Rational derivations and group actions on algebraic varieties

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Contents

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Resumen

En la presente tesis abordaremos dos tópicos importantes en álgebra conmutativa y geometría algebraica afín que son las derivaciones y los automorfismos polinomiales. Estableceremos una correspondencia entre las acciones racionales del grupo multiplicativo $\mathbb{G}_m$ sobre variedades algebraicas afines y ciertas derivaciones que llamaremos racional semisimple. Además mostraremos una forma de escribir el cuerpo de funciones racionales a través del kernel de la derivación y un elemento que llamaremos slice racional para la derivación racional semisimple. En [7] Dubouloz y Liendo definen cuando una derivación es racionalmente integrable, este tipo de derivaciones están en correspondencia con las acciones racionales del grupo aditivo $\mathbb{G}_a$, generalizaremos el concepto de racional semisimple y racionalmente integrable a través de las derivaciones racional localmente finitas, una derivación racional localmente finita satisface cumplir que la aplicación exponencial de ella se factoriza sobre un cuerpo de funciones racionales. Los automorfismos racionales localmente finitos se definen a partir de un flujo racional el cual puede ser diferenciado. Mostraremos que si tenemos una derivación racional localmente finitas la aplicación exponencial de ella da origen a un automorfismo racional localmente finito y viceversa, si tenemos un automorfismo racional localmente finito su diferenciación permite obtener una derivación racional localmente finita.

Abstract

In the present thesis, we study two important topics in commutative algebra and algebraic geometry: polynomial derivations and polynomial automorphisms. We establish a one-to-one correspondence between the rational $\mathbb{G}_m$-actions on algebraic varieties and certain derivations ∂ which we will call rational semisimple. Also, we proved that if there exists an element $s \in K_X$ such that satisfy $\partial(s) = s$, called rational slice, we can decompose the field of rational functions since $K_X \simeq K_X^{\mathbb{G}_m}(s)$.

As defined by Dubouloz and Liendo define when a derivation is rationally integrable that the type of derivation is in correspondence with the rational $\mathbb{G}_a$-actions over algebraic varieties. We generalize the concept of rational semi-simple and rationally integrable derivation through rational locally finite derivation, which coincides with the regular case with the locally finite derivations.

The rational locally finite automorphism defines a rational flow that can be differentiated. We will show if we have a rational locally finite derivation, the exponential maps associated to it, give origin to a rational locally finite automorphism, and viceversa. If we have a rational locally finite automorphism with their differentiation, we will be allowed to obtain a rational locally finite derivation.

Notation

$B^{[n]} = B[x_1, \ldots, x_n]$	Polynomial ring in n variables with coefficient in B
$(\mathbb{G}_a, +)$	Additive group $(k, +)$
$(\mathbb{G}_m, +)$	Multiplicative group $(k^*, \cdot)$
$k[X], \mathcal{O}(X)$	Ring of coordinate (ring of regular functions) of algebraic variety X
$\mathbb{A}_k^n$	Affine space n dimensional over k
B^*	Units of B
$\mathrm{Frac}(B)$	Field of rational functions of B
$K(X), K_X$	Field of rational functions associated to algebraic variety X
$\mathrm{Der}(B)$	Set of derivations over B
$\mathrm{Aut}(X)$	Automorphism groups of variety X
$[E, D]$	Lie bracket of maps E and D

Introduction

The operations of derivation and integration are widely known and have been used for a considerable amount of time as the notion of derivative in its first beginnings. This concept is generalized through what we call derivation.

Derivations occur in many different contexts in diverse areas of mathematics and, in so doing, they connect various branches of it. For instance, the theory of Lie algebras, theory of geometric invariants, commutative algebra, algebraic geometry, differential algebra, and partial differential equations among others.

The derivations are a very useful tool to study several problems in mathematics. In our context we allow ourselves to distinguish whether two varieties are not isomorphic through invariants, and to describe and understand the automorphism group of affine algebraic varieties.

Our focus is on the polynomial derivations; this has form $D = \sum_{i=1}^{n} P_i \dfrac{\partial}{\partial x_i}$ where the P_i's are polynomials in the polynomial ring in n variables with coefficients in k $k^{[n]} = k[x_1, \ldots, x_n]$ and $\dfrac{\partial}{\partial x_i}$ corresponds to a partial derivative with respect to variable x_i. Given $f \in k^{[n]}$ the element $D(f)$ is defined by

$$D(f) = P_1 \frac{\partial f}{\partial x_1} + \ldots + P_n \frac{\partial f}{\partial x_n}.$$

The locally finite derivation and automorphism have contributed to understanding and establishing equivalences to the great problems of affine algebraic geometry. Some of the problems on affine n space are:

- **Characterization problem**: Finding an algebraic and geometric characterization of the affine n dimensional space $\mathbb{A}^n$.

- **Jacobian problem**: If the map $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ has a Jacobian determinant with an element of $\mathbb{C}^*$ then is it an automorphism?

- **Automorphism problem**: Providing a description of polynomial automorphisms of $\mathbb{A}^n$. The polynomial automorphisms form a group under composition, which is rather large.

- **Linearization problem**: When is an automorphism of $\mathbb{A}^n$ linearizable?.

Also, the derivations help to classify a finitely generated ring associated to algebraic variety and this way distinguish them from each other. Makar Limanov introduces an invariant (henceforth ML invariant) that consists in the intersection of all kernels of locally nilpotent derivations. Using the ML invariant and advanced techniques of graded ring, he managed to describe the group of automorphisms of the Danielewski surfaces. Moreover, using the ML invariant, he proved that the Koras-Russell variety associated with $\{(x, y, z, t) \in \mathbb{C}^4 \mid x+x^2y+z^2+t^3=0\}$ is not isomorphic with $\mathbb{C}^3$.

Understanding the automorphism group of the affine n-dimensional space $\mathbb{A}^n$ has always been of great interest to mathematicians; however, since this group is larger and very difficult, the classification is open for $n \geq 3$. In 1942 Jung-Van der Kulk gave a description of the automorphism group for $\mathbb{A}^2$ where the automorphism group is the amalgamated product of elementary group and affine group, and therefore any automorphism is tame. At that time it was believed that for $n = 3$ the automorphisms were also tame; however, in 1972 Nagata constructed the unipotent automorphism $(x-2y(y^2+xz)-z(y^2+xz)^2, y+z(y^2+xz), z)$ (exponential maps of locally nilpotent derivation) and conjectured that was not tame, Shestakov in 2004 managed to prove that indeed this automorphism is not tame, rather it is wild. The group of automorphisms is conjectured to be generated by triangular and affine automorphisms.

Within the linearization problem, we have the case in which the automorphism comes from an action of an algebraic group. The linearization problem about regular $\mathbb{G}_m$- actions is positive for the cases $n = 1$, $n = 2$ (Gutwirth), $n = 3$ (Koras-Russell). Similarly, studying the problem of linearization of derivations for the case semisimple is analogous and more general than the problem for actions of the multiplicative group, because the semisimple derivations are in correspondence with the semisimple automorphisms, and hence particularly just with some actions of the multiplicative group.

The derivations give negative answers to Hilbert's fourteenth problem which asks whether certain algebras are finitely generated. Zariski in 1954 proves that the problem is true for $n = 1, 2$. Then in 1959 Nagata found a counterexample to Hilbert's conjecture constructed ring of invariants for the action of a linear algebraic group for $\mathbb{C}^{[n]}$ with $n \geq 32$. Daigle and Freudenburg show a counterexample for the case $n = 5$ using the kernel of locally nilpotent derivation; nevertheless, the case for $n = 4$ is open.
This thesis is divided into four chapters.

- Chapter 1 presents some known facts about algebraic geometry. These topics will help us understand the correspondence between affine algebraic varieties and finitely generated rings, the actions of algebraic groups on affine algebraic varieties, and their equivalence in the co action morphism.

- Chapter 2 Derivations over affine rings (rings finitely generated over a field of characteristic zero) are introduced. We will see the classic results of this theory and its utilities, particularly the locally nilpotent derivations and the semisimple derivations.

- In Chapter 3, I describe some results of my work during my Ph.D. We established a correspondence between the rational actions of $\mathbb{G}_m$ on X and certain derivations ∂ on the field of rational functions $K(X)$, which we will call rational semisimple; this is an analog to the work conducted by Dubouloz and Liendo [7] in regards to rational $\mathbb{G}_a$ action. We describe $K(X)$ using the kernel of derivations and an element which we will call rational slice s for ∂, following Koshevoi's idea in [12], such an element allows the decomposition $(\ker\partial)(s) = K(X)$.

- Within Chapter 4, the correspondence between rational locally finite derivations in the field of rational functions $\mathrm{Frac}(B)$ and rational locally finite automorphism in $\mathrm{Frac}(B)$ will be described. This correspondence coincides in the regular case, the correspondence between locally finite elements (regular derivations and automorphisms), the locally nilpotent derivations, and the regular $\mathbb{G}_a$ action is verified, and when the eigenvalues of a semi-simple derivation are integer numbers with the regular $\mathbb{G}_m$ action.

Chapter 1

Basic notions of algebraic geometry

In this chapter, we define the basic topics of algebraic geometry and relate it with the action of algebraic groups over algebraic varieties and the ring of regular functions, field of rational functions, these objects are in correspondence between the category of rings finitely generated and the category of affine algebraic variety.

1.1 Categories

Definition 1.1. A Category $\mathscr{C}$ consists of 3 elements.

1. A collection of objects that we denote by $\mathrm{obj}(\mathscr{C})$.
2. For all pairs of objects A, B in $\mathrm{obj}(\mathscr{C})$ a set $\mathrm{Mor}(A, B)$ where the objects will be called morphism. When we denote a morphism $\mathrm{Mor}(A, B)$, we will denote it by $f\colon A \to B$.
3. Composition rule:

$$\circ : \mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$$

where $\circ(f, g)$ denotes $g \circ f$.

In addition, the following axioms must be verified:

1. Associativity. For any map f, g, h is satisfied:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

if these compositions are well defined.

2. Identity. For all objects A in $\mathrm{obj}(\mathscr{C})$ there exists a morphism in $\mathrm{Mor}(A, A)$ which we denote by 1_A such that for all morphisms f en $\mathrm{Mor}(A, B)$ $f = 1_B \circ f$ y $f = f \circ 1_A$.

Example 1.2. 1. The category of k-vector spaces where the objects are k-vector spaces and the morphisms are the linear transformations.

2. The category of topological spaces where the objects are topological spaces and the morphism are continuous functions.

3. The category of topological spaces pointed where the objects are pairs (X, x_0) where X is a topological space and x_0 is a fixed point in X and the morphism are continuous functions that send the fixed point in fixed point.

4. The category of groups where the objects are groups and the morphisms are homomorphisms.

Definition 1.3. For a pair of categories $\mathscr{A}$, $\mathscr{B}$ we define a covariant functor $F : \mathscr{A} \to \mathscr{B}$ consisting of:

1. Identify for each $A \in \text{obj}(\mathscr{A})$, with an object $F(A) \in \text{obj}(\mathscr{B})$.

2. For every pair of objects A, B in $\text{obj}(\mathscr{A})$ associate a morphism $f \in \text{Mor}(A, B)$ a morphism $F(f) \in \text{Mor}(F(A), F(B))$ satisfying:

   (a) $F(1_A) = 1_{F(A)}$

   (b) $F(f \circ g) = F(f) \circ F(g)$

Definition 1.4. For a pair of categories $\mathscr{A}$, $\mathscr{B}$ we define a contravariant functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ that consists of:

1. Identify for each $A \in \text{obj}(\mathscr{A})$, with an object $F(A) \in \text{obj}(\mathscr{B})$.

2. For all pair of objects A, B in $\text{obj}(\mathscr{A})$ associate to a morphism $f \in \text{Mor}(A, B)$ a morphism $F(f) \in \text{Mor}(F(B), F(A))$ verifying:

   (a) $F(1_A) = 1_{F(A)}$

   (b) $F(f \circ g) = F(g) \circ F(f)$

1.2 Algebraic set and Zariski topology

In this section, we consider k as a field of characteristic zero. Let n be positive integers, we define $\mathbb{A}_k^n = \{(x_1, x_2, \ldots, x_n) \,|\, x_i \in k\}$ and consider their set of points, for simplicity, we denote by $\mathbb{A}^n$ when the field is known. We will write $k^{[n]} = k[x_1, x_2, \ldots, x_n]$ the polynomial ring with coefficients in k, $k^{[n]}$ is a Noetherian ring, if $P(x_1, \ldots, x_n)$ is a polynomial in $k^{[n]}$ and

$x = (a_1, \ldots, a_n) \in \mathbb{A}^n$, we denote $P(a_1, \ldots, a_n)$ the evaluation of the polynomial $P(x_1, \ldots, x_n)$. For simplicity we denote $P(x)$ by $P(x_1, \ldots, x_n)$.

Definition 1.5. Let S be a algebraic subset arbitrary of $k^{[n]}$

$$V(S) = \{x \in \mathbb{A}^n | \text{ for all } P(x) \in S, P(x) = 0\} \tag{1.1}$$

i.e., $V(S)$ is the set of all common zeros of the polynomials $P(x)$ in S. $V(S)$ we will call it the affine algebraic set defined by S. If $S = \{P_1, \ldots, P_r\}$ we write $V(P_1, \ldots, P_r)$ instead of $V(\{P_1, \ldots, P_r\})$.

Example 1.6.

1. For positive integers n we have $V(\{1\}) = \emptyset$ y $V(\{0\}) = \mathbb{A}^n$.
2. For $n = 2$, we have $V(x_1, x_2) = \{(0, 0)\}$ and $W(x_1x_2) = \{(a, 0)\} \cup \{(0, b)\}$ where $a, b \in k$.

Remark 1.7.

1. The function V reverses the inclusions, i.e. if $S \subset S'$, then $V(S') \subset V(S)$.
2. If S is a subset of $k^{[n]}$, we write $\langle S\rangle$ or (S) to the ideal generated by S and verify $V(S) = V(\langle S\rangle)$. We can restrict ourselves to the case where S is an ideal.
3. Since $k^{[n]}$ is Noetherian, every ideal is finitely generated, that is, $I = \langle P_1, \ldots, P_r\rangle$ hence, every affine algebraic set is determined by a finite number of polynomials and verify $V(I) = V(P_1, \ldots, P_n) = V(P_1) \cap \ldots \cap V(P_r)$.
4. The set $\{a\}$ of a point in $\mathbb{A}^n$ is an affine algebraic set given by $V(x_i - a_i)$.
5. The arbitrary intersection of affine algebraic sets is an affine algebraic set $\bigcap\limits_j V(S_j) = V\left(\bigcup\limits_j S_j\right)$. If we restrict ourselves to ideals, then replace the union by the sum of ideals.
6. The finite union of affine algebraic sets is an affine algebraic set. $V(I) \cup V(J) = V(IJ) = V(I \cap J)$.

From 5) and 6) we deduce that the family of the affine algebraic set in $\mathbb{A}^n$ is a family of the closed set for the topology in $\mathbb{A}^n$.

Definition 1.8. We define the Zariski topology over $\mathbb{A}^n$ since the topology whose closed set are the affine algebraic sets.

Definition 1.9. We consider $P \in k^{[n]}$ and let $V(P) \subset \mathbb{A}^n$. We will call the set $D(P) = \mathbb{A}^n \setminus V(P)$ standard open (is the complement of closed set).

Example 1.10. 1. For a positive integer n we have $D(1) = \mathbb{A}^n$ y $D(0) = \emptyset$.

2. For $n = 2$ and k a field of characteristic 0 we have $D(x_1x_2) = \mathbb{A}^2 \setminus \{(x_1, x_2) \in \mathbb{A}^2 | \ x_1 = 0 \text{ or } x_2 = 0\}$.

Proposition 1.11. *The standard open familiy of $\mathbb{A}^n$ is a basis for the Zariski topology.*

Proof. The proof is given in [17]. □

1.3 Ideal of a affine algebraic set

We will define an operator I that associates an ideal in the polynomial ring with a set of points.

Definition 1.12. Let V be a subset of $\mathbb{A}^n$. We define

$$I(V) = \{P \in k^{[n]} \mid P(x) = 0 \text{ for all } x \in V\} \tag{1.2}$$

is called the ideal of V.

$I(V)$ is the set of polynomial functions vanishing in V. To verify that this is an ideal, we consider the ring homomorphism

$$r : k^{[n]} \to \mathcal{F}(V, k)$$

where $\mathcal{F}(V, k)$ is the ring of polynomial functions with domain V and codomain k.

In this way $\text{Im}(r) \in \mathcal{F}(V, k)$ are the functions whose restriction in V coincides with polynomials and $\ker(r) = I(V)$, which in consequence is an ideal. The image $\text{Im}(r)$ we denote by $\Gamma(V)$ and its polynomials we will call regular functions. By the first isomorphic theorem of a ring $\Gamma(V) = \text{im}(r) \cong k^{[n]}/I(V)$

1. For a positive integer n we have $I(\emptyset) = k^{[n]}$.

2. If k is infinite, for n positive integer $I(\mathbb{A}^n_k) = 0$

1.4 Irreducibility

Definition 1.13. Let X be a topological space that is not empty, a topological space X is said to be irreducible if $X = F \cup G$, where F y G are closed sets in X, then $X = F$ or $X = G$.

Theorem 1.14. *Let V be an affine algebraic set in $\mathbb{A}^n$ endowed with the Zariski topology.*

$$V \textit{ is irreducible} \Leftrightarrow I(V) \textit{ is the ideal prime} \Leftrightarrow \Gamma(V) \textit{ integral domain.}$$

Proof. The proof of Theorem 1.14 is given in [17, Pages 13 and 14]. □

Definition 1.15. Let X be a set, a chain of subsets of X is a sequence $X_0 \subset X_1 \subset \ldots \subset X_n$ such that X_i are differents. We will say that the above chain has a large n.

Definition 1.16. Let X be a topological space. The dimension of X is the maximum of the lengths of chains of closed subsets irreducible of X. This number is a positive integer, or $+\infty$, which we denote by $\dim X$.

Theorem 1.17. *Let V be a affine algebraic subset not empty. We can write V of unique form, except rearrangement, $V = V_1 \cup \ldots \cup V_r$, where the sets V_i are irreducible affine algebraic sets and $V_i \nsubseteq V_j$. The V_i are called irreducible components of V.*

Proof. The proof of Theorem 1.17 is given in [17, Pages 14 and 15]. □

1.5 Hilbert's Nullstellensatz (zero-locus-theorem)

It is not difficult to prove that if $V \subset \mathbb{A}^n$ is an affine algebraic set, then $V(I(V)) = V$. However, it is not always true that if $I \subset k^{[n]}$ is an ideal of polynomial rings, if we verify $I(V(I)) = I$, then we just have $I \subset I(V(I))$.

Example 1.18. Let $k = \mathbb{R}$ be the real numbers and $I = (x_1^2 + x_2^2 + 1)$, then $V(I) = \emptyset$ and $I(V(I)) = \mathbb{R}^{[n]} \neq I$.

We now consider k an algebraically closed field.

Theorem 1.19 (Weak Nullstellensatz)**.** *Let $I \subset k^{[n]}$ be a proper ideal contain in $k^{[n]}$. Then $V(I)$ is not empty.*

Proof. The proof of this theorem 1.19 is in [17, Pages 15 and 16]. □

Definition 1.20. Let B be a ring, we define the radical of a ideal I in B since the ideal

$$Rad(I) = \{x \in B | \text{ there exist } r \in \mathbb{N} \text{ such that } x^r \in I\} \tag{1.3}$$

$Rad(I)$ is an ideal containing I.

Example 1.21. Let $B = \mathbb{C}^{[1]}$ be the polynomial ring in one variable with coefficients complex. We consider $I = (x^2)$ then $Rad(I) = (x)$ for all P in (x), P^2 is in (x^2), $(x) \subset Rad(I)$. On the other hand, since (X) is maximal y $Rad(I) \neq \mathbb{C}^{[1]}$ is verifying $Rad(I) = (x)$.

Theorem 1.22 (Nullstellensatz)**.** *Let I be an ideal in $k^{[n]}$. then $I(V(I)) = Rad(I)$.*

Proof. The proof of Theorem 1.22 is given in [17, Page 16]. □

Remark 1.23. The ideal $I(V)$ is radical if only if the ring $\Gamma(V)$ es reduced (i.e, has not nilpotent elements).

An application of the nullstellensatz tells us that if V is an affine algebraic set, we associate V to the ideal $I(V)$ and this one to the algebra $\Gamma(V)$, that is, a reduced k-algebra of finite type. Is reduced because $I(V)$ is a radical ideal and of finite type because it is isomorphic to $k^{[n]}/I$ ([17, page 202]).

Proposition 1.24. *There exist a correspondence bijective decrease $W \mapsto I(W)$, whose inverse is $I \mapsto V(I)$, between affine algebraic sets of $\mathbb{A}^n$ and radical ideals in $k^{[n]}$. Also, the following are equivalents :*

1. *W is irreducible $\Leftrightarrow I(W)$ prime $\Leftrightarrow \Gamma(W)$ integral.*
2. *W is a point $\Leftrightarrow I(W)$ maximal $\Leftrightarrow \Gamma(W) = k$.*

Proof. The proof of 1 is the theorem 1.14 and part 2 is deduced from theorem 1.19 and the decreasing property of I y V.

□

In general, if V is an arbitrary set of affine algebraic sets with W an affine algebraic set contained in V, $I(V) \subset I(W)$. The theorem of homomorphism of the ring, $I(W)$ determinate an ideal $I_V(W)$ of ring $\Gamma(V)$ correspond to the set $f \in \Gamma(V)$ that vanishing in W and we have the isomorphism

$$\Gamma(V)/I_V(W) \cong \Gamma(W) \tag{1.4}$$

If I is an ideal of $\Gamma(V)$, we can define $V(I)$, since the set of zeros of functions of I over V:

$$V(I) = \{x \in V \text{ for all} f \in I, f(x) = 0\} \tag{1.5}$$

Proposition 1.25. *There exists a decreasing mutually inverse bijection $W \mapsto I_V(W)$ y $I \mapsto V(I)$, between the affine algebraic sets of V and the radical ideals in $\Gamma(V)$. In addition, the following are equivalents.*

1. *W is irreducible $\Leftrightarrow I_V(W)$ is prime $\Leftrightarrow \Gamma(W)$ is integral.*
2. *W is a point $\Leftrightarrow I_V(W)$ is maximal $\Leftrightarrow \Gamma(W) = K$.*
3. *W is a irreducible component of $V \Leftrightarrow I_V(W)$ is a minimal ideal prime of $\Gamma(V)$.*

Proof. The proof of Proposition 1.25 is in [17, Page 18] and follows from Proposition 2. □

Proposition 1.26. *The points of V are in bijection with the maximal ideals of $\Gamma(V)$.*

Proof. The proof of Proposition 1.26 is in [17, Page 18]. □

Definition 1.27. Let V be a affine algebraic set and $f \in \Gamma(V)$ not zero. The set

$$D_V(f) = V \setminus V(f) = \{x \in V | f(x) \neq 0\} \tag{1.6}$$

is called a standard open set of V.

If we know V, the open standard $D_V(f)$ will be denoted by $D(f)$. This open set generates a basis of a topology on V which is also called the **Zariski topology** on V.

1.6 Category of affine algebraic sets

For this section, k is a algebraically closed field. We will define one of the basic categories in the work of algebraic geometry.

We will consider the family of objects

$$ob(\mathscr{V}) = \{V \subset \mathbb{A}^n |\ n \in \mathbb{N}, V \text{ affine algebraic set }\} \tag{1.7}$$

Definition 1.28. Given V and W elements of $obj(\mathscr{V})$ with $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ and let $\varphi : V \longrightarrow W$ be a function that can be written of form $\varphi = (\varphi_1, \ldots, \varphi_m)$, where $\varphi_i : V \longrightarrow \mathbb{A}^1$. We say that P is a regular map if $P_i \in \Gamma(V)$, and the set $\mathrm{Mor}(V, W)$ we will denote by $Reg(V, W)$.

It is not difficult to prove that considering the definition above and $ob(\mathscr{V})$ we can define a category ([17, pag 20]).

Example 1.29.

1. Every element f of $\Gamma(V)$ is a morphism. In particular, the coordinate functions are morphisms from V to $\mathbb{A}^1$.

2. If V is $V(x_2 - x_1^2)$ and let φ be the projection $\varphi : V \to \mathbb{A}^1$ given by $\varphi(x_1, x_2) = x_1$. Then φ is an isomorphism of affine algebraic sets where the inverse is given by $x_1 \mapsto (x_1, x_1^2)$.

