-2c}\otimes e\}$ if $d-2c \in \mathbf{Z}$, then*

$$[(sgn)_d] = \begin{cases} [triv] & \text{if } d-2c \text{ is odd} \\ [sgn] & \text{if } d-2c \text{ is even} \\ 0 & \text{if } d-2c \notin \mathbf{Z}.\end{cases}$$

- *If $c \in \frac{1}{2}-\mathbf{Z}_{\geq 0}$ then $L_c(sgn)$ is spanned by $\{x^{d-2c}\otimes e\}$ as long as $2c\leq d\leq 0$ and then*

$$[(sgn)_d] = \begin{cases} [triv] & \text{if } d-2c \text{ is odd} \\ [sgn] & \text{if } d-2c \text{ is even} \\ 0 & \text{if } d\geq 0 \text{ or } d<2c.\end{cases}$$

1.6. Duality. For a finite dimensional $\mathbf{C}$-vector space V, we write

$$V^\vee = \{f : V \to \mathbf{C} \mid f(av_1 + bv_2) = \overline{a} f(v_1) + \overline{b} f(v_2), \text{ for all } v_1, v_2 \in V \text{ and } a, b \in \mathbf{C}\}$$

for the conjugate linear dual of V. This defines an exact contravariant functor from $\mathcal{O}_c$ to itself, which is conjugate-linear on Hom sets. Moreover it is an equivalence and an involution, then the dual of a projective object in $\mathcal{O}_c$ is injective, and vice versa. Furthermore, simple objects remain simple under this duality.

Since our H_c-modules are graded we define a compatible grading for the dual of an object M of $\mathcal{O}_c$ with grading $M = \bigoplus M_d$ in the following way

$$M^\vee = \bigoplus M_d^\vee$$

Given $E \in \mathrm{Irr}(\mathbf{C}W)$, we define the *costandard module* $\nabla_c(E)$ to be the dual of the standard module $\Delta_c(E)$.

THEOREM 1.4 (BGG reciprocity). *For standard and costandard objects* $\Delta(E)$ *and* $\nabla(E)$, *respectively, with projective cover* $P(E)$ *and injective envelope* $I(E)$ *we have the following reciprocity formulas*

$$[\Delta(E) : L(F)] = [I(F) : \nabla(E)] \quad \textit{and} \quad [\nabla(E) : L(F)] = [P(F) : \Delta(E)] \tag{1.19}$$

PROOF. Since each layer on a good filtration of $\Delta(E)$ is autodual, we have that the number $[\Delta(E) : L(F)]$ of times $L(F)$ appears as a section of $G_j(F)/G_{j-1}(F)$ in a good filtration of $\Delta(E)$ equals the number of times that $L(F)$ appears as a section in a good filtration of $\nabla(E)$, i.e $[\Delta(E) : L(F)] = [\nabla(E) : L(F)]$ then

$$[\Delta(E) : L(F)] = [\nabla(E) : L(F)] \tag{1.20}$$

$$= \dim \mathrm{Hom}(P(F), \nabla(E)) \tag{1.21}$$

by duality we have $\dim \mathrm{Hom}(P(F), \nabla(E)) = \dim \mathrm{Hom}(\Delta(E), I(F)) = [I(F) : \nabla(E)]$, obtaining the first equality. From (1.21) and using duality again we obtain the second equality. □

2. Highest weight categories

We recall some concepts and definitions used by Cline, Parshal and Scott ([**12**]) about *highest weight categories.* Let $\mathscr{C}$ be an abelian category and let k be a field.

DEFINITION 2. *A* composition series *of an object* A *of* $\mathscr{C}$ *is a sequence of subobjects*

$$A = X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n = 0$$

such that each quotient object X_i/X_{i+1} *is simple (for* $0 \le i \le n$*).*

If A *has a composition series, the integer* n *only depends on* A *and its called the* length *of* A.

Attached to the previous definition, we have that a *composition factor* S of an object A of $\mathscr{C}$ is a composition factor of a subobject of finite length. And the *multiplicity* (possibly infinite) of S in A is

$$[A:S] = \sup\{[B:S] \mid B \text{ is a subobject of finite length of } A\}$$

We say that a poset Λ is *interval-finite* if for every $\mu \leqq \lambda$ in Λ, the "interval"

$$[\mu, \lambda] = \{\tau \in \Lambda \mid \mu \leqq \tau \leqq \lambda\}$$

is finite.

DEFINITION 3. *A* highest weight category *is a k-linear category $\mathscr{C}$ satisfying the following conditions*

- *Is locally artinian (this means that $\mathscr{C}$ admits arbitrary directed unions of subobjects and if every object is a union of its subobjects of finite length)*
- *Has enough injectives*
- *(Grothendieck condition) For a subobject B and a directed family of subobjects $\{A_\alpha\}$ of an object X*

$$B \cap (\bigcup A_\alpha) = \bigcup (B \cap A_\alpha).$$

and such that there is an integral-finite poset Λ (whose elements are called weights *of $\mathscr{C}$) satisfying the following conditions*

(1) *There is a complete collection $\{L(\lambda)\}_{\lambda \in \Lambda}$ of non-isomorphic simple objects of $\mathscr{C}$ indexed by the set Λ.*
(2) *There is a collection $\{\nabla(\lambda)\}_{\lambda \in \Lambda}$ of objects on $\mathscr{C}$ and, for each λ an embedding $L(\lambda) \subset \nabla(\lambda)$ such that the composition factors $L(\mu)$ of $\nabla(\lambda)/L(\lambda)$ satisfy $\mu < \lambda$. For $\lambda, \mu \in \Lambda$ we have $\dim_k \mathrm{Hom}_{\mathscr{C}}(\nabla(\lambda), \nabla(\mu))$ and $[\nabla(\lambda) : L(\mu)]$ are finite.*
(3) *Each simple object $L(\lambda)$ has an injective envelope $I(\lambda)$ in $\mathscr{C}$. Also, $I(\lambda)$ has a "good filtration" which begins with $\nabla(\lambda)$ – namely, an increasing (finite or infinite) filtration $0 = G_0(\lambda) \subset G_1(\lambda) \subset \cdots$ such that*
   (i) *$G_1(\lambda) \cong \nabla(\lambda)$;*
   (ii) *for $n > 1$, $G_n(\lambda)/G_{n-1}(\lambda) \simeq \nabla(\mu)$ for some $\mu = \mu(n) > \lambda$;*
   (iii) *for a given $\mu \in \Lambda$, $\mu(n) = \mu$ for finitely many n;*
   (iv) *$\cup G_i(\lambda) = I(\lambda)$.*

EXAMPLE 9. *Let U be the algebra of upper triangular matrices over k. Put $\mathscr{C} =$ mod $-U$. In one hand we know that for any ring R, R-* mod

✓ *is abelian*
✓ *has enough injectives*
✓ *satisfies Grothendieck condition*

and on the other hand the category of left modules over U

✓ *is locally artinian*

✓ *U is a k-linear category, since $\operatorname{Hom}(A,B)$ is a vector space over k for a pair of U-modules A and B*

then the conditions not involving the poset Λ are satisfied.

Let $\Lambda = \{1,\dots,n\}$ and consider k^n as a left U-module, and let $\{e_1,\dots,e_n\}$ be its standard basis as a k vector space. Now define U_i to be the span of e_i, i.e. $U_i = Ue_i = \{Me_i \mid M \in U\}$ and put $L(i) = \nabla(i) = U_i$.

(1) The set $\{L(i) \mid i \in \Lambda\}$ is a complete collection of non-isomorphic simple left U-modules, they are 1-dimensional.

(2) Clearly $1_{L(i)} : L(i) \to L(i) = \nabla(i)$ is an injection, and $\nabla(i)/L(i) \cong 0$ has no simple composition factors. For $i,j \in \lambda$ we have that $\dim \operatorname{Hom}(\nabla(i), L(i))$ is finite since $L(i)$ is finite for any $i \in \lambda$; and $[\nabla(i) : L(j)]$ is either 0 or 1.

(3) The injective hull of $L(i) = U_i/U_{i-1}$ is $I(i) = U_n/Ui-1$. Define $G_p(i) = U_{i-1+p}/U_{i-1}$ for $0 \le p \le n-i+1$, then the following increasing filtration of $I(i)$

$$0 \cong G_1(i) \subsetneq \cdots \subsetneq G_{n-i+1}(i) = I(i).$$

is a "good filtration", i.e.

(i) $G_1(i) = U_i/Ui-1 = \nabla(i)$

(ii) For $p > 1$ we have $G_p(i)/G_{p-1}(i) = (U_{i-1+p}/U_{i-1})/(U_{i-2+p}/U_{i-1}) \cong U_{i-1+p}/U_{i-2+p} = \nabla(i-1+p)$, $1+p-1 > i$

(iii) For a fixed $i \in \Lambda$, $i-1+p$ is the only value $\mu \in \Lambda$ such that $G_p(i)/G_{p-1}(i) \cong \nabla(\mu)$.

(iv) Clearly $\cup_{p=0}^{n-i+1} G_p(i) = I(i)$.

therefore the category of left modules over U is a highest weight category.

Note that $\mathscr{C}_f$ determines $\mathscr{C}$ by taking direct limits where $\mathscr{C}_f$ denotes the full subcategory of $\mathscr{C}$ consisting of all subobjects of finite length, though $\mathscr{C}_f$ rarely contains enough injectives, The following lemma lists several immediate consequences of the above axioms for a highest weight category $\mathscr{C}$.

LEMMA 2. *Let $\mathscr{C}$ be a highest weight category with poset Λ of weights. Let $\lambda, \mu \in \Lambda$. Then:*

(1) $L(\lambda)$ is the socle of $\nabla(\lambda)$.

(2) If either $\operatorname{Ext}^1_{\mathscr{C}}(\nabla(\mu), \nabla(\lambda))$ or $\operatorname{Ext}^1_{\mathscr{C}}(L(\lambda), L(\mu))$ is nonzero, then necessarily $\mu > \lambda$. If $\operatorname{Ext}^1_{\mathscr{C}}(L(\lambda), L(\mu)) \neq 0$, then λ and μ are strictly comparable (i.e. either $\lambda > \mu$ or $\mu > \lambda$).

(3) If M, N are objects in $\mathscr{C}$ of finite length, then $\operatorname{Hom}_{\mathscr{C}}(M,N)$ and $\operatorname{Ext}^1_{\mathscr{C}}(M,N)$ are finite dimensional.

(4) The filtration $\{G_n(\lambda)\}$ in axiom 3 (f) can be choosen to satisfy the additional condition:

(v) for all $i, j > 0$ if $\mu(i) < \mu(j)$, then $i < j$.

This correspond to Lemma 3.2 in [**12**]. Moreover in Lemma 3.8 they extend to Ext^n several of the results above involving Ext^1.

THEOREM 1.5. *The category $\mathcal{O}_c$ of H_c-modules is a highest weight category, with weights in $\Lambda = \mathrm{Irr}(\mathbf{C}W)$. The order considered in Λ is given by the relation for $E, F \in \mathrm{Irr}(\mathbf{C}W)$, $E < F$ if and only if $c_E - c_F$ is a positive integer.*

In [**5**] Guay, Ginzburg, Opdam and Rouquier checked that category $\mathcal{O}_c$ of H_c-modules is a highest weight category in this sense, with $\{L(\lambda)\}$ the set of simple H_c-modules defined in previous section and collection $\{\nabla(\lambda)\}$ of costandard modules by using BGG-reciprocity on properties of category $\mathcal{O}_c$ and standard modules $\Delta(E)$.

3. Homology of unitary representations

For the following definition notice that $\mathbf{C}$ is a $\mathbf{C}[\mathfrak{h}]$-module via evaluation at the origin $0 \in \mathfrak{h}$.

DEFINITION 4. *For an H_c-module M, define a right-exact functor from $\mathcal{O}_c$ to $\mathbf{C}W$-modules given by $H_0(\mathfrak{h}^*, M) = \mathbf{C} \otimes_{\mathbf{C}[\mathfrak{h}]} M$. Denote by $H_i(\mathfrak{h}^*, M)$ to the homology of its left derived functor.*

DEFINITION 5. *Similarly we define a left-exact functor from $\mathcal{O}_c$ to graded $\mathbf{C}W$-modules by*

$$H^0(\mathfrak{h}, M) = \{m \in M \mid D_y(m) = 0, \text{ for all } y \in \mathfrak{h}\}.$$

And we denote by $H^i(\mathfrak{h}, M)$ to the cohomology of its right derived funtor.

The following theorem is a synthesis of results due to Huang-Wong [**9**], Ciubotaru [**1**] and Griffeth [**7**].

THEOREM 1.6. *Let L be an irreducible object of $\mathcal{O}_c$. Then we have the following isomorphism of graded W-modules:*

$$H_i(\mathfrak{h}^*, L) \cong H^i(\mathfrak{h}, L)$$

and consequently an isomorphism of graded W-modules

$$\mathrm{Tor}_i(L, \mathbf{C}) \cong \bigoplus_{E \in \mathrm{Irr}(\mathbf{C}W)} \mathrm{Ext}^i_{\mathcal{O}_c}(\Delta_c(E), L) \otimes_{\mathbf{C}} E.$$

Moreover, for any object M of $\mathcal{O}_c$ the following equivariant purity property *holds: if the E-isotypic component of the degree d piece of the* Tor*-group* $\mathrm{Tor}_i(M, \mathbf{C})_{E,d}$ *is not zero, then $d = c_E$. Finally, if L is a unitary irreducible object of $\mathcal{O}_c$, then*

$$\mathrm{Ext}^i(\Delta_c(E), L) \cong \mathrm{Hom}_{\mathbf{C}W}(E, L_{c_E - i} \otimes \Lambda^i \mathfrak{h}^*).$$

CHAPTER 2

The cyclotomic rational Cherednik algebra

1. Combinatorics

We recall some definitions and basic notions of partitions and standard young tableaux used in further sections.

A *partition* is a non-increasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$ of positive integers. We identify partitions with their Young diagrams and visualize them as collections of boxes. The *diagram* of λ is the collection of points (x, y) for x and y integers satisfying $1 \leq y \leq \ell$ and $1 \leq x \leq \lambda_y$, which we think of as a subset of $\mathbf{R}^2$. A *skew shape* is a finite subset $D \subseteq \mathbf{R}^2$ such that whenever $(x, y) \in D$ and $(x+k, y+l) \in D$ for nonnegative integers k and l, then $(x+k', y+l') \in D$ for all integers $0 \leq l' \leq l$ and $0 \leq k' \leq k$. We think of the points of the skew shape as boxes. A skew shape D is called *integral* if $D \subseteq \mathbf{Z}^2_{>0}$. Each integral skew shape is the difference $D = \alpha \setminus \beta$ of two partitions, which are not uniquely determined by D. Boxes $b = (x, y)$ and $b' = (x', y')$ are called *adjacent* if either

(1) $x = x'$ and $y = y' \pm 1$, or
(2) $y = y'$ and $x = x' \pm 1$.

Adjacency is boxes of a skew shape is a equivalence relation, and the equivalent classes of a skew shape D are called *connected components* of D, and D is *connected* if it has inly one connected component.

For a fixed integer $\ell > 0$, an ℓ-partition is a sequence $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{\ell-1})$ of partitions, some of which may be empty. Likewise, an ℓ-skew shape is a sequence $D = (D^0, D^1, \ldots D^{\ell-1})$ of ℓ skew shapes, some of which may be empty. A box of D is a box of some D^j, thn we write

$$\beta(b) = j$$

if the box $b \in D$ is a box of D^j.

If there are n total boxes in λ then we say that λ is an *ℓ-partition of n* and we write $P_{\ell,n}$ for the set of all ℓ-partitions of n. Given $\lambda \in P_{\ell,n}$ we define its *transpose* λ^t as the ℓ-partition obtained from λ by cycling its components one spot to the left and transposing them all, i.e.

$$\lambda^t = \left((\lambda^1)^t, (\lambda^2)^t, \ldots, (\lambda^{\ell-1})^t, (\lambda^0)^t\right)$$

where μ^t denotes the classical transpose of a partition μ.

We order the boxes of each ℓ-skew shape and each ℓ-partition as follows, let $b = (x, y)$ and $b' = (x', y')$ then

$$b \leq b' \Leftrightarrow \beta(b) = \beta(b') \text{ with } x' - x \in \mathbf{Z}_{\geq 0} \text{ and } y' - y \in \mathbf{Z}_{\geq 0}$$

In this section we study reflections and representations of the complex reflection group $G(\ell,1,n)$ in order to specialize W in the definition of the rational Cherednik algebra to this group which we will call *cyclotomic rational Cherednik algebra*.

2. The group $G(\ell,1,n)$

Let $\boldsymbol{\mu}_\ell$ be the cyclic group of ℓth roots of unity and S_n the symmetric group of degree n, the group $G(\ell,1,n)$ can be defined as $\boldsymbol{\mu}_\ell \wr S_n$ the wreath product of $\boldsymbol{\mu}_\ell$ and S_n and it acts on $\mathfrak{h} = \mathbf{C}^n$ in the obvious way. Let $y_1,\dots,y_n$ be the standard basis of $\mathfrak{h}$ then this group consists of all matrices of size n by n with exactly one non-zero entry in each row and column, which is an ℓth root of unity.

EXAMPLE 10. $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in G(2,1,3)$ *and* $\begin{pmatrix} 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \in G(4,1,4)$

With this characterization, the group $G(1,1,n)$ is the group of permutation matrices of size n by n, which is isomorphic to the symmetric group S_n. And the group $G(2,1,n)$ is the group of signed permutation matrices also known as the Weyl group of type B_n.

For a fixed primitive ℓth rooth of unity ζ, we denote by ζ_i to the diagonal matrix with ones all over the diagonal except in the position i where there is ζ, s_{ij} denotes the permutation matrix which exchanges coordinates i and j, and $s_i = s_{i,i+1}$.

EXAMPLE 11. *In the group* $G(\ell,1,3)$,

$$\zeta_1 = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \zeta_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta \end{pmatrix}$$

and

$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The set of reflections of $G(\ell,1,n)$ is the set $R = R_1 \cup R_2$, where

$$R_1 = \{\zeta_i^j \,|\, 1 \le i \le n, 1 \le j \le \ell-1\} \text{ and } R_2 = \{\zeta_i^k s_{ij} \zeta_i^{-k} \,|\, 1 \le i < j \le n, 0 \le k \le \ell-1\}.$$

The group $G(\ell,1,n)$ has ℓ conjugacy classes, let $\zeta_1,\zeta_2,\dots,\zeta_{\ell-1}$ be the representatives of the $\ell-1$ conjugacy classes in R_1 and let s (some transposition) be the representative of the conjugacy class R_2.

The irreducible complex representations of $G(\ell,1,n)$ are in bijection with ℓ-partition λ of n, and for $\lambda \in P_{\ell,n}$ we write S^λ for the corresponding irreducible representation [**10**]. We write $\mathfrak{h} = \mathbf{C}^n$ for the defining representation, which is irreducible for $\ell > 1$.

With this indexing of the complex irreducible representations of $G(\ell,1,n)$, the representation indexed by λ^t is the tensor product of the representation $\Lambda^n\mathfrak{h}^*$.

There is a combinatorial formula for the dimension of λ given by

$$\begin{aligned}\dim(\lambda) &= \#\text{ of standard tableaux of shape } \lambda\\ &= n!\prod_{i=1}^{r}\prod_{b\in\lambda^{(i)}}\frac{1}{h_b}\end{aligned}$$

where h_x is the hook length at the box b. This formula follows from the corresponding result for the symmetric group which is the case $\ell = 1$.

2.1. Jucys-Murphy-Young elements. In the group algebra $\mathbf{C}G(\ell,1,n)$ we define the following analog of *Jucys-Murphy-Young elements.*

$$\phi_i = \sum_{\substack{1\le j<i\\0\le k\le \ell-1}}\zeta_i^k s_{ij}\zeta_i^{-k} = \sum_{\substack{1\le j<i\\0\le k\le \ell-1}}\zeta_j^k s_{ij}\zeta_j^{-k}\quad 1\le i\le n$$

PROPOSITION 9. *The elements $\phi_1,\phi_2,\dots,\phi_n$ are pairwise commutative.*

PROOF. Let ψ_i be the conjugacy class sum given by

$$\psi_i = \phi_1+\cdots+\phi_i = \sum_{\substack{1\le p<q\le i\\0\le k\le \ell-1}}\zeta_q^k s_{pq}\zeta_q^{-k}$$

therefore ψ_i is a central element of $\mathbf{C}G(\ell,1,i)$. Since ψ_i commutes with $\psi_1,\dots,\psi_i$ it follows that $\psi_1,\psi_2,\dots,\psi_n$ are pairwise commutative. The result follows by observing that $\phi_i = \psi_i - \psi_{i-1}$. □

PROPOSITION 10. *The* Jucys-Murphy-Young elements $\phi_1,\dots,\phi_n$ *satisfy*

(1) $\phi_i\zeta_j = \zeta_j\phi_i$ *for* $1\le i\le n$ *and* $1\le j\le n$.
(2) $\phi_i s_j = s_j\phi_i$ *for* $j\neq i-1,i$.
(3) $\phi_i s_i = s_i\phi_{i+1}-\pi_i$ *for* $1\le i\le n-1$, *where* $\pi_i = \sum_{0\le k\le\ell-1}\zeta_i^k\zeta_{i+1}^{-k}$.

PROOF. (1) Note that if $1\le i<j$ the element ζ_j commute with s_{it} for all $1\le t<i$ and then ζ_j commutes with ϕ_i. The case $i=j$ follows from

$$\zeta_i\phi_i\zeta_i^{-1} = \zeta_i\left(\sum_{\substack{1\le t<i\\0\le k\le \ell-1}}\zeta_i^k s_{it}\zeta_i^{-k}\right)\zeta_i^{-1} = \sum_{\substack{1\le t<i\\0\le k\le \ell-1}}\zeta_i^{k+1}s_{it}\zeta_i^{-(k+1)} = \phi_i.$$

Note that ψ_i,ψ_{i-1} are central in $\mathbf{C}G(\ell,1,i-1)$ then $\phi_i = \psi_i-\psi_{i-1}$ commutes with every ζ_j and s_j for $1\le j<i$, which proves (a)

(2) By the same arguments used in (a) the element ζ_j commutes s_i for $j\neq i-1,i$.

(3) For this part we compute

$$\phi_i s_i = \left(\sum_{\substack{1\le j<i\\0\le k\le \ell-1}}\zeta_i^k s_{ij}\zeta_i^{-k}\right)s_i = \sum_{\substack{1\le j<i\\0\le k\le \ell-1}}\zeta_i^k s_{ij}s_i\zeta_{i+1}^{-k}$$

$$
\begin{aligned}
&= \sum_{\substack{1\le j<i\\ 0\le k\le \ell-1}} \zeta_i^k s_i s_{i+1,j}\zeta_{i+1}^{-k} = \sum_{\substack{1\le j<i\\ 0\le k\le \ell-1}} s_i\zeta_{i+1}^k s_{i+1,j}\zeta_{i+1}^{-k}\\
&= s_i\left(\sum_{\substack{1\le j<i\\ 0\le k\le \ell-1}} \zeta_{i+1}^k s_{i+1,j}\zeta_{i+1}^{-k}\right) = s_i\left(\phi_{i+1} - \sum_{0\le k\le \ell-1}\zeta_{i+1}^k s_i \zeta_{i+1}^{-k}\right)\\
&= s_i\phi_{i+1} - \sum_{0\le k\le \ell-1}\zeta_i^k\zeta_{i+1}^{-k}
\end{aligned}
$$

□

Let $\mathfrak{u}$ be the subalgebra of $\mathbf{C}G(\ell,1,n)$ generated by $\phi_1,\ldots,\phi_n$ and $\zeta_1,\ldots,\zeta_n$. Let M be a $\mathfrak{u}$-module, a *weight* of $\mathfrak{u}$ on M is a $\mathbf{C}$-algebra homomorphism $\alpha:\mathfrak{u}\to\mathbf{C}$ such that

$$M_\alpha \neq 0$$

where $M_\alpha = \left\{m\in M \mid x\cdot m = \alpha(x)m \text{ for all } x\in\mathfrak{u}\right\}$. We identify the $\mathbf{C}$-algebra homomorphism α as above with the list

$$(\alpha(\phi_1),\ldots,\alpha(\phi_n),\alpha(\zeta_1),\ldots,\alpha(\zeta_n)).$$

In this way, given a $\mathfrak{u}$-eigenvector $m\in M$ we define the *weight* of the vector m to be the tuple

$$wt(m) = (a_1,\ldots,a_n,\zeta^{b_1},\ldots,\zeta^{b_n})$$

if $\phi_i m = a_i m$ and $\zeta_i m = \zeta^{b_i} m$ for $i\le i\le n$.

