Classical and semiclassical aspects of black holes

PhD Thesis

Olaf Baake

carried out at the Instituto de Matemáticas (INSTMAT)
and at the Centro de Estudios Científicos (CECs)

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Mathematics at the Universidad de Talca

September 2022

1. Referee:
Prof. Dr. Mokhtar Hassaïne
Instituto de Matemática y Física
Universidad de Talca

2. Referee:
Prof. Dr. Jorge Zanelli
Centro de Estudios Científicos (CECs), Valdivia

Abstract

This thesis documents our studies of different topics in black hole physics. It is split into three parts, the first part, is devoted to classical general relativity without any extensions or modifications. The second part then includes classical extensions, that is, without considering the conceptual ideas and without introducing quantum effects. In the last part of the thesis we study the semi-classical approach to gravity where we quantize the matter fields.

In the classical gravity chapter, we present a method of regularizing affine-null metric equations coupled to a scalar field and subsequently solving them using pseudo-spectral methods. Our code is entirely written in Python, which makes it accessible to anyone with a bit of experience of programming. With relatively low numerical resolution and small runtime on a standard laptop, the code allows reproducing characteristic features like mass scaling and echoing of the Choptuik critical solution. We further show that for our initial data, the time of formation of the critical solution is linearly related to one of its parameters.

The second part studies classical extensions to general relativity, and we present the three projects we have done that lead to publications. The studies are all carried out in the framework of degenerate higher order scalar tensor (DHOST) theories.

For example, we construct regular, asymptotically flat black holes using a generalized Kerr-Schild solution generating method. The solutions depend on a mass integration constant and admit a smooth core of chosen regularity. There are two types of solutions: those that have an inner and outer event horizon, and, particle-like solutions that are horizonless yet regular. Where possible, we study observational effects and compare them to the standard solutions of general relativity.

In the quantum chapter, we analytically study the semiclassical backreaction of a conformally coupled scalar field on an overspinning BTZ geometry. In particular, we extend the work that has been done on this problem before, by considering so-called overspinning geometries. These are naked singularities, where the angular momentum is greater than a factor of the mass, which classically prevents the solution of having an event horizon. Using the renormalized quantum stress-energy tensor for a conformally coupled scalar field on such a space-time, we obtain the semiclassical Einstein equations, which then can be solved perturbatively. Our results show that the quantum back-reacted solution contains an event horizon hiding all appearing singularities, in agreement with the (weak) cosmic censorship conjecture.

Acknowledgements

It is my pleasure to express my gratitude to the many people who have supported me during my time as a PhD student. First and foremost, I am extremely grateful to my two supervisors, Mokhtar Hassïne and Jorge Zanelli. Their encouragement and advice throughout my PhD, together with the numerous discussions we had, and their general support, were of considerable value. I want to take this opportunity and further thank Mokhtar Hassïne for is additional support outside of science, which greatly helped me with my transition to Chile.

Further, I would like to thank various people for the many interesting conversations and discussions we had. These include in particular Thomas Mädler, who is currently working with me on a numerical project and helped me a great deal with Python, but also other topics. I gratefully acknowledge the enlightening discussions I had with Christos Charmousis while we were working on a project together. During the period of the classes, I also had several interesting conversations with Ricardo Baeza, for which I would also like to express my gratitude.

In general, I want to say a special thank you to Eugeny Babichev, Christos Charmousis, Thomas Mädler, Ricardo Baeza, Mokhtar Hassïne and Jorge Zanelli for joining the thesis committee and for their assistance in the process of the defense.

I also want to thank my friends here in Talca and Valdivia, who made my personal life a lot more interesting. Particularly, I would like to thank Miguel San Juan for our discussions and our work together.

Several times during my PhD I visited the Centro de Estudios Científicos (CECs), and I am genuinely thankful for their repeated hospitality. In particular, I would like to thank Patricia Fernandoy, who always made sure that my visits go as smooth as possible.

Additionally, I want to gratefully acknowledge the funding through scholarship of the University of Talca, that made my PhD possible.

Last but not least, I am truly thankful for the constant support throughout all my life that I received from my family, in particular my parents.

Contents

Conventions

If not stated otherwise, the speed of light, c, the Newton constant, G_N, and the Boltzmann constant k_B are set to 1.

We write partial derivatives like $f_{,\mu} = \partial_\mu f = \frac{\partial f}{\partial x^\mu}$, or if f depends on some function, say $X(y)$, we write $f_{,X} = \frac{\partial f}{\partial X}$. Further, in the same way we write a ; to denote a covariant derivative, i.e. $V_{\nu;\mu} = \nabla_\mu V_\nu$. In the case of a scalar field, we may even, if there is no potential for confusion, omit the comma and just write something like $\phi_{\mu\nu}$ instead of $\phi_{;\mu\nu}$.

We use $(-,+,+,+,\cdots)$ as the signature of the metric.

Riemann tensor: $R^\mu_{\nu\lambda\rho} = \Gamma^\mu_{\nu\rho,\lambda} - \Gamma^\mu_{\nu\lambda,\rho} + \Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\nu\rho} - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\nu\lambda}$.

Einstein tensor: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.

We use $\dot{f} = \frac{\partial f}{\partial t}$ for the time derivative and $f' = \frac{\partial f}{\partial x}$ for any other variable (if f at most depends on t and x).

Levi-Civita tensor (totally antisymmetric): $\epsilon^{\mu\nu\rho\lambda\cdots}$, with $\epsilon^{0123\cdots} = -1/\sqrt{-\det g}$

Introduction and overview

From Newton to Einstein and beyond

In the field of gravity, arguably the most intriguing objects are black holes. In their vicinity gravitational effects are most extreme and hence they provide an outstanding theoretical playground for the different theories of gravity. Additionally, the recent groundbreaking observational results (namely the detection of gravitational waves [1] and the first direct image of the shadow of a black hole [2–9]) show that black holes are not just an important part of theoretical physics anymore, but also provide more windows to test theories of gravity now. This in turn gives rise to many interesting theoretical studies that improve our understanding of black holes and gravity in general.

Black holes have been theorized long before their observational discovery, though their mathematical description changed throughout the years, mainly because Einstein's theory of general relativity (GR) drastically changed the mathematical formulation of gravity and the way we think about it. In Newtonian physics, we treat gravity as a force between two massive objects that is proportional to their masses, and the inverse square of their distance:

$$F = G_N \frac{m_1 m_2}{r^2},$$

with Newton's gravitational constant G_N. In this framework, one can equate potential and kinetic energy in the gravitational field of an object to calculate its escape velocity, the velocity needed to escape from its surface to infinity, given by

$$v_e = \sqrt{\frac{2G_N M}{r}},$$

where r is the radius of its surface. However, we can turn this around and calculate, given a certain mass, what the radius of the object would have to be so that the escape velocity is the speed of light:

$$r_S = \frac{2G_N M}{c^2}.$$

In other words, for a sufficiently dense object, the escape velocity on its surface is so high that not even light can escape. Interestingly, even though general relativity treats gravity entirely different, this very result appears there as well. In the early 1900s, Einstein realized through a series of thought experiments that locally, there is an equivalence between gravity and acceleration [10]. After years of working this out, he concluded that space-time must be a smooth 4-dimensional manifold. The relation between curvature and matter is encoded in the field equations of general relativity,

$$G_{\mu\nu} = \kappa T_{\mu\nu},$$

where the *Einstein tensor*, $G_{\mu\nu}$, codifies the geometry, and the *energy momentum tensor* captures the dynamic of the matter interacting with the gravitational field. This was later famously summarized by Wheeler as "Spacetime tells matter how to move; matter tells spacetime how to curve."[11] A breakthrough came when the first exact solution to the field equations was discovered by Schwarzschild [12]

$$ds^2 = -\left(1 - \frac{2G_N M}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2G_N M}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2,$$

which is a spherically symmetric and static metric. One can see that the same radius as before, r_S, leads to a coordinate singularity of the metric[1]. It can be shown that this radius defines the boundary to the region from which no signal can escape anymore, the *event horizon* of the metric. Over the years more complex black hole solutions to the Einstein equations were found, most prominently, the charged Reissner-Nordström [13, 14], and the rotating Kerr solution [15]. Interestingly, all these solutions can be completely characterized by their mass, charge and angular momentum, which is the famous theorem of black holes having *no hair* [16–18]. This is to say that black holes do not present any information about what they are made up of, or what kind of matter is hiding behind the event horizon. This is inherently different from any other physical object, and one may question whether black holes exist in nature or if they are mere exotic solutions that have no physical meaning.

The observational evidence clearly indicates that black holes are not just a theoretical artifact arising as solutions to the field equations, but real astrophysical objects. This begs the question of how they form in nature. There must be a physical mechanism that is responsible for their creation. One such mechanism is that of the gravitational collapse. In other words, if the gravity of an object is strong enough to overcome its internal pressure, then it can collapse into a region of such high density that it is contained within its event horizon. Interestingly, in 1993 Choptuik studied such a collapse for the case of a massless scalar field propagating in a dynamical space-time using the Einstein-Klein-Gordon equations [19] and discovered a surprisingly simple behavior in relation to the initial data reminiscent of critical behavior in phase transitions of statistical physics. He realized that, for any one-parameter family of solutions (for example the amplitude of the initial field), there is a critical threshold value for this parameter that divides the initial data into solutions that collapse to a black hole, and those that do not. For example, considering the parameter p, there is a value p_* such that, if we choose $p > p_*$, then the initial data collapses to a black hole, while for $p < p_*$ the scalar field completely disperses, resulting in flat space as the end state. He also found that near the threshold, the masses of the resulting black holes follow a power law with a critical exponent that is independent of the parameter under consideration. After Choptuik discovered these critical phenomena in numerical relativity, there have been numerous numerical and analytical studies of different matter sources, finding similar results in spherical symmetry, and some in axial symmetry, though there is still a lot that is not fully understood yet. For example, how universal critical phenomena are regarding different matter types and beyond spherical symmetry. But with increasing computational power, and a better theoretical knowledge that has been obtained over the years, there is no doubt that our understanding of the subject will continue to improve. More details about critical collapse can be found in section 4 of chapter I.

Another aspect, that is common among all black hole solutions in the theory of general relativity,

[1] The radius was given the name *Schwarzschild radius*, named after Karl Schwarzschild.

is that they possess a singularity, at least as long as the matter satisfies reasonable energy conditions [20, 21]. This is a strong indication for a breakdown of Einstein's theory, at least close to these singularities, and cries out for new physics beyond GR. Fortunately, black holes have an event horizon, which prevents that information gets out from within the horizon. Therefore, one could argue that what happens inside the horizon is totally irrelevant, since we will never be able to interact with the inside, and it will never influence any physics that happens outside. While this may help us sleep at night, we cannot ignore the troublesome fact that the Einstein equations do not ensure the appearance of an event horizon, they admit (pathological) solutions of *naked singularities*. Even the simple case of the Schwarzschild metric is not safe: just choose the integration constant M to be negative, and the horizon is gone. Near such a naked singularity the predictability of GR is completely broken, and further, it may even be possible to violate causality. This lead Roger Penrose to conjecture the (weak) cosmic censorship hypothesis [22], which states that singularities must be hidden behind an event horizon. Unfortunately, there is no formal proof, one can even fine-tune the initial data of a collapse such that the end state is a naked singularity. This indicates that, at least in the theory of GR, it is not even possible to construct such a proof. On the other hand, so far, there is no observational evidence that such naked singularities exist in nature, so one may still hope that nature upholds it. This motivates the modification of GR to either get rid of the singularities, or at least make sure they are always hidden behind an event horizon. We will discuss possible modifications to GR in chapter II.

Another ray of hope could be the inclusion of quantum effects. After all, when matter is compressed so densely, we cannot expect that quantum theory does not play a role. Further, it is a well-known fact, that black holes behave like thermodynamic objects, which is another indication that there is an underlying theory of quantum gravity (see 2 of chapter I). Unfortunately, it is known that we cannot simply quantize general relativity and treat it as a quantum field theory. Instead, there is the need for a new, more profound theory that includes both, gravity and quantum theory. The most prominent candidates are probably string theory and loop quantum gravity. With today's technological possibilities, we are not able to reach the energies necessary to provide experimental evidence for these theories. It is therefore impossible to know whether we are on the right track or not. But there are still promising results coming from the studies of semiclassical gravity, or quantum field theory in curved space-times, where we describe gravity in the framework of a classical theory, but quantize the interacting matter fields, which will be studied in chapter III.

We organize the thesis in the following way. There are three main chapters in which we present different projects in black hole physics that we are/were working on. In all our projects, we include a scalar field into the field equations, though in each chapter the sourcing field has a different interpretation. The first chapter is purely classical, we start with a brief review of some topics that are relevant for the later part of the thesis, but using only the framework of general relativity. We then present an attempt to construct black hole solutions with torsion and explain the difficulty of achieving this. In this project, the scalar field is used as some kind of Lagrange multiplier, which results in a change of the geometry in such a way that the torsion becomes nonzero. The chapter concludes with a work in progress about the critical collapse of a scalar field that is done in collaboration with Thomas Mädler. The scalar field is a Klein-Gordon field, which can be interpreted as bosonic matter. With relatively low computation power, we can reproduce characteristic properties of the critical solutions, and even find some other surprising features of our initial data.

The second chapter studies classical modifications to GR. It contains a brief introduction and followed by a review of the *Degenerate Higher Order Scalar Tensor* (DHOST) theory that we have worked with in our projects. In these theories, the scalar field serves as a modification to general relativity; instead of gravity being solely described by a tensor, the metric tensor, gravitational interactions additionally include a scalar field. From this perspective, one can view it as something entirely different from a regular matter field. Our studies were mainly concerned with the construction of black hole solutions to this theory, which lead to three publications [23–25], which we present at the end of that chapter. We were able to show that, under certain reasonable assumptions, DHOST theories in three dimensions only allow for a BTZ-like solution, for which we then compute its thermodynamic properties. We were further able to find a non-singular black hole, and a particle-like solution with some fascinating features. Moreover, we present how to construct rotating stealth black holes with a metric given by the Myers-Perry metric with equal angular momenta. The scalar field generates a disformal transformation leaving the metric invariant which can be generalized to certain cases.

The third chapter reviews the problems that arise when one tries to quantize gravity and introduces semi-classical gravity, that is, classical gravity with quantized matter sources. We present a work in progress in collaboration with Jorge Zanelli about the quantum backreaction of a quantized scalar field conformally coupled to the Einstein equations in $(2+1)$-dimensions. We show that in our case, the consideration of quantum effects ensures the existence of an event horizon. At the end, we present a conclusion and an outlook to the topics covered.

Chapter I

Classical gravity

This chapter starts with a brief review of the BTZ solution and black hole thermodynamics. Then, after outlining the first order formalism of GR, an attempt to construct black hole solutions with torsion is studied. The chapter concludes with a work in progress on critical collapse that is done in collaboration with Thomas Mädler.

1 The BTZ solution

In 1992, Máximo Bañados, Claudio Teitelboim, and Jorge Zanelli (BTZ) showed that $(2+1)$-dimensional gravity admits a black hole solution when the cosmological constant is not zero[26]. The vacuum field equations with (negative) cosmological constant, $\Lambda = -l^{-2}$, are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{l^2} g_{\mu\nu}, \tag{1.1}$$

from which immediately follows that any solution has constant curvature, $R = -6l^{-2}$ and hence

$$R_{\mu\nu} = -\frac{2}{l^2} g_{\mu\nu}. \tag{1.2}$$

Generally, in $(2+1)$ dimensional gravity, the curvature tensor can be completely written in terms of the Ricci tensor, the Ricci scalar, and the metric, that is,

$$R_{\mu\nu\lambda\rho} = g_{\mu\lambda} R_{\nu\rho} + g_{\nu\rho} R_{\mu\lambda} - g_{\nu\lambda} R_{\mu\rho} - g_{\mu\rho} R_{\nu\lambda} - \frac{1}{2} \left(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda} \right) R. \tag{1.3}$$

Consequently, the Weyl tensor vanishes, and hence the space-time is conformally flat. The solution to the field equations in Schwarzschild-like coordinates reads

$$\mathrm{d}s^2 = -\left(\frac{r^2}{l^2} - M \right) \mathrm{d}t^2 - J \mathrm{d}t \mathrm{d}\theta + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2, \tag{1.4}$$

where M and J are integration constants that can be interpreted as the mass and angular momentum, respectively, and its coordinate ranges are: $-\infty < t < \infty$, $0 < r < \infty$ and $0 \leq \theta < 2\pi$, $\Lambda = -l^{-2}$. The solution is surprisingly similar to its $(3+1)$-dimensional counterpart, the Kerr solution, to name a few:

- It is fully classified by its mass and angular momentum parameters.
- There is an event horizon, an inner horizon and an ergosphere (see below).
- It occurs as an endpoint of a gravitational collapse [27].
- Its entropy follows an area law, and it has a non-vanishing Hawking temperature (see next section).

Though, in contrast, the BTZ metric does not have a curvature singularity at $r = 0$ (see below). The roots of $g^{rr} = 0$ determine the horizons of the BTZ metric and are given by

$$r_\pm = l \left[\frac{M}{2} \left(1 \pm \sqrt{1 - \left(\frac{J}{Ml} \right)^2} \right) \right]^{1/2} \tag{1.5}$$

Clearly, there can only be a horizon for

$$M > 0, \qquad |J| \leq Ml, \tag{1.6}$$

which defines the black hole solution. For $|J| = Ml$, we refer to it as an extremal black hole, where both horizons coincide. Now, for when either one of the conditions in (1.6) is violated, we do not have a black hole, but a naked singularity. We shall classify the remaining regions into (see [28] and references therein):

$$M < 0, \qquad |J| \leq Ml, \tag{1.7}$$

constitutes a *conical singularity*, while for,

$$-|J| \leq Ml \leq |J| \tag{1.8}$$

we call it an *overspinning geometry*, or *overspinning naked singularity*, as depicted in figure 1.1. Note that for $J = 0$, we have two special cases. Namely, $M = 0$, the *massless vacuum BTZ geometry*, and $M = -1$, which is the regular AdS vacuum. This becomes relevant for our study of quantum backreactions on the BTZ space-time in the last chapter.

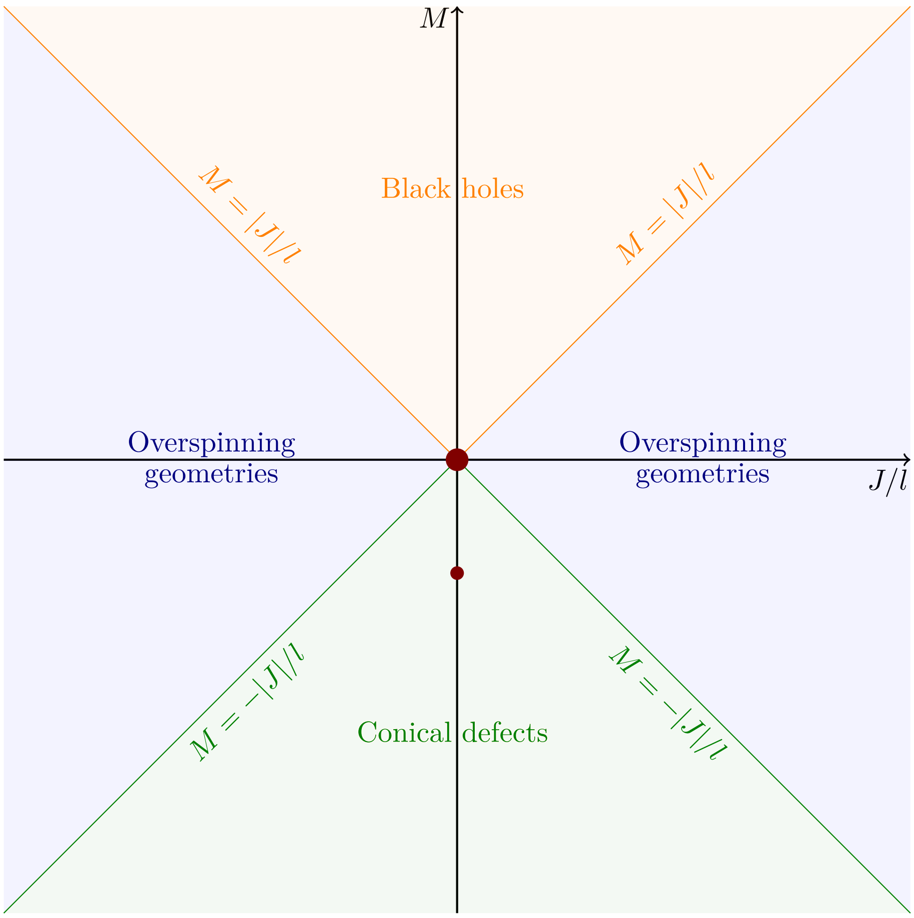


Figure 1.1: BTZ space-times in the J/l-M-plane, with their classification into different regions. The red circles denote the massless BTZ geometry at the origin, and the AdS vacuum at $M = -1$.

2 Black hole thermodynamics

In 1972 Hawking proved that the area of an event horizon never decreases [29]. This is a characteristic famously shared by the entropy and, due to its importance in thermal physics, given the catchy name *second law of thermodynamics*. This result was the first in a series of discoveries relating laws of black hole physics to those of classical thermodynamics. It may therefore be seen as the birth of black hole thermodynamics. The same year, Bekenstein conjectured a more specific relation between the area of the event horizon and a black hole's entropy [30]

$$\mathcal{S} = \eta \frac{A_{\text{hor}}}{\hbar G_N}, \tag{2.1}$$

where η is a proportionality constant. He achieved this through considering thought experiments of matter falling into black holes, annihilating entropy outside the black hole and hence violating the second law of thermodynamics. He concluded that black holes must have entropy, so that the total entropy of the system does not decrease[1], and was able to conjecture the formula (2.1). He tried to specify the proportionality constant in the year after [31] but it was eventually fixed by Hawking to the exact value of $\eta = 1/4$[32, 33], when he showed that black holes must emit thermal (Hawking) radiation corresponding to a black body of certain temperature that is proportional to its surface gravity (see below). It is therefore most commonly referenced as the *Bekenstein-Hawking entropy*. Already before the constant was fixed, relationships between black hole physics and other laws of thermodynamics had been discovered, which a complete set of laws being derived by Bardeen, Carter, and Hawking in 1973 [34]:

The four laws of thermodynamics [35]

1. The surface gravity κ (see section 2.1) is constant over the event horizon.
2. For two stationary black holes differing only by small variations in the parameters M, J, and Q,
$$\delta M = \frac{\kappa}{8\pi}\delta A_{\text{hor}} + \Omega_H \delta J + \Phi_H \delta Q, \tag{2.2}$$
where Ω_H is the angular velocity and Φ_H is the electric potential at the horizon.
3. The area of the event horizon of a black hole never decreases,
$$\delta A_{\text{hor}} \geq 0. \tag{2.3}$$
4. It is impossible by any procedure to reduce the surface gravity κ to zero in a finite number of steps.

Written this way, the analogy to thermal physics is evident with κ playing the role of the temperature, and A_{hor} being equivalent to the entropy. However, in statistical mechanics, the entropy measures the number of microstates. This begs the question, what are the microstates of a black hole? After all, the no-hair theorems suggest that classical black holes could only have a single microstate. This is a strong indication for an underlying theory of quantum gravity that provides an explanation to which states are counted here, and could serve as a consistency check for such a theory.[2]

[1] This is often referred to as *Bekenstein's generalized second law of thermodynamics*.

[2] The number of models that properly calculate the Bekenstein-Hawking entropy in the framework of some quantum theory of gravity are actually vast. For more information, see [35] and references therein.

2.1 Hawking temperature

Initially, Bardeen, Carter, and Hawking argued that the laws of black hole thermodynamics are just analogies to regular thermodynamics. They suggested, that the real temperature, and real entropy of a black hole are actually distinct from the surface gravity and horizon area, respectively [34]. Their reasoning was that heat only flows from hot to cold. But a classical black hole only absorbs energy, it does never emit radiation, independent of the temperature of its surrounding radiation. This would mean that the real temperature of a classical black hole is at absolute zero. This problem was resolved when Hawking applied the tools of quantum field theory in curved space-times to show that all black holes are black bodies emitting radiation, now called *Hawking radiation*, with a temperature proportional to their surface gravity [32, 33]. To define the surface gravity, we need to introduce another concept first. Let χ be a Killing vector field, and let us denote with Σ a null hypersurface. We say that Σ is a *Killing horizon* if the norm of χ vanishes on Σ. With every Killing horizon, we can associate a *surface gravity*, usually denoted by κ, through the geodesic equation,

$$\chi^\mu \nabla_\mu \chi^\nu = \kappa \chi_\nu. \tag{2.4}$$

One can show that the surface gravity can be calculated via[36]

$$\kappa^2 = -\frac{1}{2} \nabla^\mu \chi^\nu \nabla_\mu \chi_\nu. \tag{2.5}$$

Then the Hawking temperature in terms of the surface gravity is

$$T_H = \frac{\kappa}{2\pi}. \tag{2.6}$$

In the example of the BTZ metric, we can define

$$\chi = \partial_t + \frac{J}{2r_+^2}, \tag{2.7}$$

which leads to

$$\kappa = \frac{r_+^2 - r_-^2}{l^2 r_+}. \tag{2.8}$$

2.2 The Euclidean formalism

Gibbons and Hawking showed in 1977 how to obtain the thermodynamic properties of the Schwarzschild black hole by computing the Euclidean action of a solution of Einstein's equations by continuing the Schwarzschild solution to imaginary time [37]. To achieve that, the time coordinate of the analytic continuation has to be periodic with periodicity β, which can be identified with the inverse temperature T. Then, in analogy with ordinary quantum field theory, the Euclidean path integral for the gravitational partition function can be formally written as

$$Z(\beta) = \int \mathcal{D}g e^{-S_{\text{Euc}}}, \tag{2.9}$$

where S_{Euc} is the Einstein-Hilbert action in Euclidean signature (with imaginary time coordinate). We shall not be bothered by the non-renormalizability of general relativity here (see chapter III). Gibbons and Hawking showed that one can still obtain physically meaningful results using the saddle point approximation. For any classical solution of the Einstein field equations, the Einstein-Hilbert action vanishes, however, what Gibbons and Hawking realized was that, if one were to consider a manifold with boundary, then the action must be supplemented by a boundary term[3]. Without the boundary term, the action would not have true extrema [39]. The euclidean action is then related to the Gibbs free energy $\mathcal{F}$ through

$$S_{\text{Euc}} = \beta\mathcal{F} = \beta\mathcal{M} - \mathcal{S} - \beta\sum_i \Phi_i\mathcal{Q}_i, \tag{2.10}$$

where $\mathcal{M}$ is the mass, $\mathcal{S}$ the entropy, and the Φ_i are the thermodynamic potentials of the system with their respective charges $\mathcal{Q}_i$. From this, one can obtain the thermodynamic quantities via,

$$\mathcal{S} = \beta\frac{\partial S_{\text{Euc}}}{\partial\beta} - S_{\text{Euc}}, \tag{2.11a}$$

$$\mathcal{M} = \frac{S_{\text{Euc}}}{\partial\beta} - \sum_i \frac{\Phi_i}{\beta}\frac{\partial S_{\text{Euc}}}{\partial\Phi_i}, \tag{2.11b}$$

$$\mathcal{Q}_i = -\frac{1}{\beta}\frac{\partial S_{\text{Euc}}}{\partial\Phi_i}. \tag{2.11c}$$

In general, the euclidean action can be written in the Hamiltonian formulation, that is,

$$S_{\text{Euc}} = \beta\int d^{D-1}x\sqrt{-g}\left[N(x)\mathcal{H} + N^i(x)\mathcal{H}_i\right] + \mathcal{B}, \tag{2.12}$$

where $\mathcal{B}$ is a boundary term chosen such that the euclidean action, S_{Euc}, has a well-defined extremum. Note that, on-shell, the Hamiltonian and momentum constraints ($\mathcal{H}$ and $\mathcal{H}_i$, respectively) are zero, so that all the thermodynamic information of the Gibbs free energy is contained inside the boundary term. It is instructive to demonstrate this with an example, which we conveniently choose to be the BTZ metric (1.4). The classical action reads

$$S = \int d^3x\sqrt{-g}\left[\frac{R + 2l^{-2}}{2\kappa}\right], \tag{2.13}$$

and for this purpose, we will write the metric in the more general form

$$ds^2 = -N^2 f dt^2 + f^{-1}dr^2 + r^2\left(d\theta + N^\theta dt\right)^2, \tag{2.14}$$

where in this case we have

$$\begin{aligned} N =& 1, \\ N^\theta =& -\frac{J}{2r^2} = -\frac{r_+ r_-}{lr^2}, \\ f =& -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}. \end{aligned} \tag{2.15}$$

This allows us to write the euclidean action as

$$S_{\text{Euc}} = \beta\int dr\left[N\left(f' - 2l^{-2}r + \frac{p^2}{2r^3}\right) + N^\theta p'\right] + \mathcal{B}, \tag{2.16}$$

[3] The need for such a boundary term was actually first realized by York [38] and later refined by Gibbons and Hawking where they showed how to apply it to black hole thermodynamics.

where we have defined $p = r^3(N^\theta)'/N$. Note that from (2.16) the Hamiltonian and momentum constraints are given by

$$\begin{aligned} \mathcal{H} =& f' - 2l^{-2}r + \frac{p^2}{2r^3}, \qquad (2.17a)\\ \mathcal{H}_\theta =& p'. \qquad (2.17b) \end{aligned}$$

It is a simple matter to insert the solution (2.15) into these constraints and show that these are indeed zero. Alternatively, one could, of course, solve them and obtain the solution functions. Further, a variation with respect to $\mathcal{H}$ reveals that N indeed is constant, which we will fix to be 1 in accordance with the BTZ solution. This also highlights the fact that p is a constant, which in this case is just the angular momentum J. We can now easily read off the boundary term by demanding that $\delta S_{\mathrm{Euc}} = 0$,

$$\begin{aligned} \delta\mathcal{B} =& -\beta\left[\delta f + N^\theta \delta p\right]_{r=r_+}^{r=+\infty} \\ =& -\beta \lim_{r\to\infty} \delta f(r) + \beta\delta f(r_+) - \beta\left(\lim_{r\to\infty} N^\theta(r) + N^\theta(r_+)\right)\delta p. \end{aligned} \qquad (2.18)$$

The variation of the function f can be written as

$$\delta f = \frac{\partial f}{\partial r_+}\delta r_+ + \frac{\partial f}{\partial r_-}\delta r_-, \qquad (2.19)$$

and with $\Omega = N^\theta(r_+)$ we obtain

$$\begin{aligned} \mathcal{B} =& \beta\frac{r_+^2 + r_-^2}{l^2} - 4\pi r_+ - \beta\Omega J \\ =& \beta M - 4\pi r_+ - \beta\Omega J. \end{aligned} \qquad (2.20)$$

Now we can use the Gibbs free energy (2.10) to read off the thermodynamic quantities

$$\mathcal{M} = \frac{r_+^2 + r_-^2}{l^2} = M, \qquad \mathcal{J} = J, \qquad \mathcal{S} = 4\pi r_+. \qquad (2.21)$$

Note that the three-dimensional gravitational constant is not the same as in four dimensions, which amounts to a different factor in the area law of the entropy. It is easy to check that the first law of thermodynamics is satisfied:

$$d\mathcal{M} = Td\mathcal{S} + \Omega d\mathcal{J}. \qquad (2.22)$$

3 Gravity with torsion

The connection that is used in standard general relativity is usually assumed to by symmetric, in particular the connection that is used most commonly is the Levi-Civita connection. This in turn means that the torsion tensor of the theory is always assumed to be zero. From the standard viewpoint of the second order formalism, one may argue that there is no sufficient motivation in introducing an additional tensor. Particularly, one would have to add specific matter Lagrangians to the theory to source torsion. And for what? Classically we are only dealing with bosonic matter, so we would not even notice the effect of a non-symmetric connection when considering the trajectories of astronomical objects, as can be seen by the geodesic equation:

$$0 = \ddot{x}^\mu + \Gamma^\mu{}_{\lambda\rho}\dot{x}^\lambda\dot{x}^\rho. \tag{3.1}$$

However, even now, one might argue that considering torsion may change the geometry and the resulting equations of motion so fundamentally that they give rise to solutions with measurable differences. And further, if we are interested in quantum matter, then we cannot categorically exclude it due to the non-commutative nature of fermions.