Definition 1.30. Let $\varphi : V \to W$ be a morphism of algebraic sets. For any $f \in \Gamma(W)$ we define $\varphi^*(f) = f \circ \varphi$. Then

$$\varphi^* : \Gamma(W) \to \Gamma(V)$$

Proposition 1.31. *With the information of the above definition $\varphi^* : \Gamma(W) \to \Gamma(V)$ is a morphism of reduced k-algebras of finite type.*

Remark 1.32. Γ is a contravariant functor between the category of affine algebraic sets with the regular functions and the category of reduced k-algebras of finite type with the homomorphism of k-algebras associate to (V, φ) with $(\Gamma(V), \varphi^*)$ (See [17, Page 21]).

Proposition 1.33. *The functor Γ is completely faithful, that is, the map $\gamma \colon \varphi \mapsto \varphi^*$ from $Reg(V, W)$ to $Hom_{k\text{-}alg}(\Gamma(W), \Gamma(V))$ is biyective.*

Proof. The proof of Proposition 1.33 is in [17, Page 21] □

Remark 1.34. To further emphasize this relationship between categories and the relation between V and $\Gamma(V)$, we consider a k-algebra reduced finite dimensional n, $B \simeq k^{[n]}/I$ for some ideal I of $k^{[n]}$. Given $V = V(I)$ we will write $V = \mathrm{Spec}(B)$ in consequence, some times we will write $V = \mathrm{Spec}(\Gamma(V))$.

1.7 Ringed spaces

Definition 1.35. Let X be a topological spaces. A pre sheaf over X is given by the following information:

1. For each open set U in X, a set $\mathcal{F}(U)$.
2. For each pair of open U and V that verify $V \subset U$, a function $r_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the restriction function, which we will often denote by $r_{V,U}(f) = f|_V$ that verify the following conditions:
   (a) Si $W \subset V \subset U$, then $r_{W,U} = r_{W,V} \circ r_{V,U}$
   (b) $r_{U,U} = \mathrm{Id}_{\mathcal{F}(U)}$

If our pre-sheaf also verify:
If U is an open of X covered by open sets , $\{U_\alpha\}$ ($\alpha \in A$ a set of indexes), then for all the choice of elements $f_\alpha \in \mathcal{F}(U_\alpha)$ such that $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\beta \cap U_\alpha}$ there is only one $f \in \mathcal{F}(U)$ such that $f|_{U_\alpha} = f_\alpha$, will say that $\mathcal{F}$ is a sheaf.

Definition 1.36. If X is a topological space and $\mathcal{F}$ is a sheaf over X. If U is an open set in X, $\mathcal{F}(U)$ is a commutative ring and the restriction functions are homomorphisms of the ring, we will say that $\mathcal{F}$ is a sheaf of the ring.

Example 1.37. 1. Sheaf of continuous functions in $\mathbb{R}$ or $\mathbb{C}$.

2. Sheaf of differentiable functions in $\mathbb{R}$ or $\mathbb{C}$.

An usual notation in Sheaf's theory is $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$ and the elements of $\Gamma(U, \mathcal{F})$ are called section of $\mathcal{F}$ over U and when $U = X$ the corresponding sections are called global section.

If we fix a topological space X we can define the category of sheaf over X so it is necessary to define a set of morphisms between each pair of sheafs $\mathcal{F}$ and $\mathcal{G}$.

Definition 1.38. A The morphism of the sheaf $\phi : \mathcal{F} \to \mathcal{G}$ in a topological space X is a family of functions $\{\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)\}$ with an open set U of X that behaves well with respect to inclusions, i.e., if $U \subset V$ the following diagram is commutative

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "r_{V,U}"'] & \mathcal{G}(V) \arrow[d, "r_{V,U}"] \\
\mathcal{F}(U) \arrow[r, "\phi(U)"'] & \mathcal{G}(U)
\end{tikzcd}$$

Definition 1.39. If X is a topological space endowed with a sheaf of ring $\mathscr{O}_X$, the pair $(X, \mathscr{O}_X)$ is called a ringed space and $\mathscr{O}_X$ is called the structural sheaf.

We can define the category of ringed space where objects obviously are ringed spaces and each pair of ringed spaces $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ a morphism of a ringed space, consist in a pair (f, ϕ) where $f : X \to Y$ is a continuous function $\phi : \mathscr{O}_Y \to f^*\mathscr{O}_X$ is a sheaf morphism. $f^*\mathscr{O}_X$ is the sheaf on Y that is associated with each open set U of Y the commutative ring $\mathscr{O}_X(f^{-1}(U))$.

1.8 Affine algebraic variety

Let $V \subset \mathbb{A}^n$ be an affine algebraic set with the Zariski topology, we want to define a structural sheaf $\mathscr{O}_V$ over V to define the ringed spaces $(V, \mathscr{O}_V)$.

It is enough to define $\mathscr{O}_V$ based on open sets of V that verify certain conditions, which generate a unique sheaf in V [17, page 41].

Lemma 1.40. *Let X be a topological space, $\mathscr{B}$ be a basis of open sets in X and I a set. We suppose that for each open set U in $\mathscr{B}$ a set $\mathcal{F}(U)$ of functions from U to I satisfying the following conditions:*

1. *If V, U are in $\mathscr{B}$, $V \subset U$ y $s \in \mathcal{F}(U)$, then $s|_V \in \mathcal{F}(V)$.*
2. *If an open set $U \in \mathscr{B}$ is covered by the set U_i indexed by $i \in I$, such that $U_i \in \mathscr{B}$ and if s is a function of U to k such that, for all $i \in I$ $s|_{U_i}$ in $\mathcal{F}(U_i)$, then $s \in \mathcal{F}(U)$.*

Then there is a unique sheaf $\overline{\mathcal{F}}$ of functions on X such that, for each $U \in \mathscr{B}$, $\overline{\mathcal{F}}(U) = \mathcal{F}(U)$

Proof. The proof of lemma 1.40 is in [17, Page 41]. □

Definition 1.41. Let V an affine algebraic set , for all $f \in \Gamma(V)$, we will consider $D(f)$ and we define $\mathscr{O}_V(D(f)) = \Gamma(V)_f$. La localization of $\Gamma(V)$ en f.

Example 1.42. If $k = \mathbb{C}$, we consider $V = \mathbb{A}^n$, $I(\mathbb{A}^n) = \{0\}$, $\Gamma(V) = \mathbb{C}^{[n]}/\{0\} \simeq \mathbb{C}^{[n]}$ let $f(x_1, \ldots, x_n) = x_1 \ldots x_n \in \mathbb{C}^{[n]}$ then $\mathscr{O}_V(D(f)) = \mathbb{C}^{[n]}_{x_1 \ldots x_n}$.

The definition 1.41 confirms the conditions of Lemma 1.40 (see [17, Pages 42 and 43]) and, consequently, defines a structural sheaf over V called the sheaf of regular functions.

Definition 1.43. An affine algebraic variety is a ringed space $(X, \mathscr{O}_X)$ that is isomorphic to a ringed space $(V, \mathscr{O}_V)$, where V is an affine algebraic set and $\mathscr{O}_V$ is the sheaf of regular functions over V. A morphism of affine algebraic varieties is simply a morphism of ringed spaces.

We can define the category of affine algebraic varieties where the objects are affine algebraic varieties and for each pair of affine algebraic varieties $(X, \mathscr{O}_X)$ y $(Y, \mathscr{O}_Y)$ a morphism in $\mathrm{Hom}_{Var}(X, Y)$ is a morphism of ringed spaces.

Proposition 1.44. *Let V be affine algebraic set and we consider $f \in \Gamma(V)$ the open set $D(f)$ endowed with the restriction sheaf $\mathscr{O}_V$ to $D(f)$ is affine algebraic variety*

Proof. The proof of Proposition 1.44 is in [17, Page 43]. □

Proposition 1.45. *Let $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ be two affine algebraic varieties , there is a bijection between the following sets*

$$\mathrm{Hom}_{\mathrm{Var}}(X, Y) \simeq \mathrm{Reg}(X, Y) \simeq \mathrm{Hom}_{k\text{-alg}}(\Gamma(Y), \Gamma(X))$$

Proof. The proof of Proposition 1.45 is in [17, Page 44]. □

In the following a affine algebraic variety $(X, \mathscr{O}_X)$ will only be mentioned X and the morphisms between the varieties will be presented since the morphism between the affine algebraic set uses the proposition 1.45.

Definition 1.46. A topological space X is compact if, given any recovery of X, there is a finite sub-coverage of it.

Definition 1.47. An algebraic variety is a ringed space $(X, \mathscr{O}_X)$ such that X is compact and locally isomorphic since the ringed spaces are affine algebraic varieties. i.e., for all $x \in X$ i.e., for all x in X there is an open neighborhood U of x such that $(U, \mathscr{O}_X|_U)$ is an affine algebraic variety.

1.9 Algebraic groups

Let k be an algebraically closed field (characteristic zero). We consider affine algebraic varieties X over k, endowed with the Zariski topology. The coordinate ring of X, or the ring of a regular function, will be indicated by $k[X]$ or $\mathcal{O}(X)$. If B is an affine k domain, then $X = \mathrm{Spec}(B)$ is its corresponding affine varieties. We consider the algebraic group G over k. An algebraic group G is an algebraic variety endowed with a group structure that is compatible with its structure, since it is an algebraic variety.

Definition 1.48. Let G be algebraic variety over k and $m : G \times G \to G$ a regular morphism. The pair (G, m) is an algebraic group over k if there are regular maps

$$\mathrm{Id}_G : * \to G \ , \ \mathrm{inv} : G \to G$$

such that the following diagram commutes

$$\begin{tikzcd}
G \times G \times G \arrow[r, "Id_G \times m"] \arrow[d, "m \times Id_G"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "(Inv{,}Id_G)"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "(Id_G{,}Inv)"'] \arrow[d] \\
* \arrow[r, "e"] & G & * \arrow[l, "e"']
\end{tikzcd}$$

$$\begin{tikzcd}
* \times G \arrow[r, "e\times Id_G"] \arrow[dr, "\simeq"'] & G \times G \arrow[d, "m"] & G \times * \arrow[l, "Id_G\times e"'] \arrow[dl, "\simeq"] \\
& G &
\end{tikzcd}$$

Example 1.49.

- The additive group $\mathbb{G}_a := (k, +)$ is an algebraic group because the addition is an algebraic map and $\mathcal{O}(\mathbb{G}_a) = k[t]$.
- The multiplicative group $\mathbb{G}_m := (k \setminus \{0\}, \cdot)$ is an algebraic group with the usual multiplication and $\mathcal{O}(\mathbb{G}_m) = k[t, t^{-1}]$.
- The General linear group $\mathrm{GL}_n(k)$ with the product of matrices is an algebraic group and its coordinate ring is $\mathcal{O}(\mathrm{GL}_n(k)) = k[t_{11}, t_{12}, \ldots, t_{nn}, d^{-1}]$ with $d = \det\{\mathrm{t_{ij}}\}$.

1.10 Group actions

Definition 1.50. We denote by $\mathrm{e}_G : \mathrm{Spec}(k) \to G$ the neutral element of G and $\mathrm{m}_G : G \times G \to G$ the morphism given by the group operation law . An group action of a group G over X is a morphism $\alpha : G \times X \longrightarrow X$ such that the following diagrams are commutative.

$$\begin{tikzcd}
G \times G \times X \arrow[r, "\mathrm{Id}_G\times\alpha"] \arrow[d, "\mathrm{m}_G\times\mathrm{Id}_X"'] & G \times X \arrow[d, "\alpha"] \\
G \times X \arrow[r, "\alpha"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Spec}(k) \times X \arrow[r, "\mathrm{e}_G\times\mathrm{Id}_X"] \arrow[dr, "\mathrm{pr}_2"'] & G \times X \arrow[d, "\alpha"] \\
& X
\end{tikzcd}$$

This action can be seen in the ring of regular functions through the comorphism:

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[r, "\alpha^*"] \arrow[d, "\alpha^*"'] & \mathcal{O}(G) \otimes \mathcal{O}(X) \arrow[d, "(\mathrm{m}_G\times\mathrm{Id}_X)^*"] \\
\mathcal{O}(G) \otimes \mathcal{O}(X) \arrow[r, "(\mathrm{Id}_G\times\alpha)^*"'] & \mathcal{O}(X) \otimes \mathcal{O}(G) \otimes \mathcal{O}(G)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}(X) \arrow[r, "\alpha^*"] \arrow[dr, "\mathrm{Id}_{\mathcal{O}(X)}"'] & \mathcal{O}(G) \otimes \mathcal{O}(X) \arrow[d, "\mathrm{ev}_{e_G}"] \\
& \mathcal{O}(X)
\end{tikzcd}$$

We suppose G is algebraic group and G acts algebraically over the k-variety X and we write $B = \mathcal{O}(X)$, the ring invariants of this action is

$$B^G = \{f \in B \,|\, g \cdot f = f \;\forall\, g \in G\}$$

An element $f \in B$ is called semi-invariant for the action if there exists a character $\chi\colon G \to k^*$ such that $g \cdot f = \chi(g)f$ for all $g \in G$. In this case χ the weight of the semi-invariant f. Certain important groups, such as the lineal special group $\mathrm{SL}_2(k)$ and the additive group $\mathbb{G}_a$ have not invariant characteristics and are not trivial.

The sets of fixed points by the action are

$$X^G = \{x \in X | g \cdot x = x \;\forall g \in G\}$$

The action is free of fixed points or simply free if X^G is empty. the orbit of $x \in X$ is the set $\{g \cdot x |\; g \in G\}$, is denoted by $G \cdot x$ or $\mathcal{O}(x)$.

In terms of group actions, Our main research interests are the action of additive group ($\mathbb{G}_a$-action) and the action of multiplicative group ($\mathbb{G}_m$-action) over the field k.

1.10.1 Regular $\mathbb{G}_a$-action

Definition 1.51. A regular $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times X \to X$ is a morphism between varieties such satisfy $t \cdot (s \cdot x) = (t+s) \cdot x$ and $0 \cdot x = x$. This definition has a characterization in the category of affine algebra. A regular $\mathbb{G}_a$-action $\alpha : \mathbb{G}_a \times X \to X$ is equivalent to determining a coaction homomorphism $\alpha_t^* : \mathcal{O}(X) \longrightarrow \mathcal{O}(X)[t]$ such that the following diagrams commute:

$$\begin{tikzcd} \mathcal{O}(X) \arrow[r, "\alpha_t^*"] \arrow[d, "\alpha_s^*"'] & \mathcal{O}(X)[t] \arrow[d, "t\mapsto s+t"] \\ \mathcal{O}(X)[s] \arrow[r, "\widetilde{\alpha}"] & \mathcal{O}(X)[t,s] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}(X) \arrow[r, "\alpha_t^*"] \arrow[dr, "\mathrm{Id}_{\mathcal{O}(X)}"'] & \mathcal{O}(X)[t] \arrow[d, "\mathrm{ev}_0"] \\ & \mathcal{O}(X). \end{tikzcd} \tag{1.8}$$

Where $\widetilde{\alpha}$ is the extension of homomorphism of ring α_s^* , which fixed the element t, therefore this diagram $\alpha_t^* \circ \alpha_s^* = \alpha_{t+s}^*$ and $\mathrm{ev}_0 \circ \alpha_t^* = \mathrm{Id}_{\mathcal{O}(X)}$.

Remark 1.52. The Rensthler theorem in [19] shows that all $\mathbb{G}_a$-action, $\mathbb{G}_a \times \mathbb{C}^2 \to \mathbb{C}^2$ are conjugate to $(t, (x, y)) \mapsto (x, y + tf(x))$, with $f(x) \in \mathbb{C}^{[1]}$.

1.10.2 Regular $\mathbb{G}_m$-action

Definition 1.53. A regular $\mathbb{G}_m$-action $\alpha : \mathbb{G}_m \times X \to X$ is a morphism between varieties that satisfy $t \cdot (s \cdot x) = ts \cdot x$ and $1 \cdot x = x$. This definition has a characterization in the category of affine algebra. A regular $\mathbb{G}_m$-action $\alpha : \mathbb{G}_m \times X \to X$ is equivalent to determining a coaction homomorphism $\alpha_t^* : \mathcal{O}(X) \longrightarrow \mathcal{O}(X)[t^{\pm 1}]$ such that the following diagrams commute:

$$
\begin{tikzcd}
\mathcal{O}(X) \arrow[r, "\alpha_t^*"] \arrow[d, "\alpha_s^*"'] & \mathcal{O}(X)[t^{\pm 1}] \arrow[d, "t \mapsto st"] \\
\mathcal{O}(X)[s^{\pm 1}] \arrow[r, "\widetilde{\alpha}"] & \mathcal{O}(X)[t^{\pm 1}, s^{\pm 1}]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}(X) \arrow[r, "\alpha_t^*"] \arrow[dr, "\mathrm{Id}_{\mathcal{O}(X)}"'] & \mathcal{O}(X)[t^{\pm 1}] \arrow[d, "\mathrm{ev}_1"] \\
 & \mathcal{O}(X).
\end{tikzcd}
$$

Where $\widetilde{\alpha}$ is the extension of homomorphism of ring α_s^* , which fixed the element t, therefore this diagram $\alpha_t^* \circ \alpha_s^* = \alpha_{ts}^*$ and $\mathrm{ev}_1 \circ \alpha_t^* = \mathrm{Id}_{\mathcal{O}(X)}$.

The $\mathbb{G}_m$-actions correspond to $\mathbb{Z}$-graduations over $\mathcal{O}(X)$, therefore we can write $\mathcal{O}(X)$ since

$$\mathcal{O}(X) = \bigoplus_{i \in \mathbb{Z}} \mathcal{O}(X)_i = \{f \in \mathcal{O}(X) | t \cdot f(x) = t^i f(x)\}$$

then $\mathcal{O}(X)^{\mathbb{G}_m} = \mathcal{O}(X)_0$, this invariants ring is finitely generated because $\mathbb{G}_m$ is a reductive group. If X is a $\mathbb{G}_m$-variety, $\mathbb{G}_m$ can be seen as a subgroup of $\mathrm{Aut}(X)$ via the homomorphism ring $\mathbb{G}_m \hookrightarrow \mathrm{Aut}(X)$ given by $t \mapsto \alpha_t$.

A regular $\mathbb{G}_m$ action on $\mathbb{A}^n$ is linear if $t \cdot (x_1, x_2, \ldots x_n) = (t^{\lambda_1} x_1, t^{\lambda_2} x_2, \ldots, \ldots, t^{\lambda_n} x_n)$ with $\lambda_i \in \mathbb{Z}$. They are classified conjugate by an automorphism for $n = 1$,$n = 2$ [11], $n = 3$ [21] and are linear , for $n > 3$ is open.

Chapter 2

Derivations

Let B be a k-domain, where k is a characteristic zero field. B^* denote the group of units of B y Frac(B) denote the quotient field of B, Aut(B) denotes the automorphism group of B since k-algebra.

2.1 Basic definition for derivations

Definition 2.1. Let B be a k-algebra, $D\colon B \to B$ be a map. A **derivation** is any function D that satisfies the following conditions:
For all $a, b \in B$

1. $D(a+b) = D(a) + D(b)$
2. $D(ab) = aD(b) + bD(a)$ (Leibniz rules)

the set of all derivations over of is denoted by $\mathrm{Der}(B)$, if A is a subring of B, we denote $\mathrm{Der}_A(B)$ by the subset of all $D \in \mathrm{Der}(B)$ with $D(A) = 0$. The **kernel** of D is the set $\ker(D) = \{b \in B | D(b) = 0\}$ (also denoted by B^D the **ring of constant** of B with respect to D), some important facts.

- $\ker(D)$ is a subring of B for any $D \in \mathrm{Der}(B)$.
- The subfield $\mathbb{Q} \subset k$ has $\mathbb{Q} \subset \ker(D)$ for any $D \in \mathrm{Der}(B)$.
- $\mathrm{Aut}(B)$ acts on $\mathrm{Der}(B)$ by conjugation: $\varphi \cdot D = \varphi D \varphi^{-1}$.
- Given $b \in B$ and $D, E \in \mathrm{Der}(B)$, if $[D, E] := DE - ED$, then bD, $D + E$, and $[D, E]$ are again in $\mathrm{Der}(B)$.

We denote the nth composition $D^{(n)}$ by D^n and $D^0 = \mathrm{Id}_B$ is the identity map in B.

Definition 2.2. Let $D \in \mathrm{LND}(B)$, we said that D is **irreducible** if (DB) does not contain a proper principal ideal of B.

2.1.1 Locally finite derivations on B

Definition 2.3. Let B be a commutative k-algebra and D a k- derivation of B, D is called locally finite if for any $b \in B$ there exists a finite generated k-module $M \subset B$ such that $b \in M$ and $D(M) \subset M$.

Equivalently, if we denote by V_b, the k- submodule of B generated by the set of the n -th composition $\{b, D(b), D^2(b), D^3(b), \ldots\}$, given an element $b \in B$, the derivation D is called **locally finite**, if every module V_b is finitely generated over k for all $b \in B$. This definition indicates the existence of annihilator minimal polynomial $p(T) \in k[T]$ such that $p(D) = D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0 I = 0$. The set of locally finite derivations will be denoted by $\mathrm{LFD}(B)$, and the term "locally finite" will be abbreviated as lf. Additionally, two particular cases of lf derivations will be defined: the locally nilpotent derivations and the semisimple derivations. An element $b \in B$ is said to be semisimple in relation to D if there exists a finite D-invariant k-subspace $W \subset B$ containing b such that the k-endomorphism $D|_W$ is semisimple.

Also, we denote by $\mathrm{Nil}(D)$, $\mathrm{Sem}(D)$ and $\mathrm{Fin}(d)$ the following subset of B

$$\mathrm{Nil}(D) = \{b \in B; \exists\, n \in \mathbb{Z}_{\geq 0}\ D^n(b) = 0\}$$

$$\mathrm{Sem}(D) = \{b \in B; \exists\, M_b \subset B, b \in M_b, D|_{M_b} \text{ is semisimple}\}$$

$$\mathrm{Fin}(D) = \{b \in B; \exists\, M_b \subset B, b \in M_b, D(M_b) \subset M_b, M_b \text{ is a finite } k\text{-module}\}$$

The following containments are always followed $B^D \subset \mathrm{Nil}(D) \subset \mathrm{Fin}(D) \subset B$.

Proposition 2.4. *If D is a k-derivation of a k-algebra B, where k is a field of characteristic zero, then $\mathrm{Nil}(D), \mathrm{Sem}(D)$ and $\mathrm{Fin}(D)$ are k- sub algebra of B.*

Proof. The proof is in [14] Proposition 7.1 ,Proposition 9.5.2. □

Lemma 2.5. *Let S be a generating set for the k-algebra B. If for each $g \in S$ the vector space generated by elements $D^i(g)$ is finite-dimensional, then D is locally finite.*

Proof. The proof is based on Lemma 2.2 in [23]. □

Definition 2.6. Let $D \in \mathrm{Der}(B)$, if for each $b \in B$ there exists $j \in \mathbb{Z}_{\geq 0}$ such that $D^j(b) = 0$ (that is, we have $\mathrm{Nil}(D) = B$), D is called **locally nilpotent derivation** (LND for short) . The set of locally nilpotent derivation over B is denoted $\mathrm{LND}(B)$ and the set of locally nilpotent derivation with kernel $A \subset B$ by $\mathrm{LND}_A(B)$.

Definition 2.7. Let B be a ring and $D \in \mathrm{LND}(B)$. A **slice** of D is an element $s \in B$ that satisfies $D(s) = 1$. A **preslice** (or local slice) of D is an element $s \in B$ that satisfies $D(s) \neq 0$ and $D^2(s) = 0$.

Definition 2.8. A derivation $D \in \mathrm{Der}(B)$ is **semisimple** if there exists a basis $\{b_i\}_{i \in I}$ of B as k vector space such that $D(b_i) = \lambda_i b_i$ with $\lambda_i \in k$ (that is, we have $\mathrm{Sem}(D) = B$), the set of semisimple derivation is denoted by $\mathrm{SSD}(B)$.

A slice $s \in B$ for $D \in \mathrm{SSD}(B)$, is an element that satisfies $D(s) = s$. Note that a semisimple derivation D defines a k-graduation of B since all elements can be written as a linear combination of elements of $\{b_i\}_{i \in I}$.

Clearly, the locally nilpotent and semisimple derivations according to their definition are locally finite derivations.

Proposition 2.9. *(Decomposition Jordan- Chevalley) Any $D \in \mathrm{LFD}(B)$ admits a decomposition $D = D_N + D_S$ where $D_N \in LND(B)$,$D_S \in \mathrm{SSD}(B)$ and $[D_N, D_S] = 0$.*

Proof. For more details, see Proposition 1.3.8 in [22], Theorem 9.4.1 in [14] □

Proposition 2.10. *(Proposition 1.3.9 [22]) If $D \in \mathrm{LFD}(B)$ with above decomposition $D = D_N + D_S$, we have the following:*

- $\ker(D) = \ker(D_N) \cap \ker(D_S)$
- *For every k-subspace M of B we have: M is D-invariant if and only if M both D_S and D_N-invariant.*

If $D \in \mathrm{LND}$, the **exponential map** determined by D is $\exp(D) \colon B \to B$

$$\exp(D)(f) = \sum_{i \geq 0} \frac{1}{i!} D^i(f)$$

for any local slice $r \in B$ of D, the **Dixmier map** induced by r is $\pi_r \colon B \to B_{D(r)}$, where:

$$\pi_r(f) = \sum_{i \geq 0} \frac{(-1)^i}{i!} D^i(f) \frac{r^i}{(D(f))^i}$$

where $B_{D(r)}$ is the localization at $D(r)$

2.1.2 Locally nilpotent derivation $\mathrm{LND}(B)$

We describe some principies related with locally nilpotent derivations, it appear in Freudenburg's book [8].