LEMMA 3. *(1) The algebra $\mathfrak{u}$ acts semisimple on each $\mathbf{C}G(\ell,1,n)$-module M.*

(2) Let M be a $\mathbf{C}G(\ell,1,n)$-module and let $m\in M$ be a $\mathfrak{u}$-weight vector with

$$wt(m) = (a_1,\ldots,a_n,\zeta^{b_1},\ldots,\zeta^{b_n})$$

then

$$(a_i,\zeta^{b_i}) \neq (a_{i+1},\zeta^{b_{i+1}}) \textit{ for } 1\le i\le n-1$$

PROOF. Let M be a $\mathbf{C}G(\ell,1,n)$-module, note that ϕ_i is a self adjoint operator and ζ_i is a unitary operator with respect to any W-invariant positive definite hermitian form on M. Then $\mathfrak{u}$ is a commutative algebra, which proves (a).
For (b) suppose that $(a_i,\zeta^{b_i}) = (a_{i+1},\zeta^{b_{i+1}})$ for some $1\le i\le n-1$, then by part (c) of Proposition 10 we have

$$\phi_i s_i\cdot m = (s_i\phi_{i+1}-\pi_i)\cdot m = (s_i a_{i+1}-\pi)m = a_{i+1}s_i m - \ell m$$

hence $(\phi_i - a_i)s_i m = -\ell m \neq 0$ while $(\phi_i-a_i)^2 s_i m = -(\phi_i-a_i)\cdot\ell m = 0$ so $si_i m$ is a generalized eigenvector, which is not an eigenvector for ϕ_i which contradicts (a).

□

3. The rational Cherednik algebra of type $G(\ell,1,n)$

The *cyclotomic rational Cherednik algebra* is the algebra $H_c(G(\ell,1,n),\mathbf{C}^n)$, with set of parameters $(c_0,c_1,\dots,c_{\ell-1})$, where $c_0=c_{s_1}$ and $c_i=c_{\zeta_1^i}$.

In order to give an explicit presentation of the cyclotomic rational Cherednik algebra we use the parametrization introduced in the previous section. Let $y_1,\dots,y_n$ be the standard basis of $\mathfrak{h}$ and $x_1,\dots,x_n$ is the dual basis in $\mathfrak{h}^*$. For a reflection $r=\zeta_i^j$, the hyperplane

$$H=\operatorname{fix}(r)=\{(v_1,v_2,\dots,v_n)\in\mathbf{C}^n\,|\,\zeta^j v_i=v_i\}=\{(v_1,v_2,\dots,v_n\in\mathbf{C}^n)\,|\,v_i=0\}.$$

Note that in this case W_H and its character group $W_H^\vee$ are cyclic groups, the first one is generated by ζ_i and the second one by det, then the primitive idempotent given by equation (1.6) is

$$e_{H,\det^j}=\frac{1}{\ell}\sum_{k=0}^{\ell-1}\zeta^{-kj}\zeta_i^k. \tag{3.1}$$

and the parameter $c_{H,\chi}$ in (1.7) is

$$c_{H,\det^j}=\frac{1}{\ell}\sum_{k=1}^{\ell-1}c_k(1-\zeta^{kj}). \tag{3.2}$$

Define

$$d_j=\sum_{k=1}^{\ell-1}\zeta^{kj}c_k, \tag{3.3}$$

then $c_{H,\det^j}=\frac{1}{\ell}(d_0-d_j)$.

For $r=\zeta_i^k s_{ij}\zeta_i^{-k}$ we have

$$H=\{(v_1,v_2,\dots,v_n)\in\mathbf{C}^n\,|\,v_i=\zeta^k v_j\},$$

and in this case W_H is a group of order two generated by reflection r. Then

$$e_{H,\chi}=\begin{cases}\frac{1}{2}(1-r) & \text{if } \chi\neq 1\\ \frac{1}{2}(1+r) & \text{if } \chi=1\end{cases}\quad\text{and}\quad c_{H,\chi}=\begin{cases}c_0 & \text{if } \chi\neq 1\\ 0 & \text{if } \chi=1.\end{cases}$$

Note that

$$d_0+d_1+\cdots+d_{\ell-1}=\sum_{j=0}^{\ell-1}\sum_{k=1}^{\ell-1}\zeta^{kj}c_k=\sum_{k=1}^{\ell-1}c_k\left(\sum_{j=0}^{\ell-1}\zeta^{kj}\right)=0,$$

PROPOSITION 11. *The cyclotomic rational Cherednik algebra is generated by the polynomial rings $\mathbf{C}[y_1,\dots,y_n]$ and $\mathbf{C}[x_1,\dots,x_n]$ and the group algebra $\mathbf{C}W$, which acts by automorphisms on the two polynomial rings, subject to the relations*

$$wxw^{-1}=w(x)\quad\textit{and}\quad wyw^{-1}=w(y)\quad\textit{for } w\in W, x\in\mathfrak{h}^*\textit{ and } y\in\mathfrak{h},$$

$$y_ix_i=x_iy_i+1-c_0\sum_{\substack{1\le j\neq i\le n\\0\le t\le\ell-1}}\zeta_i^t s_{ij}\zeta_i^{-t}-\sum_{t=0}^{\ell-1}(d_t-d_{t-1})e_{it}$$

for $1\le i\le n$, where

$$e_{it}=\frac{1}{\ell}\sum_{k=0}^{\ell-1}\zeta^{-kt}\zeta_i^k$$

and

$$y_i x_j = x_j y_i + c_0 \sum_{t=0}^{\ell-1} \zeta^{-t} \zeta_i^t s_{ij} \zeta_i^{-t}$$

for $1 \le i \ne j \le n$.

PROOF. We choose

$$\alpha_H = \lambda x_i \quad \text{and} \quad \alpha_H^\vee = \lambda' y_i, \quad \text{for} \quad r = \zeta_i^j,$$

where λ and λ' are nonzero scalars, and

$$\alpha_H = x_i - \zeta^k x_j \quad \text{and} \quad \alpha_H^\vee = y_i - \zeta^{-k} y_j, \quad \text{for} \quad r = \zeta_i^k s_{ij} \zeta_i^{-k}.$$

For $i = j$ equation (1.11) turns

$$\begin{aligned}
y_i x_i - x_i y_i = \langle x_i, y_i\rangle &- \sum_{H=\{x_k=0\}} \frac{\langle \alpha_H, y_i\rangle\langle x_i, \alpha_H^\vee\rangle}{\langle \alpha_H, \alpha_H^\vee\rangle} \sum_{\chi\in W_H^\vee} (c_{H,\chi\otimes\det^{-1}} - c_{H,\chi}) n_H e_{H,\chi} \\
&- \sum_{H=\{x_p=\zeta^t x_q\}} \frac{\langle \alpha_H, y_i\rangle\langle x_i, \alpha_H^\vee\rangle}{\langle \alpha_H, \alpha_H^\vee\rangle} \sum_{\chi\in W_H^\vee} (c_{H,\chi\otimes\det^{-1}} - c_{H,\chi}) n_H e_{H,\chi} \\
&= 1 - \sum_{H=\{x_k=0\}} \frac{\langle \lambda x_k, y_i\rangle\langle x_i, \lambda' y_k\rangle}{\langle \lambda x_k, \lambda' y_k\rangle} \sum_{t=0}^{\ell-1} \left(c_{H,\det^{t-1}} - c_{H,\det^t}\right) \cdot \ell \cdot e_{H,\det^t} \\
&- \sum_{\{x_p=\zeta^t x_q\}} \frac{\langle x_p - \zeta^t x_q, y_i\rangle\langle x_i, y_p - \zeta^{-k} y_q\rangle}{\langle x_p - \zeta^k x_q, y_p - \zeta^{-k} y_q\rangle} \left((c_{H,\det} - c_{H,1}) e_{H,1} + (c_{H,1} - c_{H,\det}) e_{H,\det}\right) \\
&= 1 - \sum_{t=0}^{\ell-1} \left(\frac{1}{\ell}(d_0 - d_{t-1}) - \frac{1}{\ell}(d_0 - d_t)\right) \cdot \ell \cdot e_{H,\det^t} \\
&- c_0 \sum_{\substack{1\le p<q\le n \\ 0\le t\le \ell-1}} \Big(\langle x_p, y_i\rangle\langle x_i, y_p\rangle + \langle x_q, y_i\rangle\langle x_i, y_q\rangle\Big) \zeta_p^t s_{pq} \zeta_p^{-t} \\
&= 1 - \sum_{t=0}^{\ell-1} (d_t - d_{t-1}) e_{it} - c_0 \sum_{\substack{1\le i\ne j\le n \\ 0\le t\le \ell-1}} \zeta_i^t s_{ij} \zeta_i^{-t}
\end{aligned}$$

where $e_k i = e_{H,\det^t}$ for $H = \{x_i = 0\}$. If $i < j$

$$\begin{aligned}
y_i x_j - x_j y_i &= - \sum_{H=\{x_p=\zeta^t x_q\}} \frac{\langle \alpha_H, y_i\rangle\langle x_j, \alpha_H^\vee\rangle}{\langle \alpha_H, \alpha_H^\vee\rangle} \sum_{\chi\in W_H^\vee} (c_{H,\chi\otimes\det^{-1}} - c_{H,\chi}) n_H e_{H,\chi} \\
&= - \sum_{\substack{1\le p<q\le n \\ 0\le t\le \ell-1}} - \frac{\zeta^{-t}\langle x_p, y_i\rangle\langle x_j, y_p\rangle}{2} \cdot 2c_0 \cdot \zeta_p^t s_{pq} \zeta_p^{-t} \\
&= c_0 \sum_{t=0}^{\ell-1} \zeta^{-t} \zeta_p^t s_{pq} \zeta_p^{-t}
\end{aligned}$$

The calculation for $j < i$ is similar. □

From now on we write H_c for the rational Cherednik algebra attached to this group, where c denotes the list of central parameters, i.e.

$$c = (c_0, d_0, d_1, \dots, d_{\ell-1}) \in \mathbf{R}^{\ell+1}$$

with $d_0 + \ldots + d_{\ell-1} = 0$. These parameters give rise to statistics on ℓ-partitions as follows, given λ an ℓ-partition and a box $b \in \lambda$ we define its *charged content* $\mathrm{ct}_c(b)$ by

$$\mathrm{ct}_c(b) = d_{\beta(b)} + \ell\,\mathrm{ct}(b)c_0,$$

and we define the *charged content of* λ to be sum of the charged contents of its boxes

$$\mathrm{ct}_c(\lambda) = \sum_{b\in\lambda} \mathrm{ct}_c(b).$$

This statistic is essentially the c-function of the representation indexed by λ.

3.1. The rational Cherednik algebra of type B**.** In the examples at the end of this thesis we focus on calculations for this algebra.

The rational Cherednik algebra of type B is the cyclotomic rational Cherednik algebra attached to the Weyl group of type B, this is $W = G(2,1,n)$ and $\mathfrak{h} = \mathbf{C}^n$.

For the parameter $c = (c_r)_{r\in R}$ note that this group has only two conjugacy classes, and then we fix $d_0 = d$, $d_1 = -d$ and $c_0 = c$ as we describe before. For the deformation parameter c we use the pair (c,d) where $d_0 = d$.

There exists a bijection between irreducible H_c-modules and irreducible representations of the group $G(2,1,n)$ (or $\mathbf{C}W$-modules), which are indexed by 2-tuples of partitions $\lambda = (\lambda^0,\lambda^1)$, with n total boxes. We denote the irreducible representation by λ. There is a combinatorial formula for the dimension of λ as follows

EXAMPLE 12. *Let $n = 3$, the set of 2-partitions of 3 is*

$$\Big\{\Big(\yng(3),\emptyset\Big),\Big(\yng(2,1),\emptyset\Big),\Big(\yng(1,1,1),\emptyset\Big),\Big(\yng(2),\yng(1)\Big),\Big(\yng(1,1),\yng(1)\Big),$$
$$\Big(\yng(1),\yng(2)\Big),\Big(\yng(1),\yng(1,1)\Big),\Big(\emptyset,\yng(1,1,1)\Big)\Big(\emptyset,\yng(2,1)\Big),\Big(\emptyset,\yng(3)\Big)\Big\}$$

the dimensions of the corresponding irreducible representations are $(1,2,1,3,3,3,3,1,2,1)$*, respectively.*

CHAPTER 3

The cyclotomic degenerate affine Hecke algebra

Let H_c be the cyclotomic rational Cherednik algebra for the group $G(\ell,1,n)$. In this section we will construct a subalgebra of H_c isomorphic to the graded Hecke algebra for $G(\ell,1,n)$ defined in [**11**]. This subalgebra we call the *degenerate affine Hecke algebra of type* $G(\ell,1,n)$ was identified in [**2**] where is called *generalized graded Hecke algebra*.

The *degenerate affine Hecke algebra of type* $G(\ell,1,n)$ is the algebra $H_{\ell,n}$ generated by $\mathbf{C}[u_1,\dots,u_n]$ and the group $G(\ell,1,n)$ subject to the relations

$$\zeta_i u_j = u_j \zeta_i, \quad \text{for all } i,j \tag{0.1}$$

$$s_i u_j = u_j s_i, \quad \text{if } j \neq i, i+1 \tag{0.2}$$

$$s_i u_i = u_{i+1} s_i - \pi_i \tag{0.3}$$

where

$$\pi_i = \sum_{k=0}^{\ell-1} \zeta_i^k \zeta_{i+1}^{-k}.$$

The following lemma is a generalization of (0.2) and (0.3).

LEMMA 4. *Let* $f \in \mathbf{C}[u_1,\dots,u_n]$. *Then*

$$s_i f = s_i(f) s_i - \pi_i \frac{f - s_i(f)}{u_i - u_{i+1}}, \quad \textit{for } 1 \leq i \leq n-1. \tag{0.4}$$

where $s_i(f)$ *is the left action of* S_n *on* $\mathbf{C}[u_1,\dots,u_n]$ *given by*

$$w(u_i) = u_{w(i)}$$

PROOF. We prove by induction on $\deg(f)$. The case when $\deg(f) = 1$ is given by formulas (0.2) and (0.3). Now assuming that (0.4) holds for a polynomial $f \in \mathbf{C}[u_1,\dots,u_n]$ we prove for $g = u_j f$. If $j \neq i, i+1$ then

$$\begin{aligned} s_i g = s_i u_j f = u_j s_i f &= u_j \left(s_i(f) s_i - \pi_i \frac{f - s_i(f)}{u_i - u_{i+1}} \right) \\ &= u_j s_i(f) s_i - \pi_i \frac{u_j f - u_j s_i(f)}{u_i - u_{i+1}} \\ &= s_i(u_j) s_i(f) s_i - \pi_i \frac{u_j f - s_i(u_j) s_i(f)}{u_j - u_{i+1}} \\ &= s_i(u_j f) s_i - \pi_i \frac{u_j f - s_i(u_j f)}{u_i - u_{i+1}} \\ &= s_i(g) s_i - \pi_i \frac{g - s_i(g)}{u_i - u_{i+1}} \end{aligned}$$

If $j = i$, then

$$\begin{aligned}
s_i g = s_i u_i f &= (u_{i+1}s_i - \pi_i)f = u_{i+1}s_i f - \pi_i f \\
&= u_{i+1}\left(s_i(f)s_i - \pi_i \frac{f - s_i(f)}{u_i - u_{i+1}}\right) - \pi_i f \\
&= s_i(u_i f)s_i - \pi_i \frac{u_{i+1}f - s_i(u_i f) + u_i f - u_{i+1}f}{u_i - u_{i+1}} \\
&= s_i(u_i f)s_i - \pi_i \frac{u_i f - s_i(u_i f)}{u_i - u_{i+1}} = s_i(g)s_i - \pi_i \frac{g - s_i(g)}{u_i - u_{i+1}}
\end{aligned}$$

A similar computation handles the case $j = i+1$. □

For $1 \le i \le n$ define

$$z_i = y_i x_i + c_0 \phi_i.$$

where ϕ_i is the ith Jucys-Murphy-Young element for the group $G(\ell, 1, n)$.

PROPOSITION 12. *The elements $z_1, z_2, \ldots, z_n$ of H_c are pairwise commutative*

PROOF. Using the relations in Proposition 11 we first compute

$$\begin{aligned}
[y_i x_i, y_j x_j] &= y_i x_i y_j x_j - y_j x_j y_i x_i \\
&= y_i x_i y_j x_j - y_j x_j y_i x_i + y_i y_j x_i x_j - y_i y_j x_i x_j \\
&= y_i (x_i y_j - y_j x_i) x_j + y_j (y_i x_j - x_j y_i) x_i \\
&= y_i \left(-c_0 \sum_{k=0}^{\ell-1} \zeta^{-k} \zeta_j^k s_{ij} \zeta_j^{-k}\right) x_j + y_j \left(c_0 \sum_{k=0}^{\ell-1} \zeta^{-k} \zeta_i^k s_{ij} \zeta_i^{-k}\right) x_i \\
&= -y_i x_i \left(c_0 \sum_{k=0}^{\ell-1} \zeta_j^k s_{ij} \zeta_j^{-k}\right) + \left(c_0 \sum_{k=0}^{\ell-1} \zeta_i^k s_{ij} \zeta_i^{-k}\right) y_i x_i
\end{aligned}$$

then

$$\begin{aligned}
0 &= y_i x_i y_j x_j - y_j x_j y_i x_i + y_i x_i \left(c_0 \sum_{k=0}^{\ell-1} \zeta_j^k s_{ij} \zeta_j^{-k}\right) - \left(c_0 \sum_{k=0}^{\ell-1} \zeta_i^k s_{ij} \zeta_i^{-k}\right) y_i x_i \\
&= y_i x_i \left(y_j x_j + c_0 \sum_{k=0}^{\ell-1} \zeta_j^k s_{ij} \zeta_j^{-k}\right) - \left(y_j x_j + c_0 \sum_{k=0}^{\ell-1} \zeta_i^k s_{ij} \zeta_i^{-k}\right) y_i x_i
\end{aligned}$$

Thus

$$\left[y_i x_i, y_j x_j + c_0 \sum_{k=0}^{\ell-1} \zeta_i^k s_{ij} \zeta_i^{-k} \right] = 0 \tag{0.5}$$

We assume that $i < j$, so ϕ_i commutes with $y_j x_j$. Using formula (0.5) together with the commutativity of ϕ_i we compute

$$\begin{aligned}
[z_i, z_j] &= [y_i x_i + c_0 \phi_i, y_j x_j + c_0 \phi_j] = [y_i x_i, y_j x_j + c_0 \phi_j] \\
&= \left[y_i x_i, y_j x_j + c_0 \sum_{\substack{1 \le t < j \\ 0 \le k \le \ell-1}} \zeta_j^k s_{tj} \zeta_j^{-k} \right]
\end{aligned}$$

$$= \left[y_i x_i, y_j x_j + c_0 \left(\sum_{k=0}^{\ell-1} \zeta_j^k s_{ij} \zeta_j^{-k} + \sum_{\substack{1 \le t \ne i < j \\ 0 \le k \le \ell-1}} \zeta_j^k s_{ij} \zeta_j^{-k} \right) \right]$$

$$= \left[y_i x_i, y_j x_j + c_0 \sum_{k=0}^{\ell-1} \zeta_j^k s_{ij} \zeta_j^{-k} + c_0 \sum_{\substack{1 \le t \ne i < j \\ 0 \le k \le \ell-1}} \zeta_j^k s_{ij} \zeta_j^{-k} \right]$$

$$= \left[y_i x_i, y_j x_j + c_0 \sum_{k=0}^{\ell-1} \zeta_j^k s_{ij} \zeta_j^{-k} \right] + \left[y_i x_i, c_0 \sum_{\substack{1 \le t \ne i < j \\ 0 \le k \le \ell-1}} \zeta_j^k s_{ij} \zeta_j^{-k} \right]$$

$$= \left[y_i x_i, y_j x_j + c_0 \sum_{k=0}^{\ell-1} \zeta_i^k s_{ij} \zeta_i^{-k} \right] + \left[y_i x_i, c_0 \sum_{1 \le t \ne i < j} \sum_{k=0}^{\ell-1} \zeta_j^k s_{ij} \zeta_j^{-k} \right] = 0$$

□

PROPOSITION 13. *The elements $z_1, z_2, \ldots, z_n$ in H_c satisfy the following relations*

$$\zeta_i z_j = z_j \zeta_i \text{ for all } 1 \le i, j \le n. \tag{0.6}$$

$$s_i z_j = z_j s_i \text{ for } j \ne i, i+1. \tag{0.7}$$

$$s_i z_i = z_{i+1} s_i - c_0 \sum_{0 \le k \le \ell-1} \zeta_i^k \zeta_{i+1}^{-k} \tag{0.8}$$

PROOF. First we check that the elements ζ_i and ϕ_j commute for all $1 \le i, j \le n$. This is clear if $i > j$, since s_{tj} commutes with ζ_i for all $1 \le t \le j$. If $i = j$ then

$$\begin{aligned} \zeta_j \phi_j \zeta_j^{-1} &= \zeta_j \left(\sum_{\substack{1 \le t < j \\ 0 \le k \le \ell-1}} \zeta_j^k s_{tj} \zeta_j^{-k} \right) \zeta_j^{-1} \\ &= \sum_{\substack{1 \le t < j \\ 0 \le k \le \ell-1}} \zeta_j^{k+1} s_{tj} \zeta_j^{-k-1} = \phi_j \end{aligned}$$

Similarly, if $i < j$

$$\begin{aligned} \zeta_i \phi_j \zeta_i^{-1} &= \zeta_i \left(\sum_{\substack{1 \le t < j \\ 0 \le k \le \ell-1}} \zeta_j^k s_{tj} \zeta_j^{-k} \right) \zeta_i^{-1} \\ &= \zeta_i \left(\sum_{k=0}^{\ell-1} \zeta_j^k s_{ij} \zeta_j^{-k} + \sum_{\substack{1 \le t \ne i < j \\ 0 \le k \le \ell-1}} \zeta_j^k s_{tj} \zeta_j^{-k} \right) \zeta_i^{-1} \\ &= \sum_{k=0}^{\ell-1} \zeta_j^k \zeta_i s_{ij} \zeta_i^{-1} \zeta_j^{-k} + \sum_{\substack{1 \le t \ne i < j \\ 0 \le k \le \ell-1}} \zeta_j^k s_{tj} \zeta_j^{-k} \\ &= \sum_{k=0}^{\ell-1} \zeta_j^{k-1} s_{ij} \zeta_j^{-k+1} + \sum_{\substack{1 \le t \ne i < j \\ 0 \le k \le \ell-1}} \zeta_j^k s_{tj} \zeta_j^{-k} = \phi_j \end{aligned}$$

Then (0.6) follows from observe that the element ζ_i commutes with $y_i x_i$ for $i \neq j$ and

$$\zeta_i y_i x_i = \zeta y_i \zeta_i x_i = \zeta\zeta^{-1} y_i x_i \zeta_i = y_i x_i \zeta_i.$$

For (0.7) note that if $j \neq i, i+1$ then s_i commutes with $y_j x_j$ and ϕ_j, hence with $z_j = y_j x_j + c_o\phi_j$. Finally (0.8) follows from

$$\begin{aligned}
s_i z_i = s_i(y_i x_i + c_0\phi_i) &= s_i\left(y_i x_i + c_0 \sum_{\substack{1\le t<i\\0\le k\le \ell-1}} \zeta_i^k s_{ti}\zeta_i^{-k}\right)\\
&= y_{i+1}x_{i+1}s_i + c_0 \sum_{\substack{1\le t<i\\0\le k\le \ell-1}} \zeta_{i+1}^k s_{t,i+1}\zeta_{i+1}^{-k} s_i\\
&= \left(y_{i+1}x_{i+1} + c_0 \sum_{\substack{1\le t<i\\0\le k\le \ell-1}} \zeta_{i+1}^k s_{t,i+1}\zeta_{i+1}^{-k}\right) s_i\\
&= \left(y_{i+1}x_{i+1} + c_0 \sum_{\substack{1\le t<i+1\\0\le k\le \ell-1}} \zeta_{i+1}^k s_{t,i+1}\zeta_{i+1}^{-k} - c_0\sum_{k=0}^{\ell-1}\zeta_{i+1}^k s_i \zeta_{i+1}^{-k}\right) s_i\\
&= z_{i+1}s_i - c_0\sum_{k=0}^{\ell-1}\zeta_{i+1}^k s_i\zeta_{i+1}^{-k} s_i\\
&= z_{i+1}s_i - c_0\sum_{k=0}^{\ell-1}\zeta_{i+1}^k \zeta_i^{-k}
\end{aligned}$$

□

A consequence of the previous result the algebra $H_{\ell,n}$ can be realized as the subalgebra of H_c generated by $G(\ell,1,n)$ and $z_1, z_2, \ldots, z_n$.