But that is not all: in 1922 Élie Cartan proposed reformulating the theory into what is now called the *first-order formulation of gravity* (using Cartan geometry). This allows for a new interpretation of the geometry and reveals the vanishing of torsion to be an additional constraint on the field equations. In other words, in the first-order formalism the inclusion of torsion is actually the more natural way to treat the theory. This will be outlined below.

3.1 First order formalism of gravity

Instead of working in a coordinate basis, it is often more convenient to work in a local orthonormal frame that is constructed via basis 1-forms (tetrad) $e^a(x) = e^a{}_\mu(x)dx^\mu$, with the defining property

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = \eta_{ab}e^a(x)e^b(x), \tag{3.2}$$

and consequently

$$\begin{gathered} g_{\mu\nu}(x) = \eta_{ab}\, e^a{}_\mu(x)\, e^b{}_\nu(x), \\ \delta^a_b = e^a{}_\mu(x)e_b{}^\mu, \quad \delta^\mu_\nu = e^a{}_\nu(x)e_a{}^\mu \end{gathered} \tag{3.3}$$

All the information of the metric is contained in the tetrad, though it is not uniquely determined. We always have the freedom of rotating or boosting the frame through a local Lorentz transformation.

To obtain a consistent covariant derivative, we need to define the so-called spin connection, $\omega^a_b(x) = \omega_\mu{}^a{}_b(x)dx^\mu$:

$$DV^a{}_b = dV^a{}_b + \omega^a{}_c \wedge V^c{}_b + \omega^c{}_b \wedge V^a{}_c. \tag{3.4}$$

Note that we still assume the metric to be invariant under parallel transport, that is

$$\nabla_\lambda g_{\mu\nu} = 0. \tag{3.5}$$

In terms of the vielbein e, the spin connection ω, and the affine connection Γ this can be written as

$$\partial_\mu e^a{}_\lambda + \omega^a_{\beta\mu}e^b{}_\lambda - \Gamma^\rho_{\mu\lambda}e^a{}_\rho = 0. \tag{3.6}$$

With this in mind, we define the torsion two-from as:

$$T^a = De^a = de^a + \omega^a{}_b \wedge e^b = \frac{1}{2} e^a{}_\lambda \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right) dx^\mu \wedge dx^\nu, \tag{3.7}$$

which is zero if the affine connection is symmetric. With the spin connection, we can now construct a curvature 2-form, that is,

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu. \tag{3.8}$$

Evidently its components are essentially those of the Riemann curvature tensor.

It is instructive to write the Einstein-Hilbert action in the language of the first order formalism. But first note that from (3.3) follows that the determinant of the metric in terms of the vielbein is

$$\sqrt{-g} = \det \left(e^a{}_\mu \right), \tag{3.9}$$

and the determinant, M, of a matrix can be written as

$$\epsilon_{\mu_1 \mu_2 \dots \mu_n} M = \epsilon_{a_1 a_2 \dots a_n} M^{a_1}{}_{\mu_1} M^{a_2}{}_{\mu_2} \dots M^{a_n}{}_{\mu_n}, \tag{3.10}$$

with the Levi-Civita symbol ϵ. The Einstein-Hilbert action can then be rewritten as follows:

$$\begin{aligned} \int d^4x \sqrt{-g} R =& \frac{1}{2} \int d^4x \sqrt{-g} \left(R^{\alpha\beta}{}_{\alpha\beta} - R^{\alpha\beta}{}_{\beta\alpha} \right) \\ =& \frac{1}{2} \int d^4x \sqrt{-g} R^{\alpha\beta}{}_{\mu\nu} \left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha \right) \\ =& \frac{1}{2} \int d^4x \sqrt{-g} R^{\alpha\beta}{}_{\mu\nu} \delta^{\mu\nu}_{\alpha\beta} \\ =& \frac{1}{4} \int \left(\epsilon_{\alpha\beta\lambda\rho} \sqrt{-g} \right) R^{\alpha\beta}{}_{\mu\nu} \left(\epsilon^{\mu\nu\lambda\rho} d^4x \right) \\ =& \frac{1}{4} \int \epsilon_{abcd} e^a{}_\alpha e^b{}_\beta e^c{}_\lambda e^d{}_\rho R^{\alpha\beta}{}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho \\ =& \frac{1}{2} \int \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d. \end{aligned} \tag{3.11}$$

Let us find out how the equations of motion for the Einstein-Hilbert action look like in this language. There are two independent fields for which we can vary the action integral, e and ω. Varying the curvature form with respect to ω, one obtains

$$\delta_\omega R^{ab} = \mathrm{D}\delta\omega^{ab} = \mathrm{d}\delta\omega^{ab} + \omega^a{}_c \wedge \delta\omega^c{}_b - \omega^c{}_b \wedge \delta\omega^a{}_b. \tag{3.12}$$

Disregarding a boundary term the equation of motion coming from the ω-variation reads

$$0 = \epsilon_{abcd} e^c \wedge T^d, \tag{3.13a}$$

and varying with respect to e is straightforward and yields

$$0 = \epsilon_{abcd} e^b \wedge R^{cd}. \tag{3.13b}$$

Since the vielbein is a basis we conclude that without matter the torsion and curvature forms have to be zero. As we know, the simplest non-trivial solution to these equations is the Schwarzschild metric. We shall demonstrate this here. Consider the spherically symmetric ansatz

$$\mathrm{d}s^2 = -f(r)^2 \mathrm{d}t^2 + g(r)^2 \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2, \tag{3.14a}$$

or in terms of the vielbein:

$$e^0 = f(r)dt, \quad e^1 = g(r)dr, \quad e^2 = rd\theta, \quad e^3 = r\sin(\theta)d\varphi. \tag{3.14b}$$

Inserting this into the torsion equation, $0 = de^a + \bar{\omega}^a{}_b \wedge e^b$, one obtains the spin connection:

$$\begin{aligned} \bar{\omega}^0{}_1 &= \frac{f'}{g}dt, & \bar{\omega}^1{}_2 &= -\frac{1}{g}d\theta, \\ \bar{\omega}^2{}_3 &= -\cos(\theta)d\varphi, & \bar{\omega}^1{}_3 &= -\frac{\sin(\theta)}{g}d\varphi \end{aligned} \tag{3.15}$$

which in turn can be inserted in the curvature equation, $0 = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$, yielding

$$0 = -gf' + rf'g' - rgf'' + fg', \tag{3.16a}$$
$$0 = 2rg' + g^3 - g, \tag{3.16b}$$
$$0 = 2rf' + f - fg^2. \tag{3.16c}$$

Note that only two of these equations are independent of each other. The solution is given by

$$f(r)^2 = g(r)^{-2} = 1 - \frac{2M}{r}, \tag{3.17}$$

the commonly known Schwarzschild solution.

In general, when there is also a matter, the action reads

$$\begin{aligned} I &= I_{\mathrm{EH}} + I_{\mathrm{M}} \\ &= \kappa \int \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d + I_{\mathrm{M}}. \end{aligned} \tag{3.18}$$

Then the equations of motion are given by

$$\frac{1}{4\kappa} S_{ab} = \epsilon_{abcd} e^c \wedge T^d, \tag{3.19a}$$
$$\frac{1}{2\kappa} \tau_a = \epsilon_{abcd} e^b \wedge R^{cd}, \tag{3.19b}$$

where τ_a is a vector-valued 3-form, the *energy-momentum current of matter* that is obtained through the variation of the matter action with respect to the vielbein e. Similarly, the S_{ab} are 3-forms and called the *spin currents* of matter, obtained through the variation of the matter action with respect to the spin connection ω. Now it is easy to see that to source non-zero torsion a matter action depending on the spin connection is needed because otherwise the spin current would be zero, and therefore the torsion is identically zero due to (3.19a).
Note that we have $4 \times 4 = 16$ unknown variables in the vielbein fields and $6 \times 4 = 24$ unknowns in the spin connection fields. The total number matches exactly the number of equations (3.19).

3.2 Sourcing torsion through a scalar field

One elegant method of sourcing torsion is by making use of the Gauß-Bonnet term. Usually, it is a topological invariant in four dimensions and therefore not contributing to the equations of motion. We can circumvent this fact by using a scalar field as some kind of Lagrange multiplier so that the action reads

$$I = \int \epsilon_{abcd} \left(\varkappa R^{ab} \wedge e^c \wedge e^d + \Lambda e^a \wedge e^b \wedge e^c \wedge e^d + \phi R^{ab} \wedge R^{cd} \right), \tag{3.20}$$

where ϕ is the scalar field, Λ the cosmological constant and $\varkappa = 1/32\pi$. This action has been used in [40] to find cosmological solutions with non-zero torsion. Here we will be concerned with the quest for finding black hole solutions. The independent fields are e, ω and ϕ, so we have the following variations:

$$\begin{aligned} \delta_\omega I &= \int \epsilon_{abcd} \left(\varkappa \mathrm{D}\delta\omega^{ab} \wedge e^c \wedge e^d + 2\phi \mathrm{D}\delta\omega^{ab} \wedge R^{cd} \right) \\ &= \int 2\epsilon_{abcd} \left(\varkappa e^c \wedge T^d - \mathrm{d}\phi \wedge R^{cd} \right) \wedge \delta\omega^{ab}, \end{aligned} \tag{3.21a}$$

$$\delta_e I = -\int 2\epsilon_{abcd} \left(\varkappa e^b \wedge R^{cd} + 2\Lambda e^b \wedge e^c \wedge e^d \right) \wedge \delta e^a, \tag{3.21b}$$

$$\delta_\phi I = \int \epsilon_{abcd} R^{ab} \wedge R^{cd} \delta\phi. \tag{3.21c}$$

And therefore the equations of motion are

$$0 = \epsilon_{abcd} \left(\varkappa e^c \wedge T^d - \mathrm{d}\phi \wedge R^{cd} \right), \tag{3.22a}$$

$$0 = \epsilon_{abcd} \left(\varkappa e^b \wedge R^{cd} + 2\Lambda e^b \wedge e^c \wedge e^d \right), \tag{3.22b}$$

$$0 = \epsilon_{abcd} R^{ab} \wedge R^{cd} = \mathrm{d} \left[\epsilon_{abcd} \left(\omega^{ab} \wedge \omega^{cd} + \frac{2}{3} \omega^{ab} \wedge \omega^c{}_e \wedge \omega^{ed} \right) \right]. \tag{3.22c}$$

Note that for constant ϕ, we come back to the torsion-free case. At this point one can already see that the system has a flaw: Using a torsion-free flat metric without cosmological constant, the equations of motion are trivially satisfied regardless of the choice of the scalar field. This is worrisome, but we will still press on to study these equations further.

We have a total of 41 equations, making it a very difficult task to solve them all. It is therefore reasonable to try out the most simple non-trivial ansatz, that of a spherically symmetric and static metric:

$$\mathrm{d}s^2 = -f(r)^2 \mathrm{d}t^2 + g(r)^2 \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2 \tag{3.23a}$$

with the tetrad fields

$$e^0 = f(r)\mathrm{d}t, \quad e^1 = g(r)\mathrm{d}r, \quad e^2 = r\mathrm{d}\theta, \quad e^3 = r\sin(\theta)\mathrm{d}\varphi \tag{3.23b}$$

The Killing vectors of this metric ansatz can be written as

$$\begin{aligned} &\xi_1 = \partial_t, &&\xi_2 = \partial_\varphi, &&\xi_3 = \sin\varphi\,\partial_\theta + \frac{S_K'}{S_K}\cos\varphi\,\partial_\varphi, &&\xi_4 = \cos\varphi\,\partial_\theta - \frac{S_K'}{S_K}\sin\varphi\,\partial_\varphi, \\ &\xi_1 = \partial_t, &&\xi_2 = \partial_\varphi, &&\xi_3 = \partial_\theta, &&\xi_4 = \varphi\,\partial_\theta - \theta\,\partial_\varphi. \end{aligned}$$

We will require that the torsion and the scalar field have the same isometries as the metric, that is the isometries generated by ξ_2, ξ_3 and ξ_4. To that end, we take the Lie derivative of the torsion form with respect to each Killing vector to determine a general form, which we found to be

$$T^0 = C_0 e^0 \wedge e^1 + D_0 e^2 \wedge e^3, \tag{3.24a}$$
$$T^1 = C_1 e^0 \wedge e^1 + D_1 e^2 \wedge e^3, \tag{3.24b}$$
$$T^2 = D_2 e^0 \wedge e^2 + A_2 e^0 \wedge e^3 + D_3 e^1 \wedge e^2 + B_2 e^1 \wedge e^3, \tag{3.24c}$$
$$T^3 = D_2 e^0 \wedge e^3 - A_2 e^0 \wedge e^2 + D_3 e^1 \wedge e^3 - B_2 e^1 \wedge e^2, \tag{3.24d}$$

and the functions, as well as the scalar field ϕ, depend on r only. To make life a bit easier, let us split everything into a torsion and torsion-free part by making use of the *contorsion tensor*:

$$\kappa^a{}_b = \omega^a{}_b - \bar{\omega}^a{}_b, \tag{3.25}$$

where $\bar{\omega}$ denotes the torsion-free spin connection (3.15). This allows as to express the torsion solely in terms of the contorsion and the vielbein as

$$T^a = \kappa^a{}_b \wedge e^b, \tag{3.26}$$

and from (3.24) it follows that

$$\kappa^{01} = C_0 e^0 - C_1 e^1, \tag{3.27a}$$
$$\kappa^{02} = -\frac{1}{2} D_0 e^3 - D_2 e^2, \tag{3.27b}$$
$$\kappa^{03} = \frac{1}{2} D_0 e^2 - D_2 e^3, \tag{3.27c}$$
$$\kappa^{12} = -\frac{1}{2} D_1 e^3 + D_3 e^2, \tag{3.27d}$$
$$\kappa^{13} = \frac{1}{2} D_1 e^2 + D_3 e^3, \tag{3.27e}$$
$$\kappa^{23} = \frac{1}{2}\left(2A_2 - D_0\right) e^0 + \frac{1}{2}\left(2B_2 + D_1\right) e^1. \tag{3.27f}$$

Then, together with the torsionless part, the full spin connection is

$$\omega^{01} = \left(\frac{f'}{fg} + C_0\right) e^0 - C_1 e^1, \tag{3.28a}$$
$$\omega^{02} = -\frac{1}{2} D_0 e^3 - D_2 e^2, \tag{3.28b}$$
$$\omega^{03} = \frac{1}{2} D_0 e^2 - D_2 e^3, \tag{3.28c}$$
$$\omega^{12} = -\frac{1}{rg} e^2 - \frac{1}{2} D_1 e^3 + D_3 e^2, \tag{3.28d}$$
$$\omega^{13} = \frac{1}{2} D_1 e^2 - \frac{1}{Hg} e^3 + D_3 e^3, \tag{3.28e}$$
$$\omega^{23} = \frac{1}{2}\left(2A_2 - D_0\right) e^0 + \frac{1}{2}\left(2B_2 + D_1\right) e^1 - \frac{\cot\theta}{r} e^3. \tag{3.28f}$$

Inserting these into the equations of motion, one can show the equations only admit a solution if and only if $B_2 = C_1 = D_1 = D_2 = 0$. Hence, the torsion further simplifies to

$$T^0 = C_0 e^0 \wedge e^1 + D_0 e^2 \wedge e^3, \tag{3.29a}$$
$$T^1 = 0, \tag{3.29b}$$
$$T^2 = A_2 e^0 \wedge e^3 + D_3 e^1 \wedge e^2, \tag{3.29c}$$
$$T^3 = -A_2 e^0 \wedge e^2 + D_3 e^1 \wedge e^3. \tag{3.29d}$$

Even though we have simplified the equations a lot, there are still countless possibilities, since we have still the undetermined functions f, g, A_2, C_0, D_0, D_3, and the scalar field ϕ. It is possible to show that the following functions indeed solve all the equations for a vanishing cosmological constant, $\Lambda = 0$:

$$f(r) = \frac{M}{r^2}, \tag{3.30a}$$
$$\phi'(r) = \frac{2r\,(rf'(r) - 4f(r))}{3f(r)\,(r^2 D_0(r)^2 + 4\Lambda r^2 + 4)}, \tag{3.30b}$$
$$r^2 g(r)^2 = -\phi'(r)\left[\left(r^2 D_0(r)^2 + 4\right)\phi'(r) + 4r\right], \tag{3.30c}$$
$$D_3(r) = \frac{g(r)}{2\phi'(r)} + \frac{1}{rg(r)}, \tag{3.30d}$$
$$C_0(r) = -\frac{f'(r)}{f(r)g(r)} - \frac{g(r)}{2\phi'(r)}, \tag{3.30e}$$
$$A_2(r) = \frac{f(r)\left[(r^2 D_0(r)^2 + 2)\,\phi'(r) + r\right] - r^2 f'(r)}{r^2 D_0(r) f(r)\phi'}. \tag{3.30f}$$

Note that these functions solve the system for arbitrary D_0, it is unspecified. This is another indication that the theory may be intrinsically flawed, at least when the cosmological constant is zero. Finding a solution for non-zero Λ turns out to be a highly nontrivial task. The reason may be that the Gauß-Bonnet term has to be zero, which is very constraining. In a future work, it would therefore be interesting to study a slightly different Lagrangian, for example the one given in a different context in [41]:

$$\mathcal{L} = R + \alpha\left[\phi\mathcal{G} + 4G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi - 4\,(\nabla\phi)^2\,\Box\phi + 2\,(\nabla\phi)^4\right], \tag{3.31}$$

where $\mathcal{G}$ denotes the Gauß-Bonnet term and α is a constant. In this case, the constraint gets relaxed a little and becomes

$$2R + \alpha\mathcal{G} = 0, \tag{3.32}$$

and in [41] they have found solutions of compact objects to the equations of motion. Consequently, (3.31) could be a promising candidate to look for non-zero torsion solutions as well.

4 Gravitational collapse and critical phenomena

There are different models on how the formation of black holes could occur in nature. One particularly interesting such model is the gravitational collapse. When one has some distribution of matter, there is the possibility that regions of higher density will form causing the attraction of more and more matter, which eventually leads to the collapse with the result of an astronomical object.
When one considers initial data sets of an isolated system for such a relativistic collapse, then there are three possible final states: either the initial data leads to the formation of a stable star, collapses to a black hole, or it explodes and disperses resulting in a flat space-time. What is remarkable is that these collapses to black holes exhibit critical behavior, analogously to critical phenomena in statistical physics. Specifically, it has been observed numerically that, taking any one parameter p of the initial data, there is a sharp threshold where black hole formation occurs, and further, near the threshold the resulting black hole mass follows a universal scaling law given by

$$M_{\mathrm{BH}} \propto |p - p_*|^\gamma, \tag{4.1}$$

where M_{BH} is the black hole mass and γ is the so-called critical exponent with respect to the initial data. Interestingly, the critical exponent is independent of the particular 1-parameter family under consideration, though it depends on the different types of collapsing matter. Further, p_* is the critical parameter above which the evolution leads to a collapse and consequently the formation of a black hole.
This has been first discovered (numerically) by Choptuik[4] for a scalar field in 1993 [19]. But that is not all, in fact, Choptuik further discovered the appearance of universality and the effect of scale-echoing. The first being that for a finite length of time in a finite region of space, the evolution of all parameters close to the critical parameter (on both sides) approach the same solution, that is, they are being attracted by the critical solution. Then, after some finite time, they depart from the critical solution towards black hole formation or dispersion, depending on which side of the critical parameter they are on. The critical solution itself is unstable because when one changes the parameter slightly the evolution either takes the road towards dispersion or black hole formation.
His third discovery, the scale-echoing, is a form of discrete self-similarity. That is, zooming in towards the critical solution, the dynamical character of the solution repeats itself over and over again with some echo period Δ, for example in the case of a scalar field [42]:

$$\begin{aligned} t' &= e^{-n\Delta} t, \\ r' &= e^{-n\Delta} r, \\ ds'^2 &= e^{-2n\Delta} ds^2, \\ \phi^*(t', r') &= \phi^*(t, r), \end{aligned} \tag{4.2}$$

where $n = 1, 2, 3, \ldots$ and ϕ^* is the scalar field of the critical solution.
It is quite puzzling what the physical meaning of these dimensionless constants could be, and their discovery spiked an interest in this area, leading people to further investigate it. As has been already hinted before, these results are not just limited to scalar fields: numerous other numerical and analytical studies have shown that these critical phenomena also appear in other

[4] In 1987 D. Christodoulou posed the question to Choptuik: "Will black hole formation turn on at finite or infinitesimal mass for a generic interpolating family at threshold?", of which the discovery was a direct result.

types of matter, even for axisymmetric gravitational waves in vacuum [43]! Note that the here outlined critical phenomena, though discovered first, are coined to be of type II. The latter discovered type I critical phenomena are stationary or time-periodic instead of self-similar or scale-periodic [44]. These two types are named in analogy to first and second order phase transitions in statistical mechanics, where the order parameter is discontinuous and continuous, respectively. However, we will only deal with type II phenomena here. One can even tune the initial data of type II collapses in such a way that the evolution results in a naked singularity, which clearly gives rise to potential counterexamples to the cosmic censorship conjecture [22]. A possible way to save the conjecture may be the consideration of quantum effects (see chapter III).

Since Choptuik's discovery, various further studies have been made and the topic of critical phenomena in gravitational collapse grew rapidly into its own field of research. An excellent review can be found in [44] and we will outline some important cornerstones here. In [45] a simple algorithm is used to confirm the results of Choptuik differently. Another numerical study of the collapse of a massless scalar field has been done in [46]. They additionally observed that for subcritical and supercritical evolutions, that is, where the varied parameter is closely above or below the critical value, respectively, there is a region where the scalar curvature and field energy density can be large. In the subcritical case this would even be visible to an observer, though they point out that their equations just describe a toy model, and hence more realistic studies would have to be made. Then, in [47], they predicted a modification to the original Choptuik scaling law due to the discrete self-similarity nature of the type II critical solution. They proposed the following scaling law

$$\log\left(M_{\mathrm{BH}}\right) = \beta \log\left(p - p_*\right) + c_k + \Psi\left[\log\left(p - p_*\right)\right], \tag{4.3}$$

where c_k is a constant that depends on the parameter family under consideration, and the function, $\Psi\left[\log\left(p - p_*\right)\right]$, is a periodic function with universal period, which they verified in a numerical simulation where they also determined its period to be $\bar{\omega} \approx 4.61$. Later, it was shown in [48] that this scaling law is also satisfied by the final Bondi mass in the asymptotically flat critical collapse, and that the oscillation period is related to the leading quasi-normal mode of a black hole with rapidly decreasing mass. They further showed that the Bondi mass shows an exponential decay together with an oscillatory component. These results were further confirmed in [49], where they have utilized a new evolution algorithm in a Bondi-Sachs-like formulation. Instead of using the areal radial coordinate that is used in the Bondi-Sachs formulation, they introduce an affine parameter. This made a non-singular treatment of the event horizon possible, and further, they were able to study global features that could not be investigated before, like there is no Bondi mass gap in the transition between subcritical and supercritical evolution (and with an asymptotically flat exterior). Very recently, new open-source codes have been presented in [50], that allow the study of critical phenomena on consumer-grade computers. Their codes can reproduce the original results of Choptuik with very low computational effort, which provides a great tool to familiarize oneself with the field.

A more general model, involving a massive complex scalar field, has been studied in [51]. Additionally, to the typical behavior of a gravitational collapse, the authors find that for certain parameter ranges, the critical solutions are unstable boson stars in the ground state.

The results of Choptuik's critical phenomena can also be found in entirely different setups. For example, in [52] they considered a minimally coupled scalar field in $(2+1)$-dimensional axially symmetric, AdS space-time and found the universal scaling of the final black hole mass as well. The authors further show that outside the event-horizon, the solution approaches the BTZ solution. And even in semiclassical loop quantum gravity, critical phenomena of a collapsing scalar field were found [53].

However, there is still a lot that is unknown about critical phenomena in gravitational physics, for example it is not clear how universal these effects are with respect to different matter types, in particular when one goes beyond spherical symmetry. The next years/decades will clearly provide more interesting results, and with the constant improvement in technology, there is no doubt that the complexity of numerical simulations will constantly increase as well.

In the following, a work in progress in collaboration with Thomas Mädler is presented. We use an affine-null formulation (see [54]) of the Einstein equations coupled to a scalar field similar to [49]. The main difference is that our approach considers an alternative way of regularizing the equations to compute them numerically using spectral methods. The initial data that we use shows a particularly simple relation between a specific initial parameter and the time until the system collapses to a black hole. Our code will be publicly available and is entirely written in the interpreter language Python, which makes it accessible to anyone familiar with programming in general, and, at the time of writing this thesis, is the most popular language according to the PYPL [55] and TIOBE [56] index. With relatively low numerical resolution and small runtime on a standard computer, we can reproduce characteristic features like mass scaling and echoing of the Choptuik critical solution.

4.1 Einstein-scalar field equations

We use a spherically symmetric affine-null metric ansatz [49, 57, 58]

$$ds^2 = -V(u,\lambda)du^2 - 2du d\lambda + r^2(u,\lambda)d\Omega^2, \tag{4.4}$$

where $u = \text{const}$ are outgoing null cones parametrized with angular coordinates x^A and λ is an affine parameter along the null-hypersurface forming rays. Writing the field equations as

$$E_{ab} := R_{ab} - \Phi_{,a}\Phi_{,b}, \tag{4.5}$$

where the scalar field is massless and has been rescaled ($\Phi \to (8\pi)^{-1/2}\Phi$), and writing

$$d\Omega^2 = q_{AB}dx^A dx^B, \tag{4.6}$$

the Einstein scalar field equations read

$$E_{uu}: \qquad 0 = -\frac{2}{r}r_{,uu} + \frac{V}{2r^2}(r^2V_{,\lambda})_{,\lambda} + \frac{r_{,\lambda}V_{,u} - r_{,u}V_{,\lambda}}{r} - \Phi_{,u}^2 \tag{4.7a}$$

$$E_{u\lambda}: \qquad 0 = \frac{1}{2r^2}(r^2V_{,\lambda})_{,\lambda} - \frac{2r_{,u\lambda}}{r} - \Phi_{,u}\Phi_{,\lambda} \tag{4.7b}$$

$$E_{\lambda\lambda}: \qquad 0 = -\frac{2}{r}r_{,\lambda\lambda} - \Phi_{,\lambda}^2 \tag{4.7c}$$

$$q^{AB}E_{AB}: \qquad 0 = [V(r^2)_{,\lambda}]_{,\lambda} - 2[1 + (r^2)_{,u\lambda}] = [V(r^2)_{,\lambda} - 2\lambda - 2(r^2)_{,u}]_{,\lambda} \tag{4.7d}$$

Further, the Klein-Gordon equation, $\nabla_a\nabla^a\Phi = 0$, yields

$$0 = (r^2\Phi_{,u})_{,\lambda} + (r^2\Phi_{,\lambda})_{,u} - (r^2V\Phi_{,\lambda})_{,\lambda}. \tag{4.7e}$$

This is the system of equations that we plan to solve, but first we want to point out that one can use the twice contracted Bianchi identities to show the following[54, 59]:

(i) If the main equations ($E_{\lambda\lambda}$, $E_{u\lambda}$) are satisfied, then the component $q^{AB}E_{AB} = 0$ is trivially satisfied.

(ii) If $\nabla_a \nabla^a \Phi = 0$ holds on a null hypersurface, $u = u_0$, then $E_{u\lambda}$ holds trivially everywhere on u_0.

(iii) Further,

$$0 = \frac{1}{r^2}(r^2 E_{uu})_{,\lambda}, \tag{4.8}$$

so if $r \neq 0$ and E_{uu} holds for one value, then $\lambda = \lambda_0$, E_{uu} also holds everywhere on u_0.

This implies that we do not have to deal with all equations when solving the system (as one would hope for, given the number of equations and the number of independent functions).