Principies for Locally nilpotent derivations

Principie 2.11. We suppose $D \in \mathrm{LND}(B)$

1. $\ker(D)$ is factorially closed.
2. $B^* \subset \ker D$, in particular $\mathrm{LND}(B) = \mathrm{LND}_k(B)$
3. If $D \neq 0$, then D admits a local slice $r \in B$.
4. $\mathrm{Aut}_k(B)$ acts over $\mathrm{LND}(B)$ by conjugation.

Proof. The proof is in [8, page 22]. □

Principie 2.12. We suppose $D, E \in \mathrm{LND}(B)$.

1. $\exp(D) = \sum_{i \geq 0} \dfrac{D^i}{i!} \in \mathrm{Aut}_k(B)$.
2. If $[D, E] = 0$ then $D + E \in \mathrm{LND}(B)$ and $\exp(D + E) = \exp(D) \circ \exp(E)$.
3. The subgroup of $\mathrm{Aut}_k(B)$ generated by $\{\exp D | D \in \mathrm{LND}(B)\}$ is normal.

Proof. The proof is in [8, pag 26] . □

Corollary 2.13. *(Slice theorem) We suppose $D \in \mathrm{LND}(B)$ admits a slice $s \in B$, and let $A = \ker(D)$. Then*

1. *$B = A[s]$ and $D = \dfrac{d}{ds}$*
2. *$A = \pi_s(B)$ y $\ker(\pi_s) = sB$*
3. *If B is affine, then A is affine.*

Proof. The proof is in [8, page 28] . □

2.1.3 Correspondence between Locally nilpotent derivations and the regular $\mathbb{G}_a$ -actions

A classic result establishes the correspondence of the locally nilpotent derivations and the regular $\mathbb{G}_a$ -actions.

Theorem 2.14. *Given a regular action $\alpha_t\colon \mathbb{G}_a \times \operatorname{Spec}(B) \to \operatorname{Spec}(B)$, there exists a correspondence biyective between the regular actions of the additive group and the locally nilpotent derivations:*

$$\begin{array}{rcl} D & \rightarrow & (\exp(tD))^* \\ \operatorname{ev}_0 \circ \dfrac{d}{dt} \circ \alpha_t^* & \leftarrow & \alpha_t \end{array}$$

Proof. For details of the proof, see [5], [8]. □

2.1.4 Semisimple derivation $\operatorname{SSD}(B)$

Remark 2.15. We have the following remarks:

- The semisimple derivation allow decompose the ring B, $B = \bigoplus\limits_{\lambda \in k} B_\lambda$ with $B_\lambda = \{b \in B | D(b) = \lambda b\}$, satisfying $B_\lambda B_{\lambda'} \subset B_{\lambda+\lambda'}$ and $B_0 = \ker D$ the kernel of derivation D. If $E(D)$ is the set of all eigenvalues of D and $G(D)$ is the group generated by $E(D)$, we have $B = \bigoplus_{\lambda \in G(D)} B_\lambda$.
- Given $f \in \ker D$, we have fD is not necessarily a semi-simple derivation (unlike LND).

Definition 2.16. We define the semisimple Makar-Limanov invariant since the intersection of kernel of semisimple derivations whose eigenvalues are integers number or the intersection of all regular actions of the multiplicative group:

$$\operatorname{SML}(B) = \bigcap_{D \in \operatorname{SSD}^*(B)} \ker(D) = \bigcap_{\mathbb{G}_m \curvearrowright B} B^{\mathbb{G}_m}$$

where $\operatorname{SSD}^*(B) \subset \operatorname{SSD}(B)$ is the set of semi-simple derivations whose eigenvalues are integer numbers. If $D \in \operatorname{SSD}^*(B)$ then $\varphi D \varphi^{-1} \in \operatorname{SSD}^*(B)$, because if $\{a_i\}_{i\in I}$ is a semi-invariant basis for D, hence $\{\varphi(a_i)\}_{i \in I}$ is a semi-invariant basis for $\varphi D \varphi^{-1}$. Therefore $\varphi \operatorname{SSD}^*(B) \varphi^{-1} = \operatorname{SSD}^*(B)$ and as a consequence $\varphi(\operatorname{SML}(B)) = \operatorname{SML}(B)$

Remark 2.17.

- $\operatorname{SML}(\mathbb{C}^{[n]}) = \mathbb{C}$
- This invariant would allow us to work on rigid ring which the ML invariant does not provide information.

Example 2.18. For example, for the surface $D_p = \{(x, y, z) \in \mathbb{C}^3 | xy = p(z)\}$ with $p(z)$ no monomial, we have the $\mathbb{Z}$-graduations have form $(a, -a, 0)$ with $a \in \mathbb{Z}$, therefore, the action has form $t \cdot (x, y, z) = (t^a x, t^{-a} y, z)$ and the semisimple derivations have form $D = a(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y})$, since $\ker(D) = \ker(\lambda D)$, it is enough to compute the kernel of $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ and the kernel $\ker(D) = \mathbb{C}[xy, z] = \mathbb{C}[p(z), z] = \mathbb{C}[z]$ for all $a \in \mathbb{Z}$. Therefore $\mathrm{MLS}(\mathcal{O}(D_p)) = \mathbb{C}[z]$.

Correspondence semisimple derivations with eigenvalues integers and regular $\mathbb{G}_m$-action

Next, we demonstrate the bijective correspondence between the semisimple derivations whose eigenvalues are integer numbers over B and the regular $\mathbb{G}_m$-actions over $\mathrm{Spec}(B)$ and the $\mathbb{Z}$-graduations over B. Giving a semi-simple derivation D we define the exponential map $\exp(zD)$

$$\begin{aligned} \exp(zD)\colon B &\rightarrow B[|\tau|] \\ D &\mapsto \exp(\tau D) = \sum_{i \geq 0} \tau^i \frac{D^i}{i!} \end{aligned}$$

with the relation $t = \sum_{i \geq 0} \frac{\tau^i}{i!}$.

Lemma 2.19. *Let $D \in \mathrm{SSD}(B)$ be a semisimple derivation whose eigenvalues are integer numbers, the image of the exponential map $\exp(\tau D)(B)$ is contained in $B[t^{\pm 1}]$.*

Proof. If D is a semisimple derivation whose eigenvalues are integer numbers, there is a basis $\{a_j\}_{j \in J}$ as vector space such that for each a_j there exists $\mu_j \in \mathbb{Z}$ with $D(a_j) = \mu_j a_j$, then for any $a \in B$ we have $a = \sum_j \lambda_j a_j$ with $\lambda_j \in \mathbb{Z}$,

$$\exp(\tau D)(a) = \sum_{i \geq 0} \frac{\tau^i}{i!} D^i(\sum_j \lambda_j a_j) = \sum_j \lambda_j \sum_{i \geq 0} \frac{\tau^i}{i!} D^i(a_j) = \sum_j \lambda_j \sum_{i \geq 0} \frac{(\mu_j \tau)^i}{i!} a_j = \sum_j \lambda_j a_j t^{\mu_j}$$

with $\sum_j \lambda_j a_j t^{\mu_j} \in B[t^{\pm 1}]$. □

Lemma 2.20. *The exponential map $(\exp(\tau D))^*$ is a regular $\mathbb{G}_m$- action over $\mathrm{Spec}(B)$.*

Proof. We use the commutative diagram 1.8

1. If $t = 1$, then implies that $\tau = 0$ therefore the exponential map $\exp(\tau D)$ correspond only the first element of the serie, this equivalent to identity map.

2. Let t, t' associated with z and z' respectively, be clear that $[\tau D, \tau' D] = 0$, give $a \in B$ we have
$$\exp(\tau D) \circ \exp(\tau D)(a) = \exp((\tau + \tau')D)(a)$$
This expression is equivalent to $\displaystyle\sum_{i\geq 0} \frac{(\tau+\tau')^i}{i!} = \sum_{i\geq 0} \frac{\tau^i}{i!} \cdot \sum_{j\geq 0} \frac{\tau'^j}{j!} = tt'$, so the composition of two morphisms is equivalent to the morphism given by tt'.

$\square$

Lemma 2.21. *Let $\Sigma = \{\alpha^* \in Hom_k(B, B[t^{\pm 1}, t'^{\pm 1}])|\ \alpha_1^* = Id_B\ ;\ \alpha_t^* \circ \alpha_{t'}^* = \alpha_{tt'}^*\}$ be the set of k-homomorphisms from B to $B[t^{\pm 1}, t'^{\pm 1}]$, for each $\alpha_t^* \in \Sigma$ we have $D_\alpha := ev_1 \circ \frac{d}{dt} \circ \alpha_t^*$ is a semi-simple derivation whose eigenvalues are integer number.*

$$B \xrightarrow{\alpha_t^*} B[t^{\pm 1}] \xrightarrow{\frac{d}{dt}} B[t^{\pm 1}] \xrightarrow{\mathrm{ev}_1} B$$

Proof. 1. Linearity because the morphisms are linear, let $a, b \in B$ and $\lambda \in k$,

$$D_\alpha(\lambda a+b) = \mathrm{ev}_1 \circ \frac{d}{dt} \circ \alpha_t^*(\lambda a+b) = \lambda\, \mathrm{ev}_1 \circ \frac{d}{dt} \circ \alpha_t^*(a) + \mathrm{ev}_1 \circ \frac{d}{dt} \circ \alpha_t^*(b) = \lambda D_\alpha(a) + D_\alpha(b)$$

2. By the second diagram 1.8, for all elements $c \in B$ we have $\mathrm{ev}_1 \circ \alpha_t^*(c) = c$, then

$$\begin{aligned} D_\alpha(ab) &= \mathrm{ev}_1 \circ \frac{d}{dt} \circ \alpha_t^*(ab) \\ &= \mathrm{ev}_1 \circ \frac{d}{dt}[\alpha_t^*(a)\alpha_t^*(b)] \\ &= \mathrm{ev}_1 \circ [\alpha_t^*(a)\frac{d}{dt}\alpha_t^*(b) + \alpha_t^*(b)\frac{d}{dt}\alpha_t^*(a)] \\ D_\alpha(ab) &= aD_\alpha(b) + bD_\alpha(a) \end{aligned}$$

$\square$

Theorem 2.22. *There exists a correspondence biyective between the regular actions of the multiplicative group and the semisimple derivations whose eigenvalues are integer numbers:*

$$\begin{array}{rcl} D & \to & (\exp(\tau D))^* \\ \mathrm{ev}_1 \circ \frac{d}{dt} \circ \alpha_t^* & \leftarrow & \alpha_t \end{array}$$

with $t = \displaystyle\sum_{i\geq 0} \frac{\tau^i}{i!}$.

Proof. 1. $\rightarrow$ If D is a semi-simple derivation whose eigenvalues are integer numbers, there exists a basis of eigenvalues $\{a_i\}_{i\in I} \subset B$ such that $D(a_i) = \lambda_i a_i$ with $\lambda_i \in \mathbb{Z}$, since any element $a \in B$ can be written as follows $a = \sum_j \gamma_j a_j$, and for each a_j we have $\exp(\tau D)(a_j) = t^{\lambda_j} a_j$, we can compute $\exp(\tau D)(a)$

$$\begin{aligned}
\exp(\tau D)(a) &= \sum_j \gamma_j \exp(\tau D)(a_j) \\
&= \sum_j \gamma_j \exp(\tau D)(a_j) \\
\exp(\tau D)(a) &= \sum_j \gamma_j t^{\lambda_j} a_j
\end{aligned}$$

Now, if apply $\dfrac{d}{dt}$ and then ev_1

$$\begin{aligned}
\mathrm{ev}_1 \circ \frac{d}{dt} \circ \exp(\tau D)(a) &= \mathrm{ev}_1 \circ \frac{d}{dt} \left(\sum_j \gamma_j t^{\lambda_j} a_j \right) \\
&= \mathrm{ev}_1 \left(\sum_j \gamma_j \lambda_j t^{\lambda_j - 1} a_j \right) \\
\mathrm{ev}_1 \circ \frac{d}{dt} \circ \exp(\tau D)(a) &= \sum_i \gamma_i \lambda_i a_i = D(a)
\end{aligned}$$

Equivalent to applying D to the element a.

2. $\leftarrow$ If α_t is a $\mathbb{G}_m$-action, α_t^* is a coaction homomorphism and $D_\alpha = \mathrm{ev}_1 \circ \dfrac{d}{dt} \circ \alpha_t^*$ for lemma 2.21 is a semisimple derivation with integer eigenvalues, for any a_j we have $\alpha^*(a_j) = a_j t^{\lambda_j}$ and therefore $D_\alpha^n(a_i) = (\mathrm{ev}_1 \circ \dfrac{d}{dt} \circ \alpha_t^*)^n(a_j) = \lambda_j^n a_j$

$$
\begin{aligned}
\exp(\tau D_\alpha)(a) &= \exp(\tau D_\alpha)(\sum_j \lambda_j a_j) \\
&= \sum_j \lambda_j \sum_{n\geq 0} \frac{1}{n!}\tau^n D_\alpha^n(a_j) \\
&= \sum_j \lambda_j \sum_{n\geq 0} \frac{1}{n!}\tau^n \gamma_j^n a_j \\
&= \sum_j \lambda_j a_j \sum_{n\geq 0} \frac{1}{n!}\tau^n \gamma_j^n \\
&= \sum_j \lambda_j a_j \sum_{n\geq 0} \frac{1}{n!}(\tau\gamma_j)^n \\
&= \sum_j \lambda_j t^{\gamma_j} a_j \\
\exp(\tau D_\alpha)(a) &= \alpha_t^*(a)
\end{aligned}
$$

□

2.2 Polynomial locally finite derivation over $k^{[n]}$

In this section we consider $B = k^{[n]}$ be the ring in n variables with coefficient in k, the derivation $D \in \mathrm{Der}(k^{[n]})$ has form $D = \sum_{i=1}^{n} P_i(x_1, \ldots, x_n)\frac{\partial}{\partial x_i}$, with $P_i \in k^{[n]}$, also this derivations can be seen since vector fields V over k^n given by

$$V \colon \mathbb{A}^n \to \mathbb{A}^n$$
$$(x_1, x_2, \ldots, x_n) \mapsto (P_1, P_2, \ldots, P_n)$$

when we talks about polynomial vector field, we refer the same sense that polynomial derivation hence we have the correspondence of set $\mathrm{Der}(k^{[n]}) = \mathrm{Vec}(\mathbb{A}^n)$. Moreover, we define the **divergence** of a derivation $D \in \mathrm{Der}(k^{[n]})$, as the sum $\mathrm{div}(D) = \sum_{i=1}^{n} \frac{\partial P_i}{\partial X_i}$.

Example of polynomial derivations

Definition 2.23. A **linear derivation** has form $D = \sum_{i=1}^{n}(\sum_{j=1}^{i} a_{ij}x_j)\frac{\partial}{\partial x_i}$, this derivation we can associate to the matrix $M = [a_{ij}]_{1\leq i,j\leq n} \in M_n(k)$. This kind of derivation is locally finite because it admits a characteristic polynomial $P(T) = \det(M - TI)$ such that $P(D) = 0$. The nilpotents matrices (for example, upper triangular) are related with locally nilpotent derivations and the diagonalizable matrices with semisimple derivation.

Theorem 2.24. *(Theorem 6.2.1 in [14]). Let D be a k-derivation of $k[X]$ such that $D(x_i) = \sum_{j=1}^{n} a_{ij}x_j$ for $i = 1, \dots, n$ with $a_{ij} \in k$. If the matrix $[a_{ij}]$ is nilpotent, then the ring of constants $k[x_1, \dots, x_n]^D$ is generated finitely on k.*

Corollary 2.25. *(Theorem 6.2.2. in [14]) For any linear derivation on $k^{[n]}$ we have that the constant ring is finitely generated.*

Also, if $D_A = \sum_{i=1}^{n}(\sum_{k=1}^{i} a_{ij}x_j)\frac{\partial}{\partial x_i}$, $D_B = \sum_{i=1}^{n}(\sum_{k=1}^{i} b_{ij}x_j)\frac{\partial}{\partial x_i}$ whose associated matrix is A and B, we have $[D_A, D_B] = D_{[A,B]}$.

Definition 2.26. $D \in \mathrm{Der}(B)$ is a **triangular derivation** of B if only if $D(x_i) \in \mathbb{C}[x_1, \dots, x_{i-1}]$ for $i = 2, \dots, n-1$ and $D(x_1) \in k$.

Example 2.27. The triangular derivation $D = (x + f(y,z))\frac{\partial}{\partial x} + (y + g(z))\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}$ is a locally finite derivation on $k[x,y,z]$ for any $f(y,z) \in k[y,z]$, $g(z) \in k[z]$.

Definition 2.28. An derivation D is **Jacobian** if there exists $f_1, f_2, \dots, f_{n-1} \in k^{[n]}$, such that for $g \in k^{[n]}$, $D(g)$ is defined as follows:

$$D(g) = \Delta_{(f_1,f_2,\dots f_{n-1})}(g) = \begin{vmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_n} \\ \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{n-1}}{\partial x_1} & \frac{\partial f_{n-1}}{\partial x_1} & \cdots & \frac{\partial f_{n-1}}{\partial x_1} \end{vmatrix}$$

where $\ker(D) = k[f_1, f_2, \dots, f_{n-1}]$

2.2.1 Locally nilpotent derivation on $k^{[n]}$

Definition 2.29. Let D be a locally nilpotent k-derivation on the polynomial ring $k^{[n]}$. Then we define **rank of** D denoted by $\mathrm{rank}(D)$, is defined to be the least integers i for which there exists a coordinate system $(x_1, x_2, \dots, x_n)$ of B satisfying $k[x_{i+1}, \dots, x_n] \subset \ker(D)$.

Definition 2.30. Give $D \in \mathrm{LND}(k^{[n]})$ is **nice** if $D^2(x_i) = 0$ for all $1 \le i \le n$.

Theorem 2.31. *(Rentschler theorem [19]). Let D be a locally nilpotent k-derivation of $k[X, X]$ (where k is a field of characteristic zero). Then there exists a k-automorphism ρ of $k[x, y]$ such that*

$$\rho D \rho^{-1} = f(x)\frac{\partial}{\partial y}$$

for some $f(x) \in k[X]$.

For $n \geq 3$ the classification of LND in $k^{[n]}$ is open; however, there are interesting results regarding their characterizations; we have the following facts for $n = 3$.

- (Miyanishi, Kambayashi): For a field k of characteristic 0, if $D \in \text{LND}(k^{[3]})$, then $\ker(D) = k^{[2]}$ (polynomial ring in two variables).
- (Zurkowski): If D is a homogeneous LND on $k[x, y, z]$ with respect to a positive $\mathbb{Z}$ grading ω, then $\ker(D) = k[F, G]$ where F, G are homogeneous with respect to ω.
- (D. Daigle): If $D \in \text{LND}(k^{[n]})$ and $\ker[F_1, \ldots, F_{n-1}]$ then $D = \alpha\Delta_{(F_1,\ldots,F_{n-1})}$ for some $\alpha \in \ker(D)$.

2.2.2 Semisimple derivation on $k^{[n]}$

Definition 2.32. A derivation over $k^{[n]}$ is **diagonalizable** if $D(x_i) = a_i x_i$ with $a_i \in k$.

Proposition 2.33. *(Proposition 9.5.9., Nowicki in [16]) If k is algebraically closed, then every semisimple k- derivation of $k[x, y]$ is diagonalizable.*

Proof. The proof is given in [16, page 110]. □

This means that all semi-simple derivations of $k[x, y]$ are conjugate to the derivation of the form $ax\dfrac{\partial}{\partial x} + by\dfrac{\partial}{\partial y}$ with $a, b \in k$.

Example 2.34. $D = x\dfrac{\partial}{\partial x} + \left(f(x, y) - x\dfrac{\partial f(x, y)}{\partial x} + z\right)\dfrac{\partial}{\partial z}$ is a semi-simple derivation. Since $\text{Sem}(D)$ is a k-algebra, we observe that $x, y, f(x, y) + z$ are in $\text{Sem}(D)$ and therefore $\text{Sem}(D) = k[x, y, z]$.

Chapter 3

On rational multiplicative group actions

Abstract

We establish a one-to-one correspondence between rational $\mathbb{G}_m$-actions on an algebraic variety X and derivations $\partial\colon K_X \to K_X$ of the field of fractions K_X of X satisfying that there exists a generating set $\{a_i\}_{i\in I}$ of K_X as a field such that $\partial(a_i) = \lambda_i a_i$ with $\lambda_i \in \mathbb{Z}$ for all $i \in I$. We call such derivations rational semisimple. Furthermore, we also prove the existence of a rational slice for every rational semisimple derivation, i.e. an element $s \in K_X$ such that $\partial(s) = s$. By analogy with the case of additive group actions, we prove that $K_X \simeq K_X^{\mathbb{G}_m}(s)$ and that under this isomorphism the derivation ∂ is given by $\partial = s\frac{d}{ds}$. Here, $K_X^{\mathbb{G}_m}$ is the field of invariant of the $\mathbb{G}_m$-action.

3.1 Introduction

Let k be an algebraically closed field of characteristic zero. By a variety we mean an integral separated scheme of finite type. We let $\mathcal{O}_X$ be its structure sheaf and K_X be its field of rational functions, so that $K_X = \operatorname{Frac}\mathcal{O}_X(U)$ for any affine open set $U \subset X$.

We also let $\mathbb{G}_m$ and $\mathbb{G}_a$ be the multiplicative group and the additive group over k, respectively. It is well known that regular additive group actions on affine varieties are in one-to-one correspondence with certain derivations called locally nilpotent [5], [8]. Indeed, letting X be an affine variety a locally nilpotent derivation on X is a k-derivation $\partial\colon \mathcal{O}_X(X) \to \mathcal{O}_X(X)$ such that for every $f \in \mathcal{O}_X(X)$ there exists $i \in \mathbb{Z}_{\geq 0}$ with $\partial^i(f) = 0$. All derivations in this paper are k-derivations, so we will call them simply derivations. Given a locally nilpotent derivation ∂ on X we obtain a regular $\mathbb{G}_a$-actions $\varphi\colon \mathbb{G}_a \times X \to X$ on X via the exponential map

$$\varphi^* = \exp(z\partial)\colon \mathcal{O}_X(X) \to \mathcal{O}_X(X)[z] \quad \text{given by} \quad f \mapsto \sum_{i\geq 0} \frac{z^i \partial^i(f)}{i!}\,.$$

On the other hand, given a regular $\mathbb{G}_a$-action $\varphi\colon \mathbb{G}_a \times X \to X$ on X we obtain a locally nilpotent derivations ∂ on X via

$$\partial\colon \mathcal{O}_X(X) \to \mathcal{O}_X(X) \quad \text{given by} \quad f \mapsto \mathrm{ev}_0 \circ \frac{d}{dz} \circ \varphi^*(f)\,,$$

where $\mathrm{ev}_0\colon \mathcal{O}_X(X)[z] \to \mathcal{O}_X(X)$ is the evaluation morphism in $z = 0$.

In [7] Dubouloz and the second author introduced a class of rationally integrable derivations that generalized locally nilpotent derivations to the rational setting. In fact, a derivation $\partial\colon K_X \to K_X$ is called rationally integrable if the exponential map

$$\exp(z\partial)\colon K_X \to K_X[|z|] \quad \text{given by} \quad f \mapsto \sum_{i\geq 0} \frac{z^i \partial^i(f)}{i!}\,,$$

factors through the ring $K_X(z) \cap K_X[|z|]$. Their main theorem provides a one-to-one correspondence between rationally integrable derivations $\partial\colon K_X \to K_X$ and rational $\mathbb{G}_a$ actions $\varphi\colon \mathbb{G}_a \times X \dashrightarrow X$ on X. The correspondence is given similarly to above via

$$\varphi^* = \exp(z\partial) \quad \text{and} \quad \partial = \mathrm{ev}_0 \circ \frac{d}{dz} \circ \varphi^*(f)\,,$$

after recalling that $K_X(z) \cap K_X[|z|] = \{r(z) \in K_X(z) \mid \mathrm{ord}_0(r) \geq 0\}$ so that ev_0 is well defined.