PROPOSITION 14. *For $c_0 \neq 0$ there is a unique homomorphism $H_{\ell,n} \to H_c$ determined by*

$$u_i \mapsto \frac{1}{c_0} z_i \qquad s_i \mapsto s_i \qquad \zeta_i \mapsto \zeta_i.$$

By abuse of notation we denote the image of $H_{\ell,n}$ by the previous map by $H_{\ell,n}$.

THEOREM 3.1 (PBW). *The set $\left\{u_{i_1}u_{i_2}\ldots,u_{i_p}w \,|\, 1 \le i_1 \le \cdots \le i_p \le n,\ w \in W\right\}$ is a $\mathbf{C}$-basis for $H_{\ell,n}$.*

PROOF. By Proposition 14 we will prove that the set $\{z_1, z_2, \ldots, z_n, w \,|\, w \in W\}$ is a $\mathbf{C}$-basis for the image of $H_{\ell,n}$ in H_c. For each $J = (j_1, j_2, \ldots, j_n)$ where $j_i \in \mathbf{Z}_{\ge 0}$ for all $1 \le i \le n$, we put $|J| = j_1 + j_2 + \cdots + j_n$, $z^J = z_1^{j_1} z_2^{j_2}\cdots z_n^{j_n}$, $x^J = x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$ and $y^J = y_1^{j_1}y_2^{j_2}\cdots y_n^{j_n}$. As a consequence of the relations proved in the previous result we can write each element of $H_{\ell,n}$ in the form

$$\sum_{\substack{0\le |J|\le m\\ w\in W}} \lambda_{J,w} z^J w \tag{0.9}$$

for some scalars $\lambda_{J,w} \in \mathbf{C}$ and we need to prove that this expression is unique. Suppose that the expression (0.9) is zero, let (J,w) such that $|J| = m$ and maximal with the property $\lambda_{J,w} \neq 0$.

We claim that we can write

$$z^J = y^J x^J + \sum_{\substack{0\le |K|,|L|<|J| \\ w\in W}} \mu_{K,L,w} y^K x^L w \tag{0.10}$$

for some scalars $\mu_{K,L,w} \in \mathbf{C}$. We prove this by induction on $|J|$. In fact, by definition we have $z_i = y_i x_i + c_0\phi_i$ which proves the case $|J| = 1$. Now let j_i a component of J such that $j_i \geq 1$ and let $e_i = (0,\dots,0,1,0,\dots,0)$ where the 1 is in ith position, then $|J - e_i| = |J| - 1$ and

$$\begin{aligned}
z^J &= z^{J-e_i} z_i \\
&= \left(y^{J-e_i} x^{J-e_i} + \sum_{\substack{0\le |K|,|L|<|J|-1 \\ w\in W}} \mu_{K,L,w} y^K x^L w \right) z_i \\
&= \left(y^{J-e_i} x^{J-e_i} + \sum_{\substack{0\le |K|,|L|<|J|-1 \\ w\in W}} \mu_{K,L,w} y^K x^L w \right) (y_i x_i + c_0\phi_i) \\
&= y^{J-e_i} x^{J-e_i} y_i x_i + \sum_{\substack{0\le |K|,|L|<|J|-1 \\ w\in W}} \mu_{K,L,w} y^K x^L w y_i x_i + y_i x_i c_0 \phi_i + \sum_{\substack{0\le |K|,|L|<|J|-1 \\ w\in W}} \mu_{K,L,w} y^K x^L w c_0 \phi_i
\end{aligned}$$

by the commutation relations for H_c given in Proposition 11 we have

$$x^{J-e_i} y_i = y_i x^{J-e_i} - \sum T x^I y^I,$$

where $T x^I y^I \in H_c$ are polynomials of degree lower than $|J| - 1$.

Since none of these relations change the degree of elements x and y then the leader term of the last expression in the equation is $y^J x^J$ and every other term has degree at most $|J| - 1$, finishing the induction. Using this result expression (0.9) can be rewritten in the form

$$\sum_{\substack{|J|=m \\ w\in W}} \lambda_{J,w} y^J w x^J + \text{error term} \tag{0.11}$$

where the error term involves only polynomial with degree at most $|J| - 1$. From theorem 1.2 (PBW for the rational Cherednik algebra) we get that $\lambda_{J,w} = 0$ for all $|J| = m$, hence a contradiction. $\square$

Then $H_{\ell,n}$ is isomorphic to $\mathbf{C}[u_1,\dots,u_n] \otimes \mathbf{C}G(\ell,1,n)$ as vector spaces.

PROPOSITION 15. *The center of $H_{\ell,n}$ is $\mathbf{C}[u_1,u_2,\dots,u_n]^{S_n} \otimes \mathbf{C}[\zeta_1,\zeta_2,\dots,\zeta_n]^{S_n}$.*

PROOF. Note that by (0.1) and (0.4) we have that $\mathbf{C}[u_1,u_2,\dots,u_n]^{S_n} \otimes \mathbf{C}[\zeta_1,\zeta_2,\dots,\zeta_n]^{S_n} \subseteq Z(H_{\ell,n})$. By an inductive argument similar to that used in the proof of the PBW theorem one obtains that $a \in \mathbf{C}[u_1,u_2,\dots,u_n]^{S_n} \otimes \mathbf{C}[\zeta_1,\zeta_2,\dots,\zeta_n]^{S_n}$. See [**2**] for details. $\square$

1. Intertwining operators

For $1 \le i \le n-1$ define the intertwining operators τ_i by

$$\tau_i = s_i + \frac{c_0}{u_i - u_{i+1}}\pi_i \tag{1.1}$$

The operator τ_i is well-defined on $H_{\ell,n}$-modules since $u_i - u_{i+1}$ is invertible on the image of π_i.

PROPOSITION 16. *Let M be a $H_{\ell,n}$-module and $m \in M$ such that*

$$wt(m) = (a_1, \ldots, a_n, \zeta^{b_1}, \ldots, \zeta_{b_n}).$$

(1) *$wt(\tau_i m) = s_i(wt(m))$ where S_n acts on the set of $2n$-tuples by simultaneously permuting the subindices of a_i and b_i.*

(2) $\tau_i^2 = \dfrac{(u_i - u_{i+1} - \pi_i)(u_i - u_{i+1} + \pi_i)}{(u_i - u_i)^2}$

(3) $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$

PROOF. For (a) we use the operator

$$(u_i - u_{i+1})\tau_i = (u_i - u_{i+1})s_i + \pi_i \tag{1.2}$$

instead of (1.1) (which is an element of H_c) te check that

$$u_i \tau_i = \tau_i u_{i+1} \tag{1.3}$$

$$u_{i+1}\tau_i = \tau_i u_i \tag{1.4}$$

$$u_j \tau_i = \tau_i u_j, \quad |j-i| \neq 1 \tag{1.5}$$

$$\zeta_i \tau_i = \tau_i \zeta_{i+1} \tag{1.6}$$

$$\zeta_{i+1}\tau_i = \tau_i \zeta_i \tag{1.7}$$

$$\zeta_j \tau_i = \tau_i \zeta_j, \quad |j-i| \neq 1 \tag{1.8}$$

then

$$u_j \tau_i m = \begin{cases} \tau_i u_{i+1} m = a_{i+1}\tau_i m & j = i \\ \tau_i u_i m = a_i \tau_i m & j = i+1 \\ \tau_i u_j m = a_j \tau_i m & |j-i| \neq 1 \end{cases}$$

and

$$\zeta_j \tau_i m = \begin{cases} \tau_i \zeta_{i+1} m = \zeta^{b_{i+1}} \tau_i m, & j = i \\ \tau_i \zeta_i m = \zeta^{b_i} \tau_i m, & j = i+1 \\ \tau_i \zeta_j m = \zeta^{b_j} \tau_i m, & |j-i| \neq 1 \end{cases}$$

then $\tau_i m$ is an eigenvector with eigenvalue

$$s_i((a_1, \ldots, a_n, \zeta^{b_1}, \ldots, \zeta^{b_n})) = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n, \zeta^{b_1}, \ldots, \zeta^{b_{i+1}}, \zeta^{b_i}, \ldots, \zeta^{b_n}) \tag{1.9}$$

For (b) we compute

$$((u_i - u_{i+1})\tau_i)^2 = (u_i - u_{i+1})\tau_i (u_i - u_{i+1})\tau_i$$

$$
\begin{aligned}
&= (u_i - u_{i+1})(\tau_i u_i - \tau_i u_{i+1})\tau_i \\
&= (u_i - u_{i+1})(u_{i+1}\tau_i - u_i\tau_i)\tau_i \\
&= (u_i - u_{i+1})(u_{i+1} - u_i)\tau_i^2
\end{aligned}
$$

and using (1.2) we have that

$$
\begin{aligned}
((u_i - u_{i+1})\tau_i)^2 &= ((u_i - u_{i+1})s_i + \pi_i)^2 \\
&= (u_i - u_{i+1})s_i(u_i - u_{i+1})s_i + (u_i - u_{i+1})s_i\pi_i + \pi_i(u_i - u_{i+1})s_i + \pi_i^2 \\
&= (u_i - u_{i+1})(s_i u_i - s_i u_{i+1})s_i + 2(u_i - u_{i+1})\pi_i s_i + \pi_i^2 \\
&= (u_i - u_{i+1})(u_{i+1} - u_i)s_i^2 + \pi_i^2 \\
&= (u_i - u_{i+1})(u_{i+1} - u_i) + \pi_i^2
\end{aligned}
$$

then

$$
\begin{aligned}
(u_i - u_{i+1})(u_{i+1} - u_i)\tau_i^2 &= (u_i - u_{i+1})(u_{i+1} - u_i) + \pi_i^2 \\
-(u_i - u_{i+1})^2\tau_i^2 &= -(u_i - u_{i+1})^2 + \pi_i^2 \\
(u_i - u_{i+1})^2\tau_i^2 &= (u_i - u_{i+1})^2 - \pi_i^2 \\
\tau_i^2 &= \frac{(u_i - u_{i+1} - \pi_i)(u_i - u_{i+1} + \pi_i)}{(u_i - u_{i+1})^2}
\end{aligned}
$$

Using that $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$, part (c) follows by (a long) straightforward calculation. □

From part (b) then τ_i is invertible in m in the following cases

(1) $b_i \neq b_{i+1}$
(2) $b_i = b_{i+1}$ and $a_{i+1} \neq a_i \pm \ell$

Let M be a $H_{\ell,n}$-module, M is called $\mathfrak{u}$-diagonalizable if it has a basis consisting of simultaneous eigenvectors for $\mathfrak{u}$.

2. Automorphisms of $H_{\ell,n}$

PROPOSITION 17. *For each $\kappa \in \mathbf{C}$ there exists unique automorphisms t_κ and ρ of $H_{\ell,n}$ given by*

$$
t_\kappa(u_i) = u_i + \kappa, \quad t_\kappa(\zeta_i) = \zeta_i, \quad t_\kappa(s_i) = s_i
$$
$$
\rho(u_i) = -u_{n-i+1}, \quad \rho(\zeta_i) = \zeta_{n-i+1} \quad \rho(s_i) = s_{n-i}.
$$

PROOF. We write $A = C\langle u_1, \ldots, u_n, \zeta_1, \ldots, \zeta_n, s_1 \ldots, s_{n-1}\rangle$ for the tensor algebra on this generators, then $H_{\ell,n} \simeq A/I$ where

$$
\begin{aligned}
I = \langle u_i u_j - u_j u_i, &\quad \text{for all } 1 \leq i, j \leq n \\
\zeta_j \zeta_i - \zeta_i \zeta_j, &\quad \text{for all } 1 \leq i, j \leq n \\
s_i s_j - s_j s_i, &\quad \text{for } j \neq i, i+1 \\
s_i s_{i+1} s_i - s_{i+1} s_i s_{i+1}& \\
\zeta_i u_j - u_j \zeta_i, &\quad \text{for all } 1 \leq i, j \leq n
\end{aligned}
$$

$$s_i u_j - u_j s_i \quad \text{for } j \neq i, i+1$$
$$s_i u_i - u_{i+1} s_i + \pi_i \rangle$$

To check t_κ is an automorphism, it is enough to check that $t_\kappa(a) \in I$, for all $a \in I$. We will check only the las relation. Note that $t_\kappa(\pi_i) = \pi_i$ since $t_\kappa(\zeta_i) = \zeta_i$, hence

$$\begin{aligned} t_\kappa(s_i)t_\kappa(u_i) - t_\kappa(u_{i+1})t_\kappa(s_i) + t_\kappa(\pi_i) &= s_i(u_i+\kappa) - (u_{i+1}+\kappa)s_i + \pi_i \\ &= s_i u_i + \kappa s_i - u_{i+1}s_i - \kappa s_i + \pi_i \\ &= s_i u_i - u_{i+1} s_i + \pi_i \in I \end{aligned}$$

In the same way we did for t_κ, we will check that

$$\rho : A/I \to A/I$$

is an automorphism.

- For the first relation

$$\begin{aligned} \rho(u_i u_j - u_j u_i) &= (-u_{n-i+1})(-u_{n-j+1}) - (-u_{n-j+1})(-u_{n-i+1}) \\ &= u_{n-i+1}u_{n-j+1} - u_{n-j+1}u_{n-i+1} \in I \end{aligned}$$

- For $1 \leq i, j \leq n$ the cases $\rho(\zeta_i\zeta_j - \zeta_j\zeta_i), \rho(s_i s_{i+1} s_i - s_{i+1} s_i s_{i+1}), \rho(\zeta_i u_j - u_j \zeta_i) \in I$ and $\rho(s_i s_j - s_j s_i), \rho(s_i u_j - u_j s_i) \in I$ for $j \neq i, i+1$ are similar.
- Note that

$$\rho(\pi_i) = \rho\left(\sum_{k=0}^{\ell-1} \zeta_i^k \zeta_{i+1}^{-k}\right) = \sum_{k=0}^{\ell-1} \zeta_{n-i+1}^k \zeta_{n-i}^{-k} = \pi_{n-i}$$

hence

$$\begin{aligned} \rho(s_i u_i - u_{i+1}s_i + \pi_i) &= \rho(s_i)\rho(u_i) - \rho(u_{i+1})\rho(s_i) + \rho(\pi_i) \\ &= s_{n-i}(-u_{n-i+1} - (-u_{n-i})s_{n-i} + \pi_{n-i} \\ &= u_{n-i}s_{n-i} - s_{n-i}u_{n-i+1} + \pi_i \\ &= s_{n-i}(s_{n-i}u_{n-i} - u_{n-i+1}s_{n-i} + \pi_{n-i})s_{n-i} \in I \end{aligned}$$

then $s_{n-i}u_{n-i} - u_{n-i+1}s_{n-i} + \pi_{n-i} \in I$.

□

Both of these automorphisms preserve the group algebra $\mathbf{C}G(\ell,1,n)$ and their restrictions to $\mathbf{C}G(\ell,1,n)$ are inner. Given an $H_{\ell,n}$-module M and an automorphism a of $H_{\ell,n}$ we write M^a for the $H_{\ell,n}$-module which is equal to M as an abelian group, and with the $H_{\ell,n}$-action defined by

$$h \cdot m = a(h)m \quad \text{for } h \in H_{\ell,n} \text{ and } m \in M.$$

If $a = \rho$ or $a = t_\kappa$ then M^a is isomorphic to M as a $\mathbf{C}G(\ell,1,n)$-module, since the restrictions of this automorphisms to $G(\ell,1,n)$ are inner.

3. $H_{\ell,n}$-modules via branching for $G(\ell,1,n)$

Let $m\in\mathbf{Z}_{>0}$. Then for all $1\le i\le n$ and $1\le j\le n-1$ there exists a map $H_{\ell,n}\to\mathbf{C}G(\ell,1,m+n)$ given by

$$u_i\mapsto\phi_{m+i}$$

$$s_j\mapsto s_{m+j}$$

$$\zeta_i\mapsto\zeta_{m+i}$$

The image of this map is contained in the centralizer $C_{G(\ell,1,m+n)}G(\ell,1,m)$ so that $H_{\ell,n}$ acts on the module

$$S^{\lambda\setminus\mu}=\mathrm{Hom}_{\mathbf{C}G(\ell,1,m)}(S^{\mu},\mathrm{Res}^{m+n}_{m}(S^{\lambda}))$$

for all pairs λ,μ of ℓ-partitions, where λ is an ℓ-partition of $m+n$ and μ is an ℓ-partition of m. It follows from Young's orthogonal form that $S^{\lambda\setminus\mu}=0$ unless $\mu\subseteq\lambda$ in which case $S^{\lambda\setminus\mu}$ has a basis indexed by the set of standard Young tableaux on the skew diagram $\lambda\setminus\mu$.

Given $T\in\mathrm{SYT}(\mu)$ and $U\in\mathrm{SYT}(\lambda\setminus\mu)$ we define $T\cup U\in\mathrm{SYT}(\lambda)$ by

$$T\cup U(b)=\begin{cases}T(b), & \text{if } b\in\mu\\ U(b)+m, & \text{if } b\in\lambda\setminus\mu\end{cases}$$

Then

$$S^{\lambda\setminus\mu}=\mathrm{Hom}_{\mathbf{C}G(\ell,1,m)}(S^{\mu},\mathrm{Res}^{m+n}_{m}(S^{\lambda}))$$

and we define $\psi_U\in S^{\lambda\setminus\mu}$ by the formula

$$\psi_u(v_T)=v_{T\cup U}$$

THEOREM 3.2. *$S^{\lambda\setminus\mu}$ is an irreducible $H_{\ell,n}$-module with basis* $\{\psi_U\,|\,U\in\mathrm{SYT}(\lambda\setminus\mu)\}$

Moreover we will see that each $\mathfrak{u}$-diagonalizable $H_{\ell,n}$-module can be obtained as one of this.

Let $D\subseteq\mathbf{R}^2\times\mathbf{Z}/\ell$ be a skew shape with connected components $D_1,\ldots,D_k$. After diagonal slides, we may assume that each D_i is such that for any $(x,y)\in D_i$ we have $y_i\in\mathbf{Z}$ and moreover the set of y-coordinates of distinct D_i's are disjoint. We may choose (non-unique) $\alpha_1,\ldots,\alpha_k\in\mathbf{C}$ and integral skew shapes $\lambda_1\setminus\mu_1,\ldots,\lambda_k\setminus\mu_k$ such that

$$D_i=\lambda_i\setminus\mu_i+(\alpha_i,0)$$

so that their union is disjoint and a skew shape,

$$\lambda\setminus\mu=\coprod\lambda_i\setminus\mu_i$$

and so $\lambda_1\setminus\mu_1,\ldots,\lambda_k\setminus\mu_k$ are the connected components of their disjoint union $\lambda\setminus\mu$. Then define

$$S^D=\mathrm{Ind}_A^{H_{\ell,n}}\left(\otimes_i\,(S^{\lambda_i\setminus\mu_i})^{t_{\alpha_i}}\right)$$

where $A=H_{\ell,n_1}\otimes H_{\ell,n_2}\otimes\cdots\otimes H_{\ell,n_k}$ and $n_1,\ldots,n_k$ are certain nonnegative integers such that $\sum_{n_i}=n$.

THEOREM 3.3. *S^D defined as above is an irreducible $H_{\ell,n}$-module.*

In order to prove this result we will see some combinatorial results involving skew shaped diagrams and tableaux.

Let D be a skew diagram of shape $\lambda \setminus \mu$, for a box $b \in D$ with $b = (x, y)$ we define $R(b) = y$ and $C(b) = x$, i.e. $R(b)$ and $C(b)$ are, respectively, the row and the column of the box b in D. Let T be a standard Young tableau in D, define $I(T)$ as the set of pairs (i, j) of numbers $1 \le i < j \le n$ such that j appears in D in a row strictly above than i, i.e.

$$I(T) := \{(i,j) \mid R(T^{-1}(i)) > R(T^{-1}(j)),\ \text{for } 1 \le i < j \le n\}$$

EXAMPLE 13. *Let D be the skew shape*

$$D = \begin{array}{ccccc} \cline{3-5} & & \multicolumn{1}{|c|}{} & & \multicolumn{1}{c|}{} \\ \cline{1-5} \multicolumn{1}{|c|}{} & & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \\ \cline{1-4} \multicolumn{1}{|c|}{} & & & & \\ \cline{1-1} \end{array}$$

and let T be the tableau in D given by

$$T = \begin{array}{ccccc} \cline{3-5} & & \multicolumn{1}{|c|}{1} & 3 & \multicolumn{1}{c|}{4} \\ \cline{1-5} \multicolumn{1}{|c|}{2} & 5 & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{8} & \\ \cline{1-4} \multicolumn{1}{|c|}{7} & & & & \\ \cline{1-1} \end{array}$$

then

$$I(T) = \{(2,3),(7,8)\}$$

since the box containing the number 2 *is in a row above than the box containing the number* 3*, same occurs with boxes containing numbers* 7 *and* 8.

LEMMA 5. *Suppose $(i, i+1) \in I(T)$ and $s_i T \in \mathrm{SYT}(D)$ then $I(s_i T) = s_i(I(T) \setminus \{(i,i+1)\})$.*

PROOF. Note that $(s_i T)^{-1}(s_i(j)) = T^{-1}(j)$ then $R((s_i T)^{-1}(s_i(j))) = R(T^{-1}(j))$, then for $(j,k) \in I(T)$ such that $(j,k) \neq (i,i+1)$ we have

$$R((s_i T)^{-1}(s_i(j))) = R(T^{-1}(j)) > R(T^{-1}(k)) = ((s_i T)^{-1}(s_i(k)))$$

then $s_i(j,k) \in I(s_i T)$. And $(i,i+1) \notin I(s_i T)$ since

$$R((s_i T)^{-1}(i+1)) = R(T^{-1}(i)) > R(T^{-1}(i+1)) = ((s_i T)^{-1}(i)).$$

□

Let D be diagram with n boxes and let T be a tableau on D, we write $rw(T)$ for the *reading word* of a tableau T which is the word obtained by concatenating the rows of the diagram D, starting from the bottom row in English notation. The *row reading tableau* of shape D is the tableau T with reading word $123\cdots n$, this is a Standard Young tableau.

EXAMPLE 14. *The reading word of the tableau of the previous example es* 13425687*. The row reading tableau in $\lambda = (4,3,1)$ is*

$$T = \begin{array}{ccccc} \cline{3-5} & & \multicolumn{1}{|c|}{1} & 2 & \multicolumn{1}{c|}{3} \\ \cline{1-5} \multicolumn{1}{|c|}{4} & 5 & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{7} & \\ \cline{1-4} \multicolumn{1}{|c|}{8} & & & & \\ \cline{1-1} \end{array}$$

LEMMA 6. *If $(i, i+1) \notin I(T)$ for all $1 \le i \le n$ then T is the* row reading tableau.

PROOF. We suppose that T is *row reading* until the appearance of i, i.e. for all $j<i$ we have $\beta(T^{-1}(j+1))=\beta(T^{-1}(j))$ or $\beta(T^{-1}(j+1))=\beta(T^{-1}(j))+1$

$R(T^{-1}(j))\leq R(T^{-1}(i))$ and one of the following

(1) $R(T^{-1}(j))=R(T^{-1}(j+1))$ and $C(T^{-1}(j+1))=C(T^{-1}(j))+1$ or
(2) $R(T^{-1}(j+1))>R(T^{-1}(j))$

Now since $(i,i+1)\notin I(T)$ then $R(T^{-1}(i+1))\geq R(T^{-1}(i))$ and since T is standard we have $C(T^{-1}(i))\leq C(T^{-1}(i+1))$.