Equation (4.7d) is already conveniently written as a derivative. Integrating it yields

$$C(u) = V(r^2)_{,\lambda} - 2\lambda - 2(r^2)_{,u}, \tag{4.9}$$

where $C(u)$ is a free function at λ_0 depending solely on u. Solving for V gives us

$$V = \frac{C + 2\lambda + 2(r^2)_{,u}}{(r^2)_{,\lambda}}. \tag{4.10}$$

Requiring local flatness at the origin, $\lambda = 0$, yields the regularity conditions

$$r(u, 0) = 0, \qquad r_{,\lambda}(u, 0) = 1, \qquad V(u, 0) = 1. \tag{4.11}$$

In fact, $r(u, \lambda)$ behaves like λ around the origin, which then implies $C = 0$. Hence,

$$V = \frac{\lambda}{r r_{,\lambda}} + \frac{2 r_{,u}}{r_{,\lambda}}. \tag{4.12}$$

Note that this is singular if $r_{,\lambda} = 0$. Therefore, from $r(u, \lambda)$, we automatically know V. Then the metric is entirely determined and can be written as

$$ds^2 = -\left(\frac{\lambda}{r r_{,\lambda}} + \frac{2 r_{,u}}{r_{,\lambda}}\right) du^2 - 2 d\lambda du + r^2 d\Omega^2. \tag{4.13}$$

Given a regular solution for $r = r(u_0, \lambda)$ on some null hypersurface $u = u_0$, and integrating the hypersurface equation (4.7c) gives us a solution for the scalar field:

$$\Phi(u_0, \lambda) = \left[-2 \int_0^\lambda \frac{r_{,\tilde{\lambda}\tilde{\lambda}}(u_0, \tilde{\lambda})}{r(u_0, \tilde{\lambda})} d\tilde{\lambda} \right]^{1/2}. \tag{4.14}$$

This will be used later in section 4.4.2 to construct initial data.

4.2 Main equations as a hierarchy

To solve the equations numerically, we want to cast them into a hierarchical system. Note that the three independent main equations that we need to solve are (4.7c), (4.7d) and (4.7e), with the second one being solved in the previous section already. Now we need to deal with $r_{,u}$ in a sophisticated way, that is by introducing [49]

$$K := 2r_{,\lambda}\Phi_{,u} - 2r_{,u}\Phi_{,\lambda}, \tag{4.15}$$

with which we can write

$$\left(r^2 V\Phi_{,\lambda}\right)_{,\lambda} = \left(\frac{r\lambda\Phi_{,\lambda}}{r_{,\lambda}}\right)_{,\lambda} - r\left(\frac{rK}{r_{,\lambda}}\right)_{,\lambda} + \left(r^2\Phi_{,u}\right)_{,\lambda} + \left(r^2\Phi_{,\lambda}\right)_{,u}, \tag{4.16}$$

and, using (4.7e), it follows that

$$r\left(\frac{rK}{r_{,\lambda}}\right)_{,\lambda} = \left(\frac{r\lambda\Phi_{,\lambda}}{r_{,\lambda}}\right)_{,\lambda}, \tag{4.17}$$

which substitutes the equation of motion (4.7e) for the scalar field Φ in terms of the new variable, K. Further, equation (4.15) (the definition of K) immediately gives us an evolution equation for Φ by solving it for its time derivative, that is

$$\Phi_{,u} = \frac{K + 2r_{,u}\Phi_{,\lambda}}{2r_{,\lambda}}. \tag{4.18}$$

What remains is now to find an appropriately formulated hypersurface equation for $r_{,u}$. Therefore, we differentiate equation (4.7c) with respect to u and express $\Phi_{,u\lambda}$ in terms of the previously defined function K, which yields

$$0 = \left(\frac{r_{,u}}{r_{,\lambda}}\right)\left(\frac{r_{,\lambda\lambda}}{r}\right) - \left(\frac{r_{,u\lambda\lambda}}{r_{,\lambda}}\right) - \frac{r\Phi_{,\lambda}}{2r_{,\lambda}}\left(\frac{K}{r_{,\lambda}}\right)_{,\lambda} - \frac{r\Phi_{,\lambda}}{r_{,\lambda}}\left(\frac{r_{,u}}{r_{,\lambda}}\Phi_{\lambda}\right)_{,\lambda}. \tag{4.19}$$

Using equation (4.7c) again, one can rewrite the last term and express this equation into a more compact form with which we finally arrive at the three hierarchical hypersurface equations for r, K and $r_{,u}$, namely:

$$r_{,\lambda\lambda} = -\frac{r}{2}\Phi_{,\lambda}^2, \tag{4.20a}$$

$$\left(\frac{rK}{r_{,\lambda}}\right)_{,\lambda} = \frac{1}{r}\left(\frac{r\lambda\Phi_{,\lambda}}{r_{,\lambda}}\right)_{,\lambda}, \tag{4.20b}$$

$$\left(\frac{r_{,u}}{r_{,\lambda}}\right)_{,\lambda\lambda} = -\frac{r\Phi_{,\lambda}}{2r_{,\lambda}}\left(\frac{K}{r_{,\lambda}}\right)_{,\lambda}, \tag{4.20c}$$

and the evolution equation for the scalar field Φ

$$\Phi_{,u} = \frac{K}{2r_{,\lambda}} + \frac{r_{,u}\Phi_{,\lambda}}{r_{,\lambda}} \tag{4.21}$$

where V can be determined at any stage from (4.12), provided r is known.

The strategy can now be summarized as follows:
We start with initial data for $\Phi(u_0, \lambda)$ at the first hypersurface. This enables us to compute $r(u_0, \lambda)$ through the first of the three equations. Then, using the second we can determine K, and with the last one we find $r_{,u}$. At last, we use the evolution equation to evolve Φ to the next hypersurface, where we can repeat the previous steps. This way we can evolve the whole system and solve the Einstein equations numerically. Note that only the physical scalar field has to be specified at the initial time $u = 0$.

However, it is known that the equations that contain a division by $r_{,\lambda}$ are, in fact, not regular everywhere, which is related to the formation of an apparent horizon (cf. [49]). To see this, consider the null vectors of the metric (4.4)

$$k^a \partial_a = \partial_\lambda, \qquad n^a \partial_a = \partial_u - \frac{1}{2} V \partial_\lambda, \tag{4.22}$$

which are outgoing (k), and ingoing (n), respectively. Then their expansion rates are given by

$$\begin{aligned} \Theta^+ =& \nabla_a k^a = \frac{2r_{,\lambda}}{r}, \\ \Theta^- =& \nabla_a n^a = -\frac{2\lambda}{r^2}. \end{aligned} \tag{4.23}$$

If $r_{,\lambda} = 0$ for some $\lambda_A > 0$, then the outgoing null expansion Θ^+ vanishes, while the ingoing null expansion Θ^- is negative everywhere. This means that the corresponding 2-surface at $\lambda_A(u)$ is trapped, and an apparent horizon has formed. Thus, possible singular terms $1/r_{,\lambda}$ at finite values of λ_A in the metric and in the field equations are related to the formation of an apparent horizon, which fortunately are non-physical singularities and can therefore be regularized. This will be done in the next section, but first we shall discuss some properties of the system.

Asymptotics

Asymptotic flatness is required for the initial data, so to be consistent it needs to have an asymptotic expansion like

$$\Phi = \frac{\Phi_{[1]}(u)}{\lambda} + \frac{\Phi_{[2]}(u)}{\lambda^2} + O(1/\lambda^3). \tag{4.24}$$

And with the r-hypersurface equation (4.20a) it follows that

$$r = H(u)\lambda + 2M_B(u) + O(1/\lambda), \tag{4.25}$$

where $H = \lim_{\lambda\to\infty} r_{,\lambda}$ is a function of integration that depends solely on u, and M_B is the Bondi mass (see below). If $H = 1$, the asymptotic observer in the limit $\lambda \to \infty$ is an inertial observer, whose associated frame is called *Bondi frame*. The time coordinate in a Bondi frame, the *Bondi time*, is related to the time u at the origin via

$$\lim_{\lambda\to\infty} u_{B,u} = \frac{1}{H}. \tag{4.26}$$

Hence, for $H < 1$, the clocks of a Bondi observer go slower than those of a freely falling one along the central geodesic tracing the origin. Conversely, if, for some finite time, u_E we have that

$$\lim_{u\to u_E} H = 0, \tag{4.27}$$

then the change of Bondi time with respect to the time at the origin becomes infinite, and thus an observer in a Bondi frame does not receive further information from one at the origin (infinite redshift of a distant observer). This means that an event horizon together with a physical singularity has formed, and the origin $\lambda = 0$ is hidden behind this horizon.

Bondi mass

In spherical symmetry there exists an invariant definition of mass provided by the Misner-Sharp mass given by

$$m = \frac{r}{2}\left(1 - g^{ab} r_{,a} r_{,b}\right) = \frac{1}{2}\left(r - \lambda r_{,\lambda}\right). \tag{4.28}$$

The Misner-Sharp mass is related to the Bondi mass M_B via

$$M_B = \lim_{\lambda\to\infty} m. \tag{4.29}$$

From equation (4.25), we can determine the Bondi mass of the system through

$$M_B(u) = -\frac{1}{2}\lim_{\lambda\to\infty} \lambda^2 \left(\frac{r}{\lambda}\right), \tag{4.30}$$

and by inserting equations (4.24) and (4.25) into equation (4.7a), we find the Bondi mass loss of the system to be

$$M_{B,u}(u) = -\frac{1}{2} H(u)^3 \left(\Phi'_{[1]}(u)\right)^2 \tag{4.31}$$

4.3 Alternative regularization of the hierarchy equations

As has been mentioned before, the equations of the system (4.20) and (4.21) that have an $r_{,\lambda}$ appearing in the denominator may contain singularities. To remedy this, we propose an alternative method (to [49]) of regularizing the equations:

$$\hat{r} = 1/r_{,\lambda} \quad , \quad \kappa = \frac{rK}{r_{,\lambda}} \quad , \quad \rho = r_{,u}/r_{,\lambda}. \tag{4.32}$$

Then the hierarchical system becomes

$$r_{,\lambda\lambda} = -\frac{r}{2}\Phi_{,\lambda}^2, \tag{4.33a}$$

$$\hat{r}_{,\lambda} = \frac{1}{2} r\hat{r}^2 \Phi_{,\lambda}^2, \tag{4.33b}$$

$$\kappa_{,\lambda} = \frac{1}{r}\left(r\hat{r}\lambda\Phi_{,\lambda}\right)_{,\lambda}, \tag{4.33c}$$

$$\rho_{,\lambda\lambda} = -\frac{1}{2} r\hat{r}\Phi_{,\lambda}\left(\frac{\kappa}{r}\right)_{,\lambda}, \tag{4.33d}$$

$$\Phi_{,u} = \frac{\kappa}{2r} + \rho\Phi_{,\lambda}, \tag{4.33e}$$

where we require the regularity conditions on the physical fields: $r(u,0) = 0$, $r_{,\lambda}(u,0) = 1$, $\Phi(u,0) = f(u) =$ finite. These in turn imply

$$\hat{r}(u,0) = 1 \ \ , \ \ \kappa(u,0) = 0 \ \ , \ \ \rho(u,0) = 0. \tag{4.34}$$

Then, expanding the scalar field around the vertex $\lambda = 0$

$$\Phi(u,\lambda) = \sum_{n=0}^{N-1} \frac{\Phi_{[n]}(u)\lambda^n}{n!} + O(\lambda^N), \tag{4.35}$$

and using the hierarchical equations together with the definitions of the functions we can approximate them for small λ as well:

$$r(u,\lambda) = \lambda - \frac{1}{12}\Phi_{[1]}^2\lambda^3 + O(\lambda^4), \tag{4.36a}$$

$$\hat{r}(u,\lambda) = 1 + \frac{1}{4}\Phi_{[1]}^2\lambda^2 + \frac{1}{3}\Phi_{[1]}\Phi_{[2]}\lambda^3 + O(\lambda^4), \tag{4.36b}$$

$$\kappa(u,\lambda) = 2\Phi_{[1]}\lambda + \frac{3}{2}\Phi_{[2]}\lambda^2 + \frac{1}{3}\left(\frac{5}{6}\Phi_{[1]}^3 + 2\Phi_{[3]}\right)\lambda^3 + O(\lambda^4), \tag{4.36c}$$

$$\rho(u,\lambda) = -\frac{1}{8}\Phi_{[1]}\Phi_{[2]}\lambda^3 + O(\lambda^4), \tag{4.36d}$$

and

$$\Phi_{,u} = \Phi_{[1]} + \frac{3}{4}\Phi_{[2]}\lambda + \frac{1}{9}\left(2\Phi_{[1]}^3 + 3\Phi_{[3]}\right)\lambda^2 + \frac{1}{48}\left(16\Phi_{[1]}^2\Phi_{[2]} + 5\Phi_{[4]}\right)\lambda^3 + O(\lambda^4). \tag{4.36e}$$

Now we are ready to map the regularized equations on the numerical grid.

4.4 Numerical implementation

To map the limit $\lambda \to \infty$ to the (finite) numerical grid, we replace the affine parameter λ by a so-called grid function, that is, $\lambda = \lambda(x)$ with $x \in [-1,1]$. while requiring the properties,

$$\lambda(-1) = 0, \ \ \frac{d\lambda}{dx}\Big|_{x=0} \neq 0, \ \ \lim_{x\to 1}\lambda(x) = \infty, \ \ \lim_{x\to 1}(x-1)\lambda(x) \in \mathbb{R}. \tag{4.37}$$

Note that the last requirement essentially means that $\lambda(x)$ has a pole of order one at $x = 1$. It should be clear that the grid function is not unique, and therefore we fix it to be

$$\lambda(x) = A\frac{x+1}{1-x} \ \ , \ \ \lambda'(x) = \frac{2A}{(1-x)^2},$$

where A is an arbitrary grid parameter to be specified later for convenience.

Now is the time to do one last change of variables which is basically a conformal rescaling on the grid:

$$R = r(1-x) \ \ , \ \ \tilde{\rho} = \rho(1-x) \tag{4.38}$$

Using the notation $\frac{d\lambda}{dx} := \lambda'$ and $f_{,\lambda} = f_{,x}/\lambda'$, we are now in the position to express the system (4.33) in a very convenient way in terms of the derivatives along the grid variable x:

$$0 = R_{,xx} + \frac{1}{2}\Phi_{,x}^2 R, \tag{4.39a}$$

$$0 = \left(\frac{1}{\hat{r}}\right)_{,x} + \frac{R\Phi_{,x}^2}{2(1-x)\lambda'}, \tag{4.39b}$$

$$0 = \kappa_{,x} - \frac{1-x}{R}\left[R\hat{r}\frac{\lambda}{(1-x)\lambda'}\Phi_{,x}\right]_{,x}, \tag{4.39c}$$

$$0 = \tilde{\rho}_{,xx} - \frac{1}{2}R\hat{r}\Phi_{,x}\left[\frac{(1-x)\kappa}{R}\right]_{,x}, \tag{4.39d}$$

$$0 = \Phi_{,u} - \frac{1}{2}\frac{(1-x)\kappa}{R} - \frac{\tilde{\rho}}{(1-x)\lambda'}\Phi_{,x}. \tag{4.39e}$$

It is straightforward to show that these equations are now manifestly regular at $x = 1$. The coordinate singularity at the origin, $x = -1$, is treated by using l'Hospital's rule at $x = -1$ and imposing the boundary conditions.

This is a good point to briefly mention a few things about the numerical method. The spatial grid is given by N Gauß-Lobatto points,

$$x_i = -\cos\left(\frac{i\pi}{N-1}\right) \quad i \in \{0, ...N-1\},$$

where we can specify N as a parameter in our code suitable to our needs.
And the temporal grid is then given by

$$u = n\Delta u \quad , \quad \Delta u = \frac{1}{N^2} \quad , \quad n = 0, ..., N_u - 1$$

where N_u depends on the desired end time u_{fin}:

$$N_u = \frac{u_{fin}}{\Delta u} = u_{fin}N^2.$$

Further, we approximate the involved fields $f \in \{\Phi, R, \hat{r}, \kappa, \tilde{\rho}\}$ using a spectral base of Chebyshev polynomials of first kind, $T_n(x) = \cos[n \arccos(x)]$:

$$f(u_n, x_i) = \sum_{k=0}^{N} f_k(u_n)T_n(x_i)$$

Following [60], we then use pseudospectral methods with either a viscosity filtering [61] or an exponential filtering, and the time integration is done with a third or fourth order (depending on whether one wants faster results, or a smaller error) Runge Kutta method (see e.g., [62]) During the evolution, we monitor the trivial equation ($E_{u\lambda}$) to ensure consistency of the numerical integration scheme. To achieve the necessary numerical resolution, we designed an adaptive mesh refinement of the Gauß-Lobatto grid, to increase the numerical resolution during the runs.

4.4.1 Extraction of physical quantities

The central geodesic is being traces out by $\lambda = 0$ (or $x = -1$ in terms of the grid coordinate), and therefore its time corresponds to proper time. Consequently, $\Phi|_{x=-1}(u, x)$ gives the evolution of the scalar field at the origin with respect to proper time.

As for the determination of the fields at future null infinity, that is $x = 1$, we first need to transform equations (4.25) and (4.24) onto the grid:

$$\Phi = \frac{\Phi_{[1]}}{2A}(1-x) + \left(\frac{\Phi_{[1]}}{4A} + \frac{\Phi_{[2]}}{4A^2}\right)(1-x)^2 + O\left[(1-x)^3\right], \tag{4.40a}$$

$$R = HA(1+x) + 2M_B(1-x) + O\left[(1-x)^2\right]. \tag{4.40b}$$

Note that we are working with R defined in (4.38). From this, we immediately find

$$H = \left.\frac{R}{2A}\right|_{x=1}, \tag{4.41a}$$

and to obtain $\Phi_{[1]}$, $\Phi_{[2]}$, and M_B, we need to take derivatives of Φ and R and evaluate them at $x = 1$:

$$\Phi_{[1]} = -2A\Phi_{,x}|_{x=1}, \tag{4.41b}$$

$$\Phi_{[2]} = 2A^2\left(\Phi_{,xx}|_{x=1} + \Phi_{,x}|_{x=1}\right), \tag{4.41c}$$

$$M_B = \frac{1}{2}\left(HA - R_{,x}|_{x=1}\right). \tag{4.41d}$$

Note that these quantities will be given in terms of the code time $u_{x=1}$ at null infinity and not the Bondi time. The Bondi time axis is found by integration of

$$\frac{du_B}{du} = \frac{1}{H}, \tag{4.42}$$

using the initial condition $u_B = 0$.

4.4.2 Initial data

Following [57] we can construct initial data with help of the system (4.33). Suppose [63],

$$r(u = 0, \lambda) = \lambda - \frac{b^2\lambda^3}{(a+\lambda)^2} \tag{4.43}$$

which fulfills the flatness regularity conditions for r at the origin. The hypersurface equation (4.20a) then implies

$$\Phi_{,\lambda}(0,\lambda) = \frac{ab\sqrt{12}}{(a+\lambda)[(a+\lambda)^2 - b^2\lambda^2]^{1/2}} \tag{4.44}$$

so that the scalar field initial data on the initial null hypersurface is

$$\Phi(0,\lambda) = \sqrt{12}\arcsin\left(\frac{b\lambda}{a+\lambda}\right) - \sqrt{12}\arcsin b \tag{4.45}$$

For this data, we can integrate the hypersurface equations (4.33) and find exact expressions of the fields R, $\hat{r}$, κ, ρ and also $\Phi_{,u}$ on the initial null hypersurface $u = 0$. Having such explicit solutions of the fields on the first hypersurface will allow us to easily compare the quality of the integrator of the hypersurface equations. The full expressions for these fields are,

$$\hat{r}(0,\lambda) = \frac{(a+\lambda)^3}{(a+\lambda)^3 - 3ab^2\lambda^2 - b^2\lambda^3}, \tag{4.46a}$$

$$\kappa(0,\lambda) = \frac{\sqrt{12}b\lambda}{\sqrt{(a+\lambda)^2 - b^2\lambda^2}} \frac{(1-b^2)\lambda^3 + (4-3b^2)a\lambda^2 + 5a^2\lambda + 2a^3}{[(a+\lambda)^3 - (3a+\lambda)b^2\lambda^2]}, \tag{4.46b}$$

$$\rho(0,\lambda) = -\frac{b^2\lambda^2(3a+\lambda)^2}{4a\left[(a+\lambda)^3 - (3a+\lambda)b^2\lambda^2\right]} - \frac{9b}{8a}\lambda \log\left[\frac{\lambda\,(1-b)+a}{\lambda\,(1+b)+a}\right]. \tag{4.46c}$$

And the derivative $\Phi_{,u}(0,\lambda)$ is then an algebraic consequence of (4.45) and of the previous equations:

$$\begin{aligned}\Phi_{,u} = \frac{\sqrt{3}b}{4(a+\lambda)\sqrt{(a+\lambda)^2 - b^2\lambda^2}} \Bigg[&\frac{8a^3 + 20a^2\lambda + 2a\left(8-3b^2\right)\lambda^2 - 2\left(b^2-2\right)\lambda^3}{(a+\lambda-b\lambda)(a+\lambda+b\lambda)} \\ &-9b\lambda \log\left(\frac{a+\lambda-b\lambda}{a+\lambda+b\lambda}\right)\Bigg]\end{aligned} \tag{4.46d}$$

4.5 Numerical results

We will now present the results that we obtained from our numerical calculations. The figures that are shown in this section can be found as well in appendix 2 as bigger versions. Note that, at the time of writing this text, we are in the process of finalizing the work, so this is just a concise presentation of the upcoming results which we are currently analyzing more quantitatively. We concluded the last section with the integration of our initial data on the initial hypersurface, see equations (4.46). This is very convenient since it allows us to directly compare our solver of the hypersurface equations with their analytical solutions. For this convergence test, we therefore run an evolution for $a = 0.2$ and $b = 0.9$, corresponding to a subcritical solution, and with various numbers of grid points N. Figure 4.1 shows this comparison for the three functions R, κ, and $\Phi_{,u}$. We always plot the maximal difference between the numerical and analytical solutions by taking

$$\max_x |f_{\text{num}}(0,x) - f_{\text{ana}}(0,x)|, \tag{4.47}$$

with x running over the whole grid.

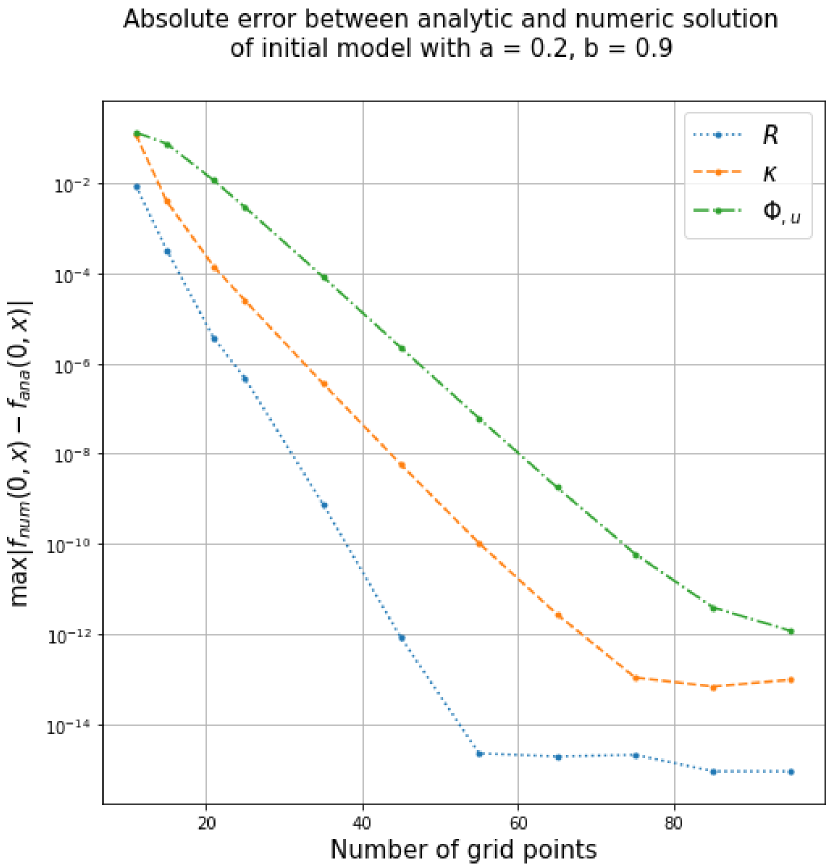


Figure 4.1: Convergence test of our hypersurface equation solver for the three functions R, κ, and $\Phi_{,u}$. For several numbers of grid points N, we compute the absolute value of the difference between the analytical and numerical solution at each point, and take the maximum over the x.

To identify the black hole formation, we need to monitor the limit $H(u) \to 0$ as $u \to u_E$ which, as we showed previously (see text around equation (4.27)), signifies the emergence of an event horizon. Scanning through the parameter b, we find that the critical value $b_* = 0.947193359375$ defines the threshold of black hole formation. To confirm Choptuik's scaling law, we compare the logarithm of the Bondi mass, with $\eta = \log(b - b_*)$, as shown in figure 4.2.

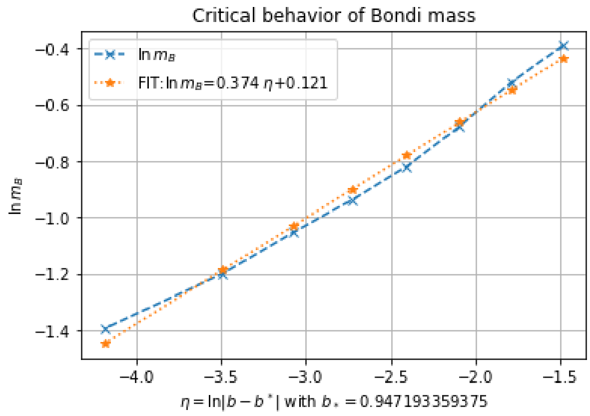


Figure 4.2: Universal scaling law for the Bondi mass of the final state, where the critical parameter is given by $b_* \approx 0.9472$, and the universal scaling exponent is $\gamma \approx 0.374$.

The fit yields

$$\log(M_B) = \gamma\eta + c = 0.374\eta + 0.121, \tag{4.48}$$

from which we can directly read of the universal scaling exponent, γ.

Another interesting feature of our initial data is revealed through the evolution of a supercritical solution for various parameter values a. For each evolution, we monitor the time of collapse and compare it with a, see figure 4.3. Surprisingly, for our initial data, the relation between a and u_c is linear. This allows us to manually decrease the computation time further by choosing a small value for a. Note that we cannot set $a = 0$, since this would make the scalar field constant along the hypersurface, as can be seen from equations (4.45) and (4.44).

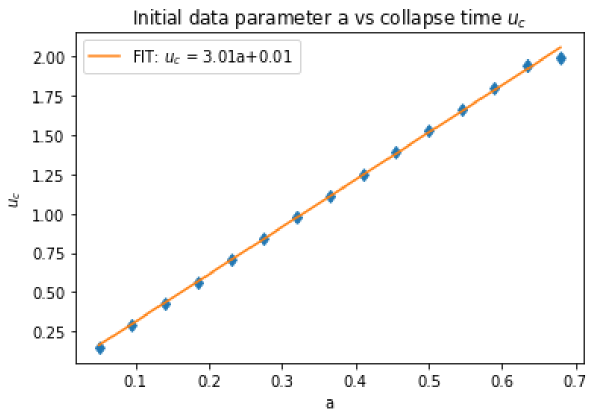
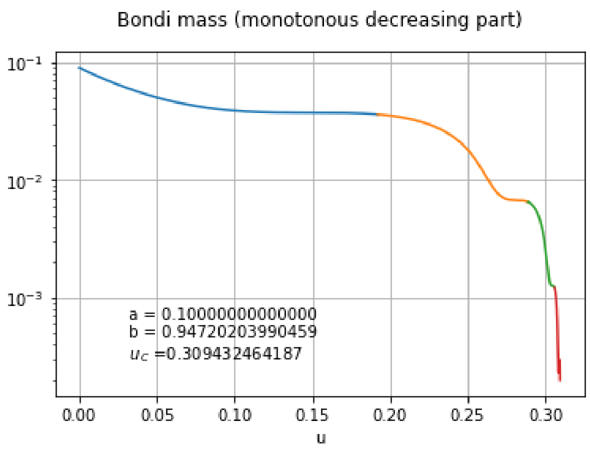
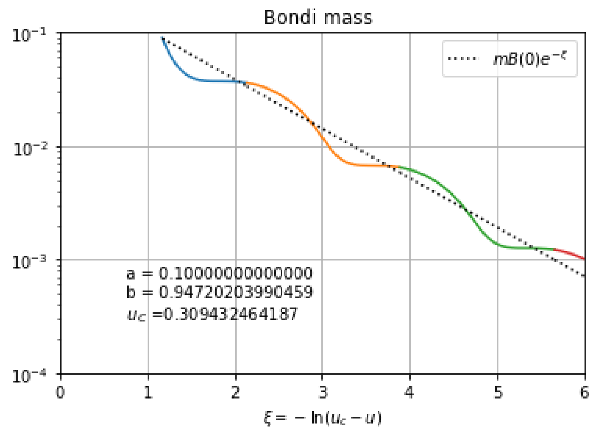


Figure 4.3: The collapse time u_c is plotted for different values of the parameter a, which shows a linear relation between the time of a collapse and a.

Plotting the Bondi mass of a supercritical solution against u and $\xi = -\log(u_c - u)$, we see its exponential decay, $M_B \approx e^{-\xi}$, together with its oscillatory component. Figure 4.4 shows these to plots next to one another. In Figure 4.4b we additionally plotted the function $M_B(0)e^{-\xi}$ so that it becomes clear how well this law is satisfied.



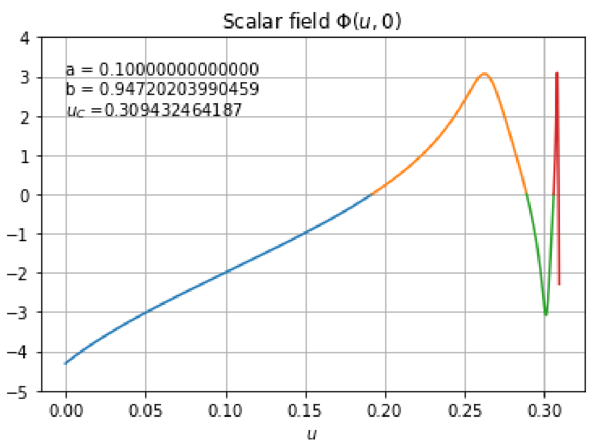
(a) Monotonously decreasing part of the Bondi mass.