In this paper, we expand and generalize the results in [7] to allow a classification of rational $\mathbb{G}_m$-action. It is well known that regular $\mathbb{G}_m$-actions on an affine variety X are in one-to-one correspondence with semisimple derivations $\partial\colon \mathcal{O}_X(X) \to \mathcal{O}_X(X)$ having integer eigenvalues. Recall that such a derivation is semisimple if there exists a basis $\{a_i\}_{i\in I}$ of A as a vector space such that $\partial(a_i) = \lambda_i a_i$ with $\lambda_i \in k$.

In Definition 3.5, we introduce rational semi-simple derivations. A derivation $\partial\colon K_X \to K_X$ on an algebraic variety X is rational semisimple if there exists a generating set $\{a_i\}_{i\in I}$ of K_X (as field) such that $\partial(a_i) = \lambda_i a_i$ with $\lambda_i \in \mathbb{Z}$. Every semisimple derivation whose eigenvalues are integer numbers is rational semisimple. Our main result in this paper is Theorem 3.14 establishing a one-to-one correspondence between rational $\mathbb{G}_m$-actions $\varphi\colon \mathbb{G}_m \times X \dashrightarrow X$ in X and rational semi-simple derivations $\partial\colon K_X \to K_X$ in X. The correspondence is as follows: we prove in Corollary 3.2 that the image of φ^* is contained in $K_X(t) \cap K_X[|t-1|] = \{r(t) \in K_X(t) \mid \mathrm{ord}_1(r) \geq 0\}$ and so we obtain ∂ from φ via

$$\partial\colon K_X \to K_X \quad \text{given by} \quad f \mapsto \mathrm{ev}_1 \circ \frac{d}{dt} \circ \varphi^*(f)\,.$$

As for the other direction, letting σ be the isomorphism of formal power series rings given by the logarithmic power series

$$\sigma\colon K_X[|z|] \to K_X[|t-1|], \quad \text{given by} \quad z \mapsto \sum_{i\geq 1}(-1)^{i+1}\frac{(t-1)^i}{i}.$$

we recover the rational action φ from ∂ via $\varphi^* = \sigma \circ \exp(z\partial)$.

As a consequence of our main result, we prove in Corollary 3.17 the existence of a rational slice for every rational semi-simple derivation, i.e., an element $s \in K_X$ such that $\partial(s) = s$. Moreover, we prove in Proposition 3.18 that $K_X = K_X^{\mathbb{G}_m}(s)$ and that under this isomorphism the derivation ∂ is given by $\partial = s\frac{d}{ds}$. Here, $K_X^{\mathbb{G}_m}$ is the field of invariant of the $\mathbb{G}_m$-action.

Finally, we provide in Proposition 3.20 a characterization of regular actions of the multiplicative group on the class of varieties that are proper over the spectrum of its ring of global regular functions. This characterization agrees with the one recalled above in the particular case of affine varieties.

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3.2 Rational $\mathbb{G}_m$-action

Let $\mu\colon \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ the morphism given by the group law $(t_1, t_2) \mapsto t_1t_2$, and let $e_{\mathbb{G}_m}\colon \operatorname{Spec}(k) \to \mathbb{G}_m$ be the neutral element map. A rational action of the multiplicative group is a rational map $\varphi\colon \mathbb{G}_m \times X \dashrightarrow X, (x,t) \mapsto t\cdot x$ such that the following diagrams are commutative:

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbb{G}_m \times X \arrow[r, dashed, "\mathrm{id}_{\mathbb{G}_m}\times\varphi"] \arrow[d, "\mu\times\mathrm{id}_X"'] & \mathbb{G}_m \times X \arrow[d, dashed, "\varphi"] \\
\mathbb{G}_m \times X \arrow[r, dashed, "\varphi"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Spec}(k) \times X \arrow[r, "e_{\mathbb{G}_m}\times\mathrm{id}_X"] \arrow[dr, "\mathrm{pr}_2"'] & \mathbb{G}_m \times X \arrow[d, dashed, "\varphi"] \\
 & X.
\end{tikzcd}
\tag{3.1}$$

We let $\operatorname{dom}(\varphi)$ be the largest open subset of $\mathbb{G}_m \times X$ where φ is well defined. If $(g,x) \in \operatorname{dom}(\varphi)$, we denote $\varphi(g,x)$ simply by $g\cdot x$. The next lemma shows that $\operatorname{dom}(\varphi) \cap (\{g\} \times X)$ is a nonempty open subset of $\{g\} \times \mathbb{G}_m$. In particular, for $g = 1$, $\operatorname{dom}(\varphi) \cap (\{1\} \times X)$ is an open subset, not empty. This will allow us to exhibit a criterion for the existence of rational action in terms of the function field of X.

Lemma 3.1. *Let X be an algebraic variety endowed with a rational $\mathbb{G}_m$-action. For every fixed $g \in \mathbb{G}_m$, we define $V_g = \mathrm{dom}(\varphi) \cap (\{g\} \times X)$. V_g is a non-empty open subset of $\{g\} \times X \simeq X$ and the morphism*

$$\begin{aligned} \varphi_g \colon V_g &\dashrightarrow X \\ x &\mapsto \varphi(g, x) \end{aligned}$$

is dominant.

Proof. Following Demazure in [6] we consider the morphism $u_g \colon \mathbb{G}_m \to \mathbb{G}_m$, $h \mapsto gh^{-1}$ whose inverse is $(u_g)^{-1} \colon \mathbb{G}_m \to \mathbb{G}_m$, $h \mapsto h^{-1}g$. We define $\beta_g = \varphi \circ (u_g \times \mathrm{id}_X) \circ (\mathrm{pr}_1, \varphi)$,where pr_1 is the projection in the first coordinate, and id_X is the identity in X. As

$$\beta_g \colon \mathbb{G}_m \times X \overset{(\mathrm{pr}_1, \varphi)}{\dashrightarrow} \mathbb{G}_m \times X \overset{u_g \times \mathrm{Id}_X}{\dashrightarrow} \mathbb{G}_m \times X \overset{\varphi}{\dashrightarrow} X$$

We have $\beta_g \colon \mathbb{G}_m \times X \dashrightarrow X$, $(h, x) \mapsto \varphi(g, x)$ is dominant for all choice of g.

Note that this map only depend of x. Then $\mathrm{dom}(\beta_g) = \mathbb{G}_m \times U$, with $U \subseteq X$ an open subset. We have $\beta_g = \varphi_g \circ \mathrm{pr}_2$, where pr_2 is the projection of $\mathbb{G}_m \times X$ into the second coordinate. Hence, $U = \mathrm{pr}_2(V_g)$. Since φ_g is a rational map and $V_g = \{g\} \times \mathrm{dom}(\varphi_g)$, we obtain $V_g \neq \emptyset$. Moreover $\beta_g = \varphi_g \circ \mathrm{pr}_2$, which means φ_g is dominant because $\mathrm{Im}(\beta_g) \subseteq \mathrm{Im}(\varphi_g)$. □

We can apply the case $g = 1$ to obtain the following characterization.

Corollary 3.2. *Let $\varphi \colon \mathbb{G}_m \times X \dashrightarrow X$ be a rational action then $\mathrm{Im}(\varphi^*) \subseteq \mathcal{O}_\nu$, where $\mathcal{O}_\nu = \{r(t) \in K(t) \mid \mathrm{ord}_1(r) \geq 0\}$ is the discrete valuation ring of $K(t)$ and ord_1 is the order of vanishing in $t = 1$.*

Proof. Let $V' \subseteq \mathbb{G}_m \times X$ and $X' \subseteq X$ be affine open sets such that $\varphi|_{V'} \colon V' \to X'$ is regular. We let $\mathcal{O}_{\mathbb{G}_m \times X}(V') = A \subseteq K(t)$ and $\mathcal{O}_X(X') = B \subseteq K$. By Lemma 3.1, we may and will assume $V' \cap V_1 \neq \emptyset$ and so it is an dense open set of $\{1\} \times X$. Now, the composition $V' \cap V_1 \to V' \to X'$ is dominant by Lemma 3.1 and induces algebra homomorphisms

$$B \xrightarrow{\varphi^*} A \longrightarrow A/A(t-1)\,,$$

and this composition is injective. This in turn shows that $\mathrm{ord}_1(\varphi^*(b)) = 0$ for every $b \in B$ different from 0. Since $K = \mathrm{Frac}\, B$ we have $\mathrm{ord}_1(\varphi^*(f)) = 0$ for every $f \in K$, which proves the corollary. □

Given a rational action $\varphi \colon \mathbb{G}_m \times X \dashrightarrow X$, for every fixed $g \in \mathbb{G}_m$ we obtain a birational automorphism

$$\varphi_g \colon X \dashrightarrow X, \qquad x \mapsto \varphi(g, x)$$

since $\varphi(1,x) = \mathrm{Id}_X$ and for every $g, g' \in \mathbb{G}_m$ we have $\varphi_g \circ \varphi_{g'}(x) = \varphi_{gg'}(x)$. Moreover, the map $\mathbb{G}_m \to \mathrm{Bir}(X)$ sending g to φ_g is a group homomorphism. Finally, a rational action $\varphi\colon \mathbb{G}_m \times X \dashrightarrow X$ such that $\mathrm{dom}(\varphi) = \mathbb{G}_m \times X$ is a regular action.

3.2.1 Criterion for existence of $\mathbb{G}_m$-rational actions

A $\mathbb{G}_m$-rational action $\varphi\colon \mathbb{G}_m \times X \dashrightarrow X$ in a variety X, is equivalent to the co-action homomorphism $\varphi_t^*\colon K_X \to k(\mathbb{G}_m \times X) = K_X(t)$ where the map φ_t^* factors through the subalgebra $\mathcal{O}_\nu = \{r(t) \in K_X(t) \mid \mathrm{ord}_1(r) \geq 0\}$, with $\mathfrak{m}_\nu = \{r(t) \in K_X(t) \mid \mathrm{ord}_1(r) > 0\}$. The co-action morphism is characterized by the commutativity of the follow diagrams:

$$\begin{tikzcd} K_X \arrow[r, "\varphi_{t_1}^*"] \arrow[d, "\varphi_{t_2}^*"'] & K_X(t_1) \arrow[d, "t_1 \mapsto t_1 t_2"] \\ K_X(t_2) \arrow[r, "\widetilde{\varphi}"] & K_X(t_1, t_2) \end{tikzcd} \qquad \begin{tikzcd} K_X \simeq \mathcal{O}_\nu/\mathfrak{m}_\nu \arrow[r, "\varphi_t^*"] \arrow[dr, "\mathrm{Id}_{K_X}"'] & \mathcal{O}_\nu \arrow[d, "\mathrm{ev}_1"] \\ & K_X \simeq \mathcal{O}_\nu/\mathfrak{m}_\nu \end{tikzcd} \tag{3.2}$$

The following proposition is classical.

Proposition 3.3. *Le X be a variety. X admits a nontrivial rational $\mathbb{G}_m$-action if only if it is birationally isomorphic to $Y \times \mathbb{P}^1$ for some k-variety Y.*

Proof. See [20, 13, 18]. $\square$

Mimicking the case of $\mathbb{G}_a$-actions on affine geometry, we we have the following definition:

Definition 3.4. Let X be a $\mathbb{G}_m$-variety, an element $s \in K_X$ such that $\varphi_t^*(s) = ts$ is called rational slice.

For each faithful rational action, a rational slice always exists. Indeed, since the action is faithful, there are two semi-invariant $a, b \in K_X$ whose weights n, m are relatively prime, i.e., $\varphi_t^*(a) = t^n a$ and $\varphi_t^*(b) = t^m b$. By Bezout theorem we have that there exist $c, d \in \mathbb{Z}$ such that $nd + mc = 1$. Hence $s = a^d b^c$ satisfies $\varphi_t^*(s) = st$ and so is a rational slice.

A derivation on k-algebra A, is a linear map $\partial\colon A \to A$ such that satisfy the Leibniz rules, $\partial(ab) = a\partial(b) + b\partial(a)$. We define the kernel of a derivation ∂ as its kernel as a linear map, i.e., $\ker(\partial) := \{a \in A | \partial(a) = 0\}$. The set of derivations over A is denoted by $\mathrm{Der}(A)$. We say ∂ is a semisimple derivation on A if there exists a basis $\{a_i\}_{i\in I}$ of A as k-vector space of eigenvalues such that $\partial(a_i) \in ka_i$, we will focus in the subset of semisimple derivations whose eigenvalues are integers numbers. For more details over semisimple derivations see [**?**, 14]. Is known that the regular $\mathbb{G}_m$-actions are in correspondence with the set of semisimple derivations whose eigenvalues are integers numbers.

Analogously with the definition of semisimple derivations over a k-algebra, we will give a definition of derivation over K_X, we will call rational semisimple.

Definition 3.5. Let K_X be the field of rational function associated to algebraic variety X, $\partial \in \mathrm{Der}(K_X)$. We say ∂ is rational semisimple if there exist a generating set $\{a_i\}_{i\in I}$ of K_X (as field) such that $\partial(a_i) = \lambda_i a_i$ with $\lambda_i \in \mathbb{Z}$.

Let $\partial\colon K_X \to K_X$ be a k-derivation. Denoting the i-th iteration of ∂ by ∂^i and ∂^0 is the identity map, we define the exponential map

$$\exp(z\partial)\colon K_X \to K_X[|z|], \quad \text{given by} \quad f \mapsto \sum_{i\geq 0} \frac{z^i\partial^i(f)}{i!}\,.$$

Furthermore, since ∂ is a k-derivation, the following proposition shows that $\exp(z\partial)$ is a k-rings homomorphism.

Proposition 3.6. *Let ∂ be a k-derivation on K_X, then exponential map $\exp(z\partial)\colon K_X \to K_X[|z|]$ is a k-ring homomorphism.*

Proof. The exponential map $\exp(z\partial)$ is k-linear since ∂ is k-linear. As for the product structure, we have

$$\begin{aligned}
\exp(z\partial)(fg) &= \sum_{i\geq 0} \frac{\partial^i(fg)z^i}{i!} \\
&= \sum_{i\geq 0} \frac{1}{i!}\left(\sum_{j=0}^{i} \binom{n}{j}\partial^j(f)\partial^{i-j}(g)\right) z^i \\
&= \sum_{i\geq 0} \left(\sum_{j=0}^{i} \frac{\partial^j(f)}{j!}\frac{\partial^{i-j}(g)}{(i-j)!}\right) z^i \\
&= \sum_{i\geq 0} \frac{\partial^i(f)z^i}{i!} \sum_{l\geq 0} \frac{\partial^l(g)z^l}{l!} \\
\exp(z\partial)(fg) &= \exp(z\partial)(f)\exp(z\partial)(g)
\end{aligned}$$

□

Furthermore, there is an isomorphism of formal power series rings

$$\sigma\colon K_X[|z|] \to K_X[|t-1|], \quad \text{given by} \quad z \mapsto \sum_{i\geq 1} (-1)^{i+1}\frac{(t-1)^i}{i}\,.$$

The inverse of this isomorphism is given by

$$\sigma^{-1}\colon K_X[|t-1|] \to K_X[|z|], \quad \text{given by} \quad t-1 \mapsto \sum_{i\geq 1} \frac{z^i}{i!}\,.$$

This corresponds to the logarithmic series and its inverse is the exponential series. In the sequel we will show that the image of $\sigma \circ \exp(z\partial)$ is contained in $K_X(t-1) = K_X(t)$ and that $\sigma \circ \exp(z\partial)$ is the comorphism of a rational $\mathbb{G}_m$-action if and only if ∂ is a rational semisimple derivation. With this in view, for every we denote $\phi_\partial^* = \sigma \circ \exp(z\partial)$ so that its comorphism corresponds to a map $\phi_\partial\colon \mathbb{G}_m \times X \dashrightarrow X$.

Lemma 3.7. *Let $\partial\colon K_X \to K_X$ be a rational derivation. Assume that the image of ϕ_∂^* is contained in $K_X(t)$, then ϕ_∂ is an action of the multiplicative group.*

Proof. From Proposition 3.6 we have ϕ_∂^* is a ring homomorphism. Let

$$t_1 = \sum_{i\geq 0} \frac{z_1^i}{i!} \quad \text{and} \quad t_2 = \sum_{i\geq 0} \frac{z_2^i}{i!}$$

We have the identity

$$t_1 t_2 = \sum_{i\geq 0} \frac{1}{i!}(z_1+z_2)^i$$

By Lemma 3.1, for every $t \in \mathbb{G}_m$ we can specialize ϕ_∂^* to obtain a field automorphism $\phi_\partial^*(t)\colon K_X \to K_X$. We extend ∂ to a derivation $\partial\colon K_X(z_1,z_2) \to K_X(z_1,z_2)$ by setting $\partial(z_1) = \partial(z_2) = 0$. Since now the derivations $z_1\partial$ and $z_2\partial$ commute, by [14, Proposition 2.4.2] we have

$$\begin{aligned}
\phi_\partial^*(t_1t_2)(f) &= \sigma \circ \exp((z_1+z_2)\partial)(f) \\
&= \sigma \circ \big(\exp(z_1\partial) \circ \exp(z_2\partial)\big)(f) \\
&= \phi_\partial^*(t_1) \circ \phi_\partial^*(t_2)(f)
\end{aligned}$$

Furthermore, by definition of $\phi_\partial^*\colon K_X \to K_X[|t-1|]$ the composition

$$\begin{aligned}
\mathrm{ev}_1 \circ \phi_\partial^* &= \mathrm{ev}_1 \circ \sigma \circ \exp(z\partial) \\
&= \mathrm{ev}_0 \circ \exp(z\partial) \\
&= \mathrm{ev}_0 \circ \sum_{i\geq 0} \frac{z^i \partial^i}{i!} \\
\mathrm{ev}_1 \circ \phi_\partial^* &= \mathrm{Id}_{K_X}\,.
\end{aligned}$$

Hence, by (3.2) we see that ϕ_∂^* is the comorphism of a multiplicative group action. $\square$

Proposition 3.8. *The following are equivalent.*

(i) $\partial\colon K_X \to K_X$ *is rational semisimple.*

(ii) The image of ϕ_∂^* *is contained in* $K_X(t)$.

Proof. First, assume that ∂ is rational semisimple. Hence, there exists a set of field generators $\{a_i\}_{i\in I}$ of K_X such that $\partial(a_j) = \lambda_j a_j$ with $\lambda_j \in \mathbb{Z}$. For these generators we have

$$\phi_\partial^*(a_j) = \sigma\circ\exp(z\partial)(a_j) = \sigma\left(\sum_{i\geq 0}\frac{z^i\partial^i(a_j)}{i!}\right) = \sigma\left(\sum_{i\geq 0}\frac{z^i\lambda_j^i a_j}{i!}\right) = \sigma\left(\sum_{i\geq 0}\frac{(z\lambda_j)^i}{i!}a_j\right) = t^{\lambda_j}a_j\,.$$

Let now $f, g \in k[a_i, i \in I]$ with $g \neq 0$. Now,

$$\begin{aligned}\phi_\partial^*\left(\frac{f}{g}\right) &= \sigma\circ\exp(z\partial)\left(\frac{f}{g}\right)\\ &= \sigma\circ\exp(z\partial)\left(\frac{f(a_i, i\in I)}{g(a_i, i\in I)}\right)\\ &= \frac{\sigma\circ\exp(z\partial)(f(a_i, i\in I))}{\sigma\circ\exp(z\partial)(g(a_i, i\in I))}\\ &= \frac{f(t^{\lambda_i}a_i, i\in I)}{g(t^{\lambda_i}a_i, i\in I)} \in K_X(t)\end{aligned}$$

This proves $(i) \to (ii)$. Inspired by Koshevoii [12], to prove the converse assertion, we let $f \in K_X$ and we let

$$\phi_\partial^*(f) = t^\ell\cdot\frac{\sum_i a_i t^i}{\sum_i b_i t^i},$$

with $\ell \in \mathbb{Z}$ and $a_0, b_0 \neq 0$. We may also assume that the representation is irreducible, meaning that $\sum_i a_i t^i$ and $\sum_i b_i t^i$ are relatively prime. Furthermore, such representations of $\phi_\partial^*(f)$ are unique if we further assume that $b_0 = 1$.

By Lemma 3.7, we have $\phi_\partial^*(s)\circ\phi_\partial^*(t)(f) = \phi_\partial^*(st)(f)$. This yields

$$t^\ell\frac{\sum_i\phi_\partial^*(s)(a_i)t^i}{\sum_i\phi_\partial^*(s)(b_i)t^i} = t^\ell\frac{\sum_i a_i(st)^i}{\sum_i b_i(st)^i} = t^\ell\frac{\sum_i(a_i s^i)t^i}{\sum_i(b_i s^i)t^i}.$$

This yields $\phi_\partial^*(a_j) = a_j t^j$ and $\phi_\partial^*(b_j) = b_j t^j$. In particular, this implies that

$$\phi_\partial^*(a_j) = \sigma\circ\exp(z\partial)(a_j) = \sigma\left(\sum_{i\geq 0}\frac{z^i\partial^i(a_j)}{i!}\right).$$

while

$$t^j a_j = \sigma\left(\sum_{i\geq 0}\frac{(zj)^i}{i!}\right)\cdot a_j = \sigma\left(\sum_{i\geq 0}\frac{(zj)^i}{i!}a_j\right) = \sigma\left(\sum_{i\geq 0}\frac{z^i j^i a_j}{i!}\right)$$

We conclude that

$$\sum_{i\geq 0} \frac{z^i \partial^i(a_j)}{i!} = \sum_{i\geq 0} \frac{z^i j^i a_j}{i!}$$

In particular, taking equality in the term with $i = 1$ we obtain that $\partial(a_j) = ja_j$. Finally, since $f = \dfrac{\sum_i a_j}{\sum_i b_i}$ we find that the set

$$\left\{ a_i, b_i \in K_X \mid t^\ell \cdot \phi_\partial^*(f) = \frac{\sum_i a_i t^i}{\sum_i b_i t^i} \text{ for some } f \in K_X \right\}$$

generates K_X and so ∂ is a rational semisimple derivation. □

Given a rational $\mathbb{G}_m$-action φ on X with comorphism $\varphi^*\colon K_X \to K_X(t)$, we define the map

$$D_\varphi\colon K_X \to K_X \quad \text{given by} \quad f \mapsto \mathrm{ev}_1 \circ \frac{d}{dt} \circ \varphi * (f),,$$

generalizing the usual definition of the infinitesimal generator of a group action of one parameter. The following lemmas will be required in our proof of our main result.

Lemma 3.9. *Let φ be a rational $\mathbb{G}_m$ action on X with comorphism $\varphi^*\colon K_X \to K_X(t)$. Then the following hold:*

[(i)]The map D_φ is a derivation. We have an equality $D_\varphi = \mathrm{ev}_1 \circ \frac{d}{dt} \circ \varphi^* = \mathrm{ev}_0 \circ \frac{d}{dz} \circ \sigma^{-1} \circ \varphi^*$.

2. *Proof.* By Corollary 3.2, we know that the image of φ^* is contained in $K_X[|t-1|]$. Given f and g in K_X, we let $\varphi^*(f) = \sum_{i\geq 0} a_i(t-1)^i$ and $\varphi^*(g) = \sum_{i\geq 0} b_i(t-1)^i$. We have $a_0 = f$ and $b_0 = g$ since φ^* is the comorphism of a $\mathbb{G}_m$-action. Moreover, we have

$$D_\varphi(f) = \mathrm{ev}_1 \circ \frac{d}{dt} \circ \varphi^*(f) = \mathrm{ev}_1 \left(\sum_{i\geq 1} a_i \cdot i(t-1)^{i-1} \right) = a_1\,, \tag{3.3}$$

so that the map $D_\varphi(f)$ corresponds to the first order term of $\varphi^*(f)$. To prove (i), remark that the composition $D_\varphi = \mathrm{ev}_1 \circ \frac{d}{dt} \circ \varphi^*$ is k-linear and maps k to 0. Furthermore, the term of the first order term of fg is $a_0 b_1 + a_1 b_0$. This yields the Leibniz rule since

$$D_\varphi(fg) = a_0 b_1 + a_1 b_0 = f D_\varphi(g) + D_\varphi(f) g\,.$$

Assertion (ii) follows by the following straightforward computation:

$$
\begin{aligned}
\mathrm{ev}_0 \circ \frac{d}{dz} \circ \sigma^{-1} \circ \varphi^*(f) &= \mathrm{ev}_0 \circ \frac{d}{dz} \circ \sigma^{-1} \left(\sum_{i \geq 0} a_i \cdot (t-1)^i \right) \\
&= \mathrm{ev}_0 \circ \frac{d}{dz} \left(\sum_{i \geq 0} a_i \left(\sum_{j \geq 1} \frac{z^j}{j!} \right)^i \right) \\
&= \mathrm{ev}_0 \left[\sum_{i \geq 1} a_i \cdot i \left(\sum_{j \geq 1} \frac{z^j}{j!} \right)^{i-1} \cdot \left(\sum_{j \geq 0} \frac{z^j}{j!} \right) \right] = a_1 = D_\varphi(f)\,.
\end{aligned}
$$

□

We will now prove that the map $\psi^* := \sigma^{-1} \circ \varphi^* \colon K_X \to K_X[|z|]$ is the germ of a rational $\mathbb{G}_a$-action in X. Indeed, letting s and w, in the following diagram put as subscript the transcendental element over K_X in the target ring.

$$
\begin{aligned}
K_X &\xrightarrow{\varphi_t^*} K_X[|t-1|] \xrightarrow{\sigma_z^{-1}} K_X[|z|] \\
K_X &\xrightarrow{\varphi_s^*} K_X[|s-1|] \xrightarrow{\sigma_w^{-1}} K_X[|w|]
\end{aligned}
$$

We also let $\psi_z^* := \sigma_z^{-1} \circ \varphi_t^*$ and $\psi_w^* := \sigma_w^{-1} \circ \varphi_s^*$. With these definitions we now proof the following lemma.