(1) If $R(T^{-1}(i))=R(T^{-1}(i+1))$ then necessarily $C(T^{-1}(i+1))=C(T^{-1}(i))+1$.
(2) If $R(T^{-1}(i+1))>R(T^{-1}(i))$ from our assumption of D being *row reading* until the appearance of i in this case we have that $T^{-1}(i+1)$ is the first box in its row. Now if we suppose that there is a box b on the right of $T^{-1}(i)$ such that $T(b)=i+k$, for some integer $k>1$ then $R(T^{-1}(i+k-1))>R(T^{-1}(i))=R(T^{-1}(i+k))$ then $(i+k-1,i+k)\in I(T)$ which is a contradiction and then $T^{-1}(i)$ is the last box in its row. Besides, there is not any box b in D in a row between i and $i+1$ (i.e. with $R(T^{-1}(i))<R(T^{-1}(b))<R(T^{-1}(i+1)))$ then $T(b)=i+k$ for some k and then $(i+1,T(b))\in I(T)$.

□

Moreover note that if T is the *row reading tableau* in D then $R(T^{-1}(i))\leq R(T^{-1}(j))$ for all $1\leq i<j\leq n-1$ then clearly $(i,j)\notin I(T)$ and $I(T)=\emptyset$. If λ is a row then there is only one standard Young tableau which is the row reading.

LEMMA 7. *Let D be a skew diagram with n boxes. Then given T and T' two standard young tableaux on D there exists a sequence of simple transpositions $s_{i_1},s_{i_2},\ldots,s_{i_p}$ such that*

$$s_{i_p}s_{i_{p-1}}\cdots s_{i_1}(T)=T'$$

and $s_{i_k}s_{i_{k-1}}\cdots s_{i_1}(T)\in \mathrm{SYT}(D)$ for all $1\leq k\leq p$.

PROOF. Without lost of generality we may assume T is the row reading tableau and proceed by induction on the number $\#I(T')$. If $\#I(T')=1$ then T' is not the reading tableau and there is exactly a pair $(i,j)\in I(T')$, by Lemma 6 necessarily $j=i+1$. Since T is standard then $rw(T')=12\cdots(i-1)(i+1)i(i+2)\cdots n$ then necessarily $T'^{-1}(i)$ is the last box on a row and $T'^{-1}(i+1)$ is the first box of the next row, then $s_i(T')=T$ which is standard.

For the general case let $T'\in\mathrm{SYT}(D)$ such that $\#I(T')=l>0$, then T' is not the row reading tableau and by Lemma 6 $(i,i+1)\in I(T')$ for some $1\leq i\leq n-1$ then $T'(b)<i$ for boxes b such that $R(b)=R(T'^{-1}(i+1))$ and $C(b)<C(T'^{-1}(i+1))$ and $T'(b)>i+1$ for boxes $R(b)=R(T'^{-1}(i+1))$ and $C(b)>C(T'^{-1}(i+1))$. Likely, $T'(b)<i$ for boxes b such that $R(b)=R(T'^{-1}(i))$ and $C(b)<C(T'^{-1}(i))$ and $T'(b)>i+1$ for boxes $R(b)=R(T'^{-1}(i))$ and $C(b)>C(T'^{-1}(i))$ then $s_iT'\in\mathrm{SYT}(D)$ and by Lemma 5 $I(s_iT')=s_i(I(T')\setminus\{(i,i+1)\})$. Thus $\#I(s_iT')=I(T')-1$, and by induction on s_iT' there exists a sequence $s_{i_2},s_{i_3},\ldots,s_{i_l}$ of simple transpositions such that $s_{i_l}s_{i_{l-1}}\cdots s_{i_2}(s_iT')=T$ and

$$s_{i_k}s_{i_{k-1}}\cdots s_{i_2}(s_iT')\in\mathrm{SYT}(D)\text{ for all }2\leq k\leq l$$

then the result follows by indexing $s_{i_1} = s_i$. □

Note that this sequence of simple transpositions given in the preceding lemma corresponds to a sequence of invertible intertwiners, and as a consequence of this lemma we can "connect" every pair of standard Young tableaux of shape D by a sequence of invertible intertwiners. Now we are in conditions to prove Theorem 3.3.

PROOF OF THEOREM 3.3. Let n_i be the number of boxes of the diagram $\lambda_i \setminus \mu_i$, and $n = \sum_i^k n_i$.

By Theorem 3.2 each $S^{\lambda_i \setminus \mu_i}$ has basis $\{\psi_U \,|\, U \in \mathrm{SYT}(\lambda_i \setminus \mu_i)\}$. Since $\lambda \setminus \mu = \coprod \lambda_i \setminus \mu_i$ then the set

$$\left\{\psi_U \,\middle|\, U \in \mathrm{SYT}(\lambda \setminus \mu) \text{ such that } U(b) \in \{n_{i-1}+1, \ldots, n_i\} \text{ if } b \in \lambda^i \setminus \mu^i\right\} \tag{3.1}$$

is a $\mathbf{C}$ basis of $\bigotimes_i S^{\lambda_i \setminus \mu_i}$, where $n_0 = 0$.

Note that $H_{\ell,n}$ can be understand as the free module over the subalgebra $H_{\ell,n_1} \otimes H_{\ell,n_2} \otimes \cdots \otimes H_{\ell,n_k}$ which by PBW-theorem for $H_{\ell,n}$ has basis indexed by the set

$$\left\{w \,\middle|\, w \in S_n / S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}\right\} \tag{3.2}$$

Then the $H_{\ell,n}$-module

$$S^D = \mathrm{Ind}_{H_{\ell,n_1} \otimes H_{\ell,n_2} \otimes \cdots \otimes H_{\ell,n_k}}^{H_{\ell,n}} \left(\bigotimes_i S^{\lambda_i \setminus \mu_i}\right) = H_{\ell,n} \otimes_{H_{\ell,n_1} \otimes H_{\ell,n_2} \otimes \cdots \otimes H_{\ell,n_k}} \left(\bigotimes_i S^{\lambda_i \setminus \mu_i}\right)^{t_{\alpha_i}}$$

has a basis $\psi_U \otimes w$ where ψ_U runs over the set (3.1) and w runs over the set (3.2), these pairs are in bijection with standard Young tableaux of shape $\lambda \setminus \mu$.

Let U as in (3.1) and w as in (3.2) with minimal length (as a word on simple transpositions). For a fixed reduced expression for $w = s_{i_1} \cdots s_{i_p}$ the correspondence given by

$$s_{i_1} \cdots s_{i_p} \to \tau_{i_1} \cdots \tau_{i_p}$$

is well defined, since the intertwining operators τ_i satisfy the braid relations (part (c) in Proposition 16). Note that $wU \in \mathrm{SYT}(\lambda \setminus \mu)$ and

$$\tau_{i_1} \cdots \tau_{i_p}(1 \otimes \psi_U) = w \otimes \psi_U + \sum v \otimes \psi_U \tag{3.3}$$

is an eigenvector with eigenvalue wU, where the length of element v is lower than the length of w. This gives rise to a basis for S^D of eigenvectors of $\mathfrak{u}$, now if S^D has a nonzero submodule, such submodule must contain one of this eigenvectors and by Lemma 7 we can connect a pair of two Young tableaux by a sequence of invertible intertwiners, hence S^D is irreducible. □

Up to isomorphism, the representation S^D is independent of the choices made in its construction and, since the automorphisms t_κ are the identity in $G(\ell, 1, n)$, its restriction to $G(\ell, 1, n)$ is isomorphic to $S^{\lambda \setminus \mu}$.

4. Littlewood-Richardson numbers

Assuming ℓ is fixed, then without causing confusion we may write $G_n = G(\ell,1,n)$ and introduce the following notation

$$\mathrm{Ind}_m^n = \mathrm{Ind}_{\mathbf{C}G_m}^{\mathbf{C}G_n}, \quad \mathrm{Res}_m^n = \mathrm{Res}_{\mathbf{C}G_m}^{\mathbf{C}G_n}, \quad \mathrm{Ind}_{m,n}^{m+n} = \mathrm{Ind}_{\mathbf{C}(G_m\times G_n)}^{\mathbf{C}G_{m+n}} \quad \text{and} \quad \mathrm{Res}_{m,n}^{m+n} = \mathrm{Res}_{\mathbf{C}(G_m\times G_n)}^{\mathbf{C}Gm+n}$$

for the appropriate functors of induction and restriction.

By $\otimes$-Hom adjunction and Frobenius reciprocity we have isomorphisms

$$\begin{aligned}\mathrm{Hom}_{G_n}(S^\nu, \mathrm{Hom}_{G_m}(S^\mu), \mathrm{Res}_m^{m+n}(S^\lambda)) &\cong \mathrm{Hom}_{G_m\times G_n}(S^\mu\otimes S^\nu, \mathrm{Res}_{m,n}^{m+n} S^\lambda)\\ &\cong \mathrm{Hom}_{G_{m+n}}(\mathrm{Ind}_{m,n}^{m+n}(S^\mu\otimes S^\nu), S^\lambda)\end{aligned}$$

Then the *(cyclotomic) Littlewood-Richardson number* $c_{\mu\nu}^\lambda$ is given by

$$c_{\mu\nu}^\lambda = \dim_{\mathbf{C}}\left(\mathrm{Hom}_{G_{m+n}}\left(\mathrm{Ind}_{m,n}^{m+n}(S^\mu\otimes S^\nu), S^\lambda\right)\right)$$

Defining

$$c_\nu^{\lambda\setminus\mu} = \dim_{\mathbf{C}}\left(\mathrm{Hom}_{G_n}\left(S^\nu, \mathrm{Hom}_{G_m}(S^\mu, \mathrm{Res}_m^{m+n}(S^\lambda))\right)\right)$$

we have $c_\nu^{\lambda\setminus\mu} = c_{\mu\nu}^\lambda$.

If D is any skew diagram such that $\lambda\setminus\mu$ may be obtained form D by translating its connected components without merging any of them, then $S^D \cong S^{\lambda\setminus\mu}$ as G_n-modules, and hence defining

$$c_\nu^D = c_\nu^{\lambda\setminus\mu}$$

we have an isomorphism of G_n-modules.

Given $\lambda = (\lambda^0,\dots,\lambda^{\ell-1})$, $\mu = (\mu^0,\dots,\mu^{\ell-1}))$ and $\nu = (\nu^0,\dots,\nu^{\ell-1})$ be ℓ-partitions. Note that S^λ can be realized as

$$S^\lambda = \mathrm{Ind}_{G_{m_0}\times G_{m_1}\times\cdots\times G_{m_{\ell-1}}}\left(S^{\lambda^0}\otimes S^{\lambda^1}\otimes\cdots\otimes S^{\lambda^{\ell-1}}\right)$$

where $m_i = |\lambda_i|$ and G_{m_i} acts on the Specht module S^{λ^i} via the surjective application $\mathbf{C}G_{m_i}\to \mathbf{C}S_{m_i}$ given by

$$\begin{aligned} s &\to s \quad \text{for all } s\in S_{m_i}\\ \zeta_j &\to \zeta^i \quad \text{for all } j.\end{aligned}$$

The Littlewood-Richardson number $c_{\lambda\mu}^\nu$ may be expressed as a product of the classical Littlewood-Richardson numbers as follows

$$c_{\lambda\mu}^\nu = \prod_{j=0}^{\ell-1} c_{\lambda_j\mu_j}^{\nu_j}$$

Now we give the most classical construction to compute the classical Littlewood-Richardson numbers.

A *Littlewood-Richardson tableau* on a skew diagram D (for $\ell = 1$) is a function $T : D \to \mathbf{Z}_{>0}$ such that

(1) The tableau T is *column strict* in the sense that $T(x,y) < T(x,y+1)$ whenever $(x,y),(x,y+1) \in D$ and $T(x,y) \le T(x+1,y)$ whenever $(x,y),(x+1,y) \in D$, and

(2) The row-reading word $T_1T_2\dots T_n$ (obtained by reading the entries from T from top to bottom and right to left) of T satisfies the *LR property*: for each integer $i \in \mathbf{Z}_{>0}$ and each $1 \le k \le n$, the number of occurrences of i in the sequence $T_1T_2\dots T_k$ is at leas as large as the number of occurrences of $i+1$.

The *weight* of a tableau T is the sequence $\nu_1,\nu_2,\dots$ where ν_i is the number of boxes $b \in D$ with $T(b) = i$. The Littlewood-Richardson coefficient c^D_ν is then the number of Littlewood-Richardson tableaux on D of weight ν.

We write $\mathfrak{h} = \mathbf{C}^n$ for the defining representation of G_n, then the Littlewood-Richradson numbers arise in the calculation of the tensor products $S^\nu \otimes \Lambda^i(\mathfrak{h}^*)$ as G_n-modules. For $\lambda = (\lambda^0,\lambda^1,\dots,\lambda^{\ell-1})$ an ℓ-partition of n we have

$$S^\lambda \otimes \Lambda^n(\mathfrak{h}^*) = S^{\lambda^t}$$

where the partition λ^t transpose of the ℓ-partition λ. When n is fixed and clear for the context, we denote by $\det^{-1}$ to the one dimensional character of G_n acting on $\Lambda^n(\mathfrak{h}^*)$. Note that $\Lambda^i(\mathfrak{h}^*)$ contains the vector $v_{n-i+1} \wedge v_{n-i+2} \wedge \cdots \wedge v_n$, which is fixed by G_{n-i} and transforms like $\det^{-1}$ under G_i embedded in G_n via the las i coordinates, and is therefore induced from a one-dimensional representation

$$\Lambda(\mathfrak{h}^*) = \mathrm{Ind}^{G_n}_{G_{n-i}\times G_i}(1 \times \det{}^{-1}).$$

Now we compute the tensor product of this representation with S^ν as follows

$$S^\nu \otimes \Lambda^i(\mathfrak{h}^*) \cong S^\nu \otimes \mathrm{Ind}^{G_n}_{G_{n-i}\times G_i}(1 \times \overset{-1}{\det}) \cong \mathrm{Ind}^{G_n}_{G_{n-i}\times G_i}\left(\mathrm{Res}^n_{n-i,i}(S^\nu) \otimes (1 \times \det{}^{-1})\right)$$

and hence

$$\begin{aligned}\mathrm{Hom}_{G_n}\left(S^\mu, S^\nu \otimes \Lambda^i(\mathfrak{h}^*)\right) &\cong \mathrm{Hom}_{G_n}\left(S^\mu, \mathrm{Ind}^n_{n-i,n}(\mathrm{Res}^n_{n-i,i}(S^\nu) \otimes (1 \det{}^{-1}))\right)\\ &\cong \mathrm{Hom}_{G_{n-i,i}}\left(\mathrm{Res}^n_{n-i,i}(S^\mu), \mathrm{Res}^n_{n-i,i}(S^\nu) \otimes (1 \times \det{}^{-1})\right).\end{aligned}$$

Taking dimensions gives

$$\dim\left(\mathrm{Hom}_{G_n}(S^\mu, S^\nu \otimes \Lambda^i(\mathfrak{h}^*))\right) = \sum_{\substack{\eta\vdash n-i\\ \chi\vdash i}} c^\nu_{\eta\chi} c^{\nu^t}_{\eta\chi} \tag{4.1}$$

5. Classification of irreducible $\mathfrak{u}$-diagonalizable $H_{\ell,n}$-modules

LEMMA 8. *Let $(a_1,\dots,a_n,\zeta^{b_1},\dots,\zeta^{b_n})$ be a sequence satisfying the property: if $i < j$ with $a_i = a_j$ and $b_i = b_j \mod \ell$, then there are $i < k, m < j$ with $b_k = b_m = b_i \mod \ell$ and*

$$a_k = a_i + \ell, \quad a_m = a_i - \ell.$$

Then there is a skew shape D and a standard Young tableau T of shape D satisfying

$$\ell\mathrm{ct}(T^{-1}(i)) = a_i \quad \textit{and} \quad \beta(T^{-1}(i)) = b_i, \quad \textit{for } 1 \le i \le n.$$

Moreover, T and D are unique up to diagonal slides of their connected components.

PROOF. We proceed by induction on n. Let $\ell = 1$ then we have the sequence $(a_1, \dots, a_n)$. In this case D has only one component, if $n = 1$ then D has a single box with content a_1 and T is the standard Young tableau that assigns to that box the number 1. For $n > 1$, by induction we suppose that the sequence $(a_1, \dots, a_{n-1})$ possesses a standard Young tableau T' on a skew diagram D'. Notice that the condition: "if $i < j$ with $a_i = a_j$ then there are $i < k, m < j$ with $a_k = a_i + \ell$ and $a_m = a_i - \ell$" implies that for boxes of D' such that $\mathrm{ct}(T^{-1}(k)) = a_i + 1$ and $\mathrm{ct}(T^{-1}(m)) = a_i - 1$ we have

$$\mathrm{ct}(T^{-1}(k)) = \mathrm{ct}(T^{-1}(i)) + 1 \quad \text{and} \quad \mathrm{ct}(T^{-1}(m)) = \mathrm{ct}(T^{-1}(i)) - 1.$$

Then the condition implies that D' is in fact a skew diagram and that D' possesses an addable box b with $\mathrm{ct}(b) = a_n$. We obtain T and D by adjoining b to D' and defining $T(b) = n$.

For the case $\ell \neq 1$ then D has ℓ components. We obtain the component D_{i_1} by considering the subsequence $(a_{i_1}, \dots, a_{i_p})$ of $(a_1, \dots, a_n)$ such that $b_{i_1} = \dots = b_{i_p} \mod \ell$ and proceeding as before. □

In Section 1 of the next Chapter we have computed explicitly in the cyclotomic rational Cherednik algebra of type B the skew diagram D and the standard Young tableau T mentioned in this Lemma.

THEOREM 3.4. *Let M be an irreducible $\mathfrak{u}$-diagonalizable $H_{\ell,n}$-module and suppose $m \in M$ satisfies*

$$u_i m = a_i m \quad \textit{and} \quad \zeta_i m = \zeta^{b_i} m \quad \textit{for } 1 \leq i \leq n.$$

Then there is a standard Young tableau T on a skew shape D such that $a_i = \ell\,\mathrm{ct}(T^{-1}(i))$ and $b_i = \beta(T^{-1}(i))$ for $1 \leq i \leq n$ and $M \cong S^D$, and moreover T and D are unique up to diagonal slides of their connected components.

PROOF. For the first part we will check that the sequence $(a_1, \dots, a_n, \zeta^{b_1}, \dots, \zeta^{b_n})$ satisfies the hypothesis of Lemma 8. Suppose that there is an index $j > i$ with $a_i = a_j$ and $b_i = b_j \mod \ell$, we claim that $j > i + 2$.

Suppose that $j = i + 1$, then

$$(a_i, \zeta^{b_i}) = (a_{i+1}, \zeta^{b_{i+1}}),$$

and by the relations between u_i and s_i we have

$$u_i s_i m = (s_i u_{i+1} - \pi_i) m = s_i u_{i+1} m - \pi_i m = s_i a_i m - \ell m = a_i s_i m - \ell m \tag{5.1}$$

$$u_{i+1} s_i m = (s_i u_i + \pi_i) m = s_i u_i m + \pi_i m = s_i a_i m + \ell m = a_i s_i m + \ell m \tag{5.2}$$

Notice that $s_i m = \pm m$, since $s_i^2 = 1$. Then if we suppose $s_i m = m$ then $u_i s_i m = u_i m = a_i m$ and if $s_i m = -m$ then $u_i s_i m = -a_i m$, both cases leads to a contradiction with (5.1). We conclude that $s_i m$ and m are linearly independent and the subspace of M generated by m and $s_i m$ is stable under u_i and u_{i+1}. Moreover $u_i = \begin{pmatrix} a_i & -\ell \\ 0 & a_i \end{pmatrix}$ and $u_{i+1} = \begin{pmatrix} a_i & \ell \\ 0 & a_i \end{pmatrix}$, then M

is not diagonalizable since there exist this Jordan blocks, which is a contradiction and then $j \neq i+1$.

For $j = i+2$, note that if τ_i was invertible then by part (a) of Proposition 16 $\tau_i(m)$ has eigenvalue

$$(a_1, \ldots, a_{i+1}, a_i, a_{i+2}, \ldots, a_n, \zeta^{b_1}, \ldots, \zeta^{b_{i+1}}, \zeta^{b_i}, \zeta^{b_{i+2}}, \ldots, \zeta^{b_n})$$

with $a_i = a_{i+1}$ and $b_i = b_{i+2}$ which contradicts the previous case. Hence τ_i is not invertible, then necessarily $b_i = b_{i+1}$ and $a_{i+1} = a_i \pm \ell$. If $a_{i+1} = a_i + \ell$ then s_i is acting by 1 on m and s_{i+1} is acting by -1 on m, which contradicts the braid relation. Analogously if $a_{i+1} = a_i - \ell$, s_i acts by -1 on m and s_{i+1} acts by 1, contradicting the braid relation again.

Since $j > i+2$ and the operators τ_i and τ_j are invertible then by Lemma 8 there is T and D as required.

It remains to prove that $M \cong S^D$. Let T_0 be the row reading tableau and ψ_{T_0} the corresponding eigenvector in S^D. Let $m \in M$ with $wt(T_0) = wt(m)$, we will show there exists a homomorphism $\varphi : S^D \to M$ such that $\varphi(\psi_{T_0}) = m$. Given $T \in \mathrm{SYT}(D)$ by Lemma 7 there exists a sequence of simple transpositions $s_{i_1}, \ldots, s_{i_p}$ such that $s_{i_p} \cdots s_{i_1} T = T_0$ with $s_{i_k} s_{i_{k-1}} \cdots s_{i_1} T_0 \in \mathrm{SYT}(D)$, this produces a sequence of invertible intertwiners $\tau_{i_1}, \ldots, \tau_{i_p}$. Choosing p minimal (as a word in the symmetric group) we define a map given by

$$\varphi(\tau_{i_p} \ldots \tau_{i_1} \psi_T) = \tau_{i_p} \cdots \tau_{i_1} m.$$

Firstly, this map is well-defined which can be proved by the solution of the word problem in the braid group since the intertwiners τ_i satisfy braid relations. Secondly, it is compatible with the action of $\mathfrak{u}$ since commutes with the action τ_i. Thus we have produced a non-zero homomorphism between the irreducible $H_{\ell,n}$-modules S^D and M, which finishes the proof. ☐

6. The Dunkl-Opdam subalgebra

The *Dunkl-Opdam* subalgebra of H_c is the subalgebra $\mathfrak{t}$ generated by the elements $z_1, \ldots, z_n$ and $\zeta_1, \ldots, \zeta_n$.

The map given in Proposition 14 is an injection of $H_{\ell,n}$ into H_c. Via this injection u_i acts on $f_{P,Q}$ by

$$u_i f_{P,Q} = \frac{1}{c_0}\Big(Q\big(P^{-1}(i)\big)+1 - \big(d_{\beta(P^{-1}(i))} - d_{\beta(P^{-1}(i))-Q(P^{-1}(i))-1}\big)\Big) - \ell\,\mathrm{ct}(P^{-1}(i)).$$

PROPOSITION 18. *Let $L_c(\lambda)$ be a $\mathfrak{t}$-diagonalizable H_c-module. Then for $Q \in \mathrm{Tab}_c(\lambda)$ fixed, the span*

$$L_Q = \mathbf{C}\{f_{P,Q} \mid (P,Q) \in \Gamma_c(\lambda)\}$$

is an irreducible $H_{\ell,n}$-module.

For a proof of this result we refer [**4**].

CHAPTER 4

Main theorem

In order to proof the main theorem of [**4**] we prove a few results. First, given a skew diagram D we define a skew diagram D^r, the *reverse*, as follows. Twisting S^D by the automorphism ρ of $H_{\ell,n}$ we obtain another $\mathfrak{u}$-diagonalizable module $(S^D)^\rho$, and D^r is the skew diagram with $S^{D^r} \cong (S^D)^\rho$.