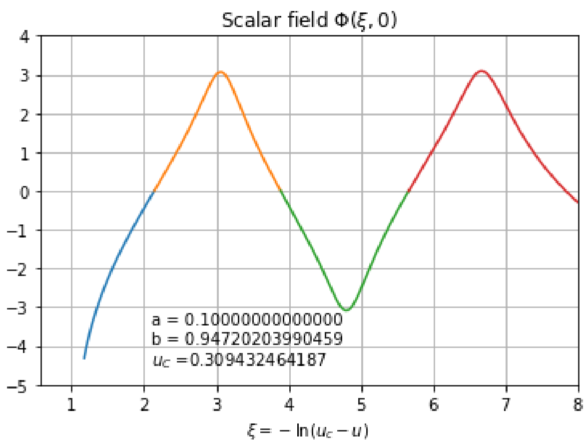
(b) Bondi mass in semilogarithmic plot to emphasize its oscillatory decay.

Figure 4.4: Bondi mass decay with its oscillations. For every mesh refinement, the curve changes its color.

For this supercritical solution, we also see the self-similarity through the echoing of the scalar field, which is shown in figure 4.5. The oscillations of the scalar field close to the collapse are better visible in the semilogarithmic plot of figure 4.5b.



(a) Scalar field at origin.



(b) Scalar field at origin in semilogarithmic plot.

Figure 4.5: Scalar field at the origin. For every mesh refinement, the curve changes its color. The echoing period is $\Delta \approx 3.48$, which is close to the high-precision result of $\Delta \approx 3.44$ in [64, 65].

The last feature we want to mention, is the universality of the critical solution. As can be seen in figures 4.6 and 4.7, the solutions of the slightly subcritical and slightly supercritical evolution are non-distinguishable until almost up to the collapse time, where the supercritical solution becomes a black hole, and the subcritical one ends up dispersing into flat space shortly after.

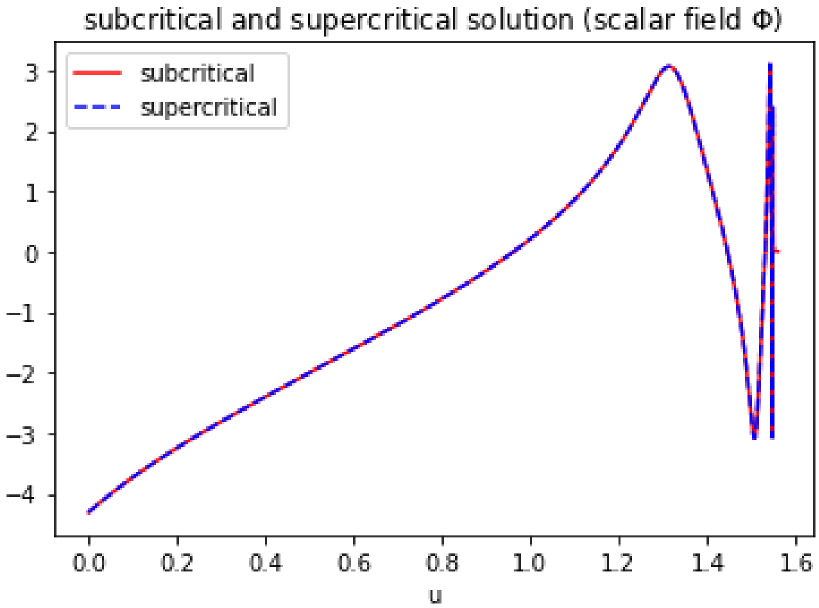


Figure 4.6: A subcritical and a supercritical solution are shown. Until close to the collapse time, they are indistinguishable (the critical solution is acting as an attractor).

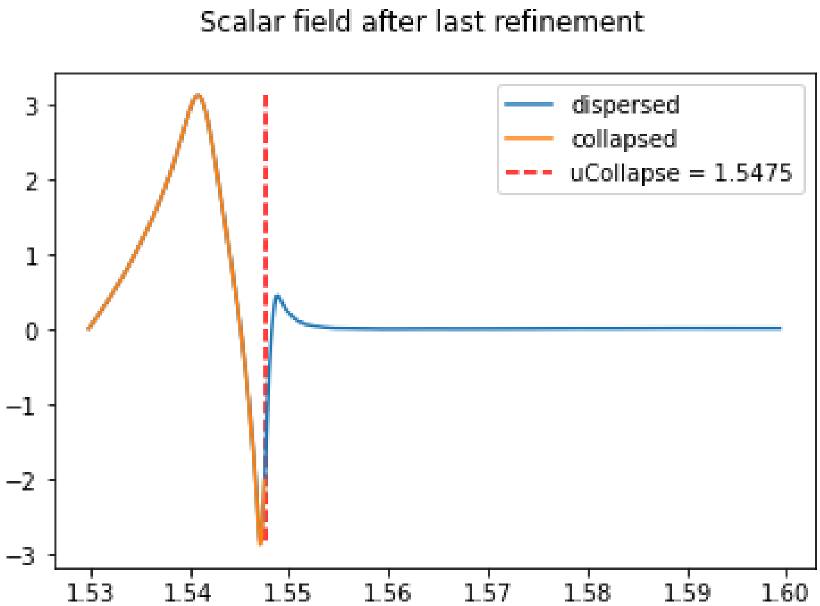


Figure 4.7: A subcritical and a supercritical solution are shown close to the collapse time. The universality of the critical solution is evident.

This concludes the documentation of our work so far, but it should be clear that the characteristic features of critical phenomena in gravitational collapse are clearly visible in our numerical data. The next step is now to analyze this more quantitatively, determining for example the oscillation periods of the Bondi mass decay, etc.

Chapter II

Classical extensions/modifications to general relativity

In this chapter, possible extensions/modifications to general relativity are studied. After a brief introduction and a description of the alternative theories that are relevant to our work, the results of our studies are shown.

1 Modifying Einstein's theory of relativity

Undeniably, Einstein's general theory of relativity (GR) is one of the greatest achievements in physics of the 20th century. It is probably the most accurate description of gravitation that we know today, and, since its publication in 1915 [10], has been the origin of numerous predictions in the fields of Astrophysics, Cosmology, and even inside our own solar system. To this day, the theory has enjoyed a vast number of experimental confirmation with remarkable precision, the most recent such being the first direct detection of gravitational waves in 2015[1], and the direct observation of the shadow of a black hole resulting in the first image ever made of a black hole in 2019[2–9].

Historically, various alternatives to GR have been developed, but with increasing experimental accuracy, and consequently the increasing success of GR, many of these theories could be ruled out experimentally. It is therefore reasonable to look at theories that incorporate GR as a special case, employing the correspondence principle. One may wonder, with all the existing experimental evidence supporting GR, why even bother looking at modifications/extensions of the theory? Apart from academic curiosity, there is another fairly simple answer to this question: There are known problems with the theory. For example, in the field of cosmology, GR alone cannot describe the effects nowadays coined as inflation, dark matter and dark energy. This alone motivates studying extensions that could explain the cosmological observations without these additional constructs. Another, more catastrophic issue is the fact that GR has been shown to not be quantizable. Consequently, one has to think about modifications to either GR, or quantum theory, or both.

Since the field of alternative theories to GR is vast, we will understand what possible modifications are allowed without changing too many underlying principles of GR and then study more or less simple extensions to see their effects compared to Einstein's theory. Our starting point will be Lovelock gravity, which is based on a theorem by David Lovelock, about the generalization of Einstein gravity. Next, the generalization of this by Gregory Horndeski will be outlined, followed by an introduction to the so-called degenerate higher-order scalar-tensor theories (DHOST). Our work in DHOST resulted in three publications that are attached at the end of this chapter.

2 Lovelock gravity

Naively, one would expect to imagine additional terms to wildly modify the Einstein-Hilbert Lagrangian with, in such a way that it is still contained as a special case. This would quickly become a Sisyphean task, as of this approach considers a huge number of possibilities, while not providing any rule on how to select which of those are physically reasonable. It is therefore advantageous to impose a set of rules and then study what is the most general theory that we can get without breaking those. This has been done in 1971 by David Lovelock [66], and we shall outline it here:
The idea is to find a generalization of Einstein's field equations,

$$G_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.1}$$

Since the right-hand side is given by matter, the problem educes to finding a more general Tensor, A, such that

$$A_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.2}$$

We know that the Stress-Energy tensor is symmetric and divergence free. Further, we want to reproduce Newtonian gravity in the appropriate limit (remember that in Newtonian gravity the equations of motion are of second order). This motivates the following conditions:

(a) $A_{\mu\nu}$ is symmetric:

$$A_{\mu\nu} = A_{\nu\mu} \tag{2.3}$$

(b) $A_{\mu\nu}$ is a function of the metric and its first two derivatives:

$$A_{\mu\nu} = A_{\mu\nu}\left(g_{\mu\nu}; g_{\mu\nu,\lambda}; g_{\mu\nu,\lambda\rho}\right) \tag{2.4}$$

(c) $A_{\mu\nu}$ is divergence free:

$$A_{\mu\nu;\lambda} = 0 \tag{2.5}$$

Note that there is no constraint on the dimension of the underlying space-time.

Theorem 2.1 (Lovelock, 1971 [66]).
The only tensor $A_{\mu\nu}$ satisfying the conditions (a), (b) and (c) is

$$A^{\mu}_{\nu} = \sum_{p=1}^{[D/2]} a_p \delta^{\mu\alpha_1\ldots\alpha_{2p}}_{\nu\beta_1\ldots\beta_{2p}} R_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \cdots R_{\alpha_{2p-1}\alpha_{2p}}{}^{\beta_{2p-1}\beta_{2p}} + a\delta^{\mu}_{\nu}. \tag{2.6}$$

Further, the associated Lagrangian can be written as

$$\mathcal{L} = \sqrt{-g}\sum_{p=1}^{[D/2]} 2a_p \delta^{\mu\alpha_1\ldots\alpha_{2p}}_{\nu\beta_1\ldots\beta_{2p}} R_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \cdots R_{\alpha_{2p-1}\alpha_{2p}}{}^{\beta_{2p-1}\beta_{2p}} + 2a\delta^{\mu}_{\nu}, \tag{2.7}$$

where a and a_p are arbitrary constants.

In particular, in four dimensions, $A_{\mu\nu}$ reduces to a linear combination of the metric and the Einstein tensor, or in other words: the only possible theory in four dimensions is the Einstein-Hilbert action with cosmological constant.

Before continuing, it is instructive to mention that the Lovelock Lagrangian can be most conveniently written using the first order formalism introduced in chapter I (cf. [67], [68]):

$$L = \sum_{p=0}^{[D/2]} a_p L^{(D,p)}, \tag{2.8}$$

where, again, the a_p are arbitrary constants, and $L^{(D,p)}$ is given by

$$L^{(D,p)} = \epsilon_{a_1\ldots a_D} R^{a_1a_2} \wedge \cdots \wedge R^{a_{2p-1}a_{2p}} \wedge e^{a_{2p+1}} \wedge \cdots \wedge e^{a_D}, \tag{2.9}$$

with R^{ab} being the curvature two-form, and e^a the vielbein (cf. section 3 in chapter I), and a sum over repeated indices is understood. In this form, one can easily see the reduction in four dimensions by noting that $R^{ab} \wedge R^{cd}$ is nothing but the Gauß-Bonnet term, which is a topological invariant in four dimensions and does not affect the equations of motion.

Now, with this result in mind, to modify GR there are essentially three options:

- work in dimensions different from four,
- consider higher than second order derivatives in the metric,
- use more fields than just the metric tensor.

At this point, it is worth mentioning an interesting work in pure Lovelock theory, [69]. However, in the following sections we will look at more general theories, consisting of the metric tensor and a scalar field.

3 Horndeski gravity

Having discussed the theorem of Lovelock, it is now time to extend the theory by a scalar field. The extension of his theorem has been done by his student, Gregory Horndeski, in 1974[70] and states that the most general Lagrangian, constructed out of the metric tensor and a scalar field, leading to second order field equations can be written as a linear combination of the following Lagrangians [71–73]:

$$\begin{aligned} L_2^{\mathrm{H}} =& G_2(\phi, X), \qquad\qquad L_3^{\mathrm{H}} = G_3(\phi, X)\Box\phi, \\ L_4^{\mathrm{H}} =& G_4(\phi, X)R - 2G_{4,X}(\phi, X)\left(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right), \\ L_5^{\mathrm{H}} =& G_5(\phi, X)G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5,X}(\phi, X)\left(\Box\phi^3 - 3\Box\phi\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}{}_{\sigma}\right), \end{aligned} \tag{3.1}$$

with $\phi_\mu = \nabla_\mu\phi$, $\phi_{\mu\nu} = \nabla_\nu\nabla_\mu\phi$, $X = \phi^\mu\phi_\mu$, and where the G_i are arbitrary functions of ϕ and X. Additionally, we denoted with $,X$ a partial derivative with respect to the function X.
Clearly G_2 can be associated with the cosmological constant, and $G_4 = 1$ casts L_4^{H} into the Einstein-Hilbert action.

An interesting side note is that in [74] a Horndeski theory has been used as an effective model for dark energy, in which the L_2^{H} has been constrained by the ATLAS experiment [75]. Further, due to the direct measurement of the speed of gravitational waves, L_4^{H} and L_5^{H} have been strongly constrained [76–80].

3.1 Going beyond Horndeski: avoiding the Ostrogradski instability

We will now address why higher order equations have not been considered. Remember that the second order nature was always a condition. This is because until recently, it was believed that Lagrangians leading to higher order equations of motion are subject to so called Ostrogradski instabilities [81] (or for a review see [82]): Ghost-like additional degrees of freedom that cause the Hamiltonian of the theory to be unbounded from below. To avoid this catastrophe, people were firmly holding on to the condition of second order equations of motion, hence the particular formulation of the Horndeski theory. It is instructive to elucidate this using a simple toy model given in [83], so we will repeat it here.

Consider the following Lagrangian of a scalar field:

$$L = \frac{a}{2}\ddot{\phi}^2 - V(\phi), \tag{3.2}$$

where a is an arbitrary constant, and we do not need to specify the potential. The important part is the second order derivative. The equations of motion are of fourth order and hence require four initial conditions, which means that there are two dynamical degrees of freedom. Now introduce an auxiliary field to cast the Lagrangian into

$$\begin{aligned} L =& a\psi\ddot{\phi} - \frac{a}{2}\psi^2 - V(\phi) \\ =& -a\dot{\psi}\dot{\phi} - \frac{a}{2}\psi^2 - V(\phi) + a\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi\dot{\phi}\right) \end{aligned} \tag{3.3}$$

On-shell this is equivalent to the first Lagrangian, since the field equation for ψ yields $\psi = \ddot{\phi}$, and inserting this into our new Lagrangian results in the first.
Now we can define new variables, $q = (\phi + \psi)/\sqrt{2}$ and $Q = (\phi - \psi)/\sqrt{2}$, and so the Lagrangian reads

$$L = -\frac{a}{2}\dot{q}^2 + \frac{a}{2}\dot{Q}^2 - U(q, Q), \tag{3.4}$$

where all terms in q and Q have been absorbed within the potential. Now it is evident that the Lagrangian contains two dynamical degrees of freedom, however, they have a relative minus sign. So independent of the sign of a, there is one ghost-like degree of freedom, which in turn gives rise to the instability. Clearly the only way out of this is to set $a = 0$, though this would render the theory meaningless. However, there is a generalization given in [71–73]: In fact, having a more general Lagrangian leading to higher order field equations, one can follow a similar procedure as has been done in this toy model using such auxiliary fields. Instead of the constant a of the toy model above, one gets a Hessian matrix defined by

$$M \equiv \left(\frac{\partial^2 L}{\partial v^a \partial v^b}\right), \tag{3.5}$$

where $v^a \equiv (\dot{Q}_i)$, the vector of velocities. Then it can be shown that the ghost-like degrees of freedom disappear if the matrix, M, is degenerate, or in other words $\det M = 0$. The corresponding Lagrangian is called degenerate as well, and, since it is without Ostrogradski instabilities, it allows for higher order extensions to the Horndeski theories.

4 Degenerate Higher-Order Scalar-Tensor theories

In the previous section, we discussed how one can construct higher order extensions to Horndeski gravity, namely by fixing the Lagrangian to be degenerate. These scalar-tensor theories yielding higher than second derivative equations of motion are then free of Ostrogradski instabilities [71–73, 84, 85] and have been dubbed *Degenerate Higher Order Scalar Tensor* (DHOST) or *Extended Scalar Tensor* (EST) theories [71–73, 84–87], and are widely studied in the literature (see for example [88–96] for compact objects in these theories and [83] for a review).

We will now give an overview of the specific theories considered in our studies. These are by far not the most general cases that one can work with, though they already fairly extend GR, as we will see shortly. The more general cases can be found in the aforementioned literature and the references therein. We restrict ourselves to a class of shift symmetric and parity preserving (quadratic[1]) DHOST theories that contain up to second order covariant derivatives of the scalar field (in the Lagrangian), whose action in four dimensions is given by

$$\begin{aligned} S[g,\phi] = \int d^4x\sqrt{-g}\Big[&Z(X) + G(X)R + A_1(X)\phi_{\mu\nu}\phi^{\mu\nu} + A_2(X)(\Box\phi)^2 \\ &+ A_3(X)\Box\phi\,\phi^{\mu}\phi_{\mu\nu}\phi^{\nu} + A_4(X)\phi^{\mu}\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho} + A_5(X)\left(\phi^{\mu}\phi_{\mu\nu}\phi^{\nu}\right)^2\Big], \end{aligned} \tag{4.1}$$

[1] Second derivatives of the scalar field appear quadratically.

where the coupling functions Z, G, A_1, A_3, A_4 and A_5 are unspecified functions of X. However, the coupling functions A_4 and A_5 are chosen to satisfy

$$\begin{aligned} A_4 =& \frac{1}{8(G-XA_1)^2}\left\{4G\left[3(-A_1+2G_X)^2-2A_3G\right]-A_3X^2(16A_1G_X+A_3G)\right.\\ &\left.+4X\left[-3A_2A_3G+16A_1^2G_X-16A_1G_X^2-4A_1^3+2A_3GG_X\right]\right\}, \end{aligned} \tag{4.2a}$$

$$A_5 = \frac{1}{8(G-XA_1)^2}(2A_1-XA_3-4G_X)\left(A_1(2A_1+3XA_3-4G_X)-4A_3G\right). \tag{4.2b}$$

These are the degeneracy conditions necessary to ensure the absence of Ostrogradski ghosts [71–73, 84, 85]. Note that these conditions look different in other dimensions than four.

It is instructive to see this in the simple case of spherical symmetry. For this purpose, set $A_1(X) = -A_2(X) \neq -G/X$ (see section 4.2.1 of this chapter) and use a static[2] ansatz of the form

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \qquad \phi(t,r) = qt + \psi(r), \tag{4.3}$$

while leaving A_4 and A_5 unconstrained. Dealing with the equations, it proved convenient to define the frequently reoccurring combinations

$$\mathcal{Z}_1 = G + XA_2, \tag{4.4a}$$
$$\mathcal{Z}_2 = 2A_2 + XA_3 + 4G_X, \tag{4.4b}$$
$$\mathcal{Z}_3 = A_3 + A_4 + XA_5. \tag{4.4c}$$

We denote the respective equations of motion with $\mathcal{E}_{\mu\nu} = 0$. Then it turns out that $\mathcal{E}_{01} \propto q\mathcal{J}^r$, the r-component of the Noether current, and so the three independent equations of motion are $\mathcal{E}_{00}$, $\mathcal{E}_{11}$ and $\mathcal{J}^r$, which are of the general form:

$$\mathcal{E}_{00} = C_{0X}(h,f,X)X'' + q^2 C_{0h}(h,f,X)h'' + C_{0f}(h,h',X,X')f' + \Delta\mathcal{E}_{00}(h,h',f,X,X'), \tag{4.5a}$$
$$\mathcal{E}_{11} = C_{1X}(h,f,X)X'' + C_{1h}(h,f,X)h'' + C_{1f}(h,h',X,X')f' + \Delta\mathcal{E}_{11}(h,h',f,X,X'), \tag{4.5b}$$
$$\mathcal{J}^r = C_{JX}(h,f,X)X'' + C_{Jh}(h,f,X)h'' + C_{Jf}(h,h',X,X')f' + \Delta\mathcal{J}^r(h,h',f,X,X'), \tag{4.5c}$$

where the coefficients also depend on q. In fact, some coefficients are proportional to each other, so it is possible to eliminate certain terms. It actually turns out that combining $\mathcal{E}_{11}$ and $\mathcal{J}^r$ allows to express the metric function f in terms of h and its derivative as

$$\begin{aligned} \frac{h}{f} =& \frac{1}{4X\left(2G+r^2Z\right)}\left\{hX\left[8\mathcal{Z}_1 + rX'\left(4\mathcal{Z}_2 + r\mathcal{Z}_3X'\right)\right] + 4rq^2X'\left(\mathcal{Z}_2 - 4\mathcal{Z}_1'\right)\right.\\ &\left.+rXX'\left(16q^2A_2' + r\mathcal{Z}_2h' + rq^2A_5X'\right) + 8rX\mathcal{Z}_1h' + 8q^2A_2\left(X+rX'\right)\right\}. \end{aligned} \tag{4.6}$$

Using this, one equation can be eliminated to cast the system into the form

$$\mathcal{E}_{00} = D_{0X}(h,h',X,X')X'' + D_{0h}(h,h',X,X')h'' + \Delta\bar{\mathcal{E}}_{00}(h,h',X,X'), \tag{4.7a}$$
$$\mathcal{J}^r = D_{JX}(h,h',X,X')X'' + D_{Jh}(h,h',X,X')h'' + \Delta\bar{\mathcal{J}}^r(h,h',X,X'). \tag{4.7b}$$

Remember the role the Hessian matrix (3.5) played, viz

$$\det M = D_{Jh}D_{0X} - D_{JX}D_{0h} \overset{!}{=} 0, \tag{4.8}$$

[2] In [23] we consider a rotating ansatz (in three dimensions), see section 5 of this chapter.

leads to

$$0 = 3\mathcal{Z}_2^2 - 8\mathcal{Z}_1\mathcal{Z}_3, \tag{4.9}$$

and

$$0 = q^2 \left\{ 8XA_5\mathcal{Z}_1^2 - \mathcal{Z}_2 \left[A_2 \left(8\mathcal{Z}_1 - X\mathcal{Z}_2 \right) + 4\mathcal{Z}_1 \left(\mathcal{Z}_2 + 4XA_{2X} - 4\mathcal{Z}_{1X} \right) \right] \right\}. \tag{4.10}$$

Writing these out in terms of the original coupling functions gives exactly the DHOST degeneracy conditions (4.2a) and (4.2b).

Interestingly, one can easily do the same analysis in different dimensions and conclude that equation (4.9) in dimension $d > 2$ reads

$$K_d\mathcal{Z}_2^2 - \mathcal{Z}_1\mathcal{Z}_3 = 0, \tag{4.11}$$

with

$$K_d = \frac{1}{4}\left(1 + \frac{1}{d-2}\right). \tag{4.12}$$

It would be interesting to see how (4.10) looks like in arbitrary dimensions, however the equations in higher dimensions become very involved when q is non-zero.

4.1 Kerr-Schild invariance

The DHOST theories we will be working with enjoy several interesting features, one of which is the way it behaves under a so-called Kerr-Schild transformation. These transformations provide a way to construct many of the vacuum black hole solutions that are known by transforming a seed metric corresponding to the asymptotic metric of the space-time. This Kerr-Schild ansatz is a geometrical way of introducing the mass parameter into a solution, while all the other parameters (like angular momentum) must be non-trivially encoded in the seed metric. Note that in general, having a matter source may prevent one from using the ansatz, since that these source terms are usually incompatible with the Kerr-Schild transformation.

However, in our case we are applying the Kerr-Schild transformation to scalar tensor theories enjoying a shift symmetry and with a kinetic term, $X = \partial_\mu\phi\,\partial^\mu\phi$, that is invariant under the Kerr-Schild transformation. The Kerr-Schild ansatz is defined as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} - \mu a(x)\, l_\mu l_\nu \tag{4.13}$$

where $g^{(0)}_{\mu\nu}$ is the seed metric, μ the mass parameter, $a(x)$ a function to be determined, and l is a null and geodesic vector field with respect to both metrics, i.e.,

$$0 = g^{\mu\nu} l_\mu l_\nu = g^{(0)\mu\nu} l_\mu l_\nu, \qquad 0 = l^\mu \nabla_\mu l_\nu = l^\mu \nabla^{(0)}_\mu l_\nu. \tag{4.14}$$

Having defined the general case, we shall look at the static case now. In spherical symmetry, the seed metric, and the null and geodesic vector field are given by

$$ds_0^2 = -h_0(r)dt^2 + \frac{dr^2}{f_0(r)} + r^2 d\Omega^2, \qquad l = dt - \frac{dr}{\sqrt{f_0(r)h_0(r)}}. \tag{4.15}$$

The Kerr-Schild metric then becomes

$$ds^2 = -(h_0(r) + \mu a(r))dt^2 + \frac{h_0(r)\, dr^2}{f_0(r)\left(h_0(r) + \mu a(r)\right)} + r^2 d\Omega^2, \tag{4.16}$$

where the time coordinate has been redefined:

$$dt \to dt + \mu a(r) \frac{dr}{\sqrt{f_0(r)h_0(r)}\,(h_0(r) + \mu a(r))}. \tag{4.17}$$

And from this one can read off the net effect of a Kerr-Schild transformation on the metric functions, namely

$$h_0(r) \to h(r) = h_0(r) + \mu a(r), \qquad f_0(r) \to f(r) = \frac{f_0(r)\,(h_0(r) + \mu a(r))}{h_0(r)}. \tag{4.18}$$

We will now apply this to the DHOST theory (4.1) of the previous chapter using (4.3) as seed metric ansatz. Remember, that we additionally require the invariance of the kinetic term:

$$X^{(0)} = g^{(0)\mu\nu}\phi^{(0)}_\mu \phi^{(0)}_\nu = g^{\mu\nu}\phi_\mu\phi_\nu = X. \tag{4.19}$$

One may expect this to be fairly restrictive, so we will now look which conditions need to be satisfied in order for the action (4.1) to be (quasi-)invariant under a Kerr-Schild transformation (4.18), (4.19). At the level of the action (after two partial integrations) its variation under the Kerr-Schild transformation (4.18) yields

$$S[g^{(0)}, \phi] - S[g, \phi] = \frac{\mu}{4}\int dr \sqrt{\frac{f(r)}{h(r)}}\Big[a(r)P(r, X) + a'(r)Q(r, X)\Big]. \tag{4.20}$$

So we require the mass function $a(r)$ to satisfy the following first-order differential equation

$$a'(r)Q(r, X) + a(r)P(r, X) = 0, \tag{4.21}$$

where we have defined

$$Q(r, X) = r\left(8\mathcal{Z}_1 + r\mathcal{Z}_2 X'\right), \tag{4.22a}$$

$$P(r, X) = 8\mathcal{Z}_1 + rX'\left(4\mathcal{Z}_2 + r\mathcal{Z}_3 X'\right). \tag{4.22b}$$

The solution is given by

$$a(r) = e^{-\int \frac{P(r,X)}{Q(r,X)} dr}, \tag{4.23}$$

and the integration constant can be absorbed in the mass parameter, μ. One can easily see from equation (4.23) that the standard GR mass term is obtained for

$$Q(r, X) = rP(r, X), \tag{4.24}$$

which is the case if $X' = 0$ or for

$$r\left[XA_5 + A_4 + A_3\right]X' + 3\left[XA_3 + 2A_2 + 4G_X\right] = 0. \tag{4.25}$$

Or in other words, solutions with $X = \text{cst}$ have to have a standard GR mass term! Further, note that setting $A_2(X) = -A_1(X)$ was a necessary condition for the action to be (quasi-)invariant under a Kerr-Schild transformation [97].

As a last remark, we want to point out that in [97] it was shown how the Kerr-Schild solution generating method can be extended to DHOST theories to construct generic and regular (non-singular) black hole solutions. In [24] we have constructed another regular solution (with very interesting properties), which we have studied in our paper (see section 6 of this chapter). This can even be reversed to construct actions from known solutions of other theories (see Appendix 1).

4.2 Disformal transformation

We will now briefly address the question of how the different DHOST theories are related to one another. This has been studied extensively in [98] where they have found that all quadratic DHOST theories are actually stable under the action of the so-called disformal transformations of the metric defined in [99]. These can be written as

$$\bar{g}_{\mu\nu} = C(X,\phi)g_{\mu\nu} + D(X,\phi)\phi_\mu\phi_\nu, \tag{4.26}$$

where C and D are arbitrary functions of X and ϕ. Then, an action $\bar{S}[\bar{g},\phi]$ induces a new action, $S[g,\phi]$, by inserting the transformation:

$$S[g_{\mu\nu},\phi] = \bar{S}[\bar{g}_{\mu\nu} = Cg_{\mu\nu} + D\phi_\mu\phi_\nu, \phi]. \tag{4.27}$$

It was then shown in [98] that applying a disformal transformation to a DHOST theory yields again a DHOST theory. Further, they found that one can define different classes of theories, which are stable under certain subclasses of disformal transformations (specified by certain constraints on C and D). This classification of DHOST theories will be outlined next. In our work, we have applied the disformal transformation on a rotating stealth black hole solution. The results were published in [25] (see 7 of this chapter).

4.2.1 Classification of DHOST theories

Now we shall briefly outline the different classes of DHOST theories given in [98, 100]. Note that classes II and III do not concern us much, so we will just state them here for completeness.