Lemma 3.10. *With the above notation, we have* $\mathrm{ev}_0 \circ \psi_z^* = \mathrm{Id}_{K_X}$ *and* $\psi_z^* \circ \psi_w^* = \psi_{z+w}^*$.

Proof. Letting $f \in K_X$ we assume

$$
\varphi^*(f) = \sum_{i \geq 0} a_i (t-1)^i \quad \text{so that} \quad \psi_z^*(f) = \sum_{i \geq 0} a_i \left(\sum_{j \geq 1} \frac{z^j}{j!} \right)^i .
$$

Since φ^* is the co morphism of a $\mathbb{G}_m$-action we have $a_0 = f$ and so $\mathrm{ev}_0 \circ \psi_z^*(f) = a_0 = f$. To prove the second assertion, remark that

$$
\begin{aligned}
\psi_z^* \circ \psi_w^*(f) &= \sigma_z^{-1} \circ \varphi_t^* \circ \sigma_w^{-1} \circ \varphi_s^* \\
&= \sigma_z^{-1} \circ \sigma_w^{-1} \circ \varphi_t^* \circ \varphi_s^*(f) \\
&= \sigma_z^{-1} \circ \sigma_w^{-1} \circ \varphi_{st}^*(f) \\
&= \sigma_z^{-1} \circ \sigma_w^{-1} \left(\sum_{i \geq 0} a_i (st-1)^i \right) \\
&= \sigma_z^{-1} \circ \sigma_w^{-1} \left(\sum_{i \geq 0} a_i \Big[\big(s-1+1\big)\big(t-1+1\big) - 1 \Big]^i \right)
\end{aligned}
$$

Applying now $\sigma_z^{-1} \circ \sigma_w^{-1}$ amounts to replace $t-1$ by $\sum_{i\geq 1} \frac{z^i}{i!}$ and $s-1$ by $\sum_{i\geq 1} \frac{w^i}{i!}$. Hence, we obtain

$$\begin{aligned}
\psi_z^* \circ \psi_w^*(f) &= \sum_{i\geq 0} a_i \left[\left(\sum_{j\geq 1} \frac{w^j}{j!} + 1 \right) \left(\sum_{j\geq 1} \frac{z^j}{j!} + 1 \right) - 1 \right]^i \\
&= \sum_{i\geq 0} a_i \left[\left(\sum_{j\geq 0} \frac{w^j}{j!} \right) \left(\sum_{j\geq 0} \frac{z^j}{j!} \right) - 1 \right]^i
\end{aligned}$$

Now, by the usual properties of the exponential sum, we obtain

$$\begin{aligned}
\psi_z^* \circ \psi_w^*(f) &= \sum_{i\geq 0} a_i \left[\left(\sum_{j\geq 0} \frac{(w+z)^j}{j!} \right) - 1 \right]^i \\
&= \sum_{i\geq 0} a_i \left[\left(\sum_{j\geq 1} \frac{(w+z)^j}{j!} \right) \right]^i \\
\psi_z^* \circ \psi_w^*(f) &= \psi_{w+z}^*(f)
\end{aligned}$$

□

Lemma 3.11. *Let φ be a rational $\mathbb{G}_m$-action on X with comorphism $\varphi^* \colon K_X \to K_X(t)$. Then the iterations D_φ^i satisfy $D_\varphi^i = \mathrm{ev}_0 \circ \dfrac{d^i}{dz^i} \circ \sigma^{-1} \circ \varphi^*$*

Proof. We can now use the argument as in [5, Proposition 4.10] applied to $\psi^* = \sigma^{-1} \circ \varphi^*$. For the convenience of the reader we copy the argument here. Letting $\psi_z^*(f) = \sum_{i\geq 0} a_i z^i$, we have

$$\begin{aligned}
\sum_{i\geq 0} \psi_z^*(a_i) w^i = \psi_z^* \circ \psi_w^*(f) = \psi_{w+z}^*(f) &= \sum_{\ell\geq 0} a_\ell (z+w)^\ell \\
&= \sum_{\ell\geq 0} a_\ell \sum_{i+j=\ell\geq 0} \binom{\ell}{i} z^j w^i \\
&= \sum_{i\geq 0} \left(\sum_{j\geq 0} a_{i+j} \binom{i+j}{i} z^j \right) w^i
\end{aligned}$$

Hence we obtain

$$\psi^*(a_i) = \sum_{j\geq 0} a_{i+j} \binom{i+j}{i} z^j \text{ and in particular } \psi^*(a_1) = \sum_{j\geq 0} a_{j+1}(j+1) z^j = \sum_{j\geq 1} a_j j z^{j-1}$$

Since $D_\varphi(f) = a_1$ by (3.3), we have $\varphi^* \circ D_\varphi = \frac{d}{dz} \circ \varphi^*$. Indeed,

$$\psi^*(D_\varphi(f)) = \psi^*(a_1) = \sum_{j \geq 1} a_j j z^{j-1} = \frac{d}{dz}\left(\sum_{j\geq 0} a_j z^j\right) = \frac{d}{dz}(\psi^*(f))$$

Hence, we have $\psi^* \circ D_\varphi^i = \frac{d^i}{dz^i} \circ \psi^*$ for all $i \geq 0$. Composing on the left with ev_0 we obtain

$$D_\varphi^i = \mathrm{ev}_0 \circ \frac{d^i}{dz^i} \circ \sigma^{-1} \circ \varphi^*\,,$$

since $\mathrm{ev}_0 \circ \psi^* = \mathrm{Id}_{K_X}$ by Lemma 3.10. □

In the following proposition, we show that the derivation D_φ is rational semisimple.

Proposition 3.12. *Let $\varphi\colon \mathbb{G}_m \times X \dashrightarrow X$ be a $\mathbb{G}_m$-rational action on X. Then the following hold*

(*i*) *D_φ is a rational semisimple derivation.*

(*ii*) *The composition $\phi^*_{D_\varphi} = \sigma \circ \exp(zD_\varphi)$ equals φ^*.*

Proof. The assertion (*i*) follows directly from (*ii*) and Proposition 3.8 since in this case the image of $\phi^*_{D_\varphi}$ equals the image of φ^* which is contained in $K_X(t)$.

To prove (*ii*), let $f \in K_X$. By Corollary 3.2, we have that $\varphi^*(f) \in K_X[|t-1|]$ and so $\sigma^{-1} \circ \varphi^*(f) \in K_X[|z|]$. Let $\sigma^{-1} \circ \varphi^*(f) = \sum_{j \geq 0} a_j z^j$. By Lemma 3.11 we have that

$$\begin{aligned}
\phi^*_{D_\varphi}(f) &= \sigma \circ \sum_{i\geq 0} \frac{z^i D_\varphi^i(f)}{i!} \\
&= \sigma \circ \sum_{i \geq 0} \frac{z^i}{i!} \cdot \mathrm{ev}_0 \circ \frac{d^i}{dz^i} \circ \sigma^{-1} \circ \varphi^*(f) \\
&= \sigma \circ \sum_{i \geq 0} \frac{z^i}{i!} \cdot \mathrm{ev}_0 \circ \frac{d^i}{dz^i}\left(\sum_{j \geq 0} a_j z^j\right) \\
&= \sigma \circ \sum_{i \geq 0} \frac{z^i}{i!} \cdot \mathrm{ev}_0 \circ \left(\sum_{j \geq i} a_j \frac{j!}{(j-i)!} z^{j-i}\right) \\
&= \sigma \circ \sum_{i \geq 0} \frac{z^i}{i!} \cdot a_i \cdot i! \\
&= \sigma \circ \sum_{i \geq 0} a_i z^i = \sigma \circ \sigma^{-1} \circ \varphi^*(f) = \varphi^*(f)
\end{aligned}$$

□

Proposition 3.13. *Let $\partial\colon K_X \to K_X$ be a rational semisimple derivation. Then the composition $D_{\phi_\partial} = \mathrm{ev}_1 \circ \frac{d}{dt} \circ \sigma \circ \exp(z\partial)$ equals ∂.*

Proof. Let ∂ be a rational semisimple derivation. Then by Lemma 3.9 (ii), we have

$$D_{\phi_\partial} = \mathrm{ev}_1 \circ \frac{d}{dt} \circ \sigma \circ \exp(z\partial) = \mathrm{ev}_0 \circ \frac{d}{dz} \circ \sigma^{-1} \circ \sigma \circ \exp(z\partial) = \mathrm{ev}_0 \circ \frac{d}{dz} \circ \exp(z\partial)$$

Letting now $f \in K_X$ we obtain

$$D_{\phi_\partial}(f) = \mathrm{ev}_0 \circ \frac{d}{dz} \circ \exp(z\partial)(f) = \mathrm{ev}_0 \circ \frac{d}{dz} \circ \sum_{i\geq 0} \frac{z^i \partial^i(f)}{i!} = \mathrm{ev}_0 \circ \sum_{i \geq 1} \frac{i z^{i-1}\partial^i(f)}{i!} = \partial(f)\,.$$

This proves the proposition. $\square$

The following is our main theorem in this paper establishing a one-to-one correspondence between rational $\mathbb{G}_m$-actions on X and rational semisimple derivations on K_X.

Theorem 3.14. *Let X be an algebraic variety. There exists a one-to-one correspondence between the rational $\mathbb{G}_m$-actions over X and rational semisimple derivations on K_X given by*

$$\begin{array}{ccc}
\Big\{\textit{Rational semisimple derivations on } K_X\Big\} & \longleftrightarrow & \Big\{\textit{Rational } \mathbb{G}_m\textit{-actions on } X\Big\} \\
\partial & \longrightarrow & \phi_\partial \\
D_\varphi & \longleftarrow & \varphi
\end{array}$$

Proof. Let X be an algebraic variety. If $\partial\colon K_X \to K_X$ is a rational semisimple derivation, then ϕ^*_∂ is a rational $\mathbb{G}_m$-action by Lemma 3.7 and Proposition 3.8. On the other hand, if φ is a rational $\mathbb{G}_m$-action on X, then, by Proposition 3.12 (i), we have that D_φ is rational semisimple. The fact that these maps are mutually inverse to each other is proven in Proposition 3.12 (ii) and Proposition 3.13. $\square$

3.3 Examples and applications

In this section we provide several examples and applications of our main theorem. To perform the computations in the sequel, we need the following technical lemma proving that conjugation of a rational semisimple derivation by an automorphism φ amounts to conjugation of the corresponding $\mathbb{G}_m$-action by the same automorphism φ.

Lemma 3.15. *Letting $\partial\colon K_X \to K_X$ be a rational semisimple derivation and $\varphi^*\colon K_X \to K_X$ be a k-automorphim, we have*

$$\phi^*_{\varphi^*\circ\partial\circ(\varphi^*)^{-1}} = \varphi^* \circ \phi^*_\partial \circ (\varphi^*)^{-1}$$

Proof. Since $(\varphi^* \circ \partial \circ (\varphi^*)^{-1})^i = \varphi^* \circ \partial^i \circ (\varphi^*)^{-1}$ and σ commutes with φ^*, we have

$$\phi_{\varphi^* \circ \partial \circ (\varphi^*)^{-1}} = \sigma \circ \sum_{i \geq 0} \frac{z^i (\varphi^* \circ \partial \circ (\varphi^*)^{-1})^i}{i!} = \sigma \circ \varphi^* \circ \left(\sum_{i \geq 0} \frac{z^i \partial^i}{i!} \right) \circ (\varphi^*)^{-1} = \varphi^* \circ \phi_\partial^* \circ (\varphi^*)^{-1}$$

□

Example 3.16. Letting $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$ we let E be the Euler derivation given by

$$E = ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} \quad \text{with} \quad a, b \in \mathbb{Z}.$$

This derivation is a regular semisimple derivation corresponding to the linear $\mathbb{G}_m$-action on X given by

$$\mathbb{G}_m \times X \to X \quad \text{where} \quad (t, (x, y)) \mapsto (t^a x, t^b y).$$

If we conjugate E with the birational map

$$\varphi \colon X \dashrightarrow X \quad \text{given by} \quad (x, y) \mapsto ((x-1)(y-1) + 1, y),$$

whose inverse is

$$\varphi^{-1} \colon X \dashrightarrow X \quad \text{given by} \quad (x, y) \mapsto \left(\frac{x-1}{y-1} + 1, y \right).$$

We define the rational semisimple derivation $\partial = \varphi^* \circ E \circ (\varphi^*)^{-1}$. A straightforward computation shows that

$$\partial = \left(\frac{a((x-1)(y-1)+1) - b(x-1)y}{y-1} \right) \frac{\partial}{\partial x} + by\frac{\partial}{\partial y}$$

By Lemma 3.15, we have $\phi_\partial^* = \varphi^* \circ \phi_E^* \circ (\varphi^*)^{-1}$. More explicitly we obtain:

$$\phi_\partial \colon \mathbb{G}_m \times X \dashrightarrow X \quad \text{given by} \quad (x, y) \mapsto \left(\frac{t^a((x-1)(y-1)+1) - 1}{t^b y - 1} + 1, t^b y \right).$$

We can recover the rational derivation ∂ by computing $\mathrm{ev}_1 \circ \frac{d}{dt} \circ \phi_\partial^*$. Indeed, a tedious computation shows that

$$\frac{d}{dt}(\phi_\partial^*(x)) = \frac{\left(at^{a-1}[(x-1)(y-1)+1]\right)(t^b y - 1) - bt^{b-1}y[t^a[(x-1)(y-1)+1] - 1]}{(t^b y - 1)^2}$$
$$\frac{d}{dt}(\phi_\partial^*(y)) = \frac{d}{dt}(t^b y) = bt^{b-1}y.$$

So that

$$\mathrm{ev}_1 \circ \frac{d}{dt} \circ \phi_\partial^*(x) = \frac{a((x-1)(y-1)+1) - b(x-1)y}{y-1}$$
$$\mathrm{ev}_1 \circ \frac{d}{dt} \circ \phi_\partial^*(y) = by,$$

recovering the initial derivation ∂.

Recall that a rational slice of a rational $\mathbb{G}_m$-action φ is a function $s \in K_X$ such that $\varphi^*(s) = ts$. We can also characterize slices in terms of the corresponding rational semisimple derivation as we show in the following lemma.

Corollary 3.17. *Letting X be an algebraic variety with fields of the rational function K_X, we let $\partial\colon K_X \to K_X$ be a rational semisimple derivation. Then $s \in K_X$ is a rational slice of $\phi_\partial^*(t)$ if and only if $\partial(s) = s$.*

Proof. Assume first that $\partial(s) = s$. Then $\partial^i(s) = s$ for every $i \geq 0$ and so we have

$$\phi_\partial^*(s) = \sigma \circ \exp(z\partial)(s) = s \cdot \sigma \left(\sum_{i \geq 0} \frac{z^i}{i!} \right) = ts\,.$$

Now if $\phi_\partial^*(s) = ts$ then $\partial(s) = \mathrm{ev}_1 \circ \frac{d}{dt} \circ \phi_\partial^*(s) = s$ $\square$

Furthermore, a slice provide a ruling of the field K_X over the field of invariants $K_X^{\mathbb{G}_m}$ of the $\mathbb{G}_m$-action.

Proposition 3.18. *If s is a rational slice for an faithful rational $\mathbb{G}_m$-action $\varphi\colon \mathbb{G}_m \times X \dashrightarrow X$, then s is transcendental over $K_X^{\mathbb{G}_m}$, $K_X = K_X^{\mathbb{G}_m}(s) = (\ker \partial)(s)$ and $\partial = s\frac{d}{ds}$ on $K_X^{\mathbb{G}_m}(s)$.*

Proof. We define the set $T = \{a \in K_X^* \mid \varphi^*(a) = t^i a \text{ with } i \in \mathbb{Z}\}$. In the proof Proposition 3.8 we proved that T generates K_X. Moreover, T is a group under multiplication. We have $\varphi^*(a) = t^0 a$ for all $a \in K_X^{\mathbb{G}_m}$. Therefore $T_0 := K_X^{\mathbb{G}_m} \setminus \{0\}$ is a subgroup of T. Let now T/T_0, given $a, b \in T$ such that $\varphi_t^*(a) = t^i a$ and $\varphi_t^*(b) = t^i a$, for some $i \in \mathbb{Z}$, we have $\varphi^*(ab^{-1}) = ab^{-1}$ implies $ab^{-1} \in T_0$. Hence, a and b differ by a element of T_0.

We define the group homomorphism $T \to \mathbb{Z}$ given by $a \mapsto i$ where $\varphi^*(a) = t^i a$. This homomorphism is surjective by the existence of a rational slice, see Definition 3.4 and below. Moreover, its kernel is T_0. We conclude that $T/T_0 \simeq \mathbb{Z}$ with a rational slice s as generator. Since T_0 generated the field $K_X^{\mathbb{G}_m}$ and T generates the field K_X we obtain that $K_X = K_X^{\mathbb{G}_m}(s)$.

Assume now that s is algebraic over $K_X^{\mathbb{G}_m}$, i.e., assume that there exists a non trivial polynomial $P \in K_X^{\mathbb{G}_m}[x]$ such that $P(s) = 0$, then $\varphi^*(P(s)) = P(ts) = 0$. This is a contradiction since t is is transcendental over K_X and so the same holds over the subfield $K_X^{\mathbb{G}_m}$. We conclude that therefore s is transcendental over $K_X^{\mathbb{G}_m}$. Finally, the structure of ∂ given as $\partial = s\frac{d}{ds}$ on $K_X^{\mathbb{G}_m}(s)$ follows since $\partial(K_X^{\mathbb{G}_m}) = 0$ and $\partial(s) = s$. $\square$

Corollary 3.19. *Let $\partial\colon K_X \to K_X$ be a rational semisimple derivation. If $\partial(s) = \lambda s$, with $\lambda \in k$, we have $K_X = \ker(\partial)(s)$*

As in the rational case treated above, given a regular $\mathbb{G}_m$-action φ on X, we define a derivation of the structure sheaf $D_\varphi\colon \mathcal{O}_X \to \mathcal{O}_X$ given over every affine open set $U \subseteq X$ by

$$D_\varphi\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U) \quad \text{given by} \quad f \mapsto \mathrm{ev}_1 \circ \frac{d}{dt} \circ \varphi^*(f)\,.$$

Any derivation $\partial\colon \mathcal{O}_X \to \mathcal{O}_X$ induces a derivation $K_X \to K_X$ simply by extending $\partial\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ to the field of fractions $\mathrm{Frac}(\mathcal{O}_X(U)) = K_X$ for any affine open set via the Leibniz rule. We denote this derivation also by the same symbol $\partial\colon K_X \to K_X$.

In the next proposition, we characterize the derivations of the structure sheaf X that come from a regular $\mathbb{G}_m$-action in the case where X is semi-affine. Recall that a variety X is called semi-affine if the canonical morphism $X \to \operatorname{Spec} \mathcal{O}_X(X)$ is proper. In this case $\mathcal{O}_X(X)$ is finitely generated and so $\operatorname{Spec} \mathcal{O}_X(X)$ is an affine variety [10, corollary 3.6]. For instance, complete or affine k-varieties are semi-affine, blow-ups of semi-affine varieties are also semi-affine.

Proposition 3.20. *Regular $\mathbb{G}_m$-actions on a semi-affine variety X are in one-to-one correspondence with rational semisimple derivations $\partial\colon \mathcal{O}_X \to \mathcal{O}_X$ such that the derivation on global sections $\partial_X\colon \mathcal{O}_X(X) \to \mathcal{O}_X(X)$ on the ring of global regular functions is semisimple with integers eigenvalues.*

Proof. By Rosenlicht theorem [20], for any regular $\mathbb{G}_m$-action on X there exists of a nonempty $\mathbb{G}_m$-invariant affine open subset U. Hence, $\partial_U\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is semisimple with integer eigenvalues and since $\mathcal{O}_X(X) \subset \mathcal{O}_X(U)$ it follows that ∂_X is a semisimple derivation of $\mathcal{O}_X(X)$ with integer eigenvalues.

Conversely, let $\partial\colon \mathcal{O}_X \to \mathcal{O}_X$ be a derivation such that $\partial_0 = \partial_X\colon \mathcal{O}_X(X) \to \mathcal{O}_X(X)$ is semisimple with integer eigenvalues. Then ∂_0 induces a possibly trivial regular $\mathbb{G}_m$-action $\varphi_0\colon \mathbb{G}_m \times X_0 \to X_0$ on $X_0 = \operatorname{Spec} \mathcal{O}_X(X)$ for which the canonical morphism $p\colon X \to X_0$ is $\mathbb{G}_m$-equivariant. In particular, for every point $x \in X$, letting $\xi = \varphi\,|_{\mathbb{G}_m \times \{x\}}\colon \mathbb{G}_m \dashrightarrow X$, $t \mapsto \varphi(t,x)$ and $\xi_0 = \varphi_0\,|_{\mathbb{G}_m \times p(x)}\colon \mathbb{G}_m \to X_0, t \mapsto \varphi_0(t,p(x))$, we have a commutative diagram

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, dashed, "\xi"] \arrow[dr, "\xi_0"'] & X \arrow[d, "p"] \\ & X_0. \end{tikzcd}$$

Since p is proper, we deduce from the valuative criterion for properness applied to the local ring of every closed point $t \in \mathbb{G}_m$ that φ is defined at every point $(x,t) \in \mathbb{G}_m \times X$ whence is a regular $\mathbb{G}_m$-action on X. $\square$

Chapter 4

Locally finite birational maps

Abstract

In this paper, we build on previous results by the authors to properly define the definition of rational locally finite derivation of an algebraic variety, which in the regular case coincide with the definition of a regular locally finite derivation.

Additionally, we define automorphism over the field of rational functions as rational locally finite attributing the existence of rational flow which can be differentiated. We show that there exists a one-to-one correspondence between the rational locally finite automorphims and the rational locally finite derivations, and the regular case coincide with the correspondence between locally finite automorphism and locally finite derivations.

Introduction

Dubouloz and Liendo define in [7] when a derivation is rationally integrable over the field of rational functions, these derivations are in correspondence with the rational action of the additive group $\mathbb{G}_a$. This idea, in conjunction with the work developed by Koshevoi [12], has allowed us to generalize in [2] the correspondence between rational $\mathbb{G}_m$ actions and certain derivations called rational semisimple. In this article, we generalize the concept of locally finite using the same idea through of an integrability condition in the exponential map, and defining when a derivation is rational locally finite, this definition is the same as completely integrable vector field. The definition of rational locally finite automorphism is defined over a generating set as field, the span linear of all compositions is a finite dimensional k-vector space, this we allow to define a rational polynomial flow which can be differentiated, and the rational locally finite automorphism are a generalization of locally finite automorphism.

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4.1 Preliminaries

Let k be an algebraically closed field of characteristic zero, B a k-algebra finitely generated $X = \mathrm{Spec}(B)$ an algebraic variety and $\mathrm{Frac}(B) = K$ the field of rational functions . If $\varphi\colon X \to X$ is a morphism over X, we obtain the equivalence of categories $\varphi^*\colon B \to B$. If B is generated by a $\{b_i\}_{i\in I}$ as k vector space, then K is generated by $\{b_i\}_{i\in I}$ as field, which means that it is generated via k- as linear combinations and quotients of these linear combinations, for example, $\mathbb{C}[x,y]$ is generated by the set $\{x^iy^i\}_{i,j\geq 0}$ then $\mathbb{C}(x,y)$ is generated as field by $\{x^iy^i\}_{i,j\geq 0}$ through k- as linear combinations and quotients of these linear combinations.