THEOREM 4.1. *Let $L_c(\lambda)$ be a $\mathfrak{t}$-diagonalizable H_c-module and let d be a positive integer. Then as a $H_{\ell,n}$-module, the degree $c_\lambda + d$ part of $L_c(\lambda)$ is semisimple and isomorphic to the direct sum*

$$L_c(\lambda)_{c_\lambda+d} \cong \bigoplus_{Q\in\mathrm{Tab}_c(\lambda),|Q|=d} S^{s_c(Q)^r}$$

PROOF. By previous Proposition L_Q is an irreducible $H_{\ell,n}$-module. Then there is a unique (up to diagonal slides of connected components) skew diagram D and a standard Young tableau T on D with

$$\mathrm{ct}(T^{-1}(i)) = \frac{1}{c_0}\Big(Q\big(P^{-1}(i)\big)+1-\big(d_{\beta(P^{-1}(i))}-d_{\beta(P^{-1}(i))-Q(P^{-1}(i))-1}\big)\Big)-\ell\,\mathrm{ct}(P^{-1}(i)).$$

It follows from this, the definition of $s_c(Q)$ and Theorem 3.4 that L_Q is isomorphic to $S^{s_c(Q)^r}$ as an $H_{\ell,n}$-module.

To establish irreducibility we use Lemma 7.4 of [**8**]. This lemma, translated into the notation we use here, shows that given P, P' with $(P,Q),(P',Q') \in \Gamma_c(\lambda)$ there is a sequence of simple transpositions $s_{i_1},\dots,s_{i_p}$ such that the $H_{\ell,n}$ submodule of $L_c(\lambda)$ generated by $f_{P,Q}$ is L_Q; together with the fact that any $H_{\ell,n}$-submodule of L_Q must contain some weight vector this finishes the proof. □

The following corollary proves the first part of Theorem 0.1.

COROLLARY 2. *Suppose $L_c(\lambda)$ is $\mathfrak{t}$-diagonalizable is a $\mathbf{C}G(\ell,1,n)$-module, the degree $c_\lambda+d$ part of $L_c(\lambda)$ is*

$$L_c(\lambda)_{c_\lambda+d} \cong \bigoplus_{\substack{Q\in Tab_c(\lambda),|Q|=d\\ \mu\in P_{\ell,n}}} \left(S^\mu\right)^{\oplus c_\mu^{s_c(Q)}}$$

PROOF. Twisting the representation $S^{s_c(Q)^r}$ by the automorphism ρ of $H_{\ell,n}$ shws that as a $\mathbf{C}G(\ell,1,n)$-module, L_Q is isomorphic to $S^{s_c(Q)}$, which proves the corollary. □

For a parameter c, each standard module has a basis $f_{P,Q}$ consisting of eigenvectors for the Dunkl-Opdam subalgebra $\mathfrak{t}$. Each irreducible quotient $L_c(\lambda)$ that is $\mathfrak{t}$-diagonalizable has a certain subset of these $f_{P,Q}$ as a basis; this subset is indexed by pairs (P,Q) of tableaux on λ satisfying the following properties:

(1) Q is a filling of the boxes of λ by non-negative integers such that $Q(b) \le Q(b')$ whenever $b \le b'$,
(2) P is a bijection from the boxes of λ to the set of integers $\{1,2,\dots,n\}$ such that if $b \le b'$ and $Q(b) \le Q(b')$ then $P(b) > P(b')$,
(3) If b is a box of λ and k is a positive integer such that
$$\mathrm{ct}_c(b) = d_{\beta(b)-k} + k$$
then $Q(b) < k$, and
(4) If b and b' are boxes of λ and k is a positive integer with $k = \beta(b) - \beta(b') \mod \ell$ and such that
$$\mathrm{ct}_c(b) - \mathrm{ct}_c(b') = k \pm \ell c_0$$
then
$$Q(b) \le Q(b') + k$$
with equality implying $P(b) > P(b')$.

We write $\Gamma_c(\lambda)$ for the set of such pairs (P,Q) and define $\mathrm{Tab}_c(\lambda)$ to be the set of Q such that there exists a P with $(P,Q) \in \Gamma_c(\lambda)$, so that $\mathrm{Tab}_c(\lambda)$ is the projection of $\Gamma_c(\lambda)$ on its second coordinate
$$\mathrm{Tab}_c(\lambda) = \pi_2(\Gamma_c(\lambda)).$$

For each $Q \in \mathrm{Tab}_c(\lambda)$ there is a skew diagram $s_c(Q)$ which is unique up to diagonal slides of its connected components. Theorem 3.4 and first line of the proof of Theorem 4.1 give an algorithm determining a standard Young tableau T with
$$\beta(T^{-1}(i)) = \beta(P^{-1}(n-i+1)) - Q(P^{-1}(n-i+1)) \tag{0.1}$$
and
$$\begin{aligned}\mathrm{ct}(T^{-1}(i)) =& \mathrm{ct}(P^{-1}(n-i+1)) - \frac{1}{\ell c_0}\Big(Q(P^{-1}(n-i+1)) \\ &- \Big(d_{\beta(P^{-1}(n-i+1))} - d_{\beta\left(P^{-1}(n-i+1)-Q(P^{-1}(n-i+1))\right)}\Big)\Big).\end{aligned} \tag{0.2}$$

Then we define $s_c(Q)$ to be the shape of T, which is independent of the choice of P. We define the *degree* $|Q|$ of Q by
$$|Q| = \sum_{b \in \lambda} Q(b).$$

Finally we deduce the second part of Theorem 0.1

COROLLARY 3. *Suppose $L_c(\lambda)$ is unitary. Let i be a non-negative integer and let μ be an ℓ-partition of n. Then*
$$\dim\Big(\mathrm{Ext}^i\Big(\Delta_c(\mu), L_c(\lambda)\Big)\Big) = \sum c_{\nu}^{S_c(Q)} c_{\eta\chi}^{\nu} c_{\eta\chi^t}^{\mu}.$$

PROOF. The formula follows from the preceding corollary and applying Theorem 1.6 and (4.1) since $L_c(\lambda)$ is unitary. ☐

1. Type B examples

Recalling definitions and notation from Chapter 2, the cyclotomic algebra of type B is the cyclotomic rational Cherednik algebra attached to Weyl group of type B with deformation parameter $(c,d) \in \mathbf{R}^2$ and $\mathbf{C}^n$ for the defining representation, i.e. the algebra $H_{(c,d)}(G(2,1,n),\mathbf{C}^n)$. With the notation we introduce in Section 3 of Chapter 2 parameters (c_0,d_0,d_1) are related with the pair (c,d) by $c = c_0$ and $d = d_0 = -d_1$. For the remainder of this section we denote this algebra simply by H_c.

For a 2-partition $\lambda = (\lambda^0,\lambda^1)$ the *charged content* of a b of λ is

$$\mathrm{ct}_c(b) = \begin{cases} d + 2\mathrm{ct}(b) & \text{if } b \in \lambda^0, \text{ and} \\ -d + 2\mathrm{ct}(b)c & \text{if } b \in \lambda^1 \end{cases}$$

Then the charged content of λ can be expressed

$$\mathrm{ct}_c(\lambda) = d(|\lambda^0| - |\lambda^1|) + 2c\sum_{b\in\lambda}\mathrm{ct}(b),$$

where $|\mu|$ is the number of boxes of the partition μ. The set of c-*admissible* tableaux $\mathrm{Tab}_c(\lambda)$ consists of all tableaux $Q : \lambda \to \mathbf{Z}_{\geq 0}$ such that

(1) $Q(b) \leq Q(b')$ whenever $b \leq b'$,

(2) $Q(b) < k$ if k is an odd positive integer and

$$d + \mathrm{ct}(b)c = k/2,$$

or if k is an even positive integer and

$$\mathrm{ct}(b)c = k/2,$$

and

(3) $Q(b) \leq Q(b') + k$ if $k = \beta(b) - \beta(b') \mod 2$ and $\mathrm{ct}_c(b) - \mathrm{ct}_c(b') = k \pm 2c$.

Define the function s_c of the boxes of λ as follows, we must distinguish two cases. By symmetry we may assume $\lambda^0 \neq \emptyset$.

Case 1. First assume either that $\lambda^1 \neq \emptyset$ or that $\lambda^1 = \emptyset$ but equation

$$d + \mathrm{ct}(b)c = 1/2$$

does not hold, where b is the removable box of λ^0 of largest content. In this case we define the c-shifting function $s_c : \mathbf{R}^2 \times \mathbf{Z}/2 \to \mathbf{R}^2 \times \mathbf{Z}/2$ by

$$s_c(x,y,0) = \left(x - \lambda^0_1,\, y - \lambda^0_1 + \frac{1}{2c} - \frac{d}{c},\, 1\right) \text{ and}$$
$$s_c(x,y,1) = \left(x - \lambda^1_1,\, y - \lambda^1_1 + \frac{1}{2c} + \frac{d}{c},\, 0\right)$$

where if $\lambda^i = \emptyset$ then we interpret $\lambda^i_1 = 0$.

Case 2. Now we assume that $\lambda^1 = \emptyset$ and the equation

$$d + \mathrm{ct}(b)c = 1/2$$

holds. In this case we put

$$s_c(x, y, 0) = \left(x - p,\ y - p + \frac{1}{2c} - \frac{d}{c},\ 1\right)$$
$$s_c(x, y, 1) = \left(x,\ y + \frac{1}{2c} + \frac{d}{c},\ 0\right)$$

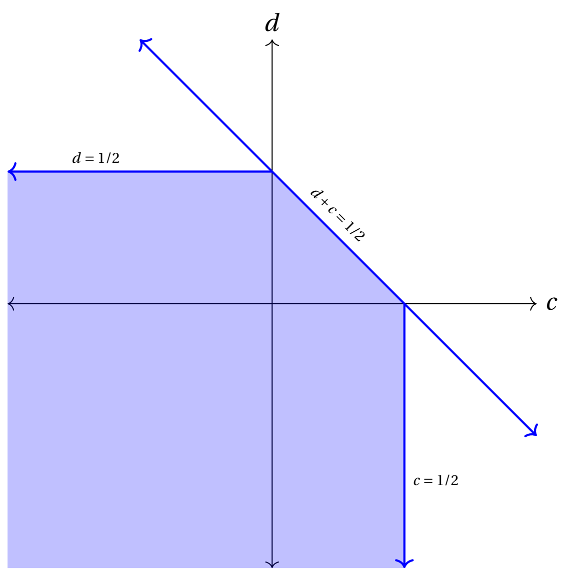
where p is the length of the second longest part of λ^0.

Then the skew shape $s_c(Q)$ is obtained by applying the kth iterate s_c^k to each box $b \in \lambda$ with $Q(b) = k$.

The classification of unitary representations is quiet inntricate and we do not state it here (see [**6**]).

Now for $\lambda \in P_{2,n}$ we give a few examples of the indexing sets $\Gamma_c(\lambda)$. And given $Q \in \mathrm{Tab}_{(c,d)}(\lambda)$ we will produce the tableau T determined by (0.1) and (0.2) and the skew shape $s_c(Q)$.

EXAMPLE 15. *For $\lambda = \left(\square\square, \emptyset\right)$, the unitary spectrum is given by*



(1) *Along the line $d + c = 1/2$ the set $\Gamma_c(\lambda)$ consists of pairs (P, Q) of tableaux on λ such that*
   (1) *Q is a filling of the boxes of λ by non-negative integers such that $Q(b) \le Q(b')$ whenever $b \le b'$.*
   (2) *P is a bijection from the boxes of λ to the set of integers $\{1, 2, \ldots, n\}$ such that if $b \le b'$ and $Q(b) = Q(b')$ then $P(b) > P(b')$.*
   (3) *$Q(b) < 1$ for boxes $b \in \lambda$ with $\mathrm{ct}(b) = 1$.*
   (4) *$Q(b_1) \le Q(b_2) + 2$ for boxes $b_1 \in \lambda^0$ and $b_2 \in \lambda^1$ with $\mathrm{ct}(b_1) - \mathrm{ct}(b_2) \pm 1 = 1$. If $Q(b_1) = Q(b_2) + 2$ implies that $P(b_1) > P(b_2)$.*

Since $\lambda^1 = \emptyset$ *condition (4) does not apply, but condition (3) forces* $Q = (\boxed{0\,|\,0\,|\,0}, \emptyset)$ *since* $\mathrm{ct}(b) = 1$

$$\lambda = \left(\boxed{\,|\,b}, \emptyset\right)$$

and

$$\Gamma_c(\lambda) = \left\{\left((\boxed{2\,|\,1}, \emptyset), (\boxed{0\,|\,0}, \emptyset)\right)\right\}.$$

We produce a standard Young tableau T *and the skew shape* $s_c(Q)$ *for this pair by the algorithm obtained before, which is determined by equations* (0.1) *and* (0.2). *We have*

$$\begin{aligned}
\beta(T^{-1}(1)) &= \beta(P^{-1}(2)) - Q(P^{-1}(2)) = 0 - 0 = 0 \\
\mathrm{ct}(T^{-1}(1)) &= \mathrm{ct}(P^{-1}(2)) - \frac{1}{2c}\big(Q(P^{-1}(2)) - (d_{\beta(P^{-1}(2))} - d_{\beta(T^{-1}(2))})\big) \\
&= 0 - \frac{1}{2c}(0 - (d_0 - d_0)) = 0 \\
\beta(T^{-1}(2)) &= \beta(P^{-1}(1)) - Q(P^{-1}(1)) = 0 - 0 \\
\mathrm{ct}(T^{-1}(2)) &= \mathrm{ct}(P^{-1}(1)) - \frac{1}{2c}\big(Q(P^{-1}(1)) - (d_{\beta(P^{-1}(1))} - d_{\beta(T^{-2}(1))})\big) = \mathrm{ct}(P^{-1}(2)) \\
&= 1 - \frac{1}{2c}(0 - (d_0 - d_0)) = 1
\end{aligned}$$

Then $T = \left(\boxed{1\,|\,2}, \emptyset\right)$ *and* $s_c(Q) = shape(T) = \left(\boxed{\,|\,}, \emptyset\right)$.

(2) *Along the line* $c = 1/2$ *the set* $\Gamma_c(\lambda)$ *consists of pairs* (P, Q) *of tableaux on* λ *such that*
   (1) Q *is a filling of the boxes of* λ *by non-negative integers such that* $Q(b) \le Q(b')$ *whenever* $b \le b'$.
   (2) P *is a bijection from the boxes of* λ *to the set of integers* $\{1, 2, \ldots, n\}$ *such that if* $b \le b'$ *and* $Q(b) = Q(b')$ *then* $P(b) > P(b')$.
   (3) $Q(b) < 2$ *for boxes* $b \in \lambda$ *with* $\mathrm{ct}(b) = 2$.
   (4) $Q(b_1) \le Q(b_2) + 2$ *for boxes* b_1, b_2 *in the same component of* λ *with* $\mathrm{ct}(b_1) - \mathrm{ct}(b_2) \pm 1 = 2$. *If* $Q(b_1) = Q(b_2) + 2$ *implies that* $P(b_1) > P(b_2)$.

Since λ *does not have any box* b *with* $\mathrm{ct}(b) = 2$ *and* $\lambda^1 = \emptyset$ *conditions (3) and (4) does not apply. We list some pairs* $(P, Q) \in \Gamma_c(\lambda)$

P	Q
$(\boxed{2}\boxed{1}, \emptyset)$	$(\boxed{0}\boxed{0}, \emptyset)$
$(\boxed{2}\boxed{1}, \emptyset)$ $(\boxed{1}\boxed{2}, \emptyset)$	$(\boxed{0}\boxed{1}, \emptyset)$
$(\boxed{2}\boxed{1}, \emptyset)$	$(\boxed{1}\boxed{1}, \emptyset)$
$(\boxed{2}\boxed{1}, \emptyset)$ $(\boxed{1}\boxed{2}, \emptyset)$	$(\boxed{0}\boxed{2}, \emptyset)$
$(\boxed{2}\boxed{1}, \emptyset)$	$(\boxed{1}\boxed{2}, \emptyset)$
$(\boxed{2}\boxed{1}, \emptyset)$ $(\boxed{1}\boxed{2}, \emptyset)$	$(\boxed{0}\boxed{3}, \emptyset)$
$\vdots$	$\vdots$

First, we will produce the tableau T and the skew shape $s_c(Q)$ for the pair $(P,Q) = \Big((\boxed{2}\boxed{1}, \emptyset), (\boxed{0}\boxed{1}, \emptyset)\Big)$.

$$\beta(T^{-1}(1)) = \beta(P^{-1}(2)) - Q(P^{-1}(2)) = 0 - 0 = 0$$

$$\begin{aligned}\mathrm{ct}(T^{-1}(1)) &= \mathrm{ct}(P^{-1}(2)) - \frac{1}{2c}\big(Q(P^{-1}(2)) - (d_{\beta(P^{-1}(2))} - d_{\beta(T^{-1}(2))})\big)\\ &= 0 - \frac{1}{2c}(0 - (d_0 - d_0)) = 0\end{aligned}$$

$$\beta(T^{-1}(2)) = \beta(P^{-1}(1)) - Q(P^{-1}(1)) = 0 - 1 = 1 \mod 2$$

$$\begin{aligned}\mathrm{ct}(T^{-1}(2)) &= \mathrm{ct}(P^{-1}(1)) - \frac{1}{2c}\big(Q(P^{-1}(1)) - (d_{\beta(P^{-1}(1))} - d_{\beta(T^{-2}(1))})\big) = \mathrm{ct}(P^{-1}(2))\\ &= 1 - \frac{1}{2c}(0 - (d_0 - d_1)) = 1 + \frac{d}{c}\end{aligned}$$

since $d < 0$ and $c = 1/3$, $\mathrm{ct}(T^{-1}(2)) = 1 + 3d$. Hence (up to diagonal slides of the box containging 2) the tableau T looks like $T = \Big(\boxed{1}, \boxed{2}\Big)$ and then $s_c(Q) = shape(T) = \Big(\square, \square\Big)$.

Now we will produce the tableau T and the skew shape $s_c(Q)$ for the pair $(P,Q) = \Big((\boxed{1}\boxed{2}, \emptyset), (\boxed{0}\boxed{2}, \emptyset)\Big)$.

$$\beta(T^{-1}(1)) = \beta(P^{-1}(2)) - Q(P^{-1}(2)) = 0 - 2 = 0 \mod 2$$

$$\begin{aligned}\mathrm{ct}(T^{-1}(1)) &= \mathrm{ct}(P^{-1}(2)) - \frac{1}{2c}\big(Q(P^{-1}(2)) - (d_{\beta(P^{-1}(2))} - d_{\beta(T^{-1}(2))})\big)\\ &= 2 - \frac{1}{2c}(2 - (d_0 - d_0)) = 1 - \frac{1}{c} = -2\end{aligned}$$

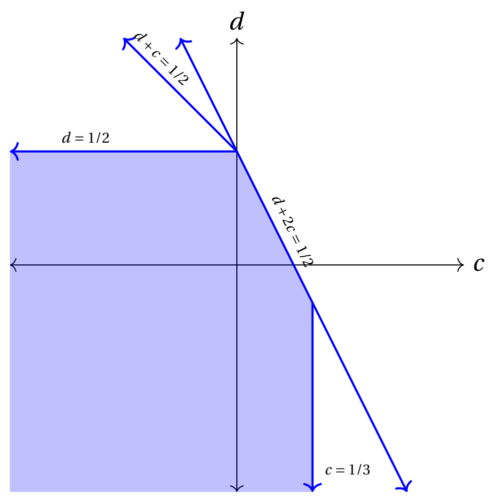
$$\beta(T^{-1}(2)) = \beta(P^{-1}(1)) - Q(P^{-1}(1)) = 0$$

$$\mathrm{ct}(T^{-1}(2)) = \mathrm{ct}(P^{-1}(1)) - \frac{1}{2c}\big(Q(P^{-1}(1)) - (d_{\beta(P^{-1}(1))} - d_{\beta(T^{-2}(1))})\big) = \mathrm{ct}(P^{-1}(2))$$

$$= 1 - \frac{1}{2c}(0 - (d_0 - d_0)) = 1$$

Hence (up to diagonal slides of the box containing 1*) the tableau T looks like*
$T = \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \emptyset\right)$ *and then* $s_c(Q) = shape(T) = \left(\begin{array}{cc} & \boxed{} \\ \boxed{} & \end{array}, \emptyset\right)$.

Example 16. *For $\lambda = \left(\begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}, \emptyset\right)$, the unitary spectrum is given by*



(1) *Along the line $d + c = 1/2$ the set $\Gamma_c(\lambda)$ consists of pairs (P, Q) of tableaux on λ such that*
  (1) *Q is a filling of the boxes of λ by non-negative integers such that $Q(b) \leq Q(b')$ whenever $b \leq b'$.*
  (2) *P is a bijection from the boxes of λ to the set of integers $\{1, 2, \ldots, n\}$ such that if $b \leq b'$ and $Q(b) = Q(b')$ then $P(b) > P(b')$.*
  (3) *$Q(b) < 1$ for boxes $b \in \lambda$ with $\mathrm{ct}(b) = 1$.*
  (4) *$Q(b_1) \leq Q(b_2) + 1$ for boxes $b_1 \in \lambda^0$ and $b_2 \in \lambda^1$ with $\mathrm{ct}(b_1) - \mathrm{ct}(b_2) \pm 1 = 1$. If $Q(b_1) = Q(b_2) + 1$ implies that $P(b_1) > P(b_2)$.*

Since $\lambda^1 = \emptyset$ condition (4) does not apply. Note that the $\mathrm{ct}(b) = 1$

$$\lambda = \left(\begin{array}{|c|c|c|}\hline & b & \\ \hline\end{array}, \emptyset\right)$$

forcing the value of Q to be 0 *in two boxes of λ, as follows* $Q = \left(\begin{array}{|c|c|c|}\hline 0 & 0 & \\ \hline\end{array}, \emptyset\right)$. *Then a list of some $(P, Q) \in \Gamma_c(\lambda)$ is given by*

P	Q
$(\boxed{3}\boxed{2}\boxed{1},\varnothing)$	$(\boxed{0}\boxed{0}\boxed{0},\varnothing)$
$(\boxed{3}\boxed{2}\boxed{1},\varnothing)$ $(\boxed{3}\boxed{1}\boxed{2},\varnothing)$ $(\boxed{2}\boxed{1}\boxed{3},\varnothing)$	$(\boxed{0}\boxed{0}\boxed{1},\varnothing)$
$(\boxed{3}\boxed{2}\boxed{1},\varnothing)$ $(\boxed{3}\boxed{1}\boxed{2},\varnothing)$ $(\boxed{2}\boxed{1}\boxed{3},\varnothing)$	$(\boxed{0}\boxed{0}\boxed{2},\varnothing)$
$(\boxed{3}\boxed{2}\boxed{1},\varnothing)$ $(\boxed{3}\boxed{1}\boxed{2},\varnothing)$ $(\boxed{2}\boxed{1}\boxed{3},\varnothing)$	$(\boxed{0}\boxed{0}\boxed{3},\varnothing)$
$\vdots$	$\vdots$

We will produce the tableau T and the skew shape $s_c(Q)$ for the pair $(P,Q) = \Big((\boxed{3}\boxed{2}\boxed{1},\varnothing),(\boxed{0}\boxed{0}\boxed{1},\varnothing)\Big)$.

$$\beta(T^{-1}(1)) = \beta(P^{-1}(3)) - Q(P^{-1}(3)) = 0$$

$$\mathrm{ct}(T^{-1}(1)) = \mathrm{ct}(P^{-1}(3)) - \frac{1}{2c}\big(Q(P^{-1}(3)) - (d_{\beta(P^{-1}(3))} - d_{\beta(T^{-1}(3))})\big) = \mathrm{ct}(P{-}1(3)) = 0$$

$$\beta(T^{-1}(2)) = \beta(P^{-1}(2) - Q(P^{-1}(2)) = 0$$

$$\mathrm{ct}(T^{-1}(2)) = \mathrm{ct}(P^{-1}(2)) - \frac{1}{2c}\big(Q(P^{-1}(2)) - (d_{\beta(P^{-1}(2))} - d_{\beta(T^{-1}(2))})\big) = \mathrm{ct}(P^{-1}(2)) = 1$$

$$\beta(T^{-1}(3)) = \beta(P^{-1}(1) - Q(P^{-1}(1)) = 0 - 1 = 1 \mod 2$$

$$\begin{aligned}\mathrm{ct}(T^{-1}(3)) &= \mathrm{ct}(P^{-1}(1)) - \frac{1}{2c}\big(Q(P^{-1}(1)) - (d_{\beta(P^{-1}(1))} - d_{\beta(T^{-1}(1))})\big)\\ &= 3 - \frac{1}{2c}(1-(d_0-d_1)) = 3 - \frac{1}{2c} + \frac{d}{c}\end{aligned}$$

since $d + c = 1/2$ *then* $\frac{1}{2c} - \frac{d}{c} = 1$ *and*

$$\mathrm{ct}(T^{-1}(3)) = 2.$$

Hence (up to diagonal slides of the box containing 3*) the tableau T looks like*

$$T = \Big(\boxed{1}\boxed{2},\qquad \boxed{3}\Big)$$

and $s_c(Q) = shape(T) = \Big(\boxed{}\boxed{},\boxed{}\Big)$.