(I) The DHOST theories of class I are defined by the relation $A_1 = -A_2$. These can be further divided into subclasses, through

$$A_1 = -\,A_2 \neq \frac{G}{X}, \tag{Ia}$$
$$A_1 = -\,A_2 = \frac{G}{X}. \tag{Ib}$$

It is evident that GR is included in Ia. Therefore, this subclass is very appealing from a physical perspective. Particularly, theories where c_g, the speed of gravitational waves equals the speed of light, $c_g = c$, we must impose that [100]

$$A_2 = 0, \tag{4.28}$$

and these particular theories are only stable under conformal transformations[100] ($D = 0$ in (4.26)). The degeneracy conditions (4.2a), (4.2b) simplify to

$$A_4 = -A_3 + \frac{1}{8G}(4G_X + A_3X)(12G_X - A_3X), \qquad A_5 = \frac{A_3}{2G}(4G_X + A_3X). \tag{4.29}$$

Inserting this into (4.22a) and (4.22b) one finds the relation

$$P(r,X) = \frac{Q(r,X)}{8Gr}\left[8G + 3r\left(4G_X + A_3X\right)X'\right], \tag{4.30}$$

and hence we get

$$-\frac{P(r,X)}{Q(r,X)} = -\frac{1}{r} - \frac{3}{2}\frac{X'}{X}R, \tag{4.31}$$

where we have defined

$$R(X) = \frac{X}{4G}(4G_X + A_3 X). \tag{4.32}$$

Now, the solution for $a(r)$ (equation (4.23)), is given by

$$a(r) = \frac{1}{r} e^{-\frac{3}{2}\int \frac{R(X)}{X}\,dX}. \tag{4.33}$$

This implies that the Coulombian form of the mass term will be valid for DHOST theories with $c_g = c$ only if $X =$ constant or $R = 0$, which is a simplification of equation (4.25).

(II) Theories that are neither in I, nor in III

(III) Theories with $G(X) = 0$.

Note that the classes II and III can be further divided in subclasses as well, though to properly describe them one would have to consider the general form of DHOST with its degeneracy conditions. These are thoroughly described in [98, 100].

Spinning black holes for generalized scalar-tensor theories in three dimensions

Olaf Baake [*]
Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca 3460000, Chile, and Centro de Estudios Cientificos (CECs), Avenida Arturo Part 514, Valdivia 5090000, Chile

Moisés Bravo-Gaete [†]
Facultad de Ciencias Básicas, Universidad Católica del Maule, Casilla 617, Talca 3460000, Chile

Mokhtar Hassaine [‡]
Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca 3460000, Chile

(Received 31 May 2020; accepted 9 July 2020; published 30 July 2020)

We consider a general class of scalar tensor theories in three dimensions whose action contains up to second-order derivatives of the scalar field with coupling functions that only depend on the standard kinetic term of the scalar field, thus ensuring the invariance under the constant shift of the scalar field. For this model, we show that the field equations for a stationary metric ansatz together with a purely radial scalar field can be fully integrated. The kinetic term of the scalar field solution is shown to satisfy an algebraic relation depending only on the coupling functions, and hence is constant while the metric solution is nothing but a Bañados-Teitelboim-Zanelli (BTZ)-like metric with an effective cosmological constant fixed in terms of the coupling functions. As a direct consequence the thermodynamics of the solution is shown to be identical to a BTZ-like one with an effective cosmological constant, despite the presence of a scalar field. Finally, the expression of the semiclassical entropy of this solution is also confirmed through a generalized Cardy-like formula involving the mass of the scalar soliton obtained from the black hole by means of a double Wick rotation.

DOI: 10.1103/PhysRevD.102.024088

I. INTRODUCTION

Since the discovery of the Bañados-Teitelboim-Zanelli (BTZ) black hole solution [1], the study of three-dimensional gravity has received considerable attention to such an extent that it is now considered an interesting laboratory to explore the many facets of the lower-dimensional physics at the classical level but also at the quantum level. By three-dimensional gravity we are referring not only to Einstein's standard action but to all of its possible variations, including, for example, its higher-order massive theories, such as the topologically massive gravity [2], or the new massive gravity [3]. The three-dimensional gravity models, with or without matter source, are likewise of importance due to the variety of their solutions, and particularly their asymptotic AdS black hole solutions whose near horizon geometry can be relevant to test some conceptual aspects of the AdS/CFT correspondence [4,5]. In this aspect, the BTZ solution is of particular interest because its in-depth study over the past three decades has considerably enhanced our knowledge on the statistical interpretation of the black hole entropy, see e.g., [6–8]. It is further fascinating that BTZ-like metrics arise as solutions of radically different three-dimensional gravity models. To illustrate this statement, we could mention for example the emergence of BTZ-like solutions in the context of massive gravity [9], in higher-order theories [10], but also in the presence of matter source, such as a scalar (dilatonic) field, see e.g., [11,12]. In the present work, we will confirm this trend by showing that the equations of motion of a general class of scalar tensor theories, enjoying a shift symmetry of the scalar field, and involving up to second-order derivatives of the scalar field, can be fully integrated and solved by a BTZ-like metric.

The interests of studying scalar tensor theories is mainly due to the fact that it constitutes one of the simplest modified gravity theories by extending general relativity with one or more scalar degrees of freedom. The dedication to scalar theories is not new and its origin may be attributed *a posteriori* to the seminal work of Horndeski [13], who presented the most general scalar tensor theory in four

[*] olaf.baake-at-inst-mat.utalca.cl
[†] mbravo-at-ucm.cl
[‡] hassaine-at-inst-mat.utalca.cl

dimensions with second order equations of motion. The requirement not to have more than two derivatives in the equations of motion is connected to the Ostrogradski theorem which states that (under certain assumptions) higher-order derivative theories have a Hamiltonian that is unbounded from below. This is related to the appearance of an extra (ghost) degree of freedom with negative energy. Thus the absence of higher time derivatives in the equations of motion guarantees the absence of the Ostrogradski ghost. Nevertheless, it has been shown recently that some particular higher-order theories of a single scalar field extension of general relativity can propagate healthy degrees of freedom and are mechanically stable. The most general such Lagrangian depending quadratically on second-order derivatives of a scalar field was constructed in [14,15], and dubbed degenerate higher order scalar tensor (DHOST) theory. This terminology indicates that the absence of Ostrogradski ghosts is mainly due to the degeneracy property of its Lagrangian. There even exists a subclass of DHOST theories where gravitational waves propagate at the speed of light, being in perfect agreement with the observed results [16]. While these attractive properties of scalar tensor theories occur in four dimensions, we nevertheless like to explore the implications of such models in three dimensions. This is precisely the aim of the present work.

Here, we will consider a general scalar tensor theory in three dimensions with a field content given by the metric g and a scalar field denoted by ϕ. The main assumption concerning the action is its invariance under the constant translation of the scalar field, i.e., $\phi \to \phi + \text{const}$ which implies the existence of a conserved Noether charge. It is known that this hypothesis considerably simplifies the integration of the equations of motion. The action will contain up to second-order covariant derivatives of the scalar field and is parity invariant, that is invariant under the discrete transformation $\phi \to -\phi$. The action is parametrized in terms of six coupling functions that depend only on the kinetic term $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ of the scalar field. Recently it has been shown that such scalar tensor theories are invariant under a Kerr-Schild symmetry, and this symmetry turns out to be extremely useful for generating black hole solutions from simple seed configurations [17]. Here we will adopt a different strategy by deriving the most general stationary solution by brute force, as it was done for the special case of Horndeski theory in three dimensions [17]. Interestingly enough, we will show that the integration of the equations of motion forces the scalar field to have a constant kinetic term while at the same time the metric functions turn out to be a BTZ-like spacetime with an effective cosmological constant expressed in terms of the coupling functions appearing in the action. We would like to emphasize that the constant value of the kinetic scalar field term results from an algebraic equation that X must satisfy, and consequently it does not correspond to any hair. Although the metric solution is given by a BTZ-like metric, it is legitimate to wonder whether the presence of the scalar field could affect the thermodynamic properties of the solution. In order to answer this question, the thermodynamics of the solution is carefully analyzed within the Euclidean method [18,19], and it is shown that the expressions of the mass, entropy and angular momentum are identical to those of a BTZ-like solution with an effective cosmological constant. In addition, since it has been pointed out that the Wald formula for the entropy [20] applied to general scalar tensor theories may be problematic [21], we have found it sensible to compute the entropy of the solution by means of a generalized Cardy formula. In this formulation the ground state is identified with a soliton whose mass is proportional to the lowest eigenvalues of the shifted Virasoro operators, see [22–26]. In order to achieve this task, we have constructed the static scalar soliton from the black hole through a double Wick rotation and computed its mass. Finally, the application of the generalized Cardy formula is shown to properly reproduce the semiclassical expression of the entropy.

The plan of the paper is organized as follows. In the next section we will present the action and derive the most general solution for a stationary ansatz for the metric together with a radial scalar field. We will show that the metric solution is nothing other than a BTZ-like metric while the kinetic term of the scalar field is constant. In Sec. III we will construct the regularized Euclidean action which allows us to identify the mass, the angular momentum and the entropy. Further, the expression of the entropy will be confirmed through a computation involving the generalized Cardy formula and the mass of the static scalar soliton. The mass of the soliton will be computed using the quasilocal formalism [27]. Finally, in Sec. IV we present our conclusions and discussions.

II. SCALAR FIELD MODEL AND THE DERIVATION OF ITS SOLUTION

In three dimensions, we are considering a scalar tensor theory whose dynamical fields are represented by a metric, g, and a scalar field, ϕ. The action reads

$$
\begin{aligned}
S &= \int d^3x\sqrt{-g}\mathcal{L} \\
&= \int d^3x\sqrt{-g}[Z(X) + G(X)R + A_3(X)\Box\phi\phi^\mu\phi_{\mu\nu}\phi^\nu \\
&\quad + A_2(X)((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}) + A_4(X)\phi^\mu\phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho \\
&\quad + A_5(X)(\phi^\mu\phi_{\mu\nu}\phi^\nu)^2], \qquad (1)
\end{aligned}
$$

where for simplicity we have defined $X = \partial_\mu\phi\partial^\mu\phi$ and $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$. Here, the six coupling functions Z, G and A_i for $i = 2, \ldots 5$ are *a priori* arbitrary functions of the kinetic term X, and contain up to second-order covariant derivatives of the scalar field. It is easy to see that the action is invariant under the shift symmetry $\phi \to \phi + \text{const}$, as well as under the discrete transformation $\phi \to -\phi$. The field equations of the action (1) are reported in the Appendix.

More general, scalar tensor theories can be considered with coupling functions that additionally depend on the scalar field, $Z = Z(\phi, X)$, $G = G(\phi, X)$ and $A_i = A_i(\phi, X)$. It is interesting to note that for Lagrangians breaking the shift symmetry $\phi \to \phi + \text{const}$, and of the form

$$\mathcal{L} = Z(\phi) + G(\phi)R,$$

there exist black hole solutions which drastically differ from the BTZ solution, see for example [28] and references therein. In contrast, as we will show below, for the shift symmetric action (1), the spectrum of black hole solutions is restricted to BTZ-like solutions.

We now look for black hole solutions of the action (1) with a stationary metric and a purely radial scalar field. The most general such ansatz can be parametrized as follows:

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + H^2(r)[d\theta - k(r)dt]^2, \\ \phi &= \phi(r). \end{aligned} \tag{2}$$

After some tedious computations one can show that the field equations associated to the action (see the Appendix) will become fully integrable for the ansatz (2) by fixing the coupling function A_5 in terms of the others through the following relation:

$$A_5 = \frac{(2A_2 + XA_3 + 4G_X)^2}{2X(G + XA_2)} - \frac{A_3 + A_4}{X}, \tag{3}$$

where $G_X = \frac{dG}{dX}$. This relation is quite similar to the four-dimensional DHOST conditions which ensure the absence of Ostrogradski ghosts [14,15]. Further, the emergence of the condition (3) is not surprising, since in the literature concerning scalar tensor theories of the type (1), most of the solutions are found for special relations between the coupling functions Z, G and the A_i's, see e.g., [29–36].

In what follows, we will consider the action (1) with the coupling function A_5 given by the relation (3), and for later convenience, we also define the following expressions:

$$\mathcal{Z}_1 = G + XA_2, \tag{4a}$$

$$\mathcal{Z}_2 = 2A_2 + XA_3 + 4G_X. \tag{4b}$$

We are now in the position to present the general derivation of the spinning solution. As a first step, we consider the following combination of the metric equations:

$$\mathcal{E}_{t\theta} + k\mathcal{E}_{\theta\theta} = 0,$$

which yields a first integral given by

$$(\mathcal{Z}_1 H^3 k')' = 0. \tag{5}$$

Further, the combination

$$2\sqrt{f^3 X}\mathcal{E}_{rr} - J^r = 0,$$

where J^r is the radial component of the conserved current (see the Appendix), permits to express the derivative of the metric function f as

$$f' = -\frac{4fH'\mathcal{Z}_1\mathcal{Z}_2X' + fH\mathcal{Z}_2^2X'^2 + 4H^3k'^2\mathcal{Z}_1^2 - 8HZ\mathcal{Z}_1}{8H'\mathcal{Z}_1^2 + 2H\mathcal{Z}_1\mathcal{Z}_2X'}. \tag{6}$$

Inserting this into the two combinations

$$\frac{\mathcal{Z}_2}{f}(\mathcal{E}_{tt} + k\mathcal{E}_{t\theta}) = 0, \tag{7a}$$

$$\frac{\mathcal{Z}_1}{\sqrt{fX}}\mathcal{E}_J + \frac{\mathcal{Z}_2}{H^2}\mathcal{E}_{t\theta} = 0, \tag{7b}$$

one obtains after some manipulations the following equation:

$$k\mathcal{Z}_2(\mathcal{Z}_1H^3k')' + 4H[(\mathcal{Z}_1Z)_X - Z\mathcal{Z}_2] = 0. \tag{8}$$

By Eq. (5) the first term vanishes, leaving

$$(\mathcal{Z}_1Z)_X - Z\mathcal{Z}_2 = 0. \tag{9}$$

It is easy to see that, for $Z = 0$, Eq. (9) is completely degenerate and gives no information about the kinetic term. Consequently in what follows we impose Z to be nonzero. Then the kinetic term X must satisfy this algebraic equation which in turn implies that X has to be constant. Moreover, we would like to stress that its constant value is not an integration constant but must be rather understood as follows: Given a scalar tensor theory (1)–(3) with specific coupling functions Z, G and the A_i's, the constant value of X will be determined by the algebraic relation (9). This restriction on the kinetic term of the scalar field significantly simplifies the equations, in particular the combination

$$\frac{2}{f}(\mathcal{E}_{tt} + 2k\mathcal{E}_{t\theta} + k\mathcal{E}_{\theta\theta}) + 2f\mathcal{E}_{rr} + \sqrt{\frac{X}{f}}J^r = 0 \tag{10}$$

implies

$$H'' = 0. \tag{11}$$

Finally, after some redefinitions of the coordinates, the metric solution can be casted in a standard BTZ-like form as

$$ds^2 = -N(r)^2 F(r) dt^2 + \frac{dr^2}{F(r)} + r^2 (d\theta + N^\theta(r) dt)^2, \qquad (12a)$$

$$N = 1, \quad F = \left(\frac{Z}{2\mathcal{Z}_1} r^2 - M + \frac{J^2}{4r^2}\right), \quad N^\theta = \frac{J}{2r^2}. \qquad (12b)$$

This metric is nothing but a BTZ-like solution with an effective cosmological cosmological constant given by $\Lambda_{\rm eff} = -Z/2\mathcal{Z}_1$. Various comments can be made concerning the emergence of a BTZ-like metric solution together with a scalar field with constant kinetic term. First of all, it is remarkable that the equations of motion of the general class of scalar tensor theories, given by the action (1) together with the condition (3), are fully integrable and yield a BTZ-like solution. It is also remarkable that, although the model is defined in terms of the coupling functions Z, G and A_2, A_3 and A_4, the resulting solution is shown to be parametrized in terms of Z and through the combinations $\mathcal{Z}_1$ and $\mathcal{Z}_2$, as defined in Eq. (4). This would also imply that scalar tensor theories of the form (1)–(3) with different coupling functions Z, G and A_i's can have the same effective cosmological constant $\Lambda_{\rm eff}$, and hence they can be solved by the same BTZ-like metric. Moreover, it is easy to see that the action (1)–(3) enjoys a Kerr-Schild symmetry as defined in [17] whose implementation on the stationary ansatz (2) can be summarized as

$$f(r) \to f(r) - a(r), \quad H(r) \to H(r), \quad k(r) \to k(r), \quad (13)$$

with a constant mass term i.e., $a(r) = M$ which is a direct consequence of the kinetic term X being constant. This can be put in analogy with the four-dimensional static case where solutions of the action (1) with constant kinetic term were shown to have the standard Coulomb mass term $a(r) = \frac{M}{r}$, see Ref. [17].

In summary, we have shown that for a stationary ansatz (2) the integration of the field equations of (1)–(3) forces X to be constant together with a BTZ-like metric (12a) and (12b). In the following section, we analyze its thermodynamics.

III. THERMODYNAMICS OF THE SPINNING SOLUTION

The thermodynamics of the solution will now be determined by means of the Euclidean method [18,19], where the Euclidean continuation of the metric is obtained by setting $t = -i\tau$ in the ansatz (12a). In order for the resulting metric to be real, one can introduce a complex constant of integration for the Euclidean momentum as $J_{\rm Eucl} = -iJ$, where J will be identified with the physical angular momentum. In order to avoid a conical singularity, the Euclidean time τ has to be made periodic with period $\beta = 1/T$, where T is the temperature that in our case is given by

$$T = \frac{F'(r)}{4\pi}\bigg|_{r=r_h} = \frac{1}{4\pi}\left(\frac{2r_h}{L^2} - \frac{J^2}{2r_h^3}\right), \qquad (14)$$

and where for simplicity we have defined the square of the effective AdS radius

$$L^2 = \frac{2\mathcal{Z}_1(X)}{Z(X)}.$$

Recall that the kinetic term X is a constant determined by the algebraic relation defined by Eq. (9). After some computations, the Euclidean action is shown to be given by

$$\begin{aligned} I_E = 2\pi\beta \int_{r_h}^{+\infty} dr \Bigg\{ N\Bigg[F'\left(\frac{1}{4} r (F(\phi')^2)' \mathcal{Z}_2 + \mathcal{Z}_1\right) \\ + \frac{1}{2} F (4(F(\phi')^2)' \mathcal{Z}_{1X} + r\mathcal{Z}_2 (F(\phi')^2)'') \\ + \frac{1}{4} F r ((F(\phi')^2)')^2 \left(2\mathcal{Z}_{2X} - \frac{\mathcal{Z}_2^2}{2\mathcal{Z}_1}\right) - Zr \\ + \frac{1}{2}\frac{p^2}{\mathcal{Z}_1 r^3}\Bigg] + N^\theta p' \Bigg\} + B_E, \end{aligned}$$

where r_h is the radius of the event horizon and

$$p(r) = \frac{r^3 (N^\theta)' \mathcal{Z}_1}{N}.$$

The Euclidean action I_E is defined up to a boundary term B_E which is fixed such that said action has an extremum, which is $\delta I_E = 0$. In the present case, the variation of this boundary term can be conveniently expressed as

$$\begin{aligned} \delta B_E = -2\pi\beta \Bigg[\left(\left(\frac{\delta I_E}{\delta F'}\right) - \left(\frac{\delta I_E}{\delta F''}\right)'\right)\delta F \\ + \left(\frac{\delta I_E}{\delta F''}\right)\delta F' + \left(\frac{\delta I_E}{\delta \phi''}\right)\delta\phi' + \left(\frac{\delta I_E}{\delta \phi'''}\right)\delta\phi'' \\ - \left(\frac{\delta I_E}{\delta \phi'''}\right)'\delta\phi' + \Bigg(\left(\frac{\delta I_E}{\delta \phi'}\right) - \left(\frac{\delta I_E}{\delta \phi''}\right)' \\ + \left(\frac{\delta I_E}{\delta \phi'''}\right)'' + 2F\phi'\left(\frac{\delta I_E}{\delta X}\right)\Bigg)\delta\phi + N^\theta \delta p \Bigg]_{r=r_h}^{r=+\infty}. \end{aligned}$$

At infinity, most of these terms cancel each other out, yielding

$$\delta B_E|_{+\infty} = 2\pi\beta\mathcal{Z}_1 \delta M \Rightarrow B_E|_{+\infty} = 2\pi\beta\mathcal{Z}_1 M,$$

while that at the horizon

$$\begin{aligned} \delta B_E|_{r_h} = 8\mathcal{Z}_1\pi^2 \delta r_h - 2\pi\beta\Omega\delta(\mathcal{Z}_1 J) \Rightarrow \\ B_E|_{r_h} = 8\mathcal{Z}_1\pi^2 r_h - 2\pi\beta\Omega\mathcal{Z}_1 J. \end{aligned}$$

In this expression, Ω represents the chemical potential, defined by

$$\Omega = \lim_{r\to+\infty} N^\theta(r) - N^\theta(r_h) = -\frac{J}{2r_h^2}.$$

With all of the above, the boundary term B_E is simply expressed as

$$\begin{aligned} B_E &= B_E|_{+\infty} - B_E|_{r_h} \\ &= 2\pi\beta\mathcal{Z}_1 M - 8\mathcal{Z}_1\pi^2 r_h + 2\pi\beta\Omega\mathcal{Z}_1 J. \end{aligned} \tag{15}$$

Finally, the thermodynamic quantities can be read off from the Gibbs free energy F:

$$I_E = \beta F = \beta\mathcal{M} - \mathcal{S} - \beta\Omega\mathcal{J}, \tag{16}$$

where $\mathcal{M}$ is the mass, $\mathcal{S}$ the entropy and, as before, Ω is the chemical potential associated with the angular momentum $\mathcal{J}$, see [18]. Finally, comparing (15) with (16), the thermodynamic parameters turn out to be given by

$$\mathcal{S} = 8\mathcal{Z}_1\pi^2 r_h, \tag{17a}$$

$$\mathcal{M} = 2\pi\mathcal{Z}_1 M = 2\pi\mathcal{Z}_1\left(\frac{r_h^2}{L^2} + \frac{J^2}{4r_h^2}\right), \tag{17b}$$

$$\mathcal{J} = -2\pi\mathcal{Z}_1 J, \qquad \Omega = -\frac{J}{2r_h^2}, \tag{17c}$$

and one can easily see that the first law holds, namely $d\mathcal{M} = Td\mathcal{S} + \Omega d\mathcal{J}$. These thermodynamic quantities (17) are identical to those of a BTZ-like solution with an effective AdS radius given by L. It is clear that in order to deal with positive mass (entropy) solutions, we have to impose that $\mathcal{Z}_1 > 0$ and $Z > 0$. It is natural to compare the scalar tensor solutions with the standard BTZ solution which would correspond to setting $\phi = 0$. In the static case, the difference between the free energies, ΔF, of the BTZ solution and the scalar solution reads

$$\Delta F = F_{\mathrm{BTZ}} - F = 16\pi^3 T^2\left[\frac{\mathcal{Z}_1^2(X)}{Z(X)} - \frac{\mathcal{Z}_1^2(0)}{Z(0)}\right]. \tag{18}$$

Hence, defining $P(X) = \frac{\mathcal{Z}_1^2(X)}{Z(X)}$, we conclude that for $P(X) < P(0)$ [respectively for $P(X) > P(0)$], the BTZ solution (respectively the scalar tensor solution) is thermodynamically favored.

We now proceed by rederiving the expression of the semiclassical entropy (17a) by means of a generalized Cardy formula. In this formulation, the entropy of the black hole solution can be microscopically computed provided the theory admits a regular scalar soliton which would be identified as the ground state of the theory, see [22,24]. In our case, the regular soliton will be obtained from the static black hole solution (12) with $J = 0$ through a double Wick rotation $t \to i\theta$ and $\theta \to it$ together with an identification for the location of the event horizon $r_h = L$ given by

$$ds^2 = -\frac{r^2}{L^2}dt^2 + \left(\frac{r^2}{L^2} - 1\right)^{-1} dr^2 + \left(\frac{r^2}{L^2} - 1\right) d\theta^2,$$

and the line element of the regular static scalar solution after a redefinition of the radial coordinate reads

$$ds^2 = -\cosh^2(\rho)dt^2 + L^2 d\rho^2 + \sinh^2(\rho)d\theta^2. \tag{19}$$

As done for example in Refs. [25,26], the mass of the soliton (19) will be computed within the quasilocal formalism defined in [27]. In order to be as self-contained as possible, we will elaborate the steps of the computations. To begin with, the variation of the action (1)–(3) can be schematically represented as

$$\delta S = \sqrt{-g}[\varepsilon_{\mu\nu}\delta g^{\mu\nu} + \varepsilon_{(\phi)}\delta\phi] + \partial_\mu\Theta^\mu(\delta g, \delta\phi), \tag{20}$$

where $\varepsilon_{\mu\nu}$ and $\varepsilon_{(\phi)}$ correspond to the equations of motions with respect to the metric $g_{\mu\nu}$ and the scalar field ϕ (see the Appendix), while Θ^μ is a surface term whose expression is given by

$$\begin{aligned} \Theta^\mu = \sqrt{-g}\Bigg[&2(P^{\mu(\alpha\beta)\gamma}\nabla_\gamma\delta g_{\alpha\beta} - \delta g_{\alpha\beta}\nabla_\gamma P^{\mu(\alpha\beta)\gamma}) \\ &+ \frac{\delta\mathcal{L}}{\delta(\phi_\mu)}\delta\phi - \nabla_\nu\left(\frac{\delta\mathcal{L}}{\delta(\phi_{\mu\nu})}\right)\delta\phi + \frac{\delta\mathcal{L}}{\delta(\phi_{\mu\nu})}\delta(\phi_\nu) \\ &- \frac{1}{2}\frac{\delta\mathcal{L}}{\delta(\phi_{\mu\sigma})}\phi^\sigma\delta g_{\sigma\rho} - \frac{1}{2}\frac{\delta\mathcal{L}}{\delta(\phi_{\sigma\mu})}\phi^\sigma\delta g_{\sigma\rho} \\ &+ \frac{1}{2}\frac{\delta\mathcal{L}}{\delta(\phi_{\sigma\rho})}\phi^\mu\delta g_{\sigma\rho}\Bigg], \end{aligned}$$

with $P^{\mu\nu\lambda\rho} = \delta\mathcal{L}/\delta R_{\mu\nu\lambda\rho}$, and $\mathcal{L}$ is the Lagrangian. Considering now the variation induced by a diffeomorphism generated by a Killing vector ξ^μ whose action on the metric and the scalar field read

$$\delta_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}, \qquad \delta_\xi\phi = \xi^\sigma(\nabla_\sigma\phi),$$
$$\delta_\xi(\nabla_\nu\phi) = \xi^\sigma\phi_{\sigma\nu} + (\nabla_\nu\xi^\sigma)\phi_\sigma,$$

we construct a Noether current given by

$$\mathcal{L}\xi^\mu + 2\varepsilon^{\mu\nu}\xi_\nu - \Theta^\mu(\delta_\xi g, \delta_\xi\phi) = \nabla_\nu K^{\mu\nu},$$

which is derived from the potential $K^{\mu\nu}$,

$$\begin{aligned} K^{\mu\nu} = \sqrt{-g}\Bigg[&2P^{\mu\nu\rho\sigma}\nabla_\rho\xi_\sigma - 4\xi_\sigma\nabla_\rho P^{\mu\nu\rho\sigma} + \frac{\delta\mathcal{L}}{\delta\phi_{\mu\sigma}}\phi^\nu\xi_\sigma \\ &- \frac{\delta\mathcal{L}}{\delta\phi_{\nu\sigma}}\phi^\mu\xi_\sigma\Bigg]. \end{aligned}$$

As shown in [27], for each Killing field a corresponding conserved quantity can be constructed as

$$Q(\xi) = \int_{\mathcal{B}} dx_{\mu\nu} \left(\delta K^{\mu\nu}(\xi) - 2\xi^{[\mu} \int_0^1 ds \Theta^{\nu]} \right). \quad (21)$$

Here, $\delta K^{\mu\nu}(\xi) = K^{\mu\nu}_{s=1}(\xi) - K^{\mu\nu}_{s=0}(\xi)$ is the difference of the Noether potential interpolating between the solutions along the path parametrized by $s \in [0, 1]$, and $dx_{\mu\nu}$ represents the integration over the two-dimensional boundary $\mathcal{B}$. For the Killing field, $\xi = \partial_t$, one obtains that

$$\delta K^{rt} = -\frac{2G}{L} \int_0^1 ds \Theta^r = -\frac{\mathcal{Z}_1}{L} + \frac{2G}{L},$$

yielding a mass for the static soliton (19) given by

$$\mathcal{M}_{\rm sol} = -2\pi \mathcal{Z}_1, \quad (22)$$

which is negative by virtue of the fact that $\mathcal{Z}_1 > 0$ in order to ensure a positive mass and entropy of the black hole solutions. As is the case for the BTZ solution, there is a phase transition between the black hole and the soliton at the critical temperature

$$T_c = \frac{\sqrt{2}}{4\pi} \sqrt{\frac{Z(X)}{\mathcal{Z}_1(X)}}.$$

We are now in position to provide a microscopic computation of the black hole entropy. As stressed in [22], the Cardy formula is more conveniently expressed in terms of the vacuum charge rather than the central charge:

$$\mathcal{S}_C = 4\pi \sqrt{-\tilde{\Delta}_0^+ \tilde{\Delta}^+} + 4\pi \sqrt{-\tilde{\Delta}_0^- \tilde{\Delta}^-}, \quad (23)$$

where ($\tilde{\Delta}_0^\pm$) $\tilde{\Delta}^\pm$ are the (lowest) eigenvalues of the shifted Virasoro operators. The eigenvalues are related to the mass and angular momentum as [6]

$$\mathcal{M} = \frac{1}{L}(\tilde{\Delta}^+ + \tilde{\Delta}^-), \qquad \mathcal{J} = \tilde{\Delta}^+ - \tilde{\Delta}^-.$$

On the other hand, since the scalar soliton is identified with the ground state of the theory, its mass (22) is proportional to the lowest eigenvalue

$$\tilde{\Delta}_0^\pm = \frac{L}{2} \mathcal{M}_{\rm sol}.$$

Finally, the Cardy formula (23) can be conveniently rewritten in terms of $\mathcal{M}$, $\mathcal{J}$ as

$$\mathcal{S}_C = 2\pi \sqrt{-L\mathcal{M}_{\rm sol}(L\mathcal{M} + \mathcal{J})} + 2\pi \sqrt{-L\mathcal{M}_{\rm sol}(L\mathcal{M} - \mathcal{J})},$$

and it can be verified that this correctly reproduces the semiclassical entropy (17), this is, $\mathcal{S}_C = \mathcal{S}$.