4.1.1 Derivations

A derivation $D\colon B \to B$, is a k linear map, satisfying the Leibniz rules for all $a, b \in B$ we have $D(ab) = aD(b) + bD(a)$, we define the kernel of D as the set $\ker(D) = \{b \in B | D(b) = 0\}$ also denoted by B^D, if $A \subset B$ we say D is a A-derivation if $A \subset \ker(D)$, for the nth composition $D^{(n)}$ we denote by D^n and $D^0 = \mathrm{Id}_B$, the set of derivations is denoted by $\mathrm{Der}(B)$. We say that D is *locally finite* if for all $b \in B$ the linear span of $\{b, D(b), D^2(b), \ldots\}$ is finite dimensional, the set of locally finite derivations is denoted by $\mathrm{LFD}(B)$ and lf means locally finite for short, the definition is associated to existence of a vanishing polynomial $P_b(T) \in k[T]$ (which is verify $P_b(D)(b) = 0$). A derivation is *locally nilpotent* if for each $b \in B$ there exists $n \in \mathbb{Z}_{\geq 0}$ such that $D^n(b) = 0$, the set of locally nilpotent over B is denoted by $\mathrm{LND}(B)$. A derivation D is called *semisimple* if there exists a basis $\{b_i\}_{i\in I}$ of B as a k-vector space such that $D(b_i) = \lambda_i b_i$ with $\lambda_i \in k$, the set of semisimple derivations is denoted by $\mathrm{SSD}(B)$. Both derivations that are locally nilpotent and semisimple are lf.

If D is an lf derivation, it admits the *Jordan decomposition* $D = D_n + D_s$ where D_n is a locally nilpotent derivation, D_s is a semisimple derivation and $D_sD_n = D_nD_s$.

4.1.2 Polynomial automorphisms

Let B be a k-algebra finitely generated, $F^*\colon B \to B$ an automorphism of ring, F^* is locally finite if for each $b \in B$ the linear span of set $\{b, F^*(b), (F^*)^2(b), \ldots\}$ is finite dimensional, F is locally finite if F^* is, the definition is associated with the existence of a vanishing polynomial

$P_b(T) \in k[T]$ (is verify $P_b(F^*)(b) = 0$). Our main object of study are the locally finite automorphisms (lf automorphism for short), and the set of lf automorphisms is denoted by $\mathrm{LFA}(B)$. An automorphism F^* is *unipotent* if $F - \mathrm{Id}$ is nilpotent. An automorphism is *semisimple* if there exists a basis $\{a_i\}_{i \in I} \subset B$ of eigenvectors such that $F^*(a_i) = \gamma_i a_i$ with $\gamma_i \in k$. Since the k-algebra B is finitely generated, for both automorphisms and derivations it is enough to prove the condition lf on a generating set.

The unipotent and semisimple automorphisms are a particular case of the lf automorphism and if F is a lf automorphism it admits the *Dunford decomposition* (Multiplicative Jordan decomposition) $F = F_u F_s$ where F_u is the unipotent automorphism and F_s is the semisimple automorphism. It is well known that there is a correspondence between unipotent automorphisms and locally nilpotent derivations, for more details see [8],[22],[5]. There also exists a correspondence between semisimple automorphisms and semisimple derivations, which we will show below.

4.1.3 Polynomial flow

Definition 4.1. Let X be an algebraic variety, a map $\varphi_\tau \colon k \times X \to X$ is said a flow if satisfy:

1. $\varphi_\tau \circ \varphi_s = \varphi_{\tau+s}$
2. $\varphi_0 = \mathrm{Id}_X$

Let $V \colon k^n \to k^n$ be a C^1-vector field, and consider the (autonomous) system of differential equations

$$\frac{\partial x(\tau)}{\partial \tau} = V(x(\tau)) \qquad \text{with initial condition} \quad x(0) = x_0 \in k^n \tag{4.1}$$

Then there exist a unique maximal local solution (flow)

$$x(\tau) = \varphi(\tau, x_0) = \varphi_\tau(x_0) \tag{4.2}$$

The flow $\varphi(\tau, x_0)$ is called a polynomial flow if $\varphi(\tau, x_0)$ is polynomial in each fixed τ, that is, $\varphi(\tau, x_0)$ depends polynomially on the initial condition x_0. If a flow $\varphi(\tau, x_0)$ is polynomial then (see [1]) it is global and even (if $k = \mathbb{C}$) entire ([3]).

Since the flow φ_τ is polynomial if depends polynomially on x for each τ, is clear that for each $\tau \in k$, define a holomorphic automorphism. We say a polynomial flow is quasi-algebraic if for each $\tau_0 \in k$ the φ_{τ_0} is an algebraic automorphism. Our case of interest for study are the algebraic and quasi algebraic because they are related with lf derivation and lf automorphism. A natural question would be for which V as above does the flow associated φ depend polynomially on the initial condition x_0, this question was answered by B. Coomes and V. Zurkowski in [4] proved (for $k = \mathbb{C}$) .

A polynomial vector field $V = (P_1, P_2 \dots, P_n)$ has a polynomial flow if and only if the derivation $P_1 \dfrac{\partial}{\partial x_1} + P_2 \dfrac{\partial}{\partial x_2} + \cdots + P_n \dfrac{\partial}{\partial x_n}$ is locally finite.

This result is very important because indicate a relation between algebraic automorphisms and derivations over k^n with this characteristics.

Example 4.2. In 1985 Bass and Meisters in [1] classify the polynomial flows and locally finite vector fields on k^2 conjugate by an automorphism in k^2, these are given by:

1. $\varphi_\tau(x, y) = (x, e^{b\tau} y)$; $D = by\frac{\partial}{\partial y}$
2. $\varphi_\tau(x, y) = (x + \tau, e^{b\tau} y)$;$D = \frac{\partial}{\partial x} + by\frac{\partial}{\partial y}$
3. $\varphi_\tau(x, y) = (x, y + f(x)\tau)$; $D = f(x)\frac{\partial}{\partial y}$
4. $\varphi_\tau(x, y) = (e^{a\tau} x, e^{b\tau} y)$;$D = ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y}$
5. $\varphi_\tau(x, y) = (e^{a\tau} x, e^{am\tau}(y + \tau x^m))$; $D = ax\frac{\partial}{\partial x} + (amy + x^m)\frac{\partial}{\partial y}$

4.1.4 Exponential map

Given $D \in \mathrm{Der}(B)$ with $D(\tau) = 0$, we define the exponential map with parameter τ, since the map:

$$\begin{aligned} \exp(\tau D) : B &\to B[|\tau|] \\ b &\mapsto \sum_{j\geq 0} \tau^j \frac{D^j(b)}{j!} \end{aligned}$$

Some times for short we denote the exponential map $\exp(\tau D) = \sum_{j\geq 0} \tau^j \dfrac{D^j}{j!}$ for $F^*_{\tau D}$.

We extend the exponential maps $\exp(\tau D)\colon B[|\tau|] \to B[|\tau|]$ fixing τ.

Proposition 4.3. *The exponential maps* $\exp(\tau D)\colon B[|\tau|] \to B[|\tau|]$ *is an isomorphism*

Proof. Clearly if $a, b \in B$ as D is linear we have $\exp(\tau D)(a + b) = \exp(\tau D)(a) + \exp(\tau D)(b)$

$$
\begin{aligned}
\exp(\tau D)(a)\exp(\tau D)(b) &= \sum_{j\geq 0}\tau^j\frac{D^j(a)}{j!}\sum_{l\geq 0}\tau^l\frac{D^l(b)}{l!}\\
&= \sum_{j+l\geq 0}\tau^{j+l}\frac{1}{j+l}\binom{j+l}{j}D^j(a)D^j(b)\\
&= \sum_{m\geq 0}\tau^m\frac{1}{m!}\left(\sum_{j+l=m}\binom{j+l}{j}D^j(a)D^j(b)\right)\\
\exp(\tau D)(a)\exp(\tau D)(b) &= \sum_{m\geq 0}\tau^m\frac{1}{m!}D^m(ab) = \exp(\tau D)(ab)
\end{aligned}
$$

Moreover if we suppose $D_1D_2 = D_2D_1$ we have $\exp(\tau(D_1+D_2)) = \exp(\tau D_1)\circ\exp(\tau D_2) = \exp(\tau D_2)\circ\exp(\tau D_1)$,

Since $D_1D_2 = D_2D_1$, we have $(D_1+D_2)^m = \sum_{i+j=m}\binom{m}{j}D_1^i\circ D_2^j$ then

$$
\begin{aligned}
\exp(\tau D_1)\circ\exp(\tau D_2) &= \sum_{j\geq 0}\frac{\tau^jD_1^j}{j!}\circ\sum_{r\geq 0}\frac{\tau^rD_2^r}{r!}\\
&= \sum_{j\geq 0}\sum_{r\geq 0}\frac{1}{j!r!}\tau^jD_1^j\circ\tau^rD_2^r\\
&= \sum_{m\geq 0}\frac{\tau^m}{m!}\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}D_1^l\circ D_2^{m-l}\\
&= \sum_{m\geq 0}\frac{\tau^m(D_1+D_2)^m}{m!}\\
\exp(\tau D_1)\circ\exp(\tau D_2) &= \exp(\tau(D_1+D_2))
\end{aligned}
$$

In particular if $D_2 = -D_1$, we have $\exp(\tau D_1)\circ\exp(\tau(-D_1)) = \exp(\tau(-D_1))\circ\exp(\tau D_1) = \mathrm{Id}$. □

4.2 Rational case

4.2.1 Rational lf derivations

Definition 4.4. Given $\partial\in\mathrm{Der}(K)$, we say ∂ rational lf derivation, if for a generating set $\{b_i\}_{i\in I}\subset K$ (as field) , V_{b_i} the linear span generated by $\{b_i,\partial(b_i),\partial^2(b_i),\ldots\}$ as k- vector space is finite dimensional.

Definition 4.5. A k-derivation $\partial\colon K\to K$ on a variety $\mathrm{Spec}(B)$ is called *rationally integrable* if the formal exponential homomorphism $\exp(\tau\partial)\colon K\to K[|\tau|]$ factors through $K(\tau)\cap K[|\tau|]$.

Definition 4.6. Let K be the field of rational functions associated to algebraic variety $\mathrm{Spec}(B)$, $\partial \in \mathrm{Der}(K)$. We say ∂ is *rational semisimple* if there exist a generating set $\{b_i\}_{i\in I}$ of K (as field) such that $\partial(b_i) = \gamma_i b_i$ with $\gamma_i \in k$. Is easy to see that exponential homomorphism $\exp(\tau\partial)\colon K \to K[[\tau]]$ factors through $K(t_1, t_2, \ldots, t_r) \cap K[|\tau|]$ where $t_i = \sum_{j\geq 0} \frac{(\gamma_i \tau)^j}{j!}$, for some a finite set $\gamma_i \in k$.

In particular if $\gamma_i \in \mathbb{Z}$, by theorem 2.14 in [2] the rational semisimple derivation are in correspondence with the rational -actions defining $t^{\gamma_i} = \sum_{j\geq 0} \frac{(\gamma_i \tau)^j}{j!}$. The rational semisimple and rationally integrable derivations are particular case of rational lf derivations.

Lemma 4.7. *If ∂ is rational lf derivations, it admits a decomposition $\partial_s + \partial_n$ where ∂_s is rational semisimple, ∂_n rational unipotent and $\partial_s\partial_n = \partial_n\partial_s$.*

Proof. Let $\Lambda \subset k$ the set of eigenvalues for ∂, and $\{b_i\}_{i\in I}$ a generating set for K such that the linear span of $\{\partial^j(b_i)\}_{j\geq 0}$ is finite dimensional, $\{b_i\}_{i\in I}$ with the linear combinations generate a k-algebra B and the condition $\{\partial^j(b_i)\}_{j\geq 0}$ finite dimensional implies $B = \bigoplus_{\lambda\in\Lambda} B_\lambda$, where $B_\lambda = \{b \in B | (\partial - \lambda \,\mathrm{Id})^m(b) = 0, \text{for some}\, m > 0\}$, for simplicity we denote $\partial - \lambda\,\mathrm{Id} = \partial - \lambda$, also we have $B_\lambda B_\mu \subset B_{\lambda+\mu}$ because if we consider $a \in B_\lambda, b \in B_\mu$ there exist l, m such that $(\partial - \lambda)^l(a) = 0$, $(\partial - \mu)^m(b) = 0$, $(\partial - (\lambda + \mu))^{l+m}(ab) = \sum_{j=0}^{l+m} \binom{l+m}{j} (\partial - \lambda)^{l+m-j}(a)(\partial - \mu)^j(b) = 0$, since we have two case the first case $l + m - j \geq l$ and $j \leq m$ this case implies $(\partial - \lambda)^{l+m-j}(a) = 0$ and the second case $l + m - j < l$ and $j > m$ then $(\partial - \mu)^j(b) = 0$, so that $B_\lambda B_\mu \subset B_{\lambda+\mu}$, for any $\lambda, \mu \in \Lambda$.

We define ∂_s as $\partial_s(b) = \sum_\lambda \lambda b_\lambda$ if $b = \sum_\lambda b_\lambda$. Clearly is a derivation, linear by definition, satisfy the Leibniz rule, if $a \in B_\lambda$, $b \in B_\mu$ we have $ab \in B_{\lambda+\mu}$, hence $\partial_s(ab) = (\lambda + \mu)ab = \lambda ab + \mu ab = \partial_s(a)b + a\partial_s(b)$, for any $b \in B$ is extended by linearity. Since $\partial_s(B_\lambda) \subset B_\lambda$ we have $\partial\partial_s = \partial_s\partial = \lambda\partial$ on B_λ, ∂_s is rational semisimple because for all $b \in B$ can be writen as $b = \sum_\lambda b_\lambda$, then if we consider $\{b_\lambda\}_{\lambda\in\Lambda}$ as generating set, K can be defined as localization of B with $S^{-1} = B \setminus \{0\}$, so that $S^{-1}B = K$. We conclude $\{b_\lambda\}_{\lambda\in\Lambda}$ is the generating set as field and satisfy $\partial_s(b_\lambda) = \lambda b_\lambda$ where $\lambda \in k$.

Let $\partial_n = \partial - \partial_s$, is a derivation because ∂ and ∂_s are, ∂_n commute with ∂, because ∂ commute with ∂ and ∂_s. Now if $b \in B_\lambda$, $\partial_n(b) = (\partial - \partial_s)(b) = (\partial - \lambda)(b)$ then there exists l sufficiently big such that $0 = (\partial - \lambda)^l(b) = \partial_n^l(b)$, then $\exp(\tau\partial_n)(b) = \sum_{j=0}^{l-1} \tau^j \partial_n^j(b)$ and in the same way as in the previous derivation if we consider the localization of B with $S^{-1} = B \setminus \{0\}$ and $\{b_\lambda\}_\lambda$ as generating set, the image of exponential map satisfy $\exp(\tau\partial_n)(K) \subset K(\tau) \cap K[|\tau|]$.

To prove the uniqueness of decomposition suppose there exist two decomposition of ∂, $\partial_s + \partial_n = \partial = \partial'_s + \partial'_n$. Since ∂ commute with $\partial_s, \partial_n, \partial'_s, \partial'_n$ we have ∂_s commute with ∂'_s and ∂_n commute with ∂'_n. A difference of commuting locally nilpotent (respectively semisimple) derivations is locally nilpotent (respectively semisimple), since $\partial_s - \partial'_s = \partial'_n - \partial_n$, $\partial_s - \partial'_s$ is semisimple and $\partial'_n - \partial_n$ is locally nilpotent derivation, on B_λ we have $(\partial_s - \partial'_s)(b) = \lambda b - \lambda b = 0$ therefore is the zero map, so that $\partial_s = \partial'_s$ and $\partial_n = \partial'_n$. ☐

Lemma 4.8. *Given $\partial \in \mathrm{Der}(K)$, with $\partial(\tau) = 0$ and the exponential map $\exp(\tau\partial)\colon K \to K[|\tau|]$, ∂ is rational locally finite if satisfy*

$$\exp(\tau\partial)(K) \subset K(\tau, t_1, \dots, t_r) \cap K[|\tau|]$$

where $t_i = \sum_{j \geq 0} \frac{(\gamma_i \tau)^j}{j!}$, for some a finite set $\gamma_i \in k$.

Proof. Since ∂ admit a decomposition $\partial_s + \partial_n$, we compute $\exp(\tau(\partial_s + \partial_n))$. For the generating set $\{b_i\}_{i \in I}$, we have $\partial_s(b_i) = \lambda_i b_i$ and $\partial_s^j(b_i) = \lambda_i^j b_i$ and $\exp(\tau\partial_s)(b_i) = t_i b_i$, if if we extend this by linear combinations and quotients we obtain $\exp(\tau\partial_s)(K) \subset K(t_1, \dots, t_r) \cap K[|\tau|]$. Since $\partial(\tau) = \partial_s(\tau) = \partial_n(\tau) = 0$, the exponential maps fix the parameter τ, we can extend the maps over τ, sending τ to τ.

$$\begin{aligned} \exp(\tau(\partial_s + \partial_n)) = \exp(\tau\partial_s) \circ \exp(\tau\partial_n)(K) &\subseteq \exp(\tau\partial_s)(K(\tau) \cap K[|\tau|]) \\ \exp(\tau\partial)(K) &\subseteq K(\tau, t_1, t_2 \dots, t_r) \cap K[|\tau|] \end{aligned}$$

☐

Corollary 4.9. *Let $\partial \in \mathrm{Der}(K)$ be a rational lf derivation, then $\exp(\partial) = \exp(\tau\partial)|_{\tau=1}$*

Proof. For lemma 4.8 we have $\exp(\tau\partial)|_{t=1}$ can be factorized as $\exp(\partial)$

$$\overbrace{K \xrightarrow{\exp(\tau\partial)} K(\tau, t_1, \dots, t_r) \xrightarrow{\tau \mapsto 1} K}^{\exp(\partial)}$$

☐

4.2.2 Rational flow

Definition 4.10. Given a algebraic variety X, we define the morphism $\varphi_\tau\colon k \times X \dashrightarrow X$, we say φ_t is a rational flow if:

- $\varphi_\tau \circ \varphi_{\tau'}(x) = \varphi_{\tau+\tau'}(x)$
- $\varphi_0(x) = x$
- For each $\tau_0 \in k$, we have φ_{τ_0} is a birational map.

We can distinguish two types of rational flow, the algebraic given by quotient of polynomials and the quasi algebraic given by quotients of polynomials and exponentials with exponent a scalar multiplied by τ.

Example 4.11. • The first example correspond to a rational flow quasi algebraic

$$\begin{array}{rcl} \varphi_\tau \colon k \times \mathbb{A}^2 & \to & \mathbb{A}^2 \\ (\tau, (x, y)) & \mapsto & (\dfrac{ae^{a\tau}x}{(1-e^{a\tau})x+a}, \dfrac{ae^{b\tau}y}{(1-e^{a\tau})x+a}) \end{array}$$

and their vector field associated $\partial = (ax + x^2)\dfrac{\partial}{\partial x} + (by + xy)\dfrac{\partial}{\partial y}$

- The second example is a rational flow algebraic

$$\begin{array}{rcl} \varphi_\tau \colon k \times \mathbb{A}^3 & \to & \mathbb{A}^3 \\ (\tau, (x, y, z)) & \mapsto & (\dfrac{x}{1+\tau x}, y + \dfrac{\tau}{p(z)}, z) \end{array}$$

and their vector field associated $-x^2\dfrac{\partial}{\partial x} + \dfrac{1}{p(z)}\dfrac{\partial}{\partial y}$

Remark 4.12. Is clear that for each $\tau_0 \in k$ the maps φ_{τ_0} is a birational map.

4.2.3 Rational lf automorphism

Definition 4.13. Given a automorphism F^* in K, we say rational locally finite, if the nth composition can be extended to rational flow $(F^*)^\tau$ which the image of a generating set $\{b_i\}_{i\in I}$ is contained in $K(\tau, t_1, t_2, \ldots, t_r)$, where $t_i = \sum_{j\geq 0} \dfrac{(\lambda_i \tau)^j}{j!}$ for some $\lambda_i \in k$.

Remark 4.14. Given an automorphism F^* in K, if there exists a generating set $\{b_i\}_{i\in I} \subset K$ (as field) such that the set V_{b_i} given by the linear span generated by $\{b_i, (F^*)(b_i), (F^*)^{(2)}(b_i), \ldots\}$ as k- vector space is finite dimensional then F^* is rational locally finite, because it allow define a rational polynomial flow whose image is contained in $K(\tau, t_1, t_2, \ldots, t_r)$, this condition is stronger than the previous definition.

Definition 4.15. Let F_u, F_s be birational morphisms over X, F_u is rational unipotent if F_u^τ comes from a rational -actions and F_s rational semisimple if there exist a generating set $\{a_i\}_{i\in I}$ (as field) such that $F_s(a_i) = \gamma_i a_i$ with $\gamma_i \in k$. In particular if $\gamma_i \in \mathbb{Z}$ the rational flow corresponds to a rational -action.

Remark 4.16. The image of comorphism of a rational -action is contained in $K(\tau)$, and the image of flow associated to semisimple comorphism above evaluated in the generating set as field is contained in $K(t_1, \dots, t_r)$ some finite set of $\{t_i\}$ therefore the rational unipotent and rational semisimple are particular case of rational locally finite automorphism.

Lemma 4.17. *If F is a rational lf automorphism then F admits a decomposition $F = F_u F_s$ where F_u rational unipotent and F_s is rational semisimple and $F_u F_s = F_s F_u$*

Proof. Let $\{b_i\}_{i \in I}$ be a generating set for K and we consider $\{e^{\lambda_i}\}_{i \in I}$ the eigenvalues for F^* and we define the k-algebra $B = \bigoplus_{\lambda \in \Omega} B_\lambda$ where $\Omega \subset k$, the B_λ is composed of set of elements of K that are fixed by $e^{-\lambda\tau} F^*$, i.e. $B_\lambda = \{b \in K | e^{-\lambda\tau}(F^*)^\tau(b) = b\}$. Also if $a \in B_\lambda$ we have $e^{-\lambda\tau}(F^*)^\tau(a) = a$ and if $b \in B_\mu$ we have $e^{-\mu\tau}(F^*)^\tau(b) = b$ then

$$ab = e^{-\lambda\tau}(F^*)^\tau(a) e^{-\mu\tau}(F^*)^\tau(b) = e^{-(\lambda+\mu)\tau}(F^*)^\tau(ab)$$

and therefore $B_\lambda B_\mu \subset B_{\lambda+\mu}$ for any $\lambda, \mu \in \Omega$. Define $(F^*)_s^\tau$ as $(F^*)_s^\tau(b) = \sum_\lambda e^{\lambda\tau} b_\lambda$ if $b = \sum_\lambda b_\lambda$. Since is linear by definition and for $b_\lambda \in B_\lambda, b_\mu \in B_\mu$ then $b_\lambda b_\mu \subset B_{\lambda+\mu}$ and we have $(F^*)_s^\tau(b_\lambda)(F^*)_s^\tau(b_\mu) = e^{\lambda\tau} b_\lambda e^{\mu\tau} b_\mu = e^{(\lambda+\mu)\tau} b_\lambda b_\mu = F_s^*(b_\lambda b_\mu)$ which can be extended for any $b \in K$ via linear combinations and quotients, their inverse for $b_\lambda \in B_\lambda$ satisfy $((F^*)_s^\tau)^{-1} = e^{-\lambda\tau} b_\lambda$. Moreover $(F^*)^\tau(B_\lambda) \subset B_\lambda$ for all λ, we have $(F^*)^\tau(F_s^*)^\tau = (F_s^*)^\tau(F^*)^\tau = e^{\lambda\tau}(F^*)^\tau$, F_s^* commutes with $(F^*)^\tau$. In particular, the automorphism F_s^* is rational semisimple since $S^{-1}B = K$ and there exist a generating set $\{b_\lambda\}_{\lambda \in \Omega}$ of K as field satisfying $F_s^*(b_\lambda) = e^\lambda b_\lambda$, where $\lambda \in k$.

Let $F_u^* = F^*(F_s^*)^{-1}$, clearly is an automorphism because is the composition of two automorphism (F^* and $(F_s^*)^{-1}$ are), for each B_λ we have $(F_u^*)^\tau$ commute with $(F^*)^\tau$, then F_u^* commute with F^* in B. Moreover, if $b_\lambda \in B_\lambda$ then for construction $(F_u^*)^\tau = e^{-\lambda\tau}(F^*)^\tau$ in B_λ

$$(F_u^*)^{\tau'}(F_u^*)^\tau = e^{-\lambda\tau}(F^*)^\tau e^{-\lambda\tau'}(F^*)^{\tau'} = e^{-\lambda(\tau+\tau')}(F^*)^{\tau+\tau'} = F_u^{\tau+\tau'}$$

and $(F_u^*)^0 = e^{-\lambda \cdot 0}(F^*)^0 = \mathrm{Id}$ therefore correspond to condition of a regular -action on B. If we extend over $K = S^{-1}B$ we obtain a $(F_u^*)^\tau$ comes from rational -action.