(2) *Along the line $c = 1/3$ the set $\Gamma_c(\lambda)$ consists of pairs (P,Q) of tableaux on λ such that*

(1) *Q is a filling of the boxes of λ by non-negative integers such that $Q(b) \le Q(b')$ whenever $b \le b'$.*

(2) *P is a bijection from the boxes of λ to the set of integers $\{1,2,\dots,n\}$ such that if $b \le b'$ and $Q(b) = Q(b')$ then $P(b) > P(b')$.*

(3) *$Q(b) < 2$ for boxes $b \in \lambda$ with $\mathrm{ct}(b) = 3$.*

(4) *$Q(b_1) \le Q(b_2) + 2$ for boxes b_1, b_2 in the same component of λ with $\mathrm{ct}(b_1) - \mathrm{ct}(b_2) \pm 1 = 3$. If $Q(b_1) = Q(b_2) + 2$ implies that $P(b_1) > P(b_2)$.*

Note that λ has no boxes b with $\mathrm{ct}(b) = 3$, and boxes b_1, b_2 as in the diagram below satisfy condition (4) since $\mathrm{ct}(b_1) = 2$ and $\mathrm{ct}(b_2) = 0$

$$\lambda = \left(\begin{array}{|c|c|c|}\hline b_2 & & b_1 \\ \hline\end{array}, \emptyset\right)$$

We list some pairs of tableaux $(P, Q) \in \Gamma_c(\lambda)$

P	Q
$\left(\begin{array}{\|c\|c\|c\|}\hline 3&2&1\\\hline\end{array},\emptyset\right)$	$\left(\begin{array}{\|c\|c\|c\|}\hline 0&0&0\\\hline\end{array},\emptyset\right)$
$\left(\begin{array}{\|c\|c\|c\|}\hline 3&2&1\\\hline\end{array},\emptyset\right)$ $\left(\begin{array}{\|c\|c\|c\|}\hline 3&1&2\\\hline\end{array},\emptyset\right)$ $\left(\begin{array}{\|c\|c\|c\|}\hline 2&1&3\\\hline\end{array},\emptyset\right)$	$\left(\begin{array}{\|c\|c\|c\|}\hline 0&0&1\\\hline\end{array},\emptyset\right)$
$\left(\begin{array}{\|c\|c\|c\|}\hline 3&2&1\\\hline\end{array},\emptyset\right)$ $\left(\begin{array}{\|c\|c\|c\|}\hline 2&3&1\\\hline\end{array},\emptyset\right)$ $\left(\begin{array}{\|c\|c\|c\|}\hline 1&3&2\\\hline\end{array},\emptyset\right)$	$\left(\begin{array}{\|c\|c\|c\|}\hline 0&1&1\\\hline\end{array},\emptyset\right)$
$\left(\begin{array}{\|c\|c\|c\|}\hline 3&2&1\\\hline\end{array},\emptyset\right)$ $\left(\begin{array}{\|c\|c\|c\|}\hline 3&1&2\\\hline\end{array},\emptyset\right)$	$\left(\begin{array}{\|c\|c\|c\|}\hline 0&0&2\\\hline\end{array},\emptyset\right)$
$\left(\begin{array}{\|c\|c\|c\|}\hline 3&2&1\\\hline\end{array},\emptyset\right)$ $\left(\begin{array}{\|c\|c\|c\|}\hline 3&1&2\\\hline\end{array},\emptyset\right)$ $\left(\begin{array}{\|c\|c\|c\|}\hline 2&3&1\\\hline\end{array},\emptyset\right)$	$\left(\begin{array}{\|c\|c\|c\|}\hline 0&1&2\\\hline\end{array},\emptyset\right)$
$\left(\begin{array}{\|c\|c\|c\|}\hline 3&2&1\\\hline\end{array},\emptyset\right)$	$\left(\begin{array}{\|c\|c\|c\|}\hline 1&1&1\\\hline\end{array},\emptyset\right)$
$\left(\begin{array}{\|c\|c\|c\|}\hline 3&2&1\\\hline\end{array},\emptyset\right)$ $\left(\begin{array}{\|c\|c\|c\|}\hline 3&2&1\\\hline\end{array},\emptyset\right)$	$\left(\begin{array}{\|c\|c\|c\|}\hline 0&2&2\\\hline\end{array},\emptyset\right)$
$\vdots$	$\vdots$

The tableau T for the pair $(P, Q) = \left(\left(\begin{array}{|c|c|c|}\hline 3&2&1\\\hline\end{array},\emptyset\right), \left(\begin{array}{|c|c|c|}\hline 0&1&2\\\hline\end{array},\emptyset\right)\right)$ satisfy

$$\beta(T^{-1}(1)) = \beta(P^{-1}(3)) - Q(P^{-1}(3)) = 0$$

$$\mathrm{ct}(T^{-1}(1)) = \mathrm{ct}(P^{-1}(3)) - \frac{1}{2c}\left(Q(P^{-1}(3)) - (d_{\beta(P^{-1}(3))} - d_{\beta(T^{-1}(3))})\right) = 0$$

$$\beta(T^{-1}(2)) = \beta(P^{-1}(2)) - Q(P^{-1}(2)) = 0 - 1 = 1 \mod 2$$

$$\begin{aligned}\mathrm{ct}(T^{-1}(2)) &= \mathrm{ct}(P^{-1}(2)) - \frac{1}{2c}\left(Q(P^{-1}(2)) - (d_{\beta(P^{-1}(2))} - d_{\beta(T^{-1}(2))})\right) \\ &= 1 - \frac{1}{2c}(1 - (d_0 - d_1)) = 1 - \frac{1}{2c} + \frac{d}{c}\end{aligned}$$

$$
\begin{aligned}
\beta(T^{-1}(3)) &= \beta(P^{-1}(1)) - Q(P^{-1}(1)) = 0 - 2 = 0 \mod 2 \\
\mathrm{ct}(T^{-1}(3)) &= \mathrm{ct}(P^{-1}(1)) - \frac{1}{2c}\big(Q(P^{-1}(1)) - (d_{\beta(P^{-1}(1))} - d_{\beta(T^{-1}(1))})\big) \\
&= 2 - \frac{1}{2c}(2 - (d_0 - d_0)) = 2 - \frac{1}{c} = -1
\end{aligned}
$$

Then (up to diagonal slides of the connected components of each component) the tableau T looks like

$$T = \left(\begin{array}{cc} & \boxed{3} \\ \boxed{1} & \end{array}, \boxed{2} \right)$$

therefore

$$s_c(Q) = shape(T) = \left(\begin{array}{cc} & \square \\ \square & \end{array}, \square \right).$$

Bibliography

[1] D. Ciubotaru, *Dirac cohomology for symplectic reflection algebras*, 2015.

[2] C. Dezelee, *Generalized graded hecke algebra for complex reflection group of type g(r,1,n)* (200606).

[3] P. Etingof and X. Ma, *Lecture notes on Cherednik algebras*, Technical Report arXiv:1001.0432, 2010. Comments: 87 pages, latex.

[4] S. Fishel, G. Griffeth, and E. Manosalva, *Unitary representations of the cherednik algebra: v^*-homology*, 2020.

[5] V. Ginzburg, N. Guay, E Opdam, and R. Rouquier, *On the category $\mathcal{O}$ for rational cherednik algebras*, Inventiones Mathematicae **154** (2003), no. 3, 617–651.

[6] S. Griffeth, *Unitary representations of cyclotomic rational cherednik algebras*, Journal of Algebra **512** (2018), 310 –356.

[7] ______, *Subspace arrangements and cherednik algebras*, 2020.

[8] Stephen Griffeth, *Orthogonal functions generalizing jack polynomials*, Transactions of the American Mathematical Society **362** (2010), no. 11, 6131–6157.

[9] J. Huang and K. D. Wong, *A casselman-osborne theorem for rational cherednik algebras*, 2017.

[10] O. V. Ogievetsky and L. Poulain dAndecy, *An inductive approach to representations of complex reflection groups g(m,1,n)*, Theoretical and Mathematical Physics **174** (2013jan), no. 1, 95–108.

[11] A. Ram and A. V. Shepler, *Classification of graded hecke algebras for complex reflection groups*, Commentarii Mathematici Helvetici **78** (200301), 308–334.

[12] L. Scott, B. Parshall, and E. Cline, *Finite dimensional algebras and highest weight categories.*, Journal für die reine und angewandte Mathematik **1988** (01 Nov. 1988), no. 391, 85 –99.

Catalog of Unitary Spectra and graded characters

For the rational Cherednik algebra of type B_n, we compute the unitary spectrum, i.e. the values for parameters (c,d) that give rise to a unitary representation of H_c for $n \le 6$. For this we use Corollaries 8.4 and 8.5 in [**6**], where unitary representations are explicitly classified. Using the main result of this thesis 0.1 , we compute the graded character for the unitary representation $L_c(\lambda)$ for λ a 2-partition of n and parameters $c > 0$.

Note that if $i = 0$ then

$$\nu, \eta \in P_{\ell,n} \text{ and } \chi \in P_{\ell,0}$$

hence $\chi = (\emptyset, \dots, \emptyset)$ and the coefficients $c^{\nu}_{\eta\chi}$, $c^{\mu}_{\eta\chi^t}$ are nonzero if $\eta = \nu = \mu$, so the only element appearing in Ext^0 is λ. For this reason we will omit the column of $i = 0$ in the tables characters.

$$n = 1$$

$\lambda = (\square, \emptyset)$. In this case the unitary spectrum is $d \le 1/2$. For $d < 1/2$ the standard module is simple. For $d = 1/2$ the only non-trivial Ext-group is Ext^1 for $\mu = (\emptyset, \square)$.

$n = 2$

$\lambda = (\square\square, \varnothing)$.

FIGURE 1. Unitary spectrum

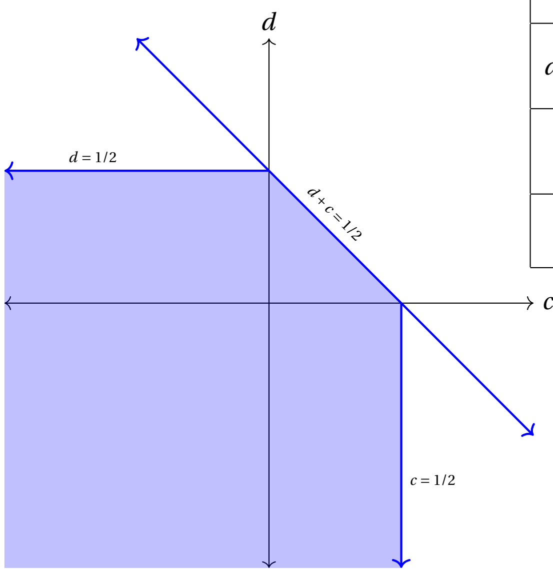


TABLE 1. Character

	$i=1$	$i=2$
$d+c=1/2$	$(\square,\square)$	$\left(\varnothing, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$
$c=1/2$	$\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \varnothing\right)$	
$d=1/2$	$(\varnothing, \square\square)$	

$\lambda = (\square, \square)$

FIGURE 2. Unitary spectrum

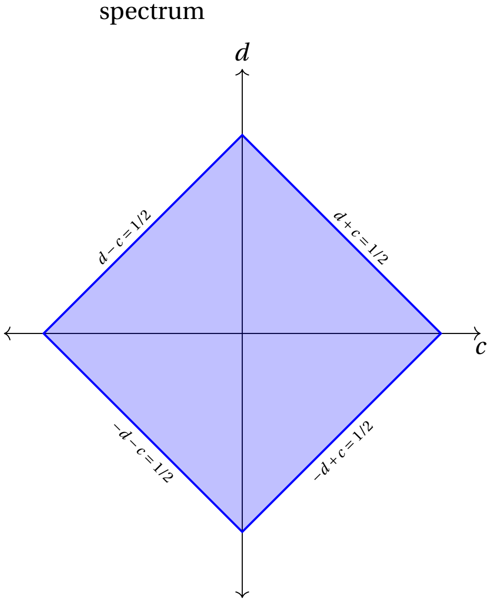


TABLE 2. Character

	$i=1$
$d+c=1/2$	$\left(\varnothing, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$
$-d+c=1/2$	$\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \varnothing\right)$
$d-c=1/2$	$(\varnothing, \square\square)$
$-d-c=1/2$	$(\square\square, \varnothing)$
$\left(\frac{1}{2}, 0\right)$	$\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \varnothing\right) + \left(\varnothing, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$
$\left(-\frac{1}{2}, 0\right)$	$(\square\square, \varnothing) + (\varnothing, \square\square)$

$$n = 3$$

$\lambda = \left(\ydiagram{3}, \varnothing\right)$.

FIGURE 3. Unitary spectrum

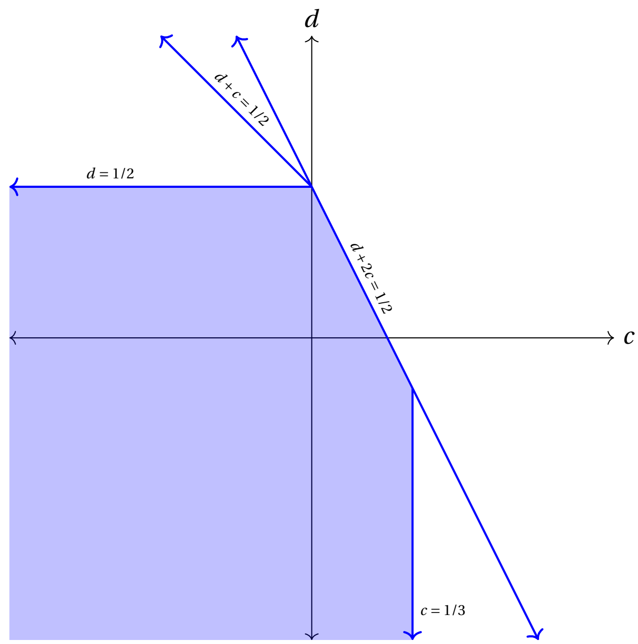


TABLE 3. Character

	$i = 1$	$i = 2$	$i = 3$
$d + 2c = 1/2$	$\left(\ydiagram{2}, \ydiagram{1}\right)$	$\left(\ydiagram{1}, \ydiagram{1,1}\right)$	$\left(\varnothing, \ydiagram{1,1,1}\right)$
$c = 1/3$	$\left(\ydiagram{2,1}, \varnothing\right)$	$\left(\ydiagram{1,1,1}, \varnothing\right)$	
$d + c = 1/2$	$\left(\ydiagram{1}, \ydiagram{2}\right)$	$\left(\varnothing, \ydiagram{2,1}\right)$	
$d = 1/2$	$\left(\varnothing, \ydiagram{3}\right)$		

$\lambda = \left(\ydiagram{2,1}, \varnothing\right)$

FIGURE 4. Unitary spectrum

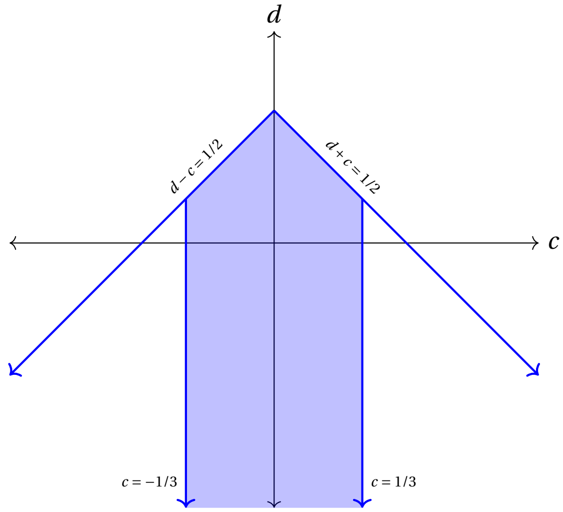


TABLE 4. Character

	$i = 1$	$i = 2$
$d + c = 1/2$	$\left(\ydiagram{1,1}, \ydiagram{1}\right)$	$\left(\varnothing, \ydiagram{1,1,1}\right)$
$c = 1/3$	$\left(\ydiagram{1,1,1}, \varnothing\right)$	
$d - c = 1/2$	$\left(\ydiagram{2}, \ydiagram{1}\right)$	$\left(\varnothing, \ydiagram{3}\right)$
$c = -1/3$	$\left(\ydiagram{3}, \varnothing\right)$	

$\lambda = \left(\ydiagram{2}, \ydiagram{1}\right)$

FIGURE 5. Unitary spectrum

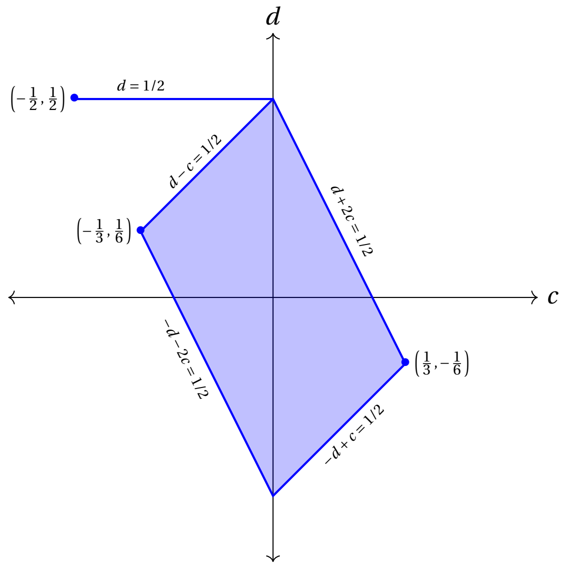


TABLE 5. Character

	$i=1$	$i=2$
$d+2c=1/2$	$\left(\ydiagram{1}, \ydiagram{1,1}\right)$	$\left(\varnothing, \ydiagram{1,1,1}\right)$
$\left(\frac{1}{3}, -\frac{1}{6}\right)$	$\left(\ydiagram{2,1}, \varnothing\right) + \left(\ydiagram{1}, \ydiagram{1,1}\right)$	$\left(\varnothing, \ydiagram{1,1,1}\right)$
$-d+c=1/2$	$\left(\ydiagram{2,1}, \varnothing\right)$	
$d=1/2$	$\left(\ydiagram{1}, \ydiagram{2}\right)$	
$\left(-\frac{1}{2}, \frac{1}{2}\right)$	$\left(\ydiagram{3}, \varnothing\right) + \left(\ydiagram{1}, \ydiagram{2}\right) + \left(\ydiagram{1}, \ydiagram{1,1}\right)$	$\left(\ydiagram{1,1}, \ydiagram{1}\right) + \left(\varnothing, \ydiagram{3}\right)$
$d-c=1/2$	$\left(\varnothing, \ydiagram{3}\right)$	
$\left(-\frac{1}{3}, \frac{1}{6}\right)$	$\left(\ydiagram{3}, \varnothing\right) + \left(\varnothing, \ydiagram{3}\right)$	
$-d-2c=1/2$	$\left(\ydiagram{3}, \varnothing\right)$	

$$n = 4$$

$\lambda = \left(\ydiagram{4}, \varnothing\right)$

FIGURE 6. Unitary spectrum

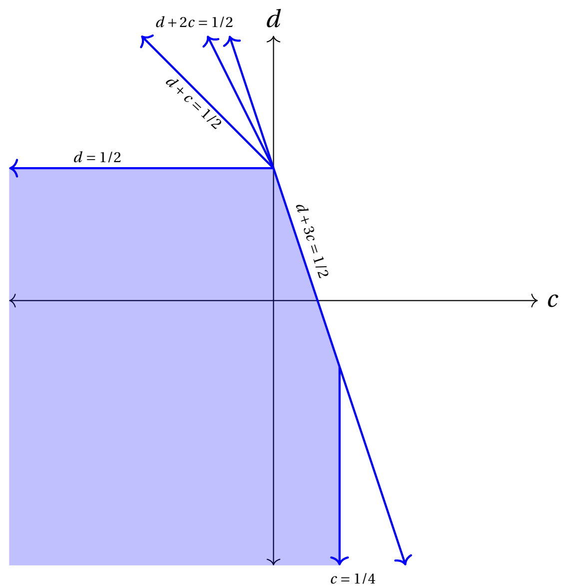


TABLE 6. Character

	$i=1$	$i=2$	$i=3$	$i=4$
$d+3c=1/2$	$\left(\ydiagram{3},\ydiagram{1}\right)$	$\left(\ydiagram{2},\ydiagram{1,1}\right)$	$\left(\ydiagram{1},\ydiagram{1,1,1}\right)$	$\left(\varnothing,\ydiagram{1,1,1,1}\right)$
$d+2c=1/2$	$\left(\ydiagram{2},\ydiagram{2}\right)$	$\left(\ydiagram{1},\ydiagram{2,1}\right)$	$\left(\varnothing,\ydiagram{2,1,1}\right)$	
$d+c=1/2$	$\left(\ydiagram{1},\ydiagram{3}\right)$	$\left(\varnothing,\ydiagram{3,1}\right)$		
$d=1/2$	$\left(\varnothing,\ydiagram{4}\right)$			
$c=1/4$	$\left(\ydiagram{3,1},\varnothing\right)$	$\left(\ydiagram{2,1,1},\varnothing\right)$	$\left(\ydiagram{1,1,1,1},\varnothing\right)$	

$\lambda = \left(\ydiagram{3,1}, \varnothing\right)$

FIGURE 7. Unitary spectrum

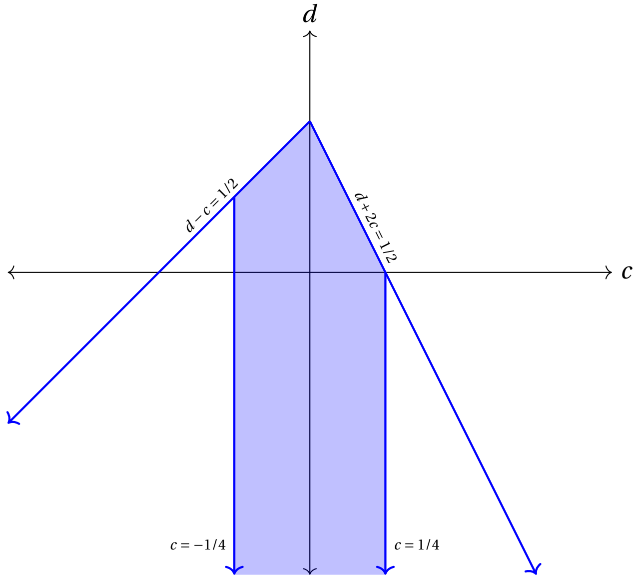


TABLE 7. Character

	$i=1$	$i=2$	$i=3$
$d+2c=1/2$	$\left(\ydiagram{2,1}, \ydiagram{1}\right)$	$\left(\ydiagram{1,1}, \ydiagram{1,1}\right)$	$\left(\varnothing, \ydiagram{1,1,1,1}\right)$
$d-c=1/2$	$\left(\ydiagram{3}, \ydiagram{1}\right)$	$\left(\varnothing, \ydiagram{4}\right)$	
$c=1/4$	$\left(\ydiagram{2,1,1}, \varnothing\right)$	$\left(\ydiagram{1,1,1,1}, \varnothing\right)$	
$c=-1/4$	$\left(\ydiagram{4}, \varnothing\right)$		