It is known from the pioneer work of Brown and Henneaux [5] that the asymptotic symmetries of the three-dimensional BTZ-like solution (12) consist in two copies of the Virasoro algebra with equal left and right moving central charges given by

$$c_+ = c_- = \frac{3L(X)}{2G_N}, \quad (24)$$

where G_N is the Newton constant. Hence, according to the AdS/CFT correspondence this family of solutions (12) would correspond to a CFT with central charges given by (24) and depending explicitly on the constant value X as determined by Eq. (9). This clearly emphasizes the difference with the BTZ solution in the sense that the scalar field, through its constant kinetic term X, will leave its mark. This would mean that, in principle, for a given CFT with equal central charges, it can be possible to adjust the form of $L(X)$ for its central charges to coincide with (24).

IV. CONCLUSIONS AND DISCUSSIONS

In the present work, we have shown that the equations of motion of a very general class of scalar tensor theories (1)–(3) can be fully integrated for a stationary metric ansatz together with a purely radial scalar field. Interestingly enough, the kinetic term of the scalar field solution was forced to be constant, while at the same time the spacetime metric resulted to be a BTZ-like metric with an effective cosmological constant expressed in terms of the coupling functions. It is somehow appealing that the spectrum of such general class of theories only consists of a BTZ-like metric with (different) effective cosmological constants. This observation is even more relevant considering that in four dimensions, theories which are much less general than that studied here admit black hole solutions that are asymptotically AdS, flat or even exhibit a rather exotic asymptotic behavior [29–36]. Even more, in four dimensions a recipe has even been given to construct black hole solutions from any simple seed metric [17]. Nevertheless, one can notice an important difference concerning the kinetic term of the scalar field solution between the three- and the four-dimensional situations. Indeed, solutions in four dimensions with a nonconstant kinetic term were shown to exist [17,35], while in our case the algebraic relation (9) forces the kinetic term to be constant. One might also have thought that the presence of a coupled scalar field should have affected the thermodynamics of the solution but this was not the case. This is essentially due to the constancy of the kinetic term of the scalar field solution. It would be nice to provide a physical explanation for the emergence of a BTZ-like metric as the solution of such a very general class of scalar tensor theories (1)–(3).

It is further intriguing that the equations of motion become fully integrable by imposing the condition (3) on the coupling function A_5. As mentioned before, this relation is quite similar to the four-dimensional DHOST conditions [14,15] which prevent the emergence of Ostrogradski ghosts. It would be compelling to explore this point more deeply. Moreover, it is worth mentioning that said BTZ-like solution remains a solution even if one replaces the scalar field ansatz with $\phi = qt + \psi(r) + L\theta$ in (2), and if X still solves the algebraic Eq. (9). Note that in this case, the vanishing of the radial component of the current $J^r = 0$ is a consequence of the field equation [37]. In [17] the uniqueness of this solution was shown for the quadratic Horndeski action. Whether or not it is unique in the general case has yet to be established.

In [17], it was shown that the solution generating method also applies for generalized Proca theories with solutions having a nonzero radial component for the potential [38]. Hence, in a complete analogy with the work done here, it will be interesting to look for black hole solutions in three dimensions for more general vector tensor theories [39].

ACKNOWLEDGMENTS

O. B. is funded by the Ph.D. scholarship of the University of Talca. M. H. gratefully acknowledges the kind support of the ECOSud Project No. C18U04. M. B. is supported by grant Programa Fondecyt de Iniciación en Investigación No. 11170037.

APPENDIX: FIELD EQUATIONS ASSOCIATED TO THE ACTION (1)

Here we report the equations of motion of the action (1) that are obtained by varying the action with respect to the metric $\mathcal{E}_{\mu\nu}$ and those with respect to the scalar field $\varepsilon_{(\phi)}$. The former are given by

$$\mathcal{E}_{\mu\nu} \coloneqq \mathcal{G}^Z_{\mu\nu} + \mathcal{G}^G_{\mu\nu} + \sum_{i=2}^{5} \mathcal{G}^{(i)}_{\mu\nu} = 0, \tag{A1}$$

where

$$\begin{aligned}
\mathcal{G}^Z_{\mu\nu} &= -\frac{1}{2}Z(X)g_{\mu\nu} + K_X\phi_\mu\phi_\nu,\\
\mathcal{G}^G_{\mu\nu} &= GG_{\mu\nu} + G_XR\phi_\mu\phi_\nu - \nabla_\nu\nabla_\mu G + g_{\mu\nu}\nabla_\lambda\nabla^\lambda G,\\
\mathcal{G}^{(2)}_{\mu\nu} &= -\phi_\mu(A_{2X}\nabla_\nu X)\Box\phi - (A_{2X}\nabla_\mu X)\phi_\nu\Box\phi - A_2\phi_{\nu\mu}\Box\phi - \phi_{\nu\mu}\phi_\lambda(A_{2X}\nabla^\lambda X) + \phi_\nu\phi_{\lambda\mu}(A_{2X}\nabla^\lambda X) + \phi_\mu\phi_{\lambda\nu}(A_{2X}\nabla^\lambda X)\\
&\quad + A_2R_{\nu\lambda}\phi_\mu\phi^\lambda + A_2R_{\mu\lambda}\phi_\nu\phi^\lambda - A_2\phi_{\lambda\nu\mu}\phi^\lambda + \frac{1}{2}A_2g_{\mu\nu}(\Box\phi)^2 + g_{\mu\nu}\phi_\lambda(A_{2X}\nabla^\lambda X)\Box\phi + A_2g_{\mu\nu}\phi^\lambda\phi^\rho_{\rho\lambda} - A_2g_{\mu\nu}R_{\lambda\rho}\phi^\lambda\phi^\rho\\
&\quad + \frac{1}{2}A_2g_{\mu\nu}\phi_{\rho\lambda}\phi^{\rho\lambda} + A_{2X}\phi_\mu\phi_\nu((\Box\phi)^2 - \phi_{\lambda\rho}\phi^{\lambda\rho}),\\
\mathcal{G}^{(3)}_{\mu\nu} &= -\frac{1}{2}A_3\phi_\mu\phi_\nu(\Box\phi)^2 - \frac{1}{2}\phi_\mu\phi_\nu\phi_\lambda(A_{3X}\nabla^\lambda X)\Box\phi + \frac{1}{2}A_3\phi_\mu\phi_{\lambda\nu}\phi^\lambda\Box\phi + \frac{1}{2}A_3\phi_\nu\phi_{\lambda\mu}\phi^\lambda\Box\phi - \frac{1}{2}A_3\phi_\mu\phi_\nu\phi^\lambda\phi^\rho_{\rho\lambda}\\
&\quad + \frac{1}{2}A_3R_{\lambda\rho}\phi_\mu\phi_\nu\phi^\lambda\phi^\rho - \frac{1}{2}\phi_\mu(A_{3X}\nabla_\nu X)\phi^\lambda\phi_{\rho\lambda}\phi^\rho - \frac{1}{2}(A_{3X}\nabla_\mu X)\phi_\nu\phi^\lambda\phi_{\rho\lambda}\phi^\rho - \frac{1}{2}A_3\phi_\nu\phi^\lambda\phi_{\rho\lambda\mu}\phi^\rho - \frac{1}{2}A_3\phi_\mu\phi^\lambda\phi_{\rho\lambda\nu}\phi^\rho\\
&\quad - A_3\phi_\nu\phi^\lambda\phi_{\rho\lambda}\phi^\rho_\mu - A_3\phi_\mu\phi^\lambda\phi_{\rho\lambda}\phi^\rho_\nu + \frac{1}{2}g_{\mu\nu}\phi_\lambda(A_{3X}\nabla^\lambda X)\phi^\rho\phi_{\sigma\rho}\phi^\sigma + \frac{1}{2}g_{\mu\nu}A_3\phi^\lambda\phi^\rho\phi_{\sigma\rho\lambda}\phi^\sigma + g_{\mu\nu}A_3\phi^\lambda\phi^\rho\phi_{\sigma\rho}\phi^\sigma_\lambda\\
&\quad + A_{3X}\phi_\mu\phi_\nu(\Box\phi)\phi^\rho\phi_{\sigma\rho}\phi^\sigma,\\
\mathcal{G}^{(4)}_{\mu\nu} &= -A_4\phi_\mu\phi_\nu\phi^\lambda\phi^\rho_{\rho\lambda} + A_4\phi_{\lambda\mu}\phi^\lambda\phi_{\rho\nu}\phi^\rho - \phi_\mu\phi_\nu(A_{4X}\nabla^\lambda X)\phi_{\rho\lambda}\phi^\rho - A_4\phi_\mu\phi_\nu\phi_{\rho\lambda}\phi^{\rho\lambda} - \frac{1}{2}A_4g_{\mu\nu}\phi^\lambda\phi^\rho\phi_{\sigma\rho}\phi^\sigma_\lambda\\
&\quad + A_{4X}\phi_\mu\phi_\nu\phi_{\lambda\rho}\phi^\lambda\phi^{\rho\sigma}\phi_\sigma,\\
\mathcal{G}^{(5)}_{\mu\nu} &= -A_5\phi_\mu\phi_\nu\phi^\lambda\phi_{\rho\lambda}\phi^\rho(\Box\phi) - \phi_\mu\phi_\nu\phi_\lambda(A_{5X}\nabla^\lambda X)\phi^\rho\phi_{\sigma\rho}\phi^\sigma + A_5\phi_\nu\phi_{\lambda\mu}\phi^\lambda\phi^\rho\phi_{\sigma\rho}\phi^\sigma + A_5\phi_\mu\phi_{\lambda\nu}\phi^\lambda\phi^\rho\phi_{\sigma\rho}\phi^\sigma\\
&\quad - A_5\phi_\mu\phi_\nu\phi^\lambda\phi^\rho\phi_{\sigma\rho\lambda}\phi^\sigma - 2A_5\phi_\mu\phi_\nu\phi^\lambda\phi^\rho\phi_{\sigma\rho}\phi^\sigma_\lambda - \frac{1}{2}A_5g_{\mu\nu}\phi^\lambda\phi_{\rho\lambda}\phi^\rho\phi^\sigma\phi_{\tau\sigma}\phi^\tau + A_{5X}\phi_\mu\phi_\nu\phi^\lambda\phi^\rho\phi_{\rho\lambda}\phi^\sigma\phi^\tau\phi_{\tau\sigma},
\end{aligned}$$

while the field equations associated to the scalar field allow to construct a current conservation equation given by

$$\varepsilon_{(\phi)} = \nabla_\mu J^\mu = \nabla_\mu\left[\frac{\delta\mathcal{L}}{\delta(\phi_\mu)} - \nabla_\nu\left(\frac{\delta\mathcal{L}}{\delta(\phi_{\mu\nu})}\right)\right] = 0,$$

where

$$J^{\mu} = J_{Z}^{\mu} + J_{G}^{\mu} + \sum_{i=2}^{5} J_{(i)}^{\mu},$$

with

$$\begin{aligned}
J_{Z}^{\mu} &= 2Z_{X}\phi^{\mu},\\
J_{G}^{\mu} &= 2G_{X}R\phi^{\mu},\\
J_{(2)}^{\mu} &= 2A_{2X}\phi^{\mu}[(\Box\phi)^{2} - \phi_{\lambda\rho}\phi^{\lambda\rho}] - 2\nabla_{\nu}[A_{2}(g^{\mu\nu} - \phi^{\mu\nu})],\\
J_{(3)}^{\mu} &= 2A_{3X}\phi^{\mu}\Box\phi\phi^{\lambda}\phi_{\lambda\rho}\phi^{\rho} + 2A_{3}\Box\phi\phi_{\lambda}^{\mu}\phi^{\lambda} - \nabla_{\nu}[A_{3}(g^{\mu\nu}\phi^{\lambda}\phi_{\lambda\rho}\phi^{\rho} + \Box\phi\phi^{\mu}\phi^{\nu})],\\
J_{(4)}^{\mu} &= 2A_{4X}\phi^{\mu}\phi^{\sigma}\phi_{\sigma\rho}\phi^{\rho\lambda}\phi_{\lambda} + A_{4}(X)[\phi_{\rho}^{\mu}\phi^{\rho\lambda}\phi_{\lambda} + \phi^{\sigma}\phi_{\sigma\rho}\phi^{\rho\mu}] - \nabla_{\nu}[A_{4}(X)(\phi^{\mu}\phi^{\nu\rho}\phi_{\rho} + \phi^{\sigma}\phi_{\sigma}^{\mu}\phi^{\nu})],\\
J_{(5)}^{\mu} &= 2A_{5X}\phi^{\mu}(\phi^{\sigma}\phi_{\sigma\rho}\phi^{\rho})^{2} + 2A_{5}(X)(\phi^{\sigma}\phi_{\sigma\rho}\phi^{\rho})(\phi^{\mu\sigma}\phi_{\sigma} + \phi^{\sigma\mu}\phi_{\sigma}) - 2\nabla_{\nu}[A_{5}(X)\phi^{\sigma}\phi_{\sigma\rho}\phi^{\rho}\phi^{\mu}\phi^{\nu}].
\end{aligned}$$

[1] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).
[2] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982).
[3] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, Phys. Rev. Lett. **102**, 201301 (2009).
[4] J. M. Maldacena, Int. J. Theor. Phys. **38**, 1113 (1999).
[5] J. D. Brown and M. Henneaux, Commun. Math. Phys. **104**, 207 (1986).
[6] A. Strominger, J. High Energy Phys. 02 (1998) 009.
[7] S. Carlip, J. Phys. Conf. Ser. **67**, 012022 (2007).
[8] E. Frodden, M. Geiller, K. Noui, and A. Perez, J. High Energy Phys. 05 (2013) 139.
[9] S. Hendi, B. Eslam Panah, and S. Panahiyan, J. High Energy Phys. 05 (2016) 029.
[10] R. Konoplya and A. Zhidenko, arXiv:2003.12171.
[11] K. Chan and R. B. Mann, Phys. Rev. D **50**, 6385 (1994).
[12] S. Hossein Hendi, B. Eslam Panah, S. Panahiyan, and M. Hassaine, Phys. Rev. D **98**, 084006 (2018).
[13] G. W. Horndeski, Int. J. Theor. Phys. **10**, 363 (1974).
[14] H. Motohashi, K. Noui, T. Suyama, M. Yamaguchi, and D. Langlois, J. Cosmol. Astropart. Phys. 07 (2016) 033.
[15] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui, and G. Tasinato, J. High Energy Phys. 12 (2016) 100.
[16] D. Langlois, R. Saito, D. Yamauchi, and K. Noui, Phys. Rev. D **97**, 061501 (2018).
[17] E. Babichev, C. Charmousis, A. Cisterna, and M. Hassaine, J. Cosmol. Astropart. Phys. 06 (2020) 049.
[18] G. Gibbons and S. Hawking, Phys. Rev. D **15**, 2752 (1977).
[19] T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974).
[20] R. M. Wald, Phys. Rev. D **48**, R3427 (1993).
[21] X. Feng, H. Liu, H. Lü, and C. Pope, J. High Energy Phys. 11 (2015) 176.
[22] F. Correa, C. Martínez, and R. Troncoso, J. High Energy Phys. 01 (2011) 034.
[23] H. A. Gonzalez, D. Tempo, and R. Troncoso, J. High Energy Phys. 11 (2011) 066.
[24] F. Correa, A. Faundez, and C. Martínez, Phys. Rev. D **87**, 027502 (2013).
[25] E. Ayon-Beato, M. Bravo-Gaete, F. Correa, M. Hassaine, and M. M. Juárez-Aubry, Phys. Rev. D **100**, 044024 (2019).
[26] M. Bravo Gaete, L. Guajardo, and M. Hassaine, J. High Energy Phys. 04 (2017) 092.
[27] W. Kim, S. Kulkarni, and S.-H. Yi, Phys. Rev. Lett. **111**, 081101 (2013).
[28] E. Ay'on-Beato, M. Hassaine, and J. A. Méndez-Zavaleta, Phys. Rev. D **92**, 024048 (2015).
[29] E. Babichev, C. Charmousis, and A. Lehebel, J. Cosmol. Astropart. Phys. 04 (2017) 027.
[30] J. Chagoya and G. Tasinato, J. Cosmol. Astropart. Phys. 08 (2018) 006.
[31] E. Babichev, C. Charmousis, G. Esposito-Farse, and A. Lehbel, Phys. Rev. Lett. **120**, 241101 (2018).
[32] T. Kobayashi and T. Hiramatsu, Phys. Rev. D **97**, 104012 (2018).
[33] C. Charmousis, M. Crisostomi, R. Gregory, and N. Stergioulas, Phys. Rev. D **100**, 084020 (2019).
[34] K. Takahashi, H. Motohashi, and M. Minamitsuji, Phys. Rev. D **100**, 024041 (2019).
[35] M. Minamitsuji and J. Edholm, Phys. Rev. D **101**, 044034 (2020).
[36] A. Lehébel, arXiv:1810.04434.
[37] E. Babichev, C. Charmousis, and M. Hassaine, J. Cosmol. Astropart. Phys. 05 (2015) 031.
[38] J. Chagoya, G. Niz, and G. Tasinato, Classical Quantum Gravity **33**, 175007 (2016); M. Minamitsuji, Phys. Rev. D **94**, 084039 (2016); E. Babichev, C. Charmousis, and M. Hassaine, J. High Energy Phys. 05 (2017) 114; A. Cisterna, M. Hassaine, J. Oliva, and M. Rinaldi, Phys. Rev. D **94**, 104039 (2016).
[39] L. Heisenberg, R. Kase, M. Minamitsuji, and S. Tsujikawa, J. Cosmol. Astropart. Phys. 08 (2017) 024.

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Regular black holes and gravitational particle-like solutions in generic DHOST theories

Olaf Baake,[a,b] **Christos Charmousis,**[c] **Mokhtar Hassaine**[a] **and Miguel San Juan**[a]

[a]Instituto de Matemáticas, Universidad de Talca,
Casilla 747, Talca 3460000, Chile
[b]Centro de Estudios Cientificos (CECs),
Av. Arturo Part 514, Valdivia 5090000, Chile
[c]Université Paris-Saclay, CNRS/IN2P3, IJCLab,
91405 Orsay, France

E-mail: olaf.baake@inst-mat.utalca.cl, christos.charmousis@th.u-psud.fr, hassaine@inst-mat.utalca.cl, miguel.sanjuan@utalca.cl

Received April 29, 2021
Accepted May 28, 2021
Published June 11, 2021

Abstract. We construct regular, asymptotically flat black holes of higher order scalar tensor (DHOST) theories, which are obtained by making use of a generalized Kerr-Schild solution generating method. The solutions depend on a mass integration constant, admit a smooth core of chosen regularity, and generically have an inner and outer event horizon. In particular, below a certain mass threshold, we find massive, horizonless, particle-like solutions. We scan through possible observational signatures ranging from weak to strong gravity and study the thermodynamics of our regular solutions, comparing them, when possible, to General Relativity black holes and their thermodynamic laws.

Keywords: GR black holes, modified gravity

ArXiv ePrint: 2104.08221

https://doi.org/10.1088/1475-7516/2021/06/021

Contents

1 Introduction

It is an undeniable fact, whose origin goes back to the Schwarzschild solution, that the notion of a black hole is intimately linked to the concept of spacetime singularities. In fact, it is well-known that, under certain energy conditions, classical solutions of general relativity exhibit singularities as a direct consequence of the so-called singularity theorems [1, 2]. The appearance of singularities is essentially due to the classical character of the theory of general relativity, and a quantum theory of gravity may be expected to cure such pathologies.

However, in the absence of a complete theory of quantum gravity one can search for black hole spacetimes with a global structure similar to the well-known solutions (like the Schwarzschild or Reissner-Nordström solutions), but in which the central singularity is absent. Such solutions are commonly known as regular black holes. These ideas originate from the pioneering works of Sakharov [3], Gilner [4] and also Bardeen [5] who presented the first example of a regular black hole as an ad-hoc metric (not originating from an action). A physical construction of the Bardeen metric as a solution of a given action was finally proposed much later in [6]. There the authors showed that the Bardeen metric can be obtained from the Einstein equations with a non-linear magnetic source. Although the Bardeen metric was the first example of a regular spacetime, the first exact regular black hole solution of a given theory was found by Ayón-Beato and Garcia [7] for the Einstein equations coupled to a specific non-linear electrodynamic source.

Models involving non-linear electrodynamics have been fruitful in the construction of regular solutions, see e.g. refs. [8–12], and also ref. [13] for a review. Many of these regular black holes present a de Sitter core at the origin, and their regularity is quantified by a regularizing parameter identified with a non-linear electrodynamic charge. It is also important to stress that the parameter of regularization is not a constant of integration but is rather an input of the matter action. This observation has important consequences, for example on the thermodynamic properties of these regular solutions. Indeed, depending on whether the regularizing parameter is considered as a varying parameter or not, the thermodynamic properties may be different. In order to illustrate this fact, one can note that for the Bardeen regular black hole, the one-quarter area law of the entropy is usually violated [14] when considering a non varying magnetic charge, while this "universal" law can be restored by promoting the magnetic charge as a variable parameter [15]. Note that in certain non-minimally coupled

Lagrangians analytic regular solutions were found where the mass and the charge truly are integration constants [16, 17].

In this work we will construct regular black hole solutions, which are asymptotically very similar to Schwarzschild, without the need of introducing an additional regularization parameter inherent to the action. For these black holes, regularity will not be enforced by the fine tuning of some action parameter, it will rather be achieved due to the functional form of the regularizing function appearing in the solutions. In other words, the fall-off of the mass term of our solutions turns out to be an analytic function with a de Sitter core at the origin as a consequence of the field equations. The degree of regularity and its strength are monitored by two parameters, one fixing the core to be de Sitter or higher and one fixing the strength of the higher order term against the mass of the black hole. The regular black holes found here are exact solutions of scalar tensor theories beyond those initially proposed by Horndeski [18]. The regularizing function sets, as one would expect, the scalar degree of freedom without any fine tuning of the theory.

The scalar tensor theories have higher than second derivative equations of motion and are (still) free of Ostrogradski type pathologies [19–22]. These general Lagrangians have been dubbed *Degenerate Higher Order Scalar Tensor* (DHOST) or *Extended Scalar Tensor* (EST) theories [19–24], and are widely studied in the literature (see for example [25–33] for their compact objects and [34] for a review). More precisely, we will consider the following class of shift symmetric and parity preserving DHOST theories that contain up to second order covariant derivatives of the scalar field (in the action),

$$\begin{aligned}S[g,\phi] = \int d^4x\sqrt{-g}\Big[&K(X) + G(X)R + A_1(X)\left[\phi_{\mu\nu}\phi^{\mu\nu} - (\Box\phi)^2\right] + A_3(X)\Box\phi\,\phi^{\mu}\phi_{\mu\nu}\phi^{\nu}\\ &+A_4(X)\phi^{\mu}\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho} + A_5(X)\left(\phi^{\mu}\phi_{\mu\nu}\phi^{\nu}\right)^2\Big],\end{aligned} \tag{1.1}$$

where the coupling functions K, G, A_1, A_3, A_4 and A_5 depend only on the kinetic term of the scalar field $X = g^{\mu\nu}\phi_\mu\phi_\nu$, and where $\phi_\mu = \partial_\mu\phi$ and $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$. The coupling functions A_4 and A_5 are chosen to satisfy

$$\begin{aligned}A_4 =& \frac{1}{8(G-XA_1)^2}\Big\{4G\left[3(-A_1+2G_X)^2 - 2A_3G\right] - A_3X^2(16A_1G_X + A_3G)\\ &+4X\left[-3A_2A_3G + 16A_1^2G_X - 16A_1G_X^2 - 4A_1^3 + 2A_3GG_X\right]\Big\},\\ A_5 =& \frac{1}{8(G-XA_1)^2}(2A_1 - XA_3 - 4G_X)\left(A_1(2A_1+3XA_3-4G_X) - 4A_3G\right)\end{aligned} \tag{1.2}$$

in order to ensure the absence of Ostrogradski ghosts [19–22]. Recently, it has been shown that regular black hole solutions for this class of theories can be constructed (including the well known cases of Bardeen [5] or Hayward metrics [35]), see ref. [36]. The algorithm of construction is a byproduct of extending the Kerr-Schild solution generating method to scalar tensor theories. The key point in extending this well known method from GR is the assumption that the kinetic term of the scalar field remains unchanged under the static (usual) Kerr-Schild transformation. Another crucial observation that we make here is that regular black holes cannot belong to Horndeski theory. We will see that the theories involving regular black holes correspond to a conformal and disformal map originating from Horndeski theory and ultimately belong to a pure DHOST theory. We will trace the reason for this to the recent interesting work discussing singularities in scalar tensor theories [37].

We would like to note that although the kinetic term of the scalar field will be assumed to be only depending on the radial coordinate, this does not exclude the fact that the scalar field can depend linearly, for example, on the time coordinate, i.e. $\phi(t,r) = \alpha t + \psi(r)$ where α is a constant. This possibility is attributed to the higher order nature of DHOST theory, and the shift invariance symmetry of the scalar field. The scalar time dependence was first used in [38] and has been found recently to be related to the geodesics of spacetime [32] whenever the kinetic term X is constant. In fact, in the case of higher order scalar tensor theories, examples of compact objects with a linear time dependent scalar field have been found, see e.g. [38–45]. In particular, stationary solutions, which are distinctively different from the Kerr spacetime [46–48], have been recently constructed.

In our search for regular black holes we will focus on a static scalar field where X will not be a constant function. This is a crucial requirement as X will also play the role of the regularizing function smoothing out the geometry near the origin. Once we obtain our regular solution we will discuss its most important properties. We will then proceed to study its possible observational characteristics scanning from weaker to stronger gravity effects.

The plan of the paper is organized as follows. In the next section, we will explicitly write the field equations associated to the variation of the DHOST action (1.1)–(1.2). The key steps of the Kerr-Schild solution generating method [36] will also be outlined, in order to explicitly construct a family of regular asymptotically flat black holes, that are solutions of some specific DHOST action (1.1)–(1.2) with coupling functions specified in appendix A. We will analyze the solutions and discuss the leading Post-Newtonian parameters, precession effects and null geodesics, scanning through observable signatures. In section 3, the thermodynamic analysis of these regular solutions will be carried out through the Euclidean method, and we will show that the regularity condition of the solutions is incompatible with the area law of the entropy. In spite of this, the first law of thermodynamics is shown to hold for the regular solutions. Our conclusions will be presented in section 4.

2 Field equations and construction of regular black holes

We will be dealing with a four-dimensional scalar tensor theory described by the metric g and a single scalar field ϕ whose dynamics is governed by the action (1.1) and whose coupling functions A_4 and A_5 are given by (1.2). We will focus on static metrics with a scalar field such that its standard kinetic term $X = g^{\mu\nu}\phi_\mu\phi_\nu$ only depends on the radial coordinate r, i.e.

$$ds^2 = -h(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin(\theta)^2 d\varphi^2\right), \qquad X = g^{\mu\nu}\phi_\mu\phi_\nu := X(r). \tag{2.1}$$

For this ansatz, the field equations associated with the DHOST action (1.1)–(1.2) are conveniently written as

$$X[2(A_1G)_X + GA_3] + r^2\left[(K\mathcal{H})_X + \frac{3}{4}K\mathcal{B}\right] = 0, \tag{2.2a}$$

$$-3(\mathcal{B}rX')^2 + 8(\mathcal{B}rX')\mathcal{H}\left(\frac{rh'}{h}+4\right) - 32\mathcal{H}\left[\frac{Kr^2+2G}{f} + 2\mathcal{H}\left(\frac{rh'}{h}+1\right)\right] = 0, \tag{2.2b}$$

$$\begin{aligned} &r^2(16\mathcal{B}_X\mathcal{H} + 3\mathcal{B}^2)X'^2 + 8\mathcal{H}X'r\left(\mathcal{B}r\frac{f'}{f} - 16\mathcal{H}_X\right) + 16r^2\mathcal{H}\mathcal{B}X'' \\ &-64\mathcal{H}^2\left[\left(\frac{rf'}{f}+1\right) + \frac{2G+r^2K}{2f\mathcal{H}}\right] = 0, \end{aligned} \tag{2.2c}$$

JCAP06(2021)021

where $(\ )'$ denotes the derivative with respect to the radial coordinate, r, while subscript X denotes the derivation with respect to the kinetic term X. To simplify the notation, we have defined the auxiliary functions of the action,

$$\begin{aligned}\mathcal{H}(X) &= A_1(X)\,X - G(X), \qquad \mathcal{B}(X) = A_3(X)\,X + 4G_X(X) - 2A_1(X),\\ \mathcal{Z}(X) &= A_3(X) + A_4(X) + X\,A_5(X).\end{aligned} \tag{2.3}$$

Another interesting note is the Horndeski limit [18] and the beyond Horndeski limit [49, 50] of our general DHOST theory equations. Indeed, (quartic) Horndeski theory, parameterized by $G_4 = G$ is attained with $2G_X = A_1 = -A_2$ and $A_3 = 0$, while quartic beyond Horndeski is given by $2G_X - XF = A_1 = -A_2$ and $A_3 = -2F$. The function F is the quartic beyond Horndeski term which is in a one to one correspondence with the disformal transformation, mapping Horndeski to beyond Horndeski theory (see for example the nice analysis in [21, 22]). In particular, we note that in both cases of quadratic Horndeski and beyond Horndeski we have $\mathcal{B} = 0$, which means that $\mathcal{B}$ in our field equations represents the conformal transformation mapping beyond Horndeski to pure DHOST theory. We will come back to this observation in a moment.