Suppose there exist two decompositions of F^*, $F_s^* F_u^* = F^* = (F')_s^*(F')_u^*$. Since F^* commute with $F_s^*, F_u^*, (F')_s^*, (F')_u^*$ we have F_s^* commute with $(F')_s^*$ and F_u^* commute with $(F')_u^*$. We have the equality $F_s^*((F')_s^*)^{-1} = (F_u^*)^{-1}(F')_u^*$. To composition of commuting rational semisimple (respectively rational unipotent) derivations is rational semisimple (respectively rational unipotent), thus must be Id, because on we have B_λ $F_s^*((F')_s^*)^{-1}(b_\lambda) = F_s^*(e^{-\lambda} b_\lambda) = e^{-\lambda} F_s^*(b_\lambda) = b_\lambda$ (rational unipotent and rational semisimple simultaneously means identity), this forces, $F_s^* = (F')_s^*$ and $F_u^* = (F')_u^*$

□

Lemma 4.18. *F^* is a rational locally finite automorphism then F^* comes from a rational polynomial flow*

$$(F^*)^\tau(b) = \frac{\sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot\tau} p_\sigma(\tau) b^\sigma}{\sum_\rho \beta_\rho \gamma^{\rho\cdot\tau} q_\rho(\tau) b^\rho}$$

where φ_τ $\sigma = (\sigma_1, \ldots, \sigma_r), \rho = (\rho_1, \ldots, \rho_r) \in \mathbb{Z}^n_{\geq 0}$, $b^\sigma = b_1^{\sigma_1} b_2^{\sigma_2} \ldots b_r^{\sigma_r}$, $b^\rho = b_1^{\rho_1} b_2^{\rho_2} \ldots b_r^{\rho_r}$, $p_\sigma(\tau), q_\rho(\tau) \in k[\tau]$, $\sigma \cdot \tau = (\sigma_1\tau, \ldots, \sigma_r\tau)$, $\rho \cdot \tau = (\rho_1\tau, \ldots, \rho_r\tau)$ i.e. $\gamma^{\sigma\cdot\tau} = \gamma_1^{\sigma_1\tau} \ldots \gamma_r^{\sigma_r\tau}$ and $\gamma^{\rho\cdot\tau} = \gamma_1^{\rho_1\tau} \ldots \gamma_r^{\rho_r\tau}$, with $\alpha_\sigma, \beta_\rho \in k$

Proof. First, we analize the particular case when F^* is rational semisimple, rational unipotent and subsequently the composition of both. If F_s^* is rational semisimple, for a generating set we have $(F_s^*)^\tau(b_i) = \gamma_i^\tau b_i$, and given $b \in K$, we have $b = \frac{\sum_\sigma \alpha_\sigma b^\sigma}{\sum_\rho \beta_\rho b^\rho}$ where $\sigma = (\sigma_1, \ldots, \sigma_r), \rho = (\rho_1, \ldots, \rho_r) \in \mathbb{Z}^n_{\geq 0}$, $b^\sigma = b_1^{\sigma_1} b_2^{\sigma_2} \ldots b_r^{\sigma_r}$, $b^\rho = b_1^{\rho_1} b_2^{\rho_2} \ldots b_r^{\rho_r}$ and since F_s^* is a homomorphism of field respect the multiplication and sum $(F_s^*)^\tau(b) = \frac{\sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot\tau} b^\sigma}{\sum_\rho \beta_\rho \gamma^{\rho\cdot\tau} b^\rho}$ where $\sigma \cdot \tau = (\sigma_1\tau, \ldots, \sigma_r\tau)$, $\rho \cdot \tau = (\rho_1\tau, \ldots, \rho_r\tau)$ i.e. $\gamma^{\sigma\cdot\tau} = \gamma_1^{\sigma_1\tau} \ldots \gamma_r^{\sigma_r\tau}$ and $\gamma^{\rho\cdot\tau} = \gamma_1^{\rho_1\tau} \ldots \gamma_r^{\rho_r\tau}$, if we extend the fixing the τ, τ''s then $(F_s^*)^{\tau'}((F_s^*)^\tau(b)) = \frac{\sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot(\tau+\tau')} b^\sigma}{\sum_\rho \beta_\rho \gamma^{\rho\cdot(\tau+\tau')} b^\rho}$ and $(F_s^*)^0(b) = \frac{\sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot 0} b^\sigma}{\sum_\rho \beta_\rho \gamma^{\rho\cdot 0} b^\rho} = b$ which is a rational quasi algebraic flow and for each $\tau_0 \in k$ F^{τ_0} is algebraic. Now if F_u^* is rationally integrable this comes from the action of the additive group, for $b = \frac{\sum_\sigma \alpha_\sigma b^\sigma}{\sum_\rho \beta_\rho b^\rho}$, we have $(F_u^*)^\tau(b) = \frac{\sum_\sigma \alpha_\sigma p_\sigma(\tau) b^\sigma}{\sum_\rho \beta_\rho q_\rho(\tau) b^\rho}$ where $p_\sigma(\tau), q_\rho(\tau) \in k[\tau]$, the coation morphism of a rational -action satisfy $(F_u^*)^\tau (F_u^*)^{\tau'} = ((F_u^*)^{\tau+\tau'}(b)) = \frac{\sum_\sigma \alpha_\sigma p_\sigma(\tau+\tau') b^\sigma}{\sum_\rho \beta_\rho q_\rho(\tau+\tau') b^\rho}$ and $(F_u^*)^0(b) = \frac{\sum_\sigma \alpha_\sigma p_\sigma(0) b^\sigma}{\sum_\rho \beta_\rho q_\rho(0) b^\rho}$ (this means $p_\sigma(0) = q_\rho(0) = 1$). We suppose F_s^* and F_u^* fix the parameter τ, we can extend the homomorphism from $K(\tau, t_1, t_2, \ldots, t_r)$ to $K(\tau, t_1, t_2, \ldots, t_r)$ leaving fixed the parameter τ as follows:

$$\begin{aligned}
(F^*)^\tau(b) &= (F_u^* F_s^*)^\tau(b) \\
&= (F_u^*)^\tau (F_s^*)^\tau(b) \\
&= (F_u^*)^\tau \left(\frac{\sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot\tau} b^\sigma}{\sum_\rho \beta_\rho \gamma^{\rho\cdot\tau} b^\rho} \right) \\
(F^*)^\tau(b) &= \frac{\sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot\tau} p_\sigma(\tau) b^\sigma}{\sum_\rho \beta_\rho \gamma^{\rho\cdot\tau} q_\rho(\tau) b^\rho}
\end{aligned}$$

satisfying the condition of a rational polynomial flow $(F^*)^{\tau'}((F^*)^\tau(b)) = \frac{\sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot(\tau+\tau')} p_\sigma(\tau+\tau') b^\sigma}{\sum_\rho \beta_\rho \gamma^{\rho\cdot(\tau+\tau')} q_\rho(\tau+\tau') b^\rho}$ and $(F^*)^0(b) = \frac{\sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot 0} p_\sigma(0) b^\sigma}{\sum_\rho \beta_\rho \gamma^{\rho\cdot 0} q_\rho(0) b^\rho} = b$.

$\square$

Definition 4.19. Given a lf rational automorphism F^* we can consider the rational flow $(F^*)^\tau$ and define a derivation associated to $(F^*)^\tau$, as follows

$$\partial_F = \mathrm{ev}_0 \circ \frac{d}{d\tau} \circ (F^*)^\tau$$

where ev_0 is the evaluation $\tau = 0$

Clearly is a derivation over K, given $a, b \in B$, $\lambda \in k$ we have

$$\begin{aligned}
\partial_F(a+\lambda b) &= \mathrm{ev}_0 \circ \frac{d}{d\tau} \circ (F^*)^\tau (a+\lambda b) \\
\partial_F(a+\lambda b) &= \mathrm{ev}_0 \circ \frac{d}{d\tau} [(F^*)^\tau (a) + \lambda (F^*)^\tau (b)] \\
\partial_F(a+\lambda b) &= \mathrm{ev}_0 \circ \frac{d}{d\tau} \circ (F^*)^\tau (a) + \lambda\, \mathrm{ev}_0 \circ \frac{d}{dt} \circ (F^*)^\tau (b) \\
\partial_F(a+\lambda b) &= \partial_F(a) + \lambda \partial_F(b)
\end{aligned}$$

The Leibniz rules

$$\begin{aligned}
(F^*)^\tau (ab) &= (F^*)^\tau (a) (F^*)^\tau (b) \quad / \frac{d}{d\tau} \\
\frac{d}{d\tau} \circ (F^*)^\tau (ab) &= \frac{d}{d\tau}((F^*)^\tau (a))(F^*)^\tau (b) + (F^*)^\tau (a) \frac{d}{d\tau}((F^*)^\tau (b)) \quad / \mathrm{ev}_0 \\
\mathrm{ev}_0 \circ \frac{d}{d\tau} \circ (F^*)^\tau (ab) &= \mathrm{ev}_0 \frac{d}{d\tau}((F^*)^\tau (a))(F^*)^0 (b) + (F^*)^0 (a)\, \mathrm{ev}_0 \frac{d}{d\tau}((F^*)^\tau (b)) \\
\partial_F(ab) &= \partial_F(a) b + a \partial_F(b)
\end{aligned}$$

Proposition 4.20. *If F is a rational lf automorphism then ∂_F is rational lf derivation*

Proof. We must prove that the image of exponential maps $\exp(\partial_F)$ factor through $K(\tau, t_1, \ldots, t_r) \cap K[|\tau|]$, We can distinguish three case

- If $F = F_u$ rational unipotent then $(F_u^*)^\tau$ comes from rational action, ∂_{F_u} is rationally integrable and the exponential map for [7], $(F_u^*)^\tau (K) = \exp(\tau \partial_{F_u})(K) \subset K(\tau) \cap K[|\tau|]$

- If $F = F_s$ rational semisimple then for a generating set as field $\{b_i\}_{i \in I}$ such that we have $F_s^*(b_i) = \gamma_i b_i$ where $\gamma_i \in k$, $(F_s^*)^\tau (b_i) = \gamma_i^\tau b_i$ hence $\mathrm{ev}_0 \circ \dfrac{d}{d\tau} \circ (F_s^*)^\tau (b_i) = \ln(\gamma_i) b_i$ and $\exp(\tau \partial_{F_s})(b_i) = \gamma_i^\tau b_i = t_i b_i$, where $t_i = \displaystyle\sum_{j \geq 0} \frac{(\ln(\gamma_i)\tau)^j}{j!}$, for some a finite set $\{\gamma_i\}_{i \in I}$, for above $(F_s^*)^\tau = \exp(\tau \partial_{F_s})$. We conclude the exponential map $\exp(\tau \partial_{F_s})(K) \subset K(t_1, \ldots, t_r) \cap K[|\tau|]$

- Using the fact $F = F_s F_u$ we have

$$\begin{aligned}
(F^*)^\tau &= (F_s^*)^\tau (F_u^*)^\tau \\
(F^*)^\tau &= \exp(\tau\partial_{F_s}) \circ \exp(\tau\partial_{F_u}) \\
(F^*)^\tau &= \exp(\tau(\partial_{F_s} + \partial_{F_u})) \quad /\frac{d}{d\tau} \\
\frac{d}{d\tau} \circ (F^*)^\tau &= \exp(\tau(\partial_{F_s} + \partial_{F_u})) \circ (\partial_{F_s} + \partial_{F_u}) \quad / \operatorname{ev}_0 \\
\operatorname{ev}_0 \circ \frac{d}{dt} \circ (F^*)^\tau &= (\partial_{F_s} + \partial_{F_u}) \\
\partial_F &= \partial_{F_s} + \partial_{F_u}
\end{aligned}$$

  We extend $\exp(\tau\partial_{F_s}), \exp(\tau\partial_{F_u})$ to $K(\tau, t_1, t_2 \ldots, t_r) \cap K[|\tau|]$ fixing the parameter τ and $\exp(\tau\partial_F)(K) = \exp(\tau\partial_{F_s}) \circ \exp(\tau\partial_{F_u})(K)$ we have

$$\begin{aligned}
\exp(\tau\partial_{F_s}) \circ \exp(\tau\partial_{F_u})(K) &\subseteq \exp(\tau\partial_{F_s})(K(\tau) \cap K[|\tau|]) \\
\exp(\tau\partial_F)(K) &\subseteq K(\tau, t_1, t_2 \ldots, t_r) \cap K[|\tau|]
\end{aligned}$$

Therefore ∂_F is lf rational derivation. □

Corollary 4.21. *If F is rational lf automorphism over* $\operatorname{Spec}(B)$ *then $\partial_F = \partial_{F_s} + \partial_{F_u}$ and $F_\partial = F_{\partial_s} F_{\partial_n}$.*

Proof. By the proof above we have $\partial_F = \partial_{F_s} + \partial_{F_u}$. It derivation commute the exponential maps of sum of derivations is the composition of maps associated $F_\partial^* = F_{\partial_s+\partial_n}^* = F_{\partial_s}^* F_{\partial_n}^*$ □

Proposition 4.22. *If ∂ is rational lf then* $\exp(\partial)$ *is rational lf automorphism.*

Proof. Clearly is an homomorphism whose inverse maps is $\exp(-\partial)$. Moreover $(\exp(\tau\partial))^\tau = \exp(\tau\partial)$ is a rational flow because if $\partial(\tau) = \partial(\tau') = 0$, $\exp(\tau\partial)$ and $\exp(\tau\partial)$ fix τ, τ' and the map can be extended over $K(\tau, \tau', t_1, \ldots, t_r, t_1', \ldots, t_r')$ then we have $\exp(\tau\partial) \circ \exp(\tau'\partial) = \exp((\tau + \tau')\partial)$ and $\exp(\tau\partial)|_{\tau=0} = \mathrm{Id}_K$ represent a rational polynomial flow which the image is contained in $K(\tau, t_1, t_2 \ldots, t_r)$ □

Theorem 4.23. *There exist a correspondence between rational semisimple derivations and rational semisimple automorphism over K.*

$$\begin{aligned}
\partial &\rightarrow \exp(\partial) \\
\operatorname{ev}_0 \circ \frac{d}{d\tau} \circ (F^*)^\tau &\leftarrow F
\end{aligned}$$

where ev_0 *consist in evaluate the parameter τ in* 0.

Proof. If ∂ is semisimple then there exist a generating set $\{b_i\}_{i\in I}$ as field such that $\partial(b_i) = \gamma_i b_i$ where $\gamma_i \in k$, $\partial^n(b_i) = \gamma_i^n b_i$ therefore $\exp(\partial)(b_i) = e^{\gamma_i} b_i$. If $b \in K$ given by $b = \dfrac{\sum_\sigma \alpha_\sigma b^\sigma}{\sum_\rho \beta_\rho b^\rho}$ then $\exp(\partial)(b) = \dfrac{\sum_\sigma \alpha_\sigma e^{\gamma\cdot\sigma} b^\sigma}{\sum_\rho \beta_\rho e^{\gamma\cdot\rho} b^\rho}$ where $\gamma \cdot \sigma = \sum_{i=1}^r \gamma_i \sigma_i$ and $\gamma \cdot \rho = \sum_{i=1}^r \gamma_i \rho_i$, which is a semisimple automorphism whose inverse is $\exp(-\partial)$. We have $(\exp(\partial))^\tau = \exp(\tau\partial)$ and for each b_i $\mathrm{ev}_0 \circ \dfrac{d}{d\tau} \circ \exp(\tau\partial)(b_i) = \mathrm{ev}_0 \circ \dfrac{d}{d\tau}(e^{\gamma_i\tau} b_i) = \mathrm{ev}_0(\gamma_i e^{\gamma_i\tau} b_i) = \gamma_i b_i = \partial(b_i)$, we conclude $\mathrm{ev}_0 \circ \dfrac{d}{d\tau} \circ \exp(\tau\partial) = \partial$, i.e $\partial_{F_\partial} = \partial$. Conversely if F is semisimple we have a basis $\{a_i\}_{i\in I}$ such that $F^*(a_i) = \lambda_i a_i$, all element $b \in B$ is written as $b = \dfrac{\sum_\sigma \alpha_\sigma a^\sigma}{\sum_\rho \beta_\rho a^\rho}$ where $a^\sigma = a_1^{\sigma_1} \cdots a_r^{\sigma_r}$, $a^\rho = a_1^{\rho_1} \cdots a_r^{\rho_r}$, $\alpha_\sigma, \beta_\rho \in k$, we have $F^*(b) = b = \dfrac{\sum_\sigma \alpha_\sigma \lambda^\sigma a^\sigma}{\sum_\rho \beta_\rho \lambda^\rho a^\rho}$ where $\lambda^\sigma = \lambda_1^{\sigma_1} \cdots \lambda_r^{\sigma_r}$, $\lambda^\rho = \lambda_1^{\rho_1} \cdots \lambda_r^{\rho_r}$. If we extend the composition over a parameter τ on the generating set, we obtain $(F^*)^\tau(a_i) = \lambda_i^\tau a_i$ and $\mathrm{ev}_0 \circ \dfrac{d}{d\tau} \circ (F^*)^\tau(a_i) = \ln(\lambda_i) a_i$. Therefore $\exp(\mathrm{ev}_0 \circ \dfrac{d}{d\tau} \circ (F^*)^\tau)(a_i) = \lambda_i a_i = F^*(a_i)$ the initial automorphism, therefore $F_{\partial_F} = F$ □

Theorem 4.24. *There exists a correspondence between the rational lf derivation over K and rational lf automorphism on* $\mathrm{Spec}(B)$

$$\begin{array}{ccc} \partial & \to & F_\partial \\ \partial_F & \leftarrow & F \end{array}$$

Proof. To prove this bijective correspondence, 1) $\partial_{F_\partial} = \partial$ and 2) $F_{\partial_F} = F$.

1. Since $\dfrac{d}{d\tau} \circ F^*_{\tau\partial} = F^*_{\tau\partial} \circ \partial$ and $\mathrm{ev}_0 \circ F^*_{\tau\partial} = \mathrm{Id}$ hence
$$\partial_{F_\partial} = \mathrm{ev}_0 \circ \frac{d}{d\tau} \circ F^*_{\tau\partial} = \mathrm{ev}_0 \circ F^*_{\tau\partial} \circ \partial = \partial$$

2. If F^* is an rational lf automorphism, by lemma 4.18 we have the existence of rational polynomial flow $(F^*)^\tau$

$$
\begin{aligned}
(F^*)^\tau(b) &= \frac{\sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot\tau} p_\sigma(\tau) b^\sigma}{\sum_\rho \beta_\rho \gamma^{\rho\cdot\tau} q_\rho(\tau) b^\rho} \quad / \frac{d}{d\tau} \\
\frac{d}{d\tau} \circ (F^*)^\tau(b) &= \frac{\frac{d}{d\tau}(\sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot\tau} p_\sigma(\tau) b^\sigma)(\sum_\rho \beta_\rho \gamma^{\rho\cdot\tau} q_\rho(\tau) b^\rho)}{(\sum_\rho \beta_\rho \gamma^{\rho\cdot\tau} q_\rho(\tau) b^\rho)^2} \\
&- \frac{(\sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot\tau} p_\sigma(\tau) b^\sigma)\frac{d}{d\tau}(\sum_\rho \beta_\rho \gamma^{\rho\cdot\tau} q_\rho(\tau) b^\rho)}{(\sum_\rho \beta_\rho \gamma^{\rho\cdot\tau} q_\rho(\tau) b^\rho)^2} \\
\frac{d}{d\tau} \circ (F^*)^\tau(b) &= \frac{(\sum_\sigma (\ln(\gamma^\sigma)\gamma^{\sigma\cdot\tau} p_\sigma(\tau) + \gamma^{\sigma\cdot\tau}\frac{dp_\sigma(\tau)}{d\tau})\alpha_\sigma b^\sigma)(\sum_\rho \beta_\rho \gamma^{\rho\cdot\tau} q_\rho(\tau) b^\rho)}{(\sum_\rho \beta_\rho \gamma^{\rho\cdot\tau} q_\rho(\tau) b^\rho)^2} \\
&- \frac{(\sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot\tau} p_\sigma(\tau) b^\sigma)(\sum_\rho (\ln(\gamma^\rho)\gamma^{\rho\cdot\tau} q_\rho(\tau) + \gamma^{\sigma\cdot\tau}\frac{dq_\rho(\tau)}{d\tau})\beta_\rho b^\rho)}{(\sum_\rho \beta_\rho \gamma^{\rho\cdot\tau} q_\rho(\tau) b^\rho)^2} \\
\mathrm{ev}_0\, \frac{d}{d\tau} \circ (F^*)^\tau(b) &= \frac{(\sum_\sigma (\ln(\gamma^\sigma) + \frac{dp_\sigma(\tau)}{d\tau}|_{\tau=0})\alpha_\sigma b^\sigma)(\sum_\rho \beta_\rho b^\rho)}{(\sum_\rho \beta_\rho b^\rho)^2} \\
&\frac{-(\sum_\sigma \alpha_\sigma b^\sigma)(\sum_\rho (\ln(\gamma^\rho) + \frac{dq_\rho(\tau)}{d\tau}|_{\tau=0})\beta_\rho b^\rho)}{(\sum_\rho \beta_\rho b^\rho)^2} \\
\mathrm{ev}_0\, \frac{d}{d\tau} \circ (F^*)^\tau(b) &= \frac{\sum_\sigma (\ln(\gamma^\sigma)\alpha_\sigma b^\sigma)(\sum_\rho \beta_\rho b^\rho) - (\sum_\sigma \alpha_\sigma b^\sigma)(\sum_\rho \ln(\gamma^\rho)\beta_\rho b^\rho)}{(\sum_\rho \beta_\rho b^\rho)^2} \\
&+ \frac{(\sum_\sigma \frac{dp_\sigma(\tau)}{d\tau}|_{\tau=0}\alpha_\sigma b^\sigma)(\sum_\rho \beta_\rho b^\rho) - (\sum_\sigma \alpha_\sigma b^\sigma)(\sum_\rho \frac{dq_\rho(\tau)}{d\tau}|_{\tau=0}\beta_\rho b^\rho)}{(\sum_\rho \beta_\rho b^\rho)^2} \\
\partial_F &= \partial_{F_s} + \partial_{F_u}
\end{aligned}
$$

Since the decomposition is unique and $\partial_{F_u} \circ \partial_{F_s} = \partial_{F_s} \circ \partial_{F_u}$ by corollary 4.21, we have $F_{\partial_{F_u}+\partial_{F_s}} = F_{\partial_{F_u}} \circ F_{\partial_{F_s}}$ and we consider for [7] $F_{\partial_{F_u}} = F_u$, and by theorem 4.23 $F_{\partial_{F_s}} = F_s$ Therefore

$$
\begin{aligned}
F_{\partial_F} &= F_{\partial_{F_u}+\partial_{F_s}} \\
F_{\partial_F} &= F_{\partial_{F_u}} \circ F_{\partial_{F_s}} \\
F_{\partial_F} &= F_u \circ F_s \\
F_{\partial_F} &= F
\end{aligned}
$$

□

Corollary 4.25. *Given a ∂ rational lf derivation, we have $K^{F^*_\partial} = \ker(\partial)$.*

Proof. 1. If $b \in K$ an element not zero of invariant ring associated to F^*, $a = F^*(a) = \sum_{i\geq 0} \frac{\partial^i(a)}{i!}$, this implies $\partial^i(a) = 0$ for $i \geq 1$, therefore $a \in \ker(\partial)$, conversely if a is in the kernel of ∂, $\frac{\partial^n(a)}{n!} = 0$ for all $n > 0$ so that $F^*(a) = a$ is an element of invariants ring of F_∂.

□

4.3 Regular case

In the following section, we will show the correspondence of lf elements in a finitely generated k-algebra.