$\lambda = \left(\ydiagram{2,2}, \varnothing\right)$

FIGURE 8. Unitary spectrum

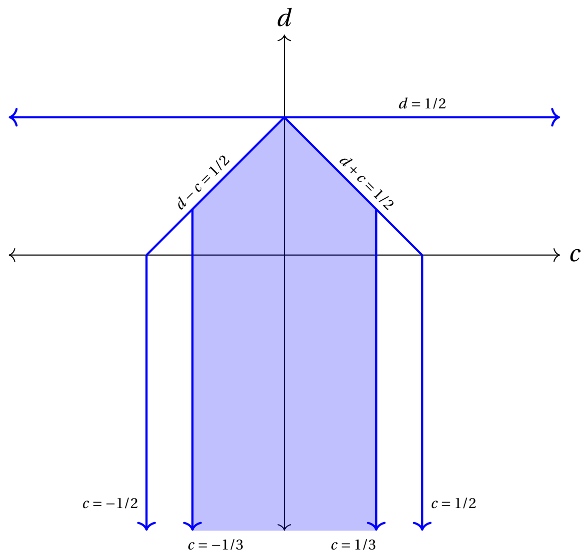


TABLE 8. Character

	$i=1$	$i=2$	$i=3$
$d=1/2$	$\left(\ydiagram{2,1},\ydiagram{1}\right)$	$\left(\ydiagram{1,1},\ydiagram{2}\right)+\left(\ydiagram{2},\ydiagram{1,1}\right)$	$\left(\ydiagram{1},\ydiagram{2,1}\right)$
$d+c=1/2$	$\left(\ydiagram{1,1},\ydiagram{1,1}\right)+\left(\ydiagram{1},\ydiagram{1,1,1}\right)$		
$\left(\frac{1}{2},0\right)$	$\left(\ydiagram{2,1,1},\varnothing\right)+\left(\ydiagram{1,1},\ydiagram{1,1}\right)$	$\left(\ydiagram{1,1,1},\ydiagram{1}\right)+\left(\ydiagram{1},\ydiagram{1,1,1}\right)$	$\left(\varnothing,\ydiagram{1,1,1,1}\right)$
$c=1/2$	$\left(\ydiagram{2,1,1},\varnothing\right)$	$\left(\ydiagram{1,1,1,1},\varnothing\right)$	
$c=1/3$	$\left(\varnothing,\ydiagram{1,1,1,1}\right)$		
$d-c=1/2$	$\left(\ydiagram{2},\ydiagram{2}\right)$	$\left(\ydiagram{1},\ydiagram{3}\right)$	
$\left(-\frac{1}{2},0\right)$	$\left(\ydiagram{3,1},\varnothing\right)+\left(\ydiagram{2},\ydiagram{2}\right)$	$\left(\ydiagram{3},\ydiagram{1}\right)+\left(\ydiagram{1},\ydiagram{3}\right)$	$\left(\varnothing,\ydiagram{4}\right)$
$c=-1/2$	$\left(\ydiagram{3,1},\varnothing\right)$	$\left(\ydiagram{4},\varnothing\right)$	
$c=-1/3$	$\left(\varnothing,\ydiagram{4}\right)$		gg

$\lambda = \left(\ydiagram{3}, \ydiagram{1}\right)$

FIGURE 9. Unitary spectrum

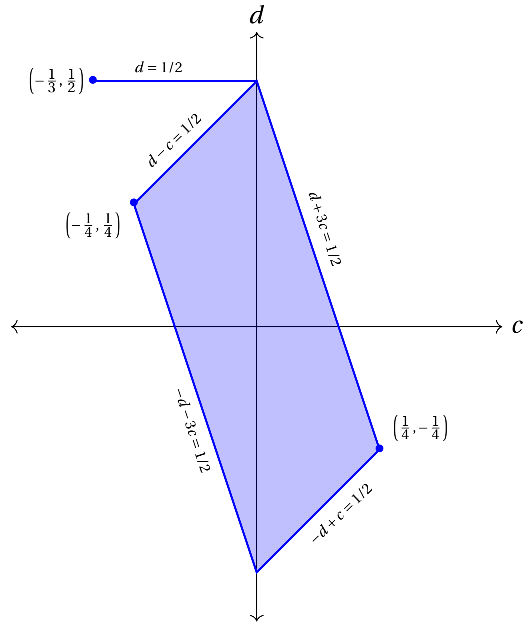


TABLE 9. Character

	$i=1$	$i=2$	$i=3$
$d+3c=1/2$	$\left(\ydiagram{2}, \ydiagram{1,1}\right)$	$\left(\ydiagram{1}, \ydiagram{1,1,1}\right)$	$\left(\emptyset, \ydiagram{1,1,1,1}\right)$
$\left(\frac{1}{4},-\frac{1}{4}\right)$	$\left(\ydiagram{3,1}, \emptyset\right)+\left(\ydiagram{2}, \ydiagram{1,1}\right)$	$\left(\ydiagram{1}, \ydiagram{1,1,1}\right)+\left(\ydiagram{2,1,1}, \emptyset\right)+\left(\ydiagram{1,1,1}, \ydiagram{1}\right)$	$\left(\emptyset, \ydiagram{1,1,1,1}\right)+\left(\ydiagram{1,1,1,1}, \emptyset\right)$
$-d+c=1/2$	$\left(\ydiagram{3,1}, \emptyset\right)$		
$d=1/2$	$\left(\ydiagram{1}, \ydiagram{3}\right)$		
$\left(-\frac{1}{3},\frac{1}{2}\right)$	$\left(\ydiagram{4}, \emptyset\right)+\left(\ydiagram{1}, \ydiagram{3}\right)$	$\left(\emptyset, \ydiagram{4}\right)$	
$d-c=1/2$	$\left(\emptyset, \ydiagram{4}\right)$		
$\left(-\frac{1}{4},\frac{1}{4}\right)$	$\left(\emptyset, \ydiagram{4}\right)+\left(\ydiagram{4}, \emptyset\right)$		
$-d-3c=1/2$	$\left(\ydiagram{4}, \emptyset\right)$		

$\lambda = \left(\ydiagram{2,1}, \ydiagram{1}\right)$

FIGURE 10. Unitary spectrum

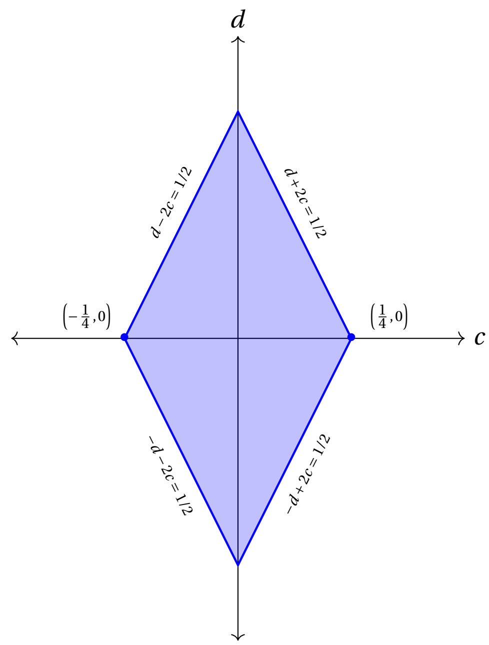


TABLE 10. Character

	$i=1$	$i=2$
$d+2c=1/2$	$\left(\ydiagram{1,1}, \ydiagram{1,1}\right)$	$\left(\varnothing, \ydiagram{1,1,1,1}\right)$
$\left(-\frac{1}{4}, 0\right)$	$\left(\ydiagram{2,1,1}, \varnothing\right) + \left(\ydiagram{1,1}, \ydiagram{1,1}\right)$	$\left(\ydiagram{1,1,1,1}, \varnothing\right) + \left(\varnothing, \ydiagram{1,1,1,1}\right)$
$-d+2c=1/2$	$\left(\ydiagram{2,1,1}, \varnothing\right)$	
$d-2c=1/2$	$\left(\ydiagram{2}, \ydiagram{2}\right)$	$\left(\varnothing, \ydiagram{4}\right)$
$\left(\frac{1}{4}, 0\right)$	$\left(\ydiagram{3,1}, \varnothing\right) + \left(\ydiagram{1,1}, \ydiagram{1,1}\right)$	$\left(\ydiagram{4}, \varnothing\right) + \left(\varnothing, \ydiagram{4}\right)$
$-d-2c=1/2$	$\left(\ydiagram{3,1}, \varnothing\right)$	

$\lambda = \left(\square\square, \square\square\right)$

FIGURE 11. Unitary spectrum

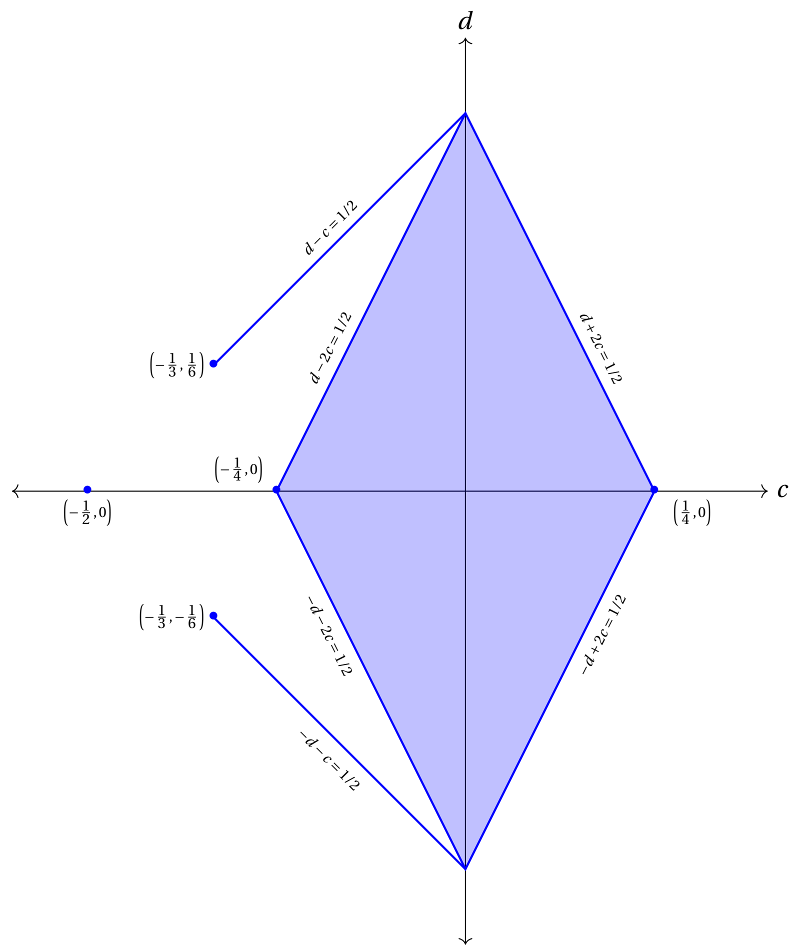


TABLE 11. Character

	$i=1$	$i=2$	$i=3$
$d+2c=1/2$	$\left(\ydiagram{1},\ydiagram{2,1}\right)$	$\left(\varnothing,\ydiagram{2,1,1}\right)$	
$\left(\frac{1}{4},0\right)$	$\left(\ydiagram{2,1},\ydiagram{1}\right)+\left(\ydiagram{1,1},\ydiagram{2}\right)$	$\left(\ydiagram{2,1,1},\varnothing\right)+\left(\ydiagram{1,1},\ydiagram{1,1}\right)+\left(\varnothing,\ydiagram{2,1,1}\right)$	$\left(\ydiagram{1,1,1,1},\varnothing\right)+\left(\varnothing,\ydiagram{1,1,1,1}\right)$
$-d+2c=1/2$	$\left(\ydiagram{2,1},\ydiagram{1}\right)$	$\left(\ydiagram{2,1,1},\varnothing\right)$	
$d-c=1/2$	$\left(\ydiagram{1},\ydiagram{3}\right)$		
$\left(-\frac{1}{3},\frac{1}{6}\right)$	$\left(\ydiagram{1},\ydiagram{3}\right)$		
$d-2c=1/2$	$\left(\ydiagram{1},\ydiagram{2,1}\right)$	$\left(\varnothing,\ydiagram{3,1}\right)$	
$\left(-\frac{1}{4},0\right)$	$\left(\ydiagram{1},\ydiagram{2,1}\right)+\left(\ydiagram{2},\ydiagram{1,1}\right)$	$\left(\varnothing,\ydiagram{2,1,1}\right)+\left(\ydiagram{1,1},\ydiagram{1,1}\right)+\left(\ydiagram{2,1,1},\varnothing\right)$	$\left(\varnothing,\ydiagram{1,1,1,1}\right)+\left(\ydiagram{1,1,1,1},\varnothing\right)$
$-d-2c=1/2$	$\left(\ydiagram{2,1},\ydiagram{1}\right)$	$\left(\ydiagram{3,1},\varnothing\right)$	
$-d-c=1/2$	$\left(\ydiagram{3},\ydiagram{1}\right)$		
$\left(-\frac{1}{3},-\frac{1}{6}\right)$	$\left(\ydiagram{3},\ydiagram{1}\right)$		
$\left(-\frac{1}{2},0\right)$	$\left(\ydiagram{3},\ydiagram{1}\right)+\left(\ydiagram{1},\ydiagram{3}\right)$	$\left(\ydiagram{4},\varnothing\right)+\left(\varnothing,\ydiagram{4}\right)$	

$$\lambda = \left(\ydiagram{2}, \ydiagram{1,1}\right)$$

FIGURE 12. Unitary spectrum

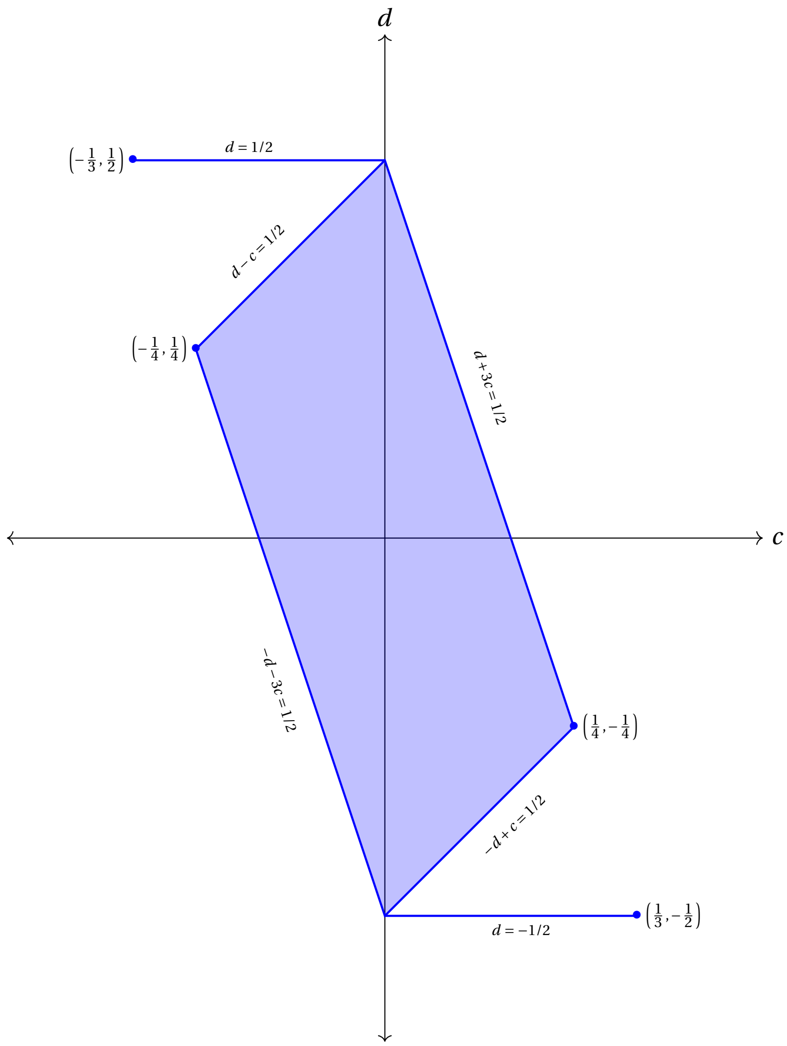


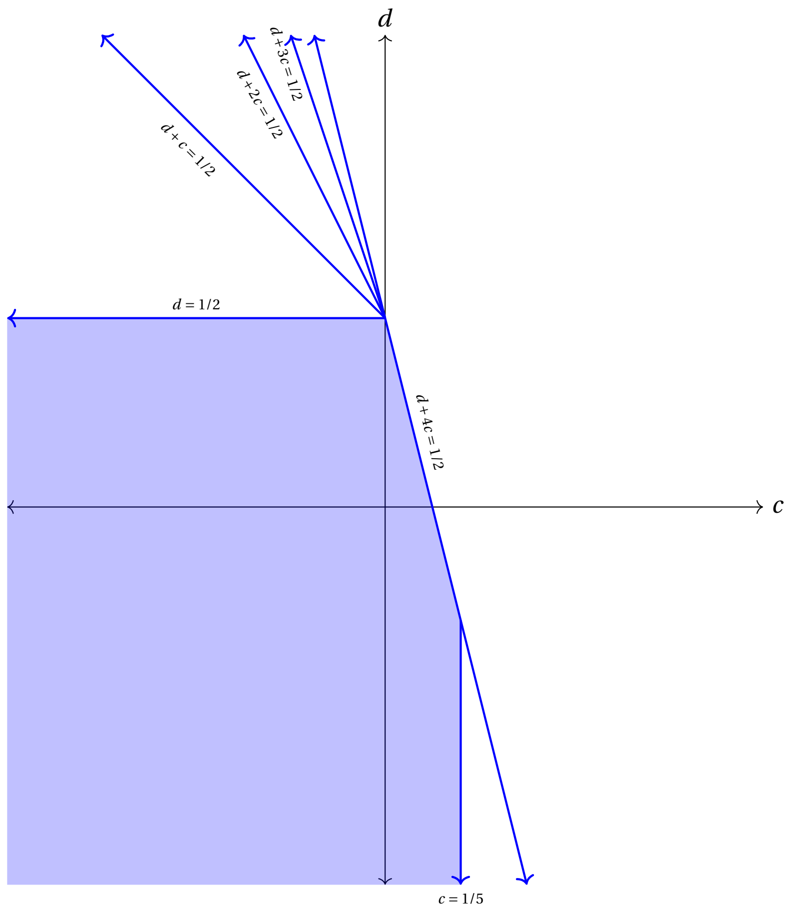
TABLE 12. Character

	$i=1$	$i=2$	$i=3$
$d+3c=1/2$	$\left(\ydiagram{1},\ydiagram{1,1,1}\right)$	$\left(\varnothing,\ydiagram{1,1,1,1}\right)$	
$\left(\frac{1}{4},-\frac{1}{4}\right)$	$\left(\ydiagram{1},\ydiagram{1,1,1}\right)+\left(\ydiagram{2,1,1},\varnothing\right)$	$\left(\varnothing,\ydiagram{1,1,1,1}\right)+\left(\ydiagram{1,1,1,1},\varnothing\right)$	
$-d+c=1/2$	$\left(\ydiagram{2,1,1},\varnothing\right)$		
$d=-1/2$	$\left(\ydiagram{2,1},\ydiagram{1}\right)$	$\left(\ydiagram{2,2},\varnothing\right)$	
$\left(\frac{1}{3},-\frac{1}{2}\right)$	$\left(\ydiagram{1},\ydiagram{1,1,1}\right)+\left(\ydiagram{2,1},\ydiagram{1}\right)$	$\left(\varnothing,\ydiagram{1,1,1,1}\right)+\left(\ydiagram{1,1,1},\ydiagram{1}\right)+\left(\ydiagram{2,2},\varnothing\right)$	$\left(\ydiagram{1,1,1,1},\varnothing\right)$
$d=1/2$	$\left(\ydiagram{1},\ydiagram{2,1}\right)$	$\left(\varnothing,\ydiagram{2,2}\right)$	
$\left(-\frac{1}{3},\frac{1}{2}\right)$	$\left(\ydiagram{3},\ydiagram{1}\right)+\left(\ydiagram{1},\ydiagram{2,1}\right)$	$\left(\ydiagram{4},\varnothing\right)+\left(\ydiagram{1},\ydiagram{3}\right)+\left(\varnothing,\ydiagram{2,2}\right)$	$\left(\varnothing,\ydiagram{4}\right)$
$d-c=1/2$	$\left(\varnothing,\ydiagram{3,1}\right)$		
$\left(-\frac{1}{4},\frac{1}{4}\right)$	$\left(\ydiagram{3},\ydiagram{1}\right)+\left(\varnothing,\ydiagram{3,1}\right)$	$\left(\ydiagram{4},\varnothing\right)+\left(\varnothing,\ydiagram{4}\right)$	
$-d-3c=1/2$	$\left(\ydiagram{3},\ydiagram{1}\right)$	$\left(\ydiagram{4},\varnothing\right)$	

$$n = 5$$

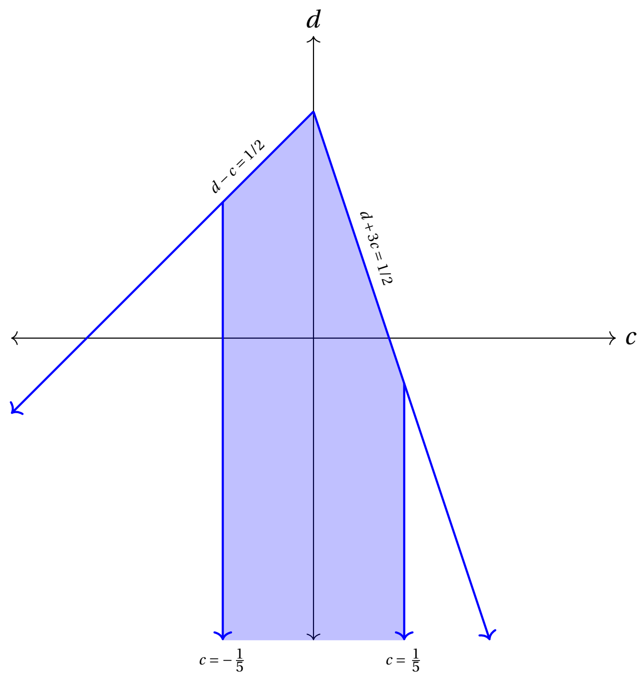
$\lambda = \left(\square\square\square\square\square, \emptyset\right))$

FIGURE 13. Unitary spectrum



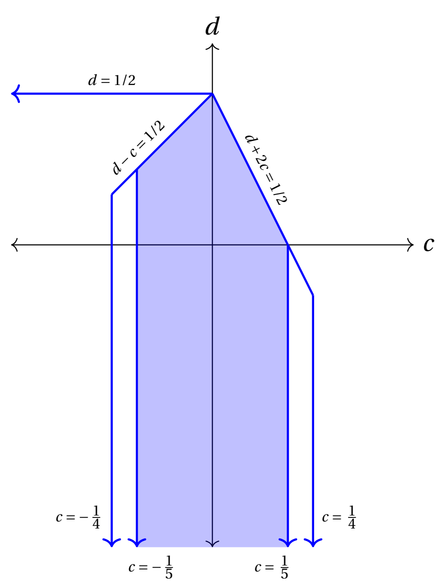
$$\lambda = \left(\ydiagram{4,1}, \varnothing \right)$$

FIGURE 14. Unitary spectrum



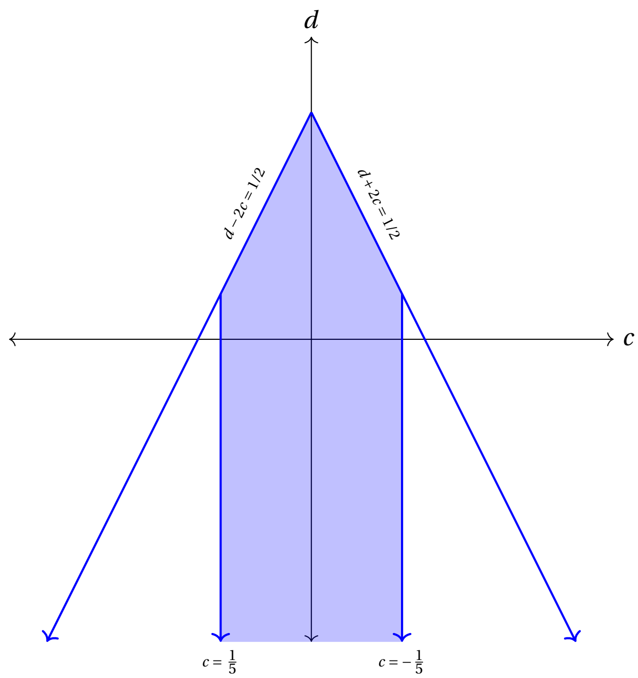
$$\lambda = \left(\ydiagram{3,2}, \varnothing \right)$$