In order to be self-contained, we will briefly recall the procedure described in [36] which allows the construction of regular black hole solutions from simple seed configurations. The first step is to look for a simple seed solution of the field equations (which does not describe a black hole) and schematically represent it by

$$ds_0^2 = -h_0(r)dt^2 + \frac{dr^2}{f_0(r)} + r^2\left(d\theta^2 + \sin(\theta)^2 d\varphi^2\right), \qquad X_0 = g^{\mu\nu}_{(0)}\phi^{(0)}_\mu\phi^{(0)}_\nu := X_0(r). \tag{2.4}$$

Now, as shown in ref. [36], the equations of motion (2.2) are invariant under a Kerr-Schild transformation of the metric, provided that the kinetic term of the scalar field is left invariant. More precisely, it is straightforward to see that the equations (2.2) are invariant under the following simultaneous transformations

$$h_0(r) \to h_0(r) - 2\mu\frac{m(r)}{r}, \quad f_0(r) \to \frac{f_0(r)}{h_0(r)}\left(h_0(r) - 2\mu\frac{m(r)}{r}\right), \quad \text{with} \quad m(r) = e^{\frac{3}{8}\int dX\,\frac{\mathcal{B}(X)}{\mathcal{H}(X)}}, \tag{2.5}$$

and X remains unchanged, i.e. $X_0(r) = X(r)$. Here μ is a constant that will be shown to be proportional to the mass of the resulting solution. Our second step is to use this Kerr-Schild symmetry (2.5) to deduce that the configuration given by,

$$\begin{aligned}ds^2 &= -\left(h_0(r) - 2\mu\,\frac{m(r)}{r}\right)dt^2 + \frac{h_0(r)\,dr^2}{f_0(r)\left(h_0(r) - 2\mu\,\frac{m(r)}{r}\right)} + r^2\left(d\theta^2 + \sin(\theta)^2 d\varphi^2\right),\\ X(r) &= g^{\mu\nu}\phi_\mu\phi_\nu = X_0(r),\end{aligned} \tag{2.6}$$

will satisfy the same equations as those satisfied by the simple seed solution (2.4), provided that the mass function $m(r)$ is given by

$$m(r) = e^{\frac{3}{8}\int dX\,\frac{\mathcal{B}(X)}{\mathcal{H}(X)}}. \tag{2.7}$$

Note that in order for the mass term to be non trivial (i.e. with a non-Newtonian fall-off) we need to venture outside of beyond Horndeski theory, where $\mathcal{B} \neq 0$. According to the observation made in the previous paragraph, $\mathcal{B}$ is related to the conformal degree of freedom for pure DHOST theory. This leads us to the conclusion that we must have a combined disformal and conformal transformation of Horndeski theory to have any hope of constructing a regular solution. The regular solutions are crucially situated in higher order DHOST theory-not in Horndeski or beyond Horndeski theory.

To keep things simple we make the following working hypothesis [36]

$$\frac{3\mathcal{B}}{8\mathcal{H}} = \frac{1}{X} \Longrightarrow m(r) = X(r), \tag{2.8}$$

Hence, starting from a seed metric, the "choice" of the mass function $m(r)$, or equivalently of the seed kinetic term (2.8) will be key in order to ensure the regularity of the final (massive) configuration (2.6) at the origin and at infinity. Moreover, once we fix the expression of $X_0(r)$ as an invertible function, we will be able to specify the corresponding DHOST theory (1.1)–(1.2), that is to determine the functions K, G, A_1 and A_3 (as functions of X only) [36]. For example, in the asymptotically flat case with a seed metric $f_0 = h_0 = 1$, the regularity at the origin will be ensured if $m(r) = \mathcal{O}(r^3)$. Indeed, in this case the solution is shown to exhibit a de Sitter core at the origin, ensuring that any invariant constructed out of the Riemann tensor will be regular at the origin. Given these preliminary requirements we see that it is essential to be in the context of DHOST theory, in order to find regular black holes in accordance with the discussion and findings in [37]. Hence, regular black holes are necessarily solutions of a pure DHOST theory. In other words, such regular solutions would be images of the mapping of a combined conformal and disformal transformation of a Horndeski solution.

2.1 Asymptotically flat regular black holes

We will first focus on the construction of asymptotically regular black holes with a flat seed metric given by $h_0 = f_0 = 1$. In this case, following the results obtained in ref. [36], one can easily express $\mathcal{H}$ and G as

$$\mathcal{H} = \frac{1}{X\left(\frac{rX'}{3X} - 1\right)}, \qquad G = \frac{1}{X}\left(1 - \frac{rX'}{X}\right) - \frac{Kr^2}{2}.$$

Now, in order to get the coupling function K, we first write

$$A_3 = -\frac{4G_X}{X} + \frac{2A_1}{X} + \frac{8\mathcal{H}}{3X^2}, \qquad A_1 = \frac{\mathcal{H} + G}{X} \tag{2.9}$$

and then inserting the expressions (2.9) into eq. (2.2a), we obtain, after some algebraic manipulations,

$$2(\mathcal{H}G)_X + r^2(K\mathcal{H})_X + \frac{2\mathcal{H}}{X}\left(\frac{4}{3}G + Kr^2\right) = 0. \tag{2.10}$$

Finally, the coupling function K is shown to be given by

$$K = -\frac{2\left[3X\left(rX'' + 2X'\right) + r^2X^{-1}X'^3 - 7rX'^2\right]}{rX\left(rX' - 3X\right)^2}.$$

We are now ready to construct an explicit family of regular black hole solutions. We will opt for a (seed) kinetic term,

$$X(r) = X_0(r) = \frac{2}{\pi}\arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right). \tag{2.11}$$

JCAP06(2021)021

The function X depends on the integer p and the bookkeeping parameter σ. In particular, the limiting case $\sigma \to 0$ gives us the usual Schwarzschild case. Our choice is motivated from three essential requirements emanating from the resulting metric function, $h(r) = 1 - \frac{2\mu X(r)}{r}$:

- First of all, for r close to the origin we have,

$$h(r) = 1 - 2\mu \left(\frac{r}{\sigma}\right)^{p-1} + O(r^{3p-1}), \tag{2.12}$$

  and hence, as shown below for $p \geq 3$, $\sigma \neq 0$, the final metric will be regular at the origin. The de Sitter core is attained for $p = 3$, and increasing regularity from there on for $p > 3$.

- Secondly, X asymptotes unity for large r, and as such gives for h a similar behavior at asymptotic infinity to the Schwarzschild solution. We have,

$$h(r) = 1 - \frac{2\mu}{r} + \frac{8\mu\sigma^{p-1}}{\pi^2 r^{p+1}} + O(r^{3p+1}), \tag{2.13}$$

- Last but not least, the function $X(r)$ is bijective for our coordinate range $r \in [0, \infty[$.

Using the latter property one can see that the seed configuration, $h_0 = f_0 = 1$, with a kinetic term given by (2.11), is a solution of the DHOST action (1.1)–(1.2) with coupling functions reported in appendix A. Crucially, the action functionals are only functions of X, and the theory parameters, σ and p. The power, p, fixes the solution's core regularity at the origin. Once p is fixed, the solution is regular without any fine-tuning of the parameter σ, which has been inserted so as to track down differences from GR at $\sigma \to 0$. Using therefore the generalized Kerr-Schild transformation, one determines that the solution given by

$$\begin{aligned} ds^2 &= -\left(1 - \frac{4\mu \arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right)}{r\pi}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{4\mu \arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right)}{r\pi}\right)} + r^2\left(d\theta^2 + \sin(\theta)^2 d\varphi^2\right), \\ X(r) &= \frac{2}{\pi} \arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right), \end{aligned} \tag{2.14}$$

satisfies the field equations of the DHOST action (1.1)–(1.2) with coupling functions given in appendix A, which has been additionally verified by inserting this solution directly into the equations of motion.

Let us now make some comments on the properties of (2.14). First of all, for $p > 0$, the metric solution will behave asymptotically ($r \to \infty$) as the Schwarzschild spacetime. For $\mu > 0$ and $p > 0$, the metric solution has an inner and an outer event horizon as we see from the plot in figure 1. The outer horizon is an event and Killing horizon (for the Killing vector ∂_t), which is manifest by preforming the usual Eddington-Finkestein coordinate transformation. The inner horizon is a Cauchy horizon for any timelike hypersurface situated in the exterior spacetime where ∂_t is timelike. The solution has a central curvature singularity for $0 < p < 3$. However, for $p = 3$, the metric solution (2.14) is regular with a de Sitter core, while for $p > 3$, the family of solutions are again regular black holes with an increasingly regular core [51]. The region internal to the inner horizon is spacelike and completely regular at the origin. Setting $p = 3$ for definiteness and $2\sigma^2 = \pi$ we find that for $\mu_{\text{ext}} \sim 1.13$ we

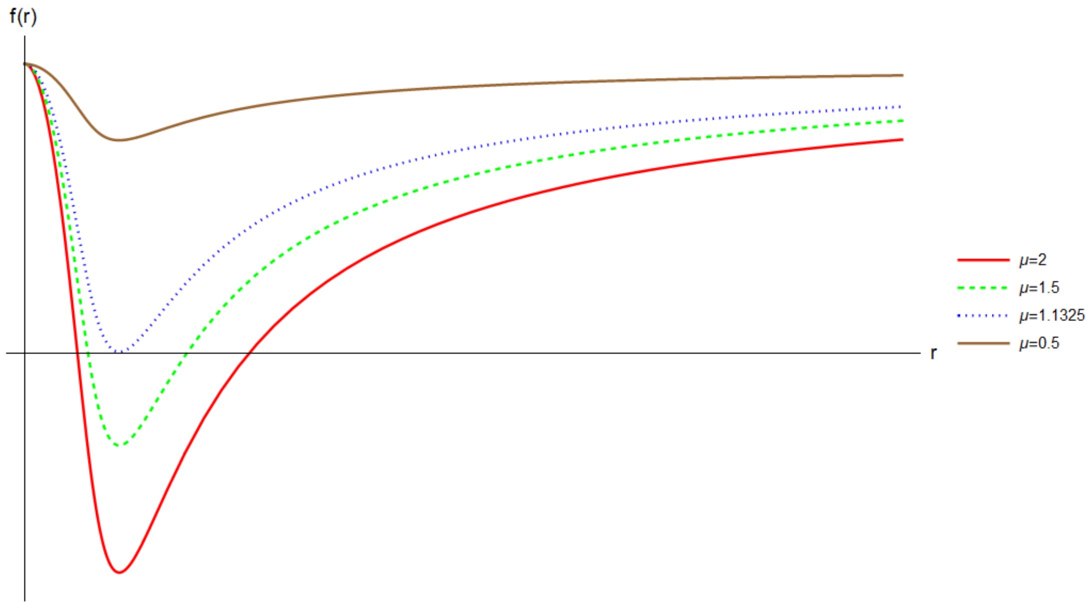


Figure 1. Metric function g_{00} for $p = 3$ and $2\sigma^2 = \pi$. The inner and outer horizons correspond to the roots of the function, while for smaller masses than $\mu_{\rm ext}$ (blue dotted curve) the solution has no horizon.

have an extremal black hole. For $\mu_{\rm ext} \leq \mu$ we have a sequence of regular black holes whereas for smaller masses than $\mu_{\rm ext}$ we have a regular solution without horizon; spacetime is curved but not sufficiently in order to create an event horizon. These solutions are gravitational particle-like solutions akin to dark matter, provided they are stable.

We now proceed to scan, starting from weak up to strong gravity, the possible notable differences of our regular solution, as compared to standard GR. We do not aim to be extensive here, we rather give a first approach that is useful for future studies. Let us first seek the leading PPN parameters of this solution in order to effectively see how it compares with GR. In order to do this we effectively find a Cartesian distance coordinate $\rho = \sqrt{x^2 + y^2 + z^2}$ where (x, y, z) are harmonic coordinates suited for a Newtonian gauge. As an example take $p = 3$ whereupon we get,

$$r = \rho + M - \frac{4\mu\sigma^2}{\rho^3} + O(1/\rho^4). \tag{2.15}$$

This coordinate system is harmonic for large distances compared to the size of the outer event horizon. Furthermore, to leading order, it agrees with the harmonic radial coordinate of Schwarzschild (see [52] for clarification on coordinate issues in higher PN calculations). Such distances of the order of some 1400 Schwarzschild radii correspond to the orbits of stars like S2 orbiting Sgr*A. Using these coordinates we can quite easily obtain the leading (see for example [53]) PN parameters, $\beta = \gamma = 1$, which end up identical to GR for $p \geq 3$.

We can try to go a step further and evaluate directly the precession of a star like S2 orbiting the massive compact object identified with Sgr A* (see [54] and references within). Star S2 orbits the central, regular for our purposes, black hole, following timelike geodesics at the equator $\theta = \pi/2$. Using the Killing symmetries for rest energy per unit rest mass E

and angular momentum per unit rest mass L we have the standard relations,

$$E = h(r)\frac{dt}{d\tau}, \qquad L = r^2\frac{d\phi}{d\tau}, \tag{2.16}$$

where τ is the geodesic parameter. Transforming to $u = 1/r$ coordinates and using the above, it is straightforward to obtain the Binet's modified equation governing the trajectory of $S2$,

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{L^2}(uX_u + X) + 3\mu u^2 X + \mu u^3 X_u, \tag{2.17}$$

where now u is a function the angular coordinate ϕ. The above equation gives us precisely the GR case of Schwarzschild for $X = 1$. Binet's original equation, valid for the Newtonian limit, is obtained if we take $X = 1$ and we additionally neglect the higher order $3\mu u^2$ term. This orbital equation is valid for any regular black hole we choose in the face of X and for classical precession tests of solar system planets. As an example, we can set $p = 3$ for our regular solution and Taylor expand for small u (or large r),

$$X = 1 - \frac{4\sigma^2}{\pi^2}u^3 + O(u^9). \tag{2.18}$$

We get the approximate equation,

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{L^2} + \frac{\epsilon L^2}{\mu}u^2 - \frac{16\sigma^2\epsilon}{3\mu\pi^2}u^3 + O(u^5). \tag{2.19}$$

Here we have introduced $\epsilon = \frac{3\mu^2}{L^2}$ as our small[1] dimensionless parameter [55]. We are using the same expansion parameter as for the case of Schwarzschild as we want to point out the difference with the case of GR. Now expanding $u = u_0 + \epsilon u_1$, we obtain to zeroth order the elliptic Kepler trajectory $u_0 = \frac{\mu}{L^2}(1 + e\cos\phi)$, where e is the eccentricity. To linear order in ϵ, keeping only the term with growing contribution we find at the end,

$$u \sim \frac{\mu}{L^2}\Big[1 + e\cos[\phi(1 - \epsilon f_{SP})]\Big], \tag{2.20}$$

where $f_{SP} = 1 - 8\frac{\mu\sigma^2}{L^4\pi^2}\left(1 + \frac{e^2}{4}\right)$ denotes our correction beyond the GR $f_{SP} = 1$ value. Constraints from GRAVITY place $f_{SP} \sim 1.1 \pm 0.2$ which in turn constrains our action parameter σ. Note however, that given our expansion in ϵ we are assuming that our parameter σ^2 is big enough so as to be of the same order as the Schwarzschild correction. If we adapt our calculation to the orbit characteristics of the S2 star orbit there will be fine-tuning involved. Generically $f_{SP} = 1$ since $\beta = \gamma = 1$ for our background. A similar calculation can be undertaken using null geodesics for time delay effects akin to pulsars for example (see the review by Johannsen [56]).

A last interesting point is to consider our solution in the strong field regime. For our generic purposes we will pursue here the light trajectories of photons or massless particles such as neutrinos in presence of our regular black hole. Again we follow the standard text book procedure for equatorial geodesics but now we focus on light rays, defining $b = L/E$, the apparent impact parameter, for an observer in the asymptotically flat region. The parameter b can vary up to the closest distance photons get to the black hole without being necessarily

[1]In our geometrized units we have $G = c^2 = 1$ and therefore $\mu(cm) = 0.742 \times 10^{-28}\,\frac{cm}{g}\,\mu(g)$.

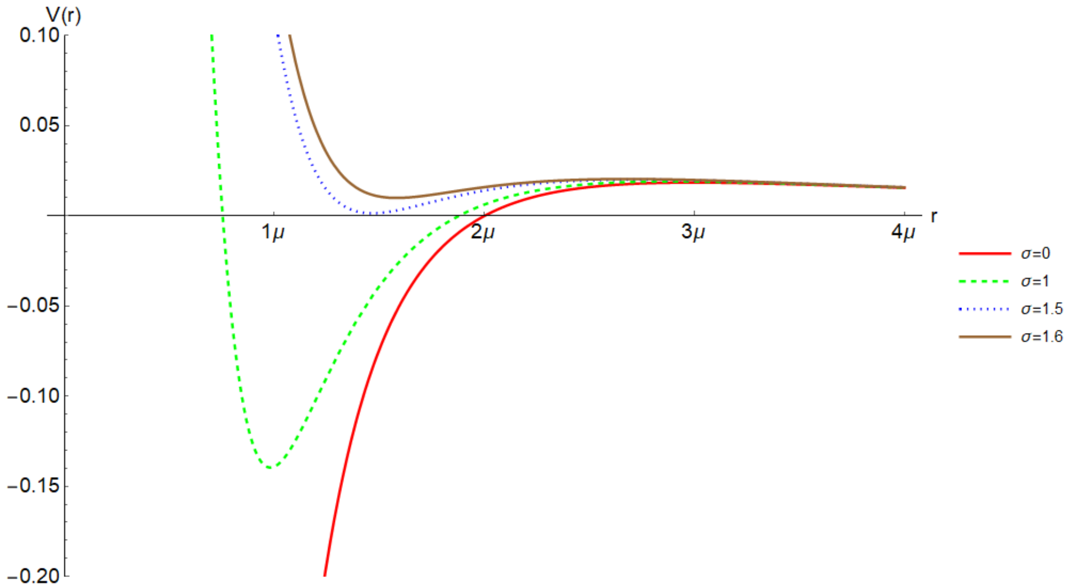


Figure 2. Effective potential (2.22) for different values of σ our theory parameter. In particular, $\sigma = 0$ corresponds to the effective potential of the Schwarzschild solution for which $X = 1$. Varying $\sigma > 0$ changes the root of the potential and a non-zero value actually changes the singularity to a minimum. Increasing the value of σ further can even remove the root corresponding to the absence of an event horizon altogether. The height of the potential maximum marks $1/b^2_{\rm crit}$ for each curve of the potential.

eaten up by the gravitational well of the black hole. The geodesic equation takes a familiar (particle in a potential) form,

$$\frac{1}{2}\left(\frac{dr}{d\tilde{\tau}}\right)^2 + \frac{h(r)}{2r^2} = \frac{1}{2b^2}, \tag{2.21}$$

where we have rescaled $\tilde{\tau} = L\tau$. Therefore the effective potential takes the form,

$$V_{\rm eff} = \frac{1}{2r^2}\left(1 - \frac{2\mu}{r}X(r)\right), \tag{2.22}$$

and critical light rings occur at the zeroes of $V'_{\rm eff} = 0$ which are the zeroes of the equation,

$$r + \mu X' - 3\mu X = 0. \tag{2.23}$$

The effective potential and its derivative are depicted in figures 2 and 3 respectively. Note the familiar light ring solution at $r_R = 3\mu$ for Schwarzschild when we set $X = 1$. Once we have a zero of (2.23), $r = r_R$ we get the maximal impact parameter using (2.21),

$$b_{\rm crit} = \frac{r_R}{\sqrt{h(r_R)}}. \tag{2.24}$$

The critical impact factor can be as well formulated as

$$b_{\rm crit} = b_{\rm Schwar.}\frac{\left(X(r_R) - \frac{1}{3}X'(r_R)\right)^{\frac{3}{2}}}{\sqrt{X(r_R) - X'(r_R)}} = b_{\rm Schwar.}\left(\frac{r_R}{3\mu}\right)\sqrt{\frac{4\sigma^4 + \pi^2 r_R^6}{\pi^2 r_R^6 - 24\mu r_R\sigma^2 + 4\sigma^4}}, \tag{2.25}$$

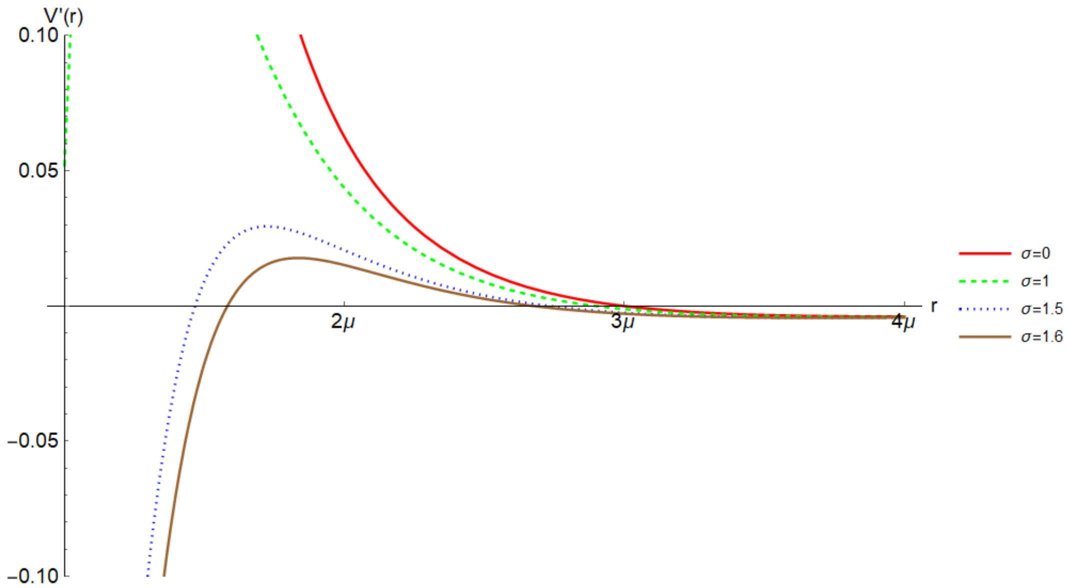


Figure 3. Derivative of the effective potential. One can see a small but finite shift of its root, r_R, for different values of σ as a decreasing function of σ.

where the impact factor for the Schwarzschild solution is given by $b_{\text{Schwar.}} = 3^{3/2}\mu$. It is easy to see that

$$\left(\frac{r_R}{3\mu}\right) b_{\text{Schwar.}} \leq b_{\text{crit}} \leq \left(\frac{r_R^2}{3\mu}\right)\sqrt{\frac{\pi}{\pi r_R^2 - 6\mu}}$$

and the lower bound is achieved for $\sigma = 0$ (the Schwarzschild limit) and at the limit $\sigma \to \infty$, corresponding to the flat limit.

The determination of the light ring sets the size of the black hole shadow. The Event Horizon Telescope (EHT) has obtained the first image of the supermassive M87 black hole. For M87 the size of the shadow was used as a test for GR, estimating the black hole mass [57, 58] and comparing to the independent calculation for M87's mass given by stellar dynamics [59]. There are a number of caveats with this calculation as a test of GR that have primarily to do with the little knowledge of the illuminating accretion flow for M87 or the sheer mass of the object (see in particular the critical analysis presented in [60]). Rather than putting in the numbers we will choose here to sketch the different cases for our regular solution as opposed to Schwarzschild. For definiteness let us fix the mass of the black hole to $\mu = 1$ and vary the theory parameter σ instead, in order to see how the characteristics of the effective potential change as we sweep through our theory. Indeed we find that for $0 < \sigma < \sigma_{\text{ext}}$ our effective potential always has a photon ring (outside of the event horizon) and as σ is increased we have $r_R^\sigma < 3$, the GR photon ring case. At the same time, increasing σ, the height of the potential maximum increases and therefore the critical impact parameter $b_{\text{crit}}^\sigma < b_{\text{crit}}^0$ is always below the Schwarzschild one (again see [60]). Note also that once $\sigma > 0$ we always have a minimum of the potential. This scheme continues until we arrive at σ_{ext}, the case where (for unit mass) we have an extremal black hole. Beyond this point there is no event horizon anymore, for $\mu = 1$, and our theories present now two visible critical points, one stable and one unstable. For a region of impact parameters in between the critical values of the potential, we have bound light orbits for local light sources at $r < 3$ or so. This is

JCAP06(2021)021

a distinctive feature of the particle-like solutions and is something that differentiates them from the regular black hole case. Furthermore, note that photons starting out from infinity can probe into the gravitational solution to all distances. Therefore, for $\sigma > \sigma_{\rm ext}$ there is no longer a central shadow, but rather enhanced light rings very close to the $r = 0$ center. In summary, for each given theory (where p and σ are fixed) we will have particle-like solutions for $\mu < \mu_{\rm ext}$ and regular black holes for $\mu > \mu_{\rm ext}$.

3 Thermodynamics of asymptotically flat regular black holes with a scalar field source

We now turn to the study of the thermodynamic properties of the regular class of black hole solutions (2.14). The thermodynamics of regular solutions is one of the aspects that is widely studied in the literature, see e.g. [61–65]. We start by pointing out a difference of our DHOST solution in comparison to regular black holes with non-linear electrodynamics. In the latter case the regularization parameter is actually part of the theory, and is usually associated with a magnetic charge. This means that the latter solution exists for a fixed value of the magnetic charge, and that to change this value corresponds to changing the theory. A direct consequence of this is that the regularization parameter cannot be considered as a variable parameter, and hence must not appear in the equation of the first law of thermodynamics. This aspect obscures the thermodynamic interpretation of regular solutions. On the contrary in our case, the regularity of the solution (2.14) is not inherent to the presence of our action bookkeeping parameter σ, but rather in the presence of the regularizing arctangent function rendering the metric function smooth at the origin. In addition, as it can be seen in eq. (2.14), the regularizing function comes with a constant μ which is an integration constant, and hence its interpretation as a thermodynamical variable is not ambiguous.

The thermodynamic analysis of the regular solution (2.14) will be carried out with the Euclidean approach in which the partition function is identified with the Euclidean path integral in the saddle point around the classical solution. In practice, we consider a mini superspace with the following ansatz

$$ds^2 = N(r)^2 f(r) d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_2^2, \qquad \phi = \phi(r), \tag{3.1}$$

where τ (in this section) is the Euclidean (periodic) time with $0 < \tau \leq \beta$ and, where β is the inverse of the temperature

$$\beta^{-1} = T = \frac{1}{4\pi} N(r) f'(r)|_{r_h}, \tag{3.2}$$

with r_h being the radius of the horizon. In the mini superspace defined by the ansatz (3.1), the Euclidean action I_E (using the proper normalization factor) reads

$$I_E = -\frac{1}{4}\beta \int N \left[(\mathcal{P} - 2\mathcal{Q}') f - \mathcal{Q} f' + 2G + r^2 K \right] + B_E, \tag{3.3}$$

where $\mathcal{H}, \mathcal{B}$ and $\mathcal{Z}$ are given in (2.3), and where for simplicity we have defined,

$$\mathcal{Q} = \frac{\mathcal{B}}{4} r^2 X' - 2r\mathcal{H}, \qquad \mathcal{P} = r X' \mathcal{B} + \frac{r^2}{4} (X')^2 \mathcal{Z} - 2\mathcal{H}. \tag{3.4}$$

In the Euclidean action (3.3), the term B_E is an appropriate boundary term ensuring that the solution corresponds to an extremum of the action, and at the same time it codifies all the thermodynamic properties. After some algebraic manipulations we get,

$$B_E = \frac{\beta}{4} \lim_{r\to\infty} \left\{ \frac{N(r)\mathcal{Q}(r)X(r)}{r} \right\} \mu - \pi \int \mathcal{Q}(r_h) dr_h. \tag{3.5}$$

On the other hand, since the Euclidean action is related to the Gibbs free energy $\mathcal{G}$ through

$$I_E = \beta\,\mathcal{G} = \beta\,\mathcal{M} - \mathcal{S},$$

one can easily read off the expressions of the mass $\mathcal{M}$ and of the entropy $\mathcal{S}$ from the boundary term,

$$\mathcal{M} = \frac{1}{4} \lim_{r\to\infty} \left\{ \frac{N(r)\mathcal{Q}(r)X(r)}{r} \right\} \mu, \qquad \mathcal{S} = \pi \int \mathcal{Q}(r_h) dr_h. \tag{3.6}$$

For the specific regular black hole solution (2.14), these expressions reduce to

$$\mathcal{M} = \frac{1}{6} \frac{r_h}{\arctan\left(\frac{1}{2}\pi r_h^p \sigma^{1-p}\right)}, \qquad \mathcal{S} = \frac{2}{3} \int \frac{\pi r_h}{\arctan\left(\frac{1}{2}\pi r_h^p \sigma^{1-p}\right)} dr_h, \tag{3.7}$$

while the temperature is given by

$$T = \frac{1}{4\pi r_h} \left(1 - \frac{2\pi\sigma^{p-1} p r_h^p}{\left(\pi^2 r_h^{2p} + 4\sigma^{2p-2}\right) \arctan\left(\frac{1}{2}\pi r_h^p \sigma^{1-p}\right)} \right).$$

It is clear from these relations that the mass and the entropy of the regular solution are positive, and although we do not have a closed form of the entropy we can nonetheless verify the validity of the first law $d\mathcal{M} = T\,d\mathcal{S}$. We also note that the entropy of the regular solution does not satisfy the area law. In fact, from the generic expression as obtained in (3.6), the only way for the entropy to satisfy the area law is that the function $\mathcal{Q}$, as defined in (3.4), must be proportional to $\mathcal{Q}(r) \propto r$. However, it is a simple matter to check that the solutions of the field equations given by (2.2), and for an ansatz of the form (2.6) will necessarily imply that

$$\mathcal{Q}(r) \propto \frac{r}{X(r)},$$

and, consequently the entropy will be proportional to one-quarter of the area only for a constant kinetic term. On the other hand, our analysis shows that a constant kinetic term is incompatible with the regularity of the solution. Hence, we deduce that for the DHOST theories considered here the regularity of the solutions fitting our ansatz (2.6) will not be compatible with the one-quarter area law for the entropy. This is not uncommon for modified gravity theories and is understood geometrically in certain cases such as Einstein-Gauss-Bonnet theory (see for example [66]).