Proposition 4.26. *Let $D \in \mathrm{LFD}(B)$ then the image of exponential map $\exp(\tau D) : B \to B[|\tau|]$ is $B[\tau, t_1, \ldots, t_r]$, where $t = \sum_{j\geq 0} \frac{\tau^j}{j!}$ for some $\gamma_i \in k$ and $t_i = t^{\lambda_i}$.*

Proof. Since D is a lf derivation admits a decomposition $D = D_n + D_s$, where $D_n \in \mathrm{LND}(B)$, $D_s \in \mathrm{SSD}(B)$. We can distinguish three cases to analyze:

- If $D = D_n \in \mathrm{LND}(B)$, for all element $b \in B$ there exist $l \in \mathbb{Z}_{\geq 0}$ such that $D^l(b) = 0$, therefore $\exp(\tau D)(b) = b + D(b)\tau + \dfrac{D^2(b)}{2}\tau^2 + \cdots + \dfrac{D^{l-1}(b)}{l!}\tau^{l-1}$ and the image of exponential map is contained $B[\tau]$.
- If $D = D_s \in \mathrm{SSD}(B)$ there exist a basis of eigenvectors $\{b_i\}_{i\in I}$ such that $D(b_i) = \gamma_i b_i$ hence $D^j(b_i) = \gamma_i^j b_i$. therefore $\exp(\tau D)(b_i) = b_i \sum_{j\geq 0} \dfrac{(\gamma_i \tau)^j}{j!} = b_i t^{\gamma_i}$, where for each i we write $t_i = t^{\gamma_i} := \sum_{j\geq 0} \dfrac{(\gamma_i t)^j}{j!}$. Every element $b \in B$ can be written as $b = \sum_{j=1}^{m} \beta_j b_j$ we can extend for linearity the exponential map for b of the following way $\exp(\tau D)(b) = \sum_{j=1}^{m} \beta_j a_j t_j$, we conclude $\exp(\tau D)(B) \subset B[t_1, \ldots, t_r]$.
- The last case is when $D = D_s + D_s$. The exponential maps $\exp(\tau(D_s + D_n))$ is equal to $\exp(\tau D_s) \circ \exp(\tau D_n) = \exp(\tau D_n) \circ \exp(\tau D_s)$ because $\tau D_s(\tau D_n) = \tau D_s(\tau D_n)$. The exponential map fix τ and also in consequence the t_i's. we can extend the composition as follows

$$\overbrace{B \xrightarrow{\exp(\tau D_n)} B[\tau] \xrightarrow{\exp(\tau D_s)} B[\tau, t_1, \ldots, t_r]}^{\exp(\tau D)}$$

□

Remark 4.27. When we evaluate $\tau = 0$ we obtain $t_i = 1$, and if $\tau = 1$ we obtain $t_i = e^{\gamma_i}$. Also, $\dfrac{d\tau}{d\tau} = 1$ and $\dfrac{dt_i}{d\tau} = \gamma_i t_i$

Corollary 4.28. *Let $D \in \mathrm{LFD}(B)$ then the image of exponential map $\exp(\tau D): K \to K[|\tau|]$ is contain in $K(\tau, t_1, \ldots, t_r)$, where $t_i = \sum_{j\geq 0} \dfrac{(\gamma_i \tau)^j}{j!}$ for some $\gamma_i \in k$ and $1 \leq i \leq r$.*

Proof. if $a, b \in B$ with $b \neq 0$ we have

$$\exp(\tau D)\left(\frac{a}{b}\right) = \exp(\tau D)(ab^{-1}) = \exp(\tau D)(a)\exp(\tau D)(b^{-1}) = \frac{\exp(\tau D)(a)}{\exp(\tau D)(b)}$$

where $\exp(\tau D)(a), \exp(\tau D)(b) \in B[\tau, t_1, \ldots, t_r]$ □

Corollary 4.29. *Let $D \in \mathrm{LFD}(B)$, then $\exp(D) = \exp(\tau D)|_{\tau=1}$*

Proof. For the proposition 4.26 we have $\exp(\tau D)|_{t=1}$ can be factorized as $\exp(D)$

$$\overbrace{B \xrightarrow{\exp(\tau D)} B[\tau, t_1, \ldots, t_r] \xrightarrow{\tau \mapsto 1} B}^{\exp(D)}$$

□

Theorem 4.30. *There exist a correspondence between semisimple derivations and semisimple automorphism over B.*

$$\begin{array}{ccc} D & \to & \exp(D) \\ \mathrm{ev}_0 \circ \dfrac{d}{d\tau} \circ (F^*)^\tau & \leftarrow & F \end{array}$$

where ev_0 consist in evaluate the parameter τ in 0.

Proof. If D is semisimple then there exist a basis $\{b_i\}_{i\in I}$ such that $D(b_i) = \gamma_i b_i$ where $\gamma_i \in k$, $D^n(b_i) = \gamma_i^n b_i$ therefore $\exp(D)(b_i) = e^{\gamma_i} b_i$. If $b \in B$ then $b = \sum_i \alpha_i b_i$ then for linearity $\exp(D)(b) = \sum_i \alpha_i e^{\gamma_i} b_i$, which is a semisimple automorphism whose inverse is $\exp(-D)$. We have $(\exp(D))^\tau = \exp(\tau D)$ and $\mathrm{ev}_0 \circ \dfrac{d}{d\tau} \circ \exp(\tau D)(b_i) = \mathrm{ev}_0 \circ \dfrac{d}{d\tau}(e^{\gamma_i \tau} b_i) = \mathrm{ev}_0(\gamma_i e^{\gamma_i \tau} b_i) = \gamma_i b_i = D(b_i)$, we conclude $\mathrm{ev}_0 \circ \dfrac{d}{d\tau} \circ \exp(\tau D) = D$, i.e $D_{F_D} = D$. Conversely if F is semisimple we have a basis $\{a_i\}_{i\in I}$ such that $F^*(a_i) = \lambda_i a_i$, all element $b \in B$ is written as $b = \sum_i \beta_i a_i$ we have $F^*(b) = \sum_i \beta_i \lambda_i a_i$. If we extend the composition over a parameter τ we obtain $(F^*)^\tau(a_i) = \lambda_i^\tau a_i$ and $\mathrm{ev}_0 \circ \dfrac{d}{d\tau} \circ (F^*)^\tau(a_i) = \ln(\lambda_i) a_i$. Therefore $\exp(\mathrm{ev}_0 \circ \dfrac{d}{d\tau} \circ (F^*)^\tau)(a_i) = \lambda_i a_i = F^*(a_i)$ the initial automorphism, therefore $F_{D_F} = F$ □

Proposition 4.31. *Let $D \in \mathrm{LFD}(B)$, then $\exp(D)\colon B \to B$ is a lf automorphism.*

Proof. We first prove that it is a homomorphism, invertible and then locally finite

1. Since $D^j(a+b) = D^j(a) + D^j(b)$ we have $\exp(D)(a) + \exp(D)(b)$

$$
\begin{aligned}
\exp(D)(a)\exp(D)(b) &= \sum_{j\geq 0} \frac{D^j(a)}{j!} \sum_{l\geq 0} \frac{D^l(b)}{l!} \\
&= \sum_{j+l\geq 0} \frac{1}{j+l} \binom{j+l}{j} D^j(a) D^j(b) \\
&= \sum_{m\geq 0} \frac{1}{m!} \left(\sum_{j+l=m} \binom{j+l}{j} D^j(a) D^j(b) \right) \\
\exp(D)(a)\exp(D)(b) &= \sum_{m\geq 0} \frac{D^m(ab)}{m} = \exp(D)(ab)
\end{aligned}
$$

2. $\exp(D)$ is an homomorphism and $\exp(-D)$ is their inverse.

3. If D_s is semisimple then $\exp(D_s)$ is semisimple and if D_n is locally nilpotent then $\exp(D_n)$ is unipotent, also if D admit a decomposition $D = D_n + D_s$ where $D_s D_n = D_n D_s$ we have the Dunford decomposition is given by $\exp(D) = \exp(D_s + D_n) = \exp(D_s) \circ \exp(D_n)$, it is unique, therefore is locally finite.

□

Theorem 4.32. *There exist a correspondence between lf derivations and lf automorphism over B.*

$$
\begin{aligned}
D &\rightarrow \exp(D) \\
\mathrm{ev}_0 \circ \frac{d}{d\tau} \circ (F^*)^\tau &\leftarrow F
\end{aligned}
$$

where ev_0 *consist in evaluate the parameter* τ *in* 0.

Proof. To prove this bijective correspondence, 1) $\partial_{F_D} = \partial$ and 2) $F_{D_F} = F$.

1. Since $\dfrac{d}{d\tau} \circ F^*_{\tau D} = F^*_{\tau D} \circ D$ and $\mathrm{ev}_0 \circ F^*_{\tau D} = \mathrm{Id}$ hence

$$
D_{F_D} = \mathrm{ev}_0 \circ \frac{d}{d\tau} \circ F^*_{\tau D} = \mathrm{ev}_0 \circ F^*_{\tau D} \circ D = D
$$

2. If $b = \sum_\sigma \alpha_\sigma \gamma^\sigma b^\sigma$ we define $(F^*)^\tau(b) = \sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot\tau} p_\sigma(\tau) b^\sigma$ with $p_i(0) = 1$

$$
\begin{aligned}
(F^*)^\tau(b) &= \sum_\sigma \alpha_\sigma \gamma^{\sigma\cdot\tau} p_\sigma(\tau) b^\sigma \\
\frac{d}{d\tau} \circ (F^*)^\tau(b) &= \sum_\sigma (\ln(\gamma^\sigma)\gamma^{\sigma\cdot\tau} p_\sigma(\tau) + \gamma^{\sigma\cdot\tau}\frac{dp_\sigma(\tau)}{d\tau})\alpha_\sigma b^\sigma \\
\mathrm{ev}_0 \circ (F^*)^\tau(b) &= \sum_\sigma (\ln(\gamma^\sigma) + \frac{dp_\sigma(\tau)}{d\tau}|_{\tau=0})\alpha_\sigma b^\sigma \\
\mathrm{ev}_0 \circ (F^*)^\tau(b) &= \sum_\sigma \ln(\gamma^\sigma)\alpha_\sigma b^\sigma + \sum_\sigma \frac{dp_\sigma(\tau)}{d\tau}|_{\tau=0}\alpha_\sigma b^\sigma \\
\mathrm{ev}_0 \circ (F^*)^\tau(b) &= D_{F_s}(b) + D_{F_n}(b)
\end{aligned}
$$

Hereinafter

$$F^*_{D_F} = \exp(D_F) = \exp(D_{F_s} + D_{F_u}) = \exp(D_{F_s})\exp(D_{F_u}) = F^*_s F^*_u = F^*$$

, therefore $F_{D_F} = F$

$\square$

Nowicki theorem showed "if G is a connected algebraic group which acts algebraically on the polynomial ring B, then there exists $D \in \mathrm{Der}(B)$ with $\ker(D) = B^G$. In particular, this means B^G is an algebraically closed subring of B".

Corollary 4.33. *Let $B = k^{[n]}$ the polynomial ring in n variables, $D \in \mathrm{LFD}(B)$*

There exist a locally finite derivation D such that $\ker(D) = A$ if only if there exist an algebraic group G such that $B^G = A$

Proof. If D is lf we have F_D is an algebraic automorphism and for theorem 0.12.1 in [9] there exist $G = \overline{\langle F_D \rangle} \subset \mathrm{Aut}(B)$ such that $\ker(D) = B^G$. If $B^G = A$, since we have $G \hookrightarrow \mathrm{Aut}(B)$, we choose some $g \in G$, $D = \mathrm{ev}_0 \circ \frac{d}{d\tau} \circ g^\tau$ where $\ker(D) = A$

$\square$

Corollary 4.34. *Let $D, D_1, D_2 \in \mathrm{LFD}(B)$, $\varphi \in \mathrm{Aut}(B)$, we have the following facts:*

1. *If D_1, D_2 are commutative, then $D_1 + D_2$ is locally finite and $F^*_{D_1+D_2} = F^*_{D_1} \circ F^*_{D_2}$.*
2. *$F^*_D \in \mathrm{LFAut}(B)$ and $(F^*)^{-1}_D = F^*_{-D}$.*
3. *$H = \{F^*_D | D \in \mathrm{LFD}(B)\}$ then $\langle H \rangle$ is a normal subgroup of $\mathrm{Aut}(B)$.*
4. *$F^*_D \circ D = D \circ F^*_D$ and $\frac{d}{d\tau} \circ F^*_{\tau D} = F^*_{\tau D} \circ D$.*

5. $\ker(D) = B^{F_D^*}$.

6. *If D is locally finite, this can be obtained by differentiating $F_{\tau D}^*$ and then evaluating at τ at zero, i.e.*

$$D = \left(\frac{F_{\tau D}^* - \mathrm{Id}}{\tau}\right)\Big|_{\tau=0}$$

. In particular if If D is locally nilpotent we have the equivalence $D = \log(I + F^*) = \sum_{i \geq 1} (-1)^{i+1} \frac{(F_D^* - \mathrm{Id})^i}{i}$.

7. *F_D is linearizable if only if D is linearizable.*

8. *$B^D = B^{D_s} \cap B^{D_n}$*

Proof. 1.

$$\begin{aligned}
F_{D_1}^* \circ F_{D_2}^* &= \exp D_1 \circ \exp D_2 \\
&= \sum_{j \geq 0} \frac{D_1^j}{j!} \circ \sum_{r \geq 0} \frac{D_2^r}{r!} \\
&= \sum_{j \geq 0} \sum_{r \geq 0} \frac{1}{j!r!} D_1^j \circ D_2^r \\
&= \sum_{m \geq 0} \frac{1}{m!} \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} D_1^l \circ D_2^{m-l} \\
&= \sum_{m \geq 0} \frac{(D_1 + D_2)^m}{m!} \\
F_{D_1}^* \circ F_{D_2}^* &= \exp(D_1 + D_2) = F_{D_1 + D_2}^*
\end{aligned}$$

2. Since $D = D_s + D_n$ and commute, and we have $F_{D_s}^*$ is a semisimple automorphism and $F_{D_n}^*$ unipotent we have F_D is lf automorphism . Since $D(-D) = (-D)D$ we have $F_D^* \circ F_{-D}^* = F_{-D}^* \circ F_D^* = F_0^* = \mathrm{Id}_B$

3. Let $\varphi \in \mathrm{Aut}(B)$, since $\varphi \circ D^j \circ \varphi^{-1} = (\varphi D \varphi^{-1})^j$, we have the exponential maps satisfy $\varphi F_D^* \varphi^{-1} = F_{\varphi D \varphi^{-1}}^*$ and the conjugation of lf derivations is a lf derivation $\varphi D \varphi^{-1} \in \mathrm{LFD}(B)$ then $\varphi \circ F_D^* \circ \varphi^{-1} \in H$ and therefore is in $\langle H \rangle$.

4.

$$D \circ F_D^* = D \circ \sum_{j \geq 0} \frac{D^j}{j!} = \sum_{j \geq 0} \frac{D^{j+1}}{j!} = \sum_{j \geq 0} \frac{D^j}{j!} \circ D = F_D^* \circ D$$

and

$$\frac{d}{d\tau} \circ F^*_{\tau D} = \frac{d}{d\tau} \circ \sum_{j \geq 0} \tau^j \frac{D^j}{j!} = \sum_{j \geq 1} \tau^{j-1} \frac{D^j}{j-1!} = \sum_{j \geq 0} \tau^j \frac{D^{j+1}}{j!} = F^*_{\tau D} \circ D$$

5. If $b \in \ker(D)$ for $j \geq 1$ we have $D^j(b) = 0$ thus $\sum_{j \geq 0} \frac{D^j(b)}{j!} = b$ this implies $\ker(D) \subset B^{F^*_D}$. Conversely if $b \in B^{F^*_D}$ we have $F_D(b) = b$, this means $D^j(b) = 0$ for $j \leq 1$, in particular for $j = 1$ then $B^{F^*_D} \subset \ker(D)$.

6.

$$\begin{aligned}
F^*_{\tau D} &= \sum_{j \geq 0} \tau^j \frac{D^j}{j!} \quad / - D^0 \\
F^*_{\tau D} - \mathrm{Id} &= \sum_{j \geq 1} \tau^j \frac{D^j}{j!} \quad / \cdot \frac{1}{\tau} \\
\frac{F^*_{\tau D} - \mathrm{Id}}{\tau} &= \sum_{j \geq 1} \tau^{j-1} \frac{D^j}{j!} \quad / \, \mathrm{ev}_0 \\
\left(\frac{F^*_{\tau D} - \mathrm{Id}}{\tau} \right) \Big|_{\tau = 0} &= D
\end{aligned}$$

The proof of $D = \log(I + F) = \sum_{i \geq 1} (-1)^{i+1} \frac{(F^*_D - \mathrm{Id})^i}{i!}$ can be seen in Van de Essen book [22].

7. Using the fact $F_{\varphi D \varphi^{-1}} = \varphi F_D \varphi^{-1}$, we can conclude if D is linearizable then $\varphi D \varphi^{-1}$ is linear and $F_{\varphi D \varphi^{-1}}$ is linear and thus $\varphi F \varphi^{-1}$, the same applies if F_D is linearizable.

8. The proof is in Corollary 2.3 [15]

□

4.3.1 Examples

Example 4.35. Let F be the automorphism

$$\begin{aligned}
F \colon \mathbb{C}^2 &\to \mathbb{C}^2 \\
(x, y) &\mapsto (4x + 4y^2, 2y)
\end{aligned}$$

This maps can be decomposed as $(4x+4y^2, 2y) = (4x, 2y) \circ (x+y^2, y)$ where $F_u = (x+y^2, y)$ is unipotent and $F_s = (4x, 2y)$ is semisimple and both commute.

- $F_s^\tau = (4^\tau x, 2^\tau y)$ then $D_{F_s} = \ln(4) x \frac{\partial}{\partial x} + \ln(2) y \frac{\partial}{\partial y}$.

- $F_u^\tau = (x+\tau y^2, y)$ then $D_{F_u} = y^2\dfrac{\partial}{\partial x}$.

Therefore $F_{D_s} = F_s$, $D_{F_s} = D_s$ and $D_{F_u} = D_n$, $D_{F_s} = D_s$ and

$$D = (\ln(4)x + y^2)\frac{\partial}{\partial x} + \ln(2)y\frac{\partial}{\partial x}$$

Example 4.36. In the linear case, the derivation and automorphism are lf, one very obvious reason is due to the existence of the characteristic polynomial. If $M \in \mathrm{GL}_n$, it is diagonalizable then there exist $P, D \in \mathrm{GL}_n$ such that $M = PDP^{-1}$ where $D = \mathrm{diag}(a_1, \dots, a_n)$ is a diagonal matrix, then

$$\begin{aligned}
(PDP^{-1})^\tau &= PD^\tau P^{-1} \\
&= P\,\mathrm{diag}(a_1^\tau, \dots, a_n^\tau)P^{-1} \quad /\frac{d}{d\tau} \\
\frac{d}{d\tau}((PDP^{-1})^\tau) &= P\,\mathrm{diag}(\ln(a_1)a_1^\tau, \dots, \ln(a_n)a_n^\tau)P^{-1} \quad /\,\mathrm{ev}_0 \\
\mathrm{ev}_0\,\frac{d}{d\tau}((PDP^{-1})^\tau) &= P\,\mathrm{diag}(\ln(a_1), \dots, \ln(a_n))P^{-1}
\end{aligned}$$

The derivation conjugate to linear automorphism is $P\,\mathrm{diag}(\ln(a_1)x_1\dfrac{\partial}{\partial x_1} \dots + \ln(a_n)x_n\dfrac{\partial}{\partial x_n})P^{-1}$.

Conversely if $A \in M_n$ $A = N + S$ where $NS = NS$, S is diagonazable and N is nilpotent, then $\exp(A) = \exp(M+N) = \exp(M)\exp(N)$ is a linear automorphism.

Example 4.37. Let $D \in \mathrm{SSD}(k^{[3]})$ given by $D = x\dfrac{\partial}{\partial x} + (f(x,y) - x\dfrac{\partial f(x,y)}{\partial x} + z)\dfrac{\partial}{\partial z}$ where $f(x,y) \in k[x,y]$. We compute F_D where it is sufficient to calculate it for x, y and z. We obtain $F_D(x) = ex$, $F_D(y) = y$. By above we have $F_D(f(x,y)) = f(F_D(x), F_D(y)) = f(ex, y)$ and as $D(z + f(x,y))$,

$$\begin{aligned}
F_D(z) &= F_D(z + f(x,y) - f(x,y)) \\
&= F_D(z + f(x,y)) - F_D(f(x,y)) \\
&= e(z + f(x,y)) - f(ex, y) \\
F_D(z) &= ez + ef(x,y) - f(ex,y)
\end{aligned}$$

Example 4.38. We consider the following flow:

$$\begin{aligned}
F_{\tau D}\colon k \times \mathbb{A}^2 &\to \mathbb{A}^2 \\
(\tau, (x,y)) &\mapsto (\cos(\tau)x - \sin(\tau)y, \sin(\tau)x + \cos(\tau)y)
\end{aligned}$$

is quasi-algebraic linear polynomial flow. We have a not algebraic action give by

$$F(x,y) = \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

If we derivate respecto to τ we obtain

$$\frac{d}{d\tau} F(x,y) = \begin{bmatrix} -\sin(\tau) & -\cos(\tau) \\ \cos(\tau) & -\sin(\tau) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

Finally we evaluate τ in 0 and obtain $D_F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ equivalent to the derivation

$$D_F = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Note that de action is quasi algebraic and the derivation associated is locally finite.

Example 4.39. Is known the Lie algebra $\mathfrak{sl}_2$ is generated by the matrices

$$D_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } D_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

every one can be related with the derivations

$$\mathfrak{sl}_2 \simeq \langle Y\frac{\partial}{\partial X}, X\frac{\partial}{\partial Y}, X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}\rangle = \mathbb{C}Y\frac{\partial}{\partial X} \bigoplus \mathbb{C}X\frac{\partial}{\partial Y} \bigoplus \mathbb{C}(X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})$$

and the algebraic group is SL_2 is generated by $F_{D_1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, F_{D_2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $F_{D_3} = \begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}$

$$\mathrm{SL}_2 = \langle (X, Y+X), (X+Y, Y), (eX, e^{-1}Y) \rangle$$

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Bibliography

[1] Bass, Hyman and Meisters, Gary, *Polynomial flows in the plane*, Adv. in Math., Advances in Mathematics, 55, 1985, 2, 173–208.

[2] Luis Cid and Alvaro Liendo, *On rational multiplicative group actions*,
2022, 2208.05024, arXiv, math.AG

[3] B.A Coomes , *Polynomial flow on* $\mathbb{C}^n$, Trans. Amer. Math. Soc, 320, 493-506, 1985

[4] Coomes, Brian and Zurkowski, Victor, *Linearization of polynomial flows and spectra of derivations*, J. Dynam. Differential Equations, Journal of Dynamics and Differential Equations, 3, 1991, 1, 29–66, 1040-7294.

[5] Daigle, Daniel, *Locally nilpotent derivations*, 2003, Lecture notes for the ""September School"" of Algebraic Geometry Łukęcin, Poland, September

[6] Michel Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup. (4), 1970, 3, 507–588, Annales Scientifiques de l'École Normale Supérieure. Quatrième Série.

[7] Dubouloz, Adrien and Liendo, Alvaro, *Rationally integrable vector fields and rational additive group actions*, Internat. J. Math., International Journal of Mathematics, 27, 2016, 8, 1650060, 19, 0129-167X.

[8] Freudenburg, Gene, *Algebraic theory of locally nilpotent derivations*,Encyclopaedia of Mathematical Sciences,136, Second,Invariant Theory and Algebraic Transformation Groups, VII, Springer-Verlag, Berlin, 2017.

[9] Furter Jean-Philippe, and Kraft Hanspeter, *On the geometry of the automorphism groups of affine varieties*, 2018, 1809.04175

[10] Goodman, Jacob Eli and Landman, Alan, *Varieties proper over affine schemes*, Invent. Math., Inventiones Mathematicae, 20, 1973, 267–312.

[11] Gutwirth, A.,*The action of an algebraic torus on the affine plane*,Trans. Amer. Math. Soc., Transactions of the American Mathematical Society, 105, 1962, 407-414, 0002-9947,

[12] Koševoĭ, È. G., Birational representations of a multiplicative and an additive group, Sibirsk. Mat. Ž., Akademija Nauk SSSR. Sibirskoe Otdelenie. Sibirskiĭ Matematičeskiĭ Žurnal, 8, 1967, 1339–1345.

[13] H. Matsumura, On algebraic groups of birational transformations, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur (8), 1963, 34, 151–155.

[14] Nowicki, Andrzej,*Polynomial derivations and their rings of constants*, Uniwersytet Mikołaja Kopernika, Toruń, 1994.

[15] Nowicki, Andrzej and Tyc, Andrzej, *On semisimple derivations in characteristic zero*, Comm. Algebra, Communications in Algebra, 29, 2001, 11, 5115–5130.

[16] Nowicki,Andrzej , Some remarks on polynomial flows and locally finite derivations , J. Lie Theory, The lecture at International Workshop on Affine Algebraic Geometry 8 14 December 1993, Technion, Haifa, Israel , 2003, 2, 116(3):861-861

[17] Perrin, Daniel, *Algebraic geometry*, Springer Verlag London, Ltd., London, EDP Sciences, Les Ulis, 2008.

[18] Popov, Vladimir L., Birational splitting and algebraic group actions, Eur. J. Math., European Journal of Mathematics, 2, 2016, 1, 283–290, 2199-675X.

[19] Rentschler, Rudolf *Opérations du groupe additif sur le plan affine*, C. R. Acad. Sci. Paris Sér. A-B,

[20] Rosenlicht, Maxwell,Some basic theorems on algebraic groups, Amer. J. Math., American Journal of Mathematics, 78, 1956, 401–443.

[21] Shulim Kaliman and Mariusz Koras and Leonid G. Makar-Limanov and Peter Russell *$\mathbb{C}^*$-actions on $\mathbb{C}^3$ are linearizable*, Electronic Research Announcements of The American Mathematical Society, 1997, 3, 63-71

[22] Van den Essen, Arno, Polynomial automorphisms and the Jacobian conjecture, Progress in Mathematics, Birkhäuser Verlag, Basel,2000.

[23] Van den Essen,Arno , *Locally finite and locally nilpotent derivations with applications to polynomial flows and polynomial morphisms* , J. Lie Theory, Proceedings of the American Mathematical Society , 5,1992,2, 116(3):861-861,