FIGURE 15. Unitary spectrum



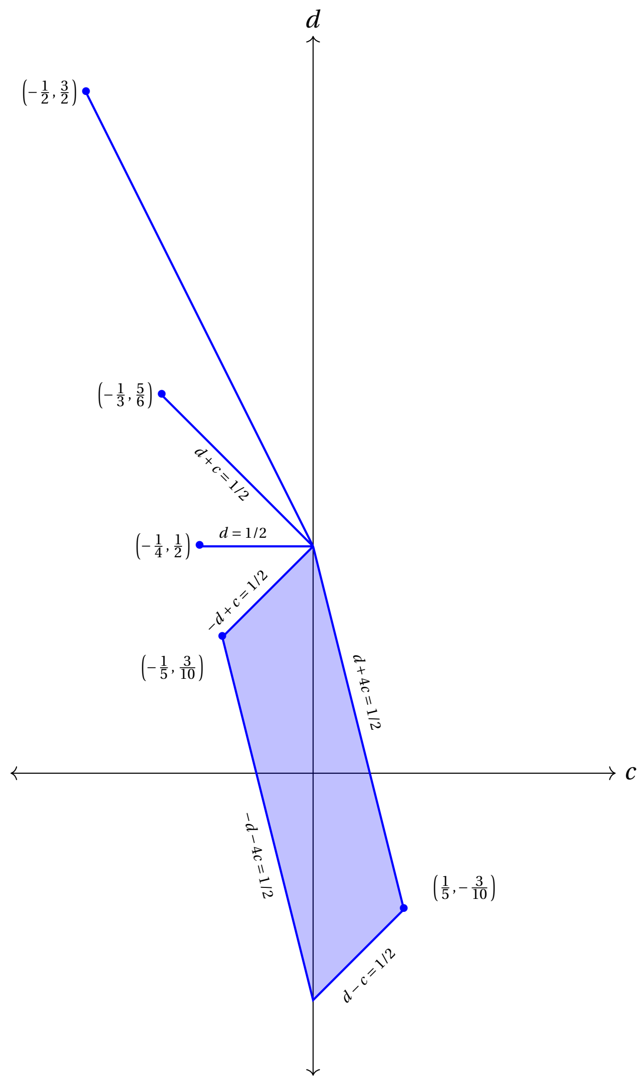
$$\lambda = \left(\ydiagram{3,1,1}, \varnothing \right)$$

FIGURE 16. Unitary spectrum



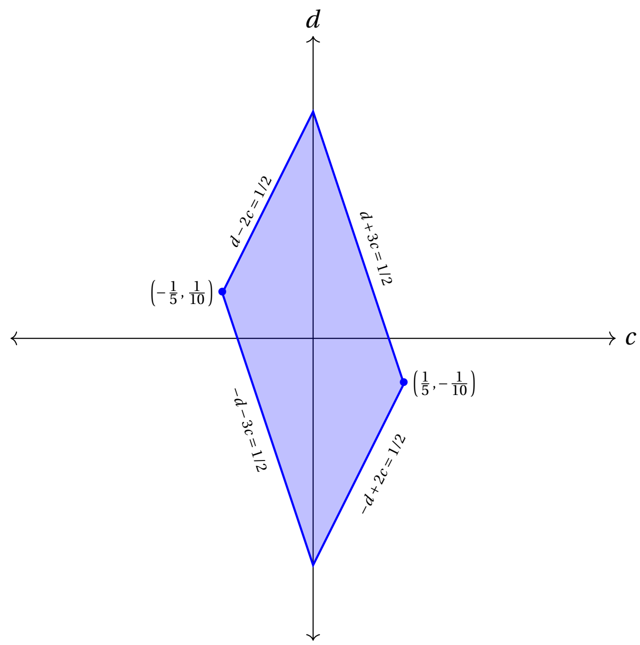
$\lambda = (\square\square\square\square, \square)$

FIGURE 17. Unitary spectrum



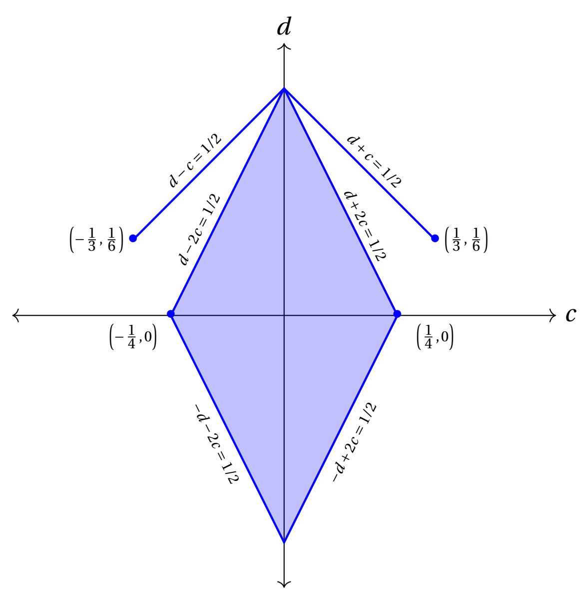
$\lambda = \left(\ydiagram{3,1}, \ydiagram{1}\right)$

FIGURE 18. Unitary spectrum



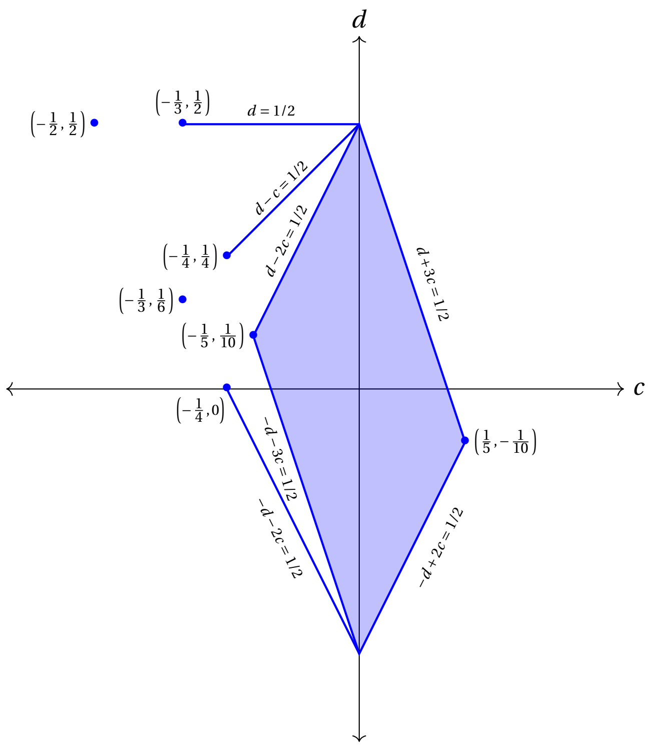
$\lambda = \left(\ydiagram{2,2}, \ydiagram{1}\right)$

FIGURE 19. Unitary spectrum



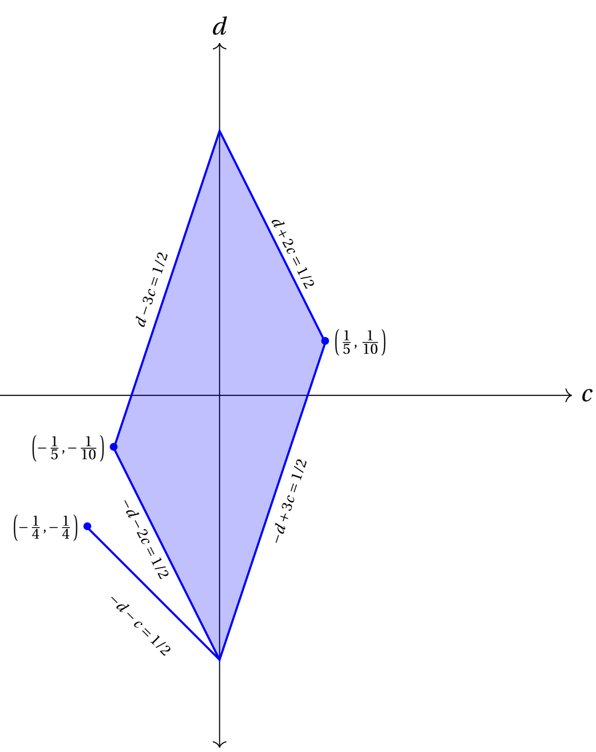
$\lambda = \left(\ydiagram{3}, \ydiagram{2}\right)$

FIGURE 20. Unitary spectrum



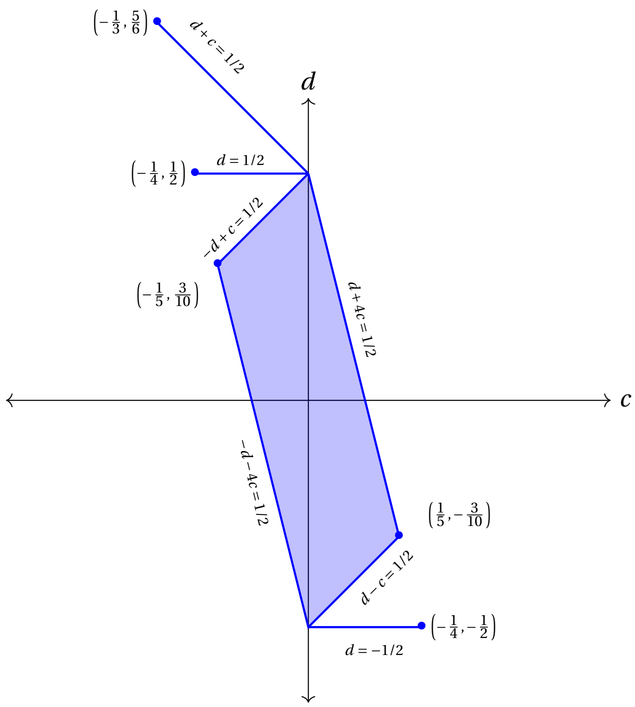
$\lambda = \left(\ydiagram{2,1}, \ydiagram{2}\right)$

FIGURE 21. Unitary spectrum



$\lambda = \left(\square\square\square , \begin{smallmatrix}\square\\\square\end{smallmatrix} \right)$

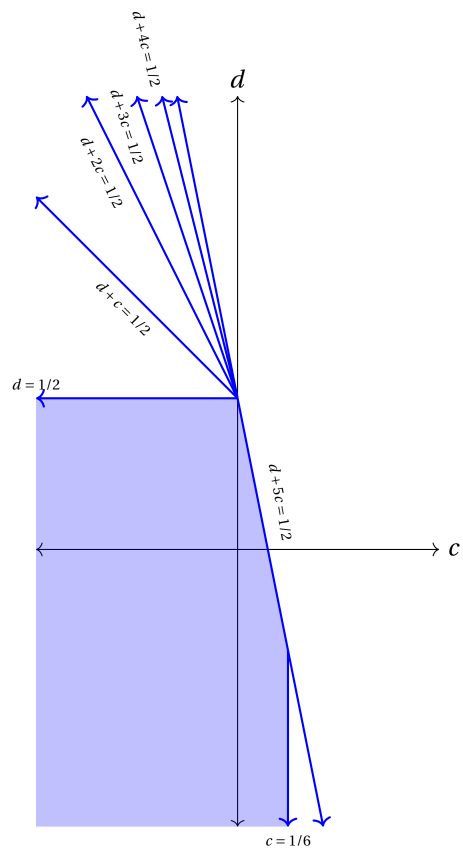
FIGURE 22. Unitary spectrum



$$n = 6$$

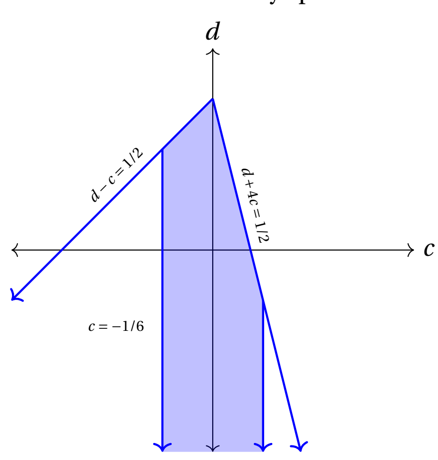
$\lambda = \left(\ydiagram{6}, \varnothing\right)$

FIGURE 23. Unitary spectrum



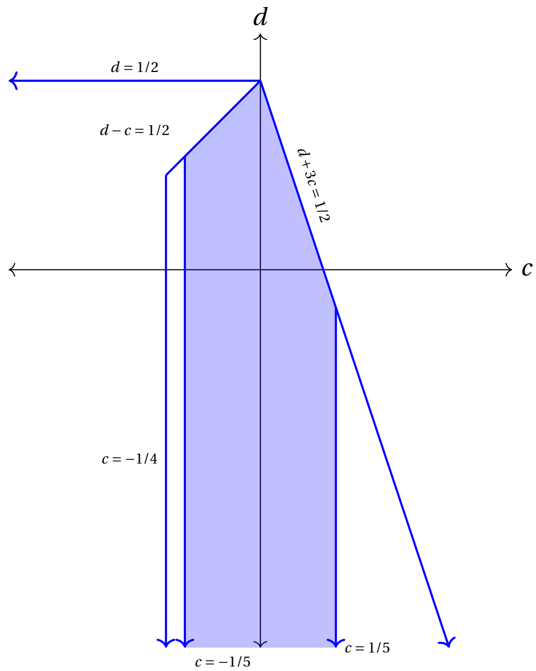
$\lambda = \left(\ydiagram{5,1}, \varnothing\right)$

FIGURE 24. Unitary spectrum



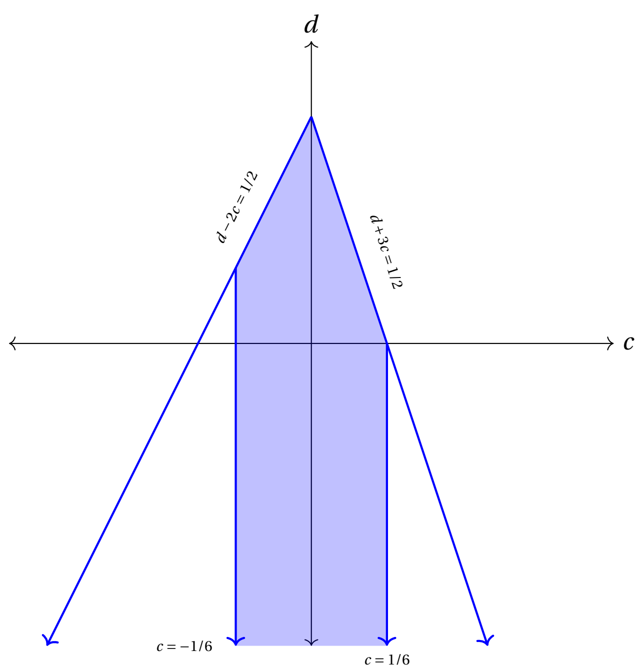
$\lambda = \left(\ydiagram{4,2}, \varnothing\right)$

FIGURE 25. Unitary spectrum



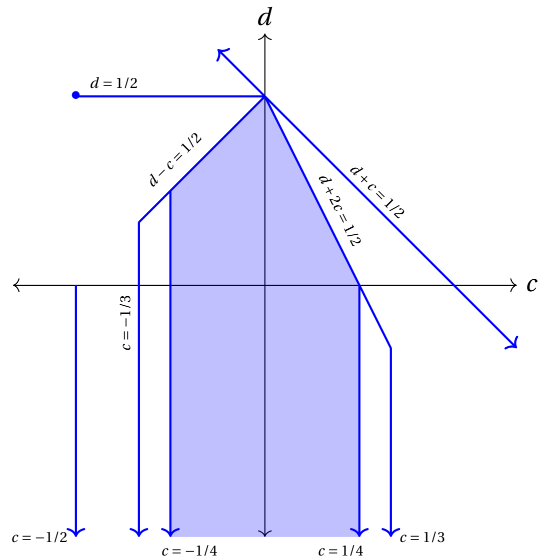
$\lambda = \left(\ydiagram{4,1,1}, \varnothing\right)$

FIGURE 26. Unitary spectrum



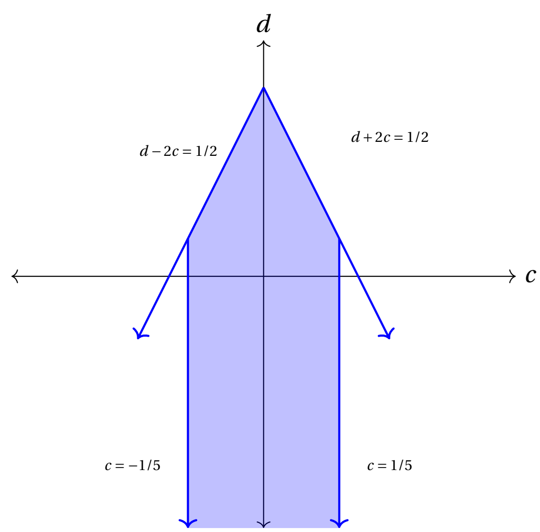
$$\lambda = \left(\ydiagram{3,3}, \varnothing\right)$$

FIGURE 27. Unitary spectrum



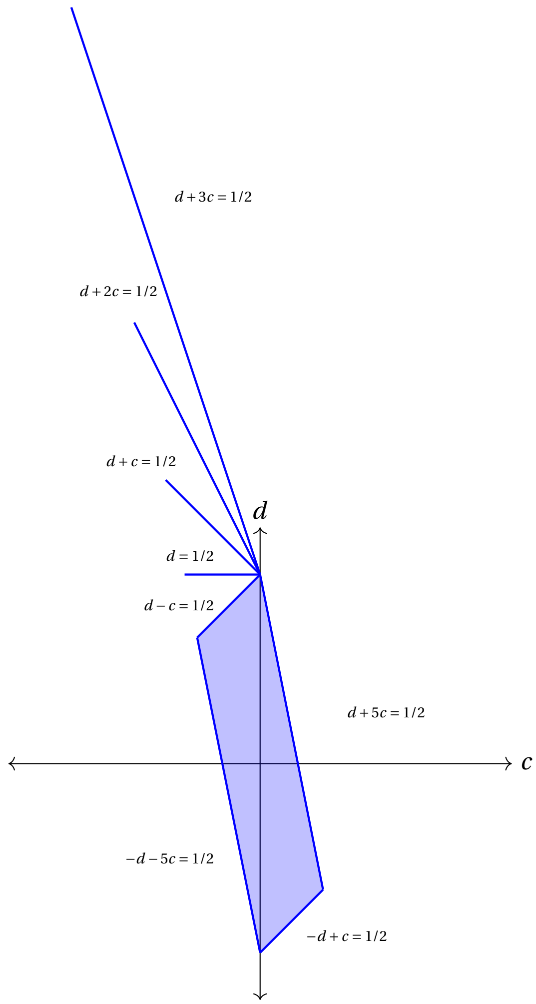
$$\lambda = \left(\ydiagram{3,2,1}, \varnothing\right)$$

FIGURE 28. Unitary spectrum



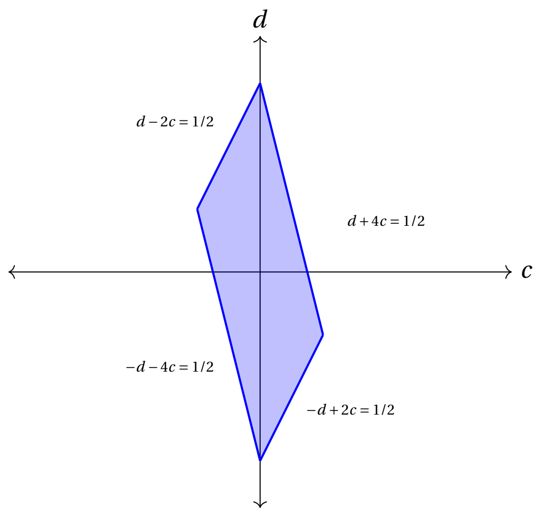
$\lambda = \left(\square\square\square\square\square, \square\right)$

FIGURE 29. Unitary spectrum



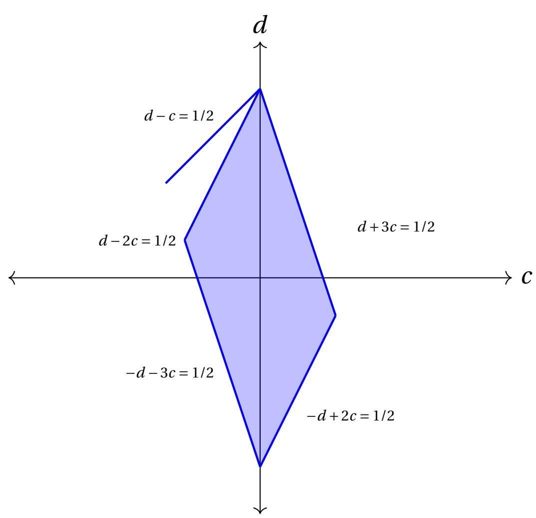
$$\lambda = \left(\yng(4,1), \yng(1)\right)$$

FIGURE 30. Unitary spectrum



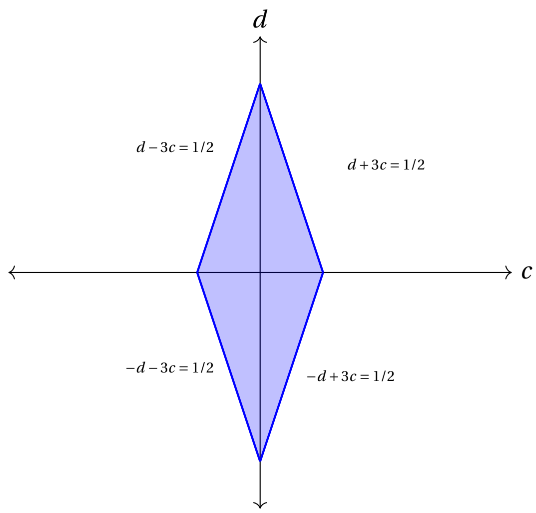
$$\lambda = \left(\yng(3,2), \yng(1)\right)$$

FIGURE 31. Unitary spectrum



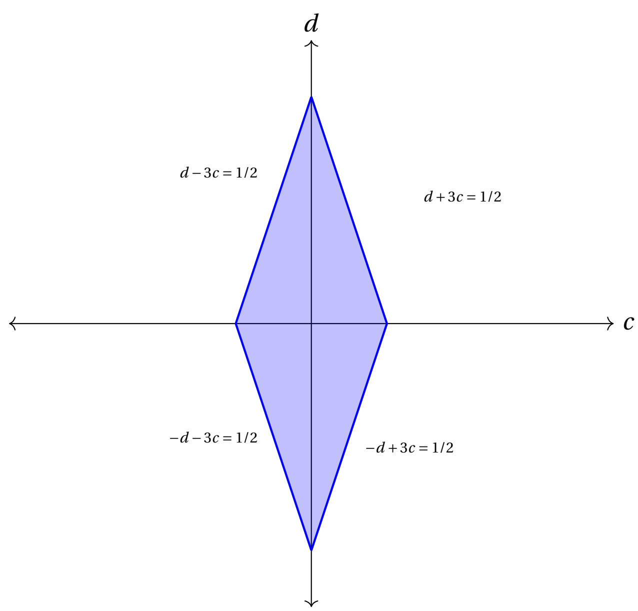
$$\lambda = \left(\ydiagram{3,1,1}, \ydiagram{1}\right)$$

FIGURE 32. Unitary spectrum



$$\lambda = \left(\ydiagram{4}, \ydiagram{1,1}\right)$$

FIGURE 33. Unitary spectrum



$\lambda = \left(\square\square\square\square, \square\square\right)$

FIGURE 34. Unitary spectrum