Thermodynamic stability of the regular solution is addressed by computing the heat capacity $C_H = T\frac{\partial\mathcal{S}}{\partial T}$. From this definition it becomes clear that the heat capacity will provide information about the thermal stability with respect to the temperature fluctuations, and that a positive heat capacity is a necessary condition to ensure the local stability of the system. Also, the critical hypersurfaces, that is those where C_H vanishes or diverges, will

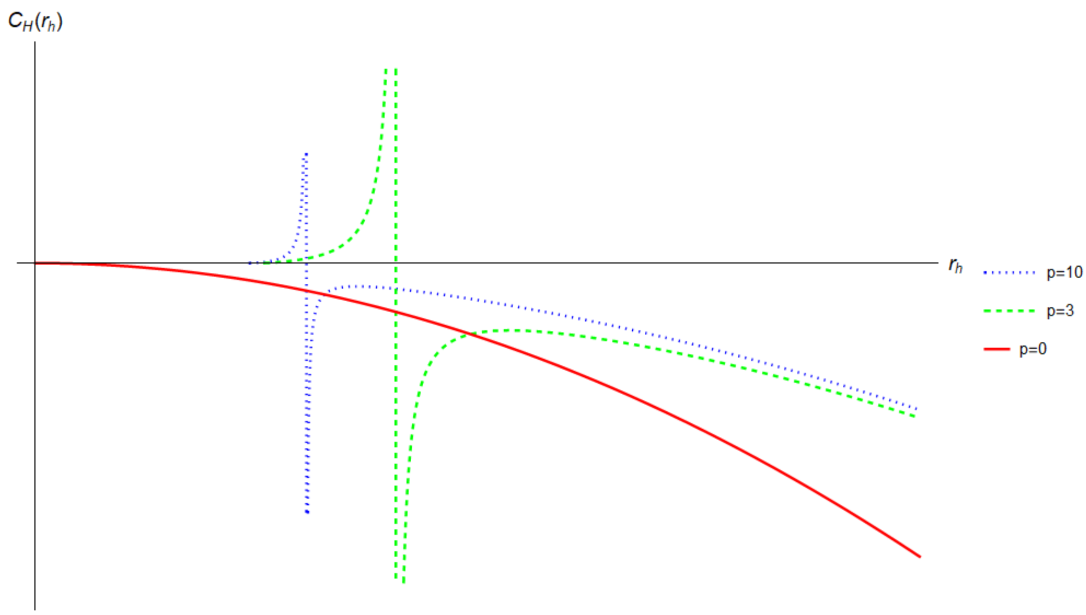


Figure 4. Heat capacity of the (A.1) black hole for different values of p and σ such that $2\sigma^{p-1} = \pi$ starting at r_{Extremal} respectively. Note that these correspond to different theories. There is a second order phase transition at r_{PT}. The asymptotic behavior is like $\propto -r^2$ at infinity. Setting $p = 0$ corresponds to the Schwarzschild solution, which has no phase transition.

correspond to the extrema of the temperature with respect to the entropy. For technical reasons, it is more convenient to express the heat capacity as

$$C_H = T\frac{\partial \mathcal{S}}{\partial T} = T\left(\frac{\partial \mathcal{S}}{\partial r_h}\right)\left(\frac{\partial T}{\partial r_h}\right)^{-1},$$

and, for the regular black hole solution (2.14) we get

$$C_{\text{H}} = \frac{2\pi C r_h^2\left(r_h^{2p} + \frac{4}{\pi^2}\sigma^{2p-2}\right)\left[\left(r_h^{2p} + \frac{4}{\pi^2}\sigma^{2p-2}\right)\arctan\left(\frac{1}{2}\pi r_h^p\sigma^{1-p}\right) - \frac{2}{\pi}\sigma^{p-1}pr_h^p\right]}{\mathcal{C}},$$

with

$$\begin{aligned}\mathcal{C} =\, & 3\arctan\left(\frac{1}{2}\pi r_h^p\sigma^{1-p}\right)\left[\frac{2}{\pi}\sigma^{p-1}p\left(\frac{4}{\pi^2}\sigma^{2p-2}(p-1) - (p+1)r_h^{2p}\right)r_h^p\right.\\ & \left.+\left(r_h^{2p} + \frac{4}{\pi^2}\sigma^{2p-2}\right)^2\arctan\left(\frac{1}{2}\pi r_h^p\sigma^{1-p}\right)\right] - \frac{12}{\pi^2}\sigma^{2p-2}p^2r_h^{2p}.\end{aligned}$$

Due to its lengthy form it is insightful to plot the heat capacities. The heat capacities are shown in figure 4, where we have excluded the part that corresponds to negative temperatures (akin to the presence of an internal horizon). From this picture, one can see that only small black holes are locally stable and a critical hypersurface will emerge at some positive radius revealing the existence of a second order phase transition, as it is the case for the non-linear electrodynamical regular black holes, see e.g. [61–65].

Before closing this section, we would like to address the following question: for the DHOST theory as defined in appendix A, does there exist another solution, and if so, would this allow for a thermodynamic stability comparison of the two solutions? In order to answer this question, we notice that the first equation (2.2a) gives,

$$0 = \frac{16\left[\frac{2}{\pi}\sigma^{p-1}\sin\left(\frac{\pi}{2}X\right)\right]^{-\frac{2}{p}}}{3\pi^2\left[\frac{2}{\pi}p\sin\left(\frac{\pi}{2}X\right)-6X\right]^4 X}\left[-r^2\cos\left(\frac{\pi}{2}X\right)^{\frac{2}{p}}+\left(\frac{2}{\pi}\sigma^{p-1}\sin\left(\frac{\pi}{2}X\right)\right)^{\frac{2}{p}}\right]F[X], \tag{3.8}$$

with $F[X]$ being an algebraic equation in X given by

$$\begin{aligned} F[X] = {} & 72X^2\left[p^2\cos(2\pi X)-p\cos(\pi X)-2\right]-\frac{32}{\pi^2}p^2\sin^2(\pi X)\left[p\cos(\pi X)-4\right] \\ & +\frac{12}{\pi}pX\sin(\pi X)\left[p^2\cos(2\pi X)+3p^2-26p\cos(\pi X)+26\right]. \end{aligned}$$

From this it is easy to see that there are only two possibilities: either X is given by the previous form (2.14), or X is a constant solving the constraint $F[X]=0$. On the other hand, taking the difference between (2.2b)–(2.2c) yields $f(r)=h(r)$, so in the first case we end up with the regular black hole. After some straightforward computations, we can establish that only the DHOST theory defined in appendix A with $p=1$ will admit two different solutions, and one of these is a stealth Schwarzschild black hole configuration given by

$$h(r)=f(r)=1-\frac{\mu}{r}, \qquad X=1+2n, \tag{3.9}$$

where n is an integer number. The thermodynamic quantities of this stealth solution are given by

$$\mathcal{M}=\frac{r_h}{3\pi}, \quad \mathcal{S}=\frac{2}{3}r_h^2, \quad T=\frac{1}{4\pi r_h}, \quad C_{\mathrm{H}}=-\frac{4}{3}r_h^2, \tag{3.10}$$

and as stressed before the entropy satisfies the area law because of the constant value of the kinetic term (3.9). The comparison of the respective heat capacities can be seen in figure 5. We can now compare the arctan −solution (2.14) for $p=1$ with the stealth solution (3.9). Using the free energy, defined as $\mathcal{F}=\mathcal{M}-T\mathcal{S}$, one can calculate the difference of the respective solutions at equal temperatures

$$\begin{aligned} \Delta\mathcal{F} &= F_{\text{regular}}-F_{\text{stealth}} = T\int\mathcal{F}(r_h)dr_h, \\ \mathcal{F}(r) &= \frac{r\left[-4\left(r^2+1\right)\arctan(r)^2+\pi\left(r^2+1\right)\arctan(r)-\pi r\right]\left[-2r^3\arctan(r)-r^2+\left(r^2+1\right)^2\arctan(r)^2\right]}{\arctan(r)\left[\left(r^2+1\right)\arctan(r)-r\right]^3} \end{aligned}$$

It is easy to notice that the integrand $\mathcal{F}(r)$, goes to $+\infty$ for $r\to 0$ and to $-\infty$ for $r\to\infty$. Hence, one would expect the stealth solution to be thermodynamically favored for small r_h, and there is the possibility that this changes for sufficiently large r_h. However, because of its lengthy integral form it is not possible to make any exact statements about this.

JCAP06(2021)021

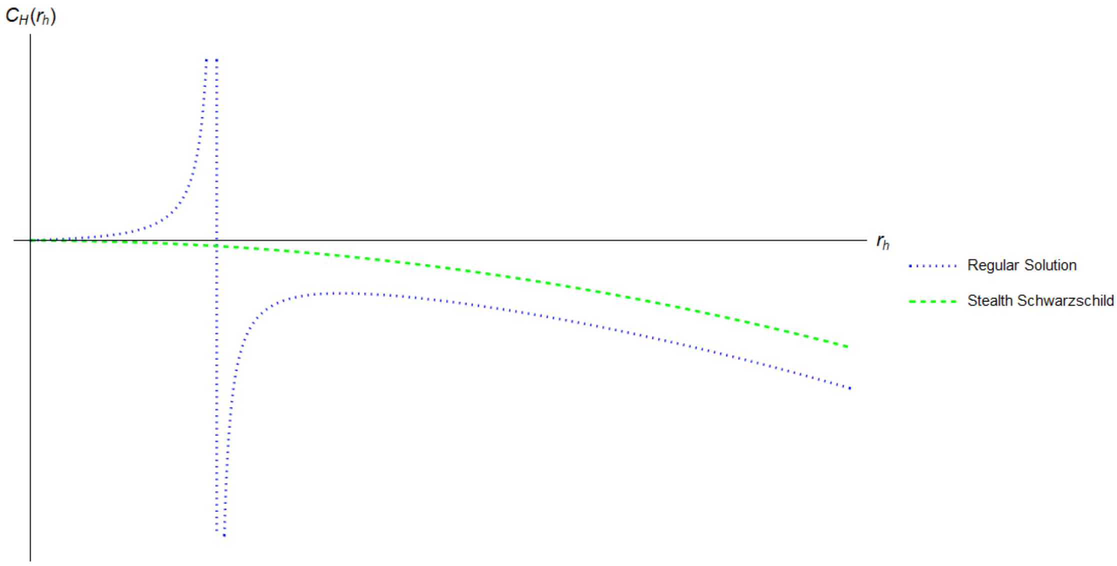


Figure 5. Heat capacity of the (A.1) black hole for $p = 1$ and the stealth Schwarzschild solution. This time they correspond to the same theories, even though their behaviour looks identical to before. Further the temperature is positive everywhere, so there is no extremal value of r and the heat capacities can be plotted from $r = 0$.

4 Conclusions

Making use of a generalized Kerr-Schild solution generating method, as described in [36], we have constructed a family of regular black holes, namely solutions without curvature singularities. They are characterized by the presence of an arctangent regularizing function, and are regular solutions of specific higher-order scalar tensor theories known as DHOST theories. The solutions are asymptotically flat and are accompanied by a regular scalar field. They are characterized by a de Sitter or, increasingly regular core, inner and outer event horizons and particle-like regular solutions. The latter appear depending on a certain theory strength parameter σ (related to the mass) and could have a distinct phenomenology as compared to black holes due to the absence of the horizon. Indeed we examined a number of observable consequences of our solutions ranging from weaker to stronger gravity: from the leading post-Newtonnian Eddington parameters to leading precession effects up to enhanced geodesic light rings. It would be interesting to go beyond our initial calculations and check for example echoes of our particle-like solutions as predicted in [67–69]. Very recent similar studies have shown such effects in the case of Einstein-Gauss-Bonnet theories [70] and it would be interesting to apply known methods for our analytic explicit solutions.

Our regular black hole solutions differ from existing models of regular solutions in several ways. First of all, it is important to stress that the DHOST models for which regular black holes exist are not finetuned by some regularizing parameter, which is usually the case for regular black holes. Regularity of the solution is achieved directly by the form of the kinetic $X(r)$ function. As a direct consequence the regular solutions (once regularity of the core is fixed) only depend on a unique integration constant, mass and a bookkeeping parameter σ which measures the magnitude of the higher order effects (the limiting case $\sigma \to 0$ gives GR). This is a major difference with respect to the regular black holes of non-linear electrodynamic models, since in those cases the mass, as well as the regularizing parameter (usually associated

JCAP06(2021)021

to a magnetic charge), are part of the non-linear electrodynamic Lagrangian. In the present case, the regular solutions only depend on a unique integration constant, which is shown to be proportional to the mass. We also note that the "usual" area law for the entropy is not compatible with the regularity of our solution (2.6)–(2.8) and this is due to the theory's modified nature of gravity. This is quite common and understood in certain cases due to the higher order nature of the theory (see for example [66]). In spite of the violation of the area law, we have shown that the first law of thermodynamics is always satisfied. The regular black hole solutions have a mass fall-off of the form $\frac{\arctan(r^p)}{r}$, where $p > 0$ is a parameter of the theory. Note that examples of black hole solutions with such regular terms at the origin have been encountered [71] as AdS solitons. We have seen that the small regular black holes are thermodynamically stable since their heat capacity turns out to be positive and for the range of values of the parameter ensuring the regularity solution, we have observed the existence of second order phase transitions for all our regular black holes.

It would be interesting to question if regularity of such solutions in DHOST theories persists once these are rotating. Given the recent progress in this direction [46–48] there may be hope in such a direction, even analytically. Furthermore, it would be an interesting first step to extend regular solutions to the presence of a time dependent scalar field in order to understand how the picture of geodesics is altered with regularity. These are some of the possible directions in this exciting field that we hope to pursue in the near future.

Acknowledgments

We would like to thank Tim Anson, Eloy Ayón-Beato, Eugeny Babichev, Thanasis Bakopoulos, Alessandro Fabbri, Panagiota Kanti, Antoine Lehébel and Georgios Pappas for many enlightening discussions. The authors also gratefully acknowledge the kind support of the PROGRAMA DE COOPERACIÓN CIENTÍFICA ECOSud-CONICYT 180011/C18U04. OB is funded by the PhD scholarship of the University of Talca. The work of MSJ is funded by the National Agency for Research and Development (ANID) / Scholarship Program/ DOCTORADO BECA NACIONAL/ 2019 – 21192009

A DHOST models for the regular solution (2.14)

Along the lines of [36], one can show that the DHOST action defined by

$$
\begin{aligned}
\mathcal{H}(X) &= -\frac{2}{3\pi X - p\sin(\pi X)},\\
G(X) &= \frac{p^2\sin(2\pi X) - 8p\sin(\pi X) + 6\pi X}{(p\sin(\pi X) - 3\pi X)^2},\\
A_1(X) &= \frac{2p\sin(\pi X)(p\cos(\pi X) - 3)}{X(p\sin(\pi X) - 3\pi X)^2},\\
K(X) &= \frac{p\sin(\frac{\pi}{2}X)^{\frac{p-2}{p}}\cos(\frac{\pi}{2}X)^{\frac{p+2}{p}}\left(B^2p^2\cos(2\pi X) - B^2p^2 - 24pX^2\cos(\pi X) + 28BpX\sin(\pi X) - 24X^2\right)}{3X^2A^{\frac{2}{p}}(p\sin(\pi X) - 3\pi X)^2},
\end{aligned}
$$

and

$$
A_3(X) = \frac{B\left(2p^2(5B^2+144X^2)\cos(2\pi X)+3p(B^2p^2-192X^2)\cos(\pi X)-3B^2p^3\cos(3\pi X)-10B^2p^2+24BpX\sin(\pi X)(-23p\cos(\pi X)+2p^2+43)-288X^2\right)}{3X^2(Bp\sin(\pi X)-6X)^3},
$$

where $A = \frac{2\sigma^{p-1}}{\pi}$ and $B = \frac{2}{\pi}$ and σ an unspecified constant, admits the following regular black hole solution

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2\mu \arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right)}{\pi r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2\mu \arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right)}{\pi r}\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \\
X(r) &= \frac{2}{\pi} \arctan\left(\frac{\pi r^p}{2\sigma^{p-1}}\right). \qquad \text{(A.1)}
\end{aligned}
$$

References

[1] R. Penrose, *Gravitational collapse and space-time singularities*, *Phys. Rev. Lett.* **14** (1965) 57 [INSPIRE].

[2] S.W. Hawking and R. Penrose, *The Singularities of gravitational collapse and cosmology*, *Proc. Roy. Soc. Lond. A* **314** (1970) 529 [INSPIRE].

[3] A.D. Sakharov, *Nachal'naia stadija rasshirenija Vselennoj i vozniknovenije neodnorodnosti raspredelenija veshchestva*, *Sov. Phys. JETP* **22** (1966) 241 [INSPIRE].

[4] E. Gliner, *Algebraic Properties of the Energy-momentum Tensor and Vacuum-like States of Matter*, *Sov. Phys. JETP* **22** (1966) 378.

[5] J. Bardeen, *Non-singular general-relativistic gravitational collapse*, in *Proceedings of International Conference GR5*, Tbilisi USSR (1968), pg. 174.

[6] E. Ayon-Beato and A. Garcia, *The Bardeen model as a nonlinear magnetic monopole*, *Phys. Lett. B* **493** (2000) 149 [gr-qc/0009077] [INSPIRE].

[7] E. Ayon-Beato and A. Garcia, *Regular black hole in general relativity coupled to nonlinear electrodynamics*, *Phys. Rev. Lett.* **80** (1998) 5056 [gr-qc/9911046] [INSPIRE].

[8] E. Ayon-Beato and A. Garcia, *New regular black hole solution from nonlinear electrodynamics*, *Phys. Lett. B* **464** (1999) 25 [hep-th/9911174] [INSPIRE].

[9] E. Ayon-Beato and A. Garcia, *Four parametric regular black hole solution*, *Gen. Rel. Grav.* **37** (2005) 635 [hep-th/0403229] [INSPIRE].

[10] K.A. Bronnikov, *Regular magnetic black holes and monopoles from nonlinear electrodynamics*, *Phys. Rev. D* **63** (2001) 044005 [gr-qc/0006014] [INSPIRE].

[11] I. Dymnikova, *Regular electrically charged structures in nonlinear electrodynamics coupled to general relativity*, *Class. Quant. Grav.* **21** (2004) 4417 [gr-qc/0407072] [INSPIRE].

[12] L. Balart and E.C. Vagenas, *Regular black holes with a nonlinear electrodynamics source*, *Phys. Rev. D* **90** (2014) 124045 [arXiv:1408.0306] [INSPIRE].

[13] S. Ansoldi, *Spherical black holes with regular center: A Review of existing models including a recent realization with Gaussian sources*, in *Conference on Black Holes and Naked Singularities*, Milan Italy (2008) [arXiv:0802.0330] [INSPIRE].

[14] M. Sharif and W. Javed, *Quantum Corrections for a Bardeen Regular Black Hole*, *J. Korean Phys. Soc.* **57** (2010) 217.

[15] M.-S. Ma and R. Zhao, *Corrected form of the first law of thermodynamics for regular black holes*, *Class. Quant. Grav.* **31** (2014) 245014 [arXiv:1411.0833] [INSPIRE].

[16] P.A. Cano and Á. Murcia, *Resolution of Reissner-Nordström singularities by higher-derivative corrections*, *Class. Quant. Grav.* **38** (2021) 075014 [arXiv:2006.15149] [INSPIRE].

[17] P.A. Cano and Á. Murcia, *Electromagnetic Quasitopological Gravities*, *JHEP* **10** (2020) 125 [arXiv:2007.04331] [INSPIRE].

[18] G.W. Horndeski, *Second-order scalar-tensor field equations in a four-dimensional space*, *Int. J. Theor. Phys.* **10** (1974) 363 [INSPIRE].

[19] D. Langlois and K. Noui, *Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability*, *JCAP* **02** (2016) 034 [arXiv:1510.06930] [INSPIRE].

[20] D. Langlois and K. Noui, *Hamiltonian analysis of higher derivative scalar-tensor theories*, *JCAP* **07** (2016) 016 [arXiv:1512.06820] [INSPIRE].

[21] M. Crisostomi, M. Hull, K. Koyama and G. Tasinato, *Horndeski: beyond, or not beyond?*, *JCAP* **03** (2016) 038 [arXiv:1601.04658] [INSPIRE].

[22] M. Crisostomi, K. Koyama and G. Tasinato, *Extended Scalar-Tensor Theories of Gravity*, *JCAP* **04** (2016) 044 [arXiv:1602.03119] [INSPIRE].

[23] H. Motohashi, K. Noui, T. Suyama, M. Yamaguchi and D. Langlois, *Healthy degenerate theories with higher derivatives*, *JCAP* **07** (2016) 033 [arXiv:1603.09355] [INSPIRE].

[24] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui and G. Tasinato, *Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order*, *JHEP* **12** (2016) 100 [arXiv:1608.08135] [INSPIRE].

[25] E. Babichev, C. Charmousis and A. Lehébel, *Asymptotically flat black holes in Horndeski theory and beyond*, *JCAP* **04** (2017) 027 [arXiv:1702.01938] [INSPIRE].

[26] E. Babichev and A. Lehébel, *The sound of DHOST*, *JCAP* **12** (2018) 027 [arXiv:1810.09997] [INSPIRE].

[27] J. Chagoya and G. Tasinato, *Compact objects in scalar-tensor theories after GW170817*, *JCAP* **08** (2018) 006 [arXiv:1803.07476] [INSPIRE].

[28] T. Kobayashi and T. Hiramatsu, *Relativistic stars in degenerate higher-order scalar-tensor theories after GW170817*, *Phys. Rev. D* **97** (2018) 104012 [arXiv:1803.10510] [INSPIRE].

[29] M. Minamitsuji and J. Edholm, *Black holes with a nonconstant kinetic term in degenerate higher-order scalar tensor theories*, *Phys. Rev. D* **101** (2020) 044034 [arXiv:1912.01744] [INSPIRE].

[30] E. Babichev, C. Charmousis, G. Esposito-Farèse and A. Lehébel, *Stability of Black Holes and the Speed of Gravitational Waves within Self-Tuning Cosmological Models*, *Phys. Rev. Lett.* **120** (2018) 241101 [arXiv:1712.04398] [INSPIRE].

[31] E. Babichev, C. Charmousis, G. Esposito-Farèse and A. Lehébel, *Hamiltonian unboundedness vs stability with an application to Horndeski theory*, *Phys. Rev. D* **98** (2018) 104050 [arXiv:1803.11444] [INSPIRE].

[32] C. Charmousis, M. Crisostomi, R. Gregory and N. Stergioulas, *Rotating Black Holes in Higher Order Gravity*, *Phys. Rev. D* **100** (2019) 084020 [arXiv:1903.05519] [INSPIRE].

[33] E. Babichev and G. Esposito-Farese, *Cosmological self-tuning and local solutions in generalized Horndeski theories*, *Phys. Rev. D* **95** (2017) 024020 [arXiv:1609.09798] [INSPIRE].

[34] T. Kobayashi, *Horndeski theory and beyond: a review*, *Rept. Prog. Phys.* **82** (2019) 086901 [arXiv:1901.07183] [INSPIRE].

[35] S.A. Hayward, *Formation and evaporation of regular black holes*, *Phys. Rev. Lett.* **96** (2006) 031103 [gr-qc/0506126] [INSPIRE].

[36] E. Babichev, C. Charmousis, A. Cisterna and M. Hassaine, *Regular black holes via the Kerr-Schild construction in DHOST theories*, *JCAP* **06** (2020) 049 [arXiv:2004.00597] [INSPIRE].

[37] G. Domènech, A. Naruko, M. Sasaki and C. Wetterich, *Could the black hole singularity be a field singularity?*, *Int. J. Mod. Phys. D* **29** (2020) 2050026 [arXiv:1912.02845] [INSPIRE].

[38] E. Babichev and C. Charmousis, *Dressing a black hole with a time-dependent Galileon*, *JHEP* **08** (2014) 106 [arXiv:1312.3204] [INSPIRE].

[39] T. Kobayashi and N. Tanahashi, *Exact black hole solutions in shift symmetric scalar–tensor theories*, *PTEP* **2014** (2014) 073E02 [arXiv:1403.4364] [INSPIRE].

[40] M. Bravo-Gaete and M. Hassaine, *Lifshitz black holes with a time-dependent scalar field in a Horndeski theory*, *Phys. Rev. D* **89** (2014) 104028 [arXiv:1312.7736] [INSPIRE].

[41] A. Cisterna, T. Delsate and M. Rinaldi, *Neutron stars in general second order scalar-tensor theory: The case of nonminimal derivative coupling*, *Phys. Rev. D* **92** (2015) 044050 [arXiv:1504.05189] [INSPIRE].

[42] O. Baake, M.F. Bravo Gaete and M. Hassaine, *Spinning black holes for generalized scalar tensor theories in three dimensions*, *Phys. Rev. D* **102** (2020) 024088 [arXiv:2005.10869] [INSPIRE].

[43] A. Anabalon, A. Cisterna and J. Oliva, *Asymptotically locally AdS and flat black holes in Horndeski theory*, *Phys. Rev. D* **89** (2014) 084050 [arXiv:1312.3597] [INSPIRE].

[44] C. Charmousis and D. Iosifidis, *Self tuning scalar tensor black holes*, *J. Phys. Conf. Ser.* **600** (2015) 012003 [arXiv:1501.05167] [INSPIRE].

[45] A. Lehébel, *Compact astrophysical objects in modified gravity*, other thesis, 10, 2018 [arXiv:1810.04434] [INSPIRE].

[46] T. Anson, E. Babichev, C. Charmousis and M. Hassaine, *Disforming the Kerr metric*, *JHEP* **01** (2021) 018 [arXiv:2006.06461] [INSPIRE].

[47] J. Ben Achour, H. Liu, H. Motohashi, S. Mukohyama and K. Noui, *On rotating black holes in DHOST theories*, *JCAP* **11** (2020) 001 [arXiv:2006.07245] [INSPIRE].

[48] T. Anson, E. Babichev and C. Charmousis, *Deformed black hole in Sagittarius A*, arXiv:2103.05490 [INSPIRE].

[49] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, *Exploring gravitational theories beyond Horndeski*, *JCAP* **02** (2015) 018 [arXiv:1408.1952] [INSPIRE].

[50] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, *Healthy theories beyond Horndeski*, *Phys. Rev. Lett.* **114** (2015) 211101 [arXiv:1404.6495] [INSPIRE].

[51] A. Simpson and M. Visser, *Regular black holes with asymptotically Minkowski cores*, *Universe* **6** (2019) 8 [arXiv:1911.01020] [INSPIRE].

[52] A. Tucker and C.M. Will, *Pericenter advance in general relativity: Comparison of approaches at high post-Newtonian orders*, *Class. Quant. Grav.* **36** (2019) 115001 [arXiv:1809.05163] [INSPIRE].

[53] C.M. Will, *The Confrontation between General Relativity and Experiment*, *Living Rev. Rel.* **17** (2014) 4 [arXiv:1403.7377] [INSPIRE].

[54] GRAVITY collaboration, *Detection of the Schwarzschild precession in the orbit of the star S2 near the Galactic centre massive black hole*, *Astron. Astrophys.* **636** (2020) L5 [arXiv:2004.07187] [INSPIRE].

[55] R. d'Inverno, *Introducing Einstein's Relativity*, Clarendon Press, Oxford U.K. (1992).

[56] T. Johannsen, *Sgr A* and General Relativity*, *Class. Quant. Grav.* **33** (2016) 113001 [arXiv:1512.03818] [INSPIRE].

[57] EVENT HORIZON TELESCOPE collaboration, *First M87 Event Horizon Telescope Results. VI. The Shadow and Mass of the Central Black Hole*, *Astrophys. J. Lett.* **875** (2019) L6 [arXiv:1906.11243] [INSPIRE].

[58] EVENT HORIZON TELESCOPE collaboration, *Gravitational Test Beyond the First Post-Newtonian Order with the Shadow of the M87 Black Hole*, Phys. Rev. Lett. **125** (2020) 141104 [arXiv:2010.01055] [INSPIRE].

[59] K. Gebhardt et al., *The Black-Hole Mass in M87 from Gemini/NIFS Adaptive Optics Observations*, Astrophys. J. **729** (2011) 119 [arXiv:1101.1954] [INSPIRE].

[60] K. Glampedakis and G. Pappas, *Can supermassive black hole shadows test the Kerr metric?*, arXiv:2102.13573 [INSPIRE].

[61] Y.S. Myung, Y.-W. Kim and Y.-J. Park, *Entropy of an extremal regular black hole*, Phys. Lett. B **659** (2008) 832 [arXiv:0705.2478] [INSPIRE].

[62] Y.S. Myung, Y.-W. Kim and Y.-J. Park, *Thermodynamics of regular black hole*, Gen. Rel. Grav. **41** (2009) 1051 [arXiv:0708.3145] [INSPIRE].

[63] L. Balart, *Quasilocal Energy, Komar Charge and Horizon for Regular Black Holes*, Phys. Lett. B **687** (2010) 280 [arXiv:0912.0334] [INSPIRE].

[64] R. Tharanath, J. Suresh and V.C. Kuriakose, *Phase transitions and Geometrothermodynamics of Regular black holes*, Gen. Rel. Grav. **47** (2015) 46 [arXiv:1406.3916] [INSPIRE].

[65] S.H. Hendi, S.N. Sajadi and M. Khademi, *Physical properties of a regular rotating black hole: Thermodynamics, stability, and quasinormal modes*, Phys. Rev. D **103** (2021) 064016 [arXiv:2006.11575] [INSPIRE].

[66] C. Charmousis, *Higher order gravity theories and their black hole solutions*, Lect. Notes Phys. **769** (2009) 299 [arXiv:0805.0568] [INSPIRE].

[67] V. Cardoso, E. Franzin and P. Pani, *Is the gravitational-wave ringdown a probe of the event horizon?*, Phys. Rev. Lett. **116** (2016) 171101 [*Erratum ibid.* **117** (2016) 089902] [arXiv:1602.07309] [INSPIRE].

[68] V. Cardoso, S. Hopper, C.F.B. Macedo, C. Palenzuela and P. Pani, *Gravitational-wave signatures of exotic compact objects and of quantum corrections at the horizon scale*, Phys. Rev. D **94** (2016) 084031 [arXiv:1608.08637] [INSPIRE].

[69] V. Cardoso and P. Pani, *Tests for the existence of black holes through gravitational wave echoes*, Nature Astron. **1** (2017) 586 [arXiv:1709.01525] [INSPIRE].

[70] B. Kleihaus, J. Kunz and P. Kanti, *Properties of ultracompact particlelike solutions in Einstein-scalar-Gauss-Bonnet theories*, Phys. Rev. D **102** (2020) 024070 [arXiv:2005.07650] [INSPIRE].

[71] E. Babichev, C. Charmousis and M. Hassaine, *Black holes and solitons in an extended Proca theory*, JHEP **05** (2017) 114 [arXiv:1703.07676] [INSPIRE].

JCAP06(2021)021

Eur. Phys. J. C (2021) 81:642
https://doi.org/10.1140/epjc/s10052-021-09449-2

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Rotating stealth black holes with a cohomogeneity−1 metric

Olaf Baake[1,2,a], **Mokhtar Hassaine**[1,b]

[1] Instituto de Matemática (INSTMAT), Universidad de Talca, Casilla 747, 3460000 Talca, Chile
[2] Centro de Estudios Cientificos (CECs), Av. Arturo Part 514, 5090000 Valdivia, Chile

Received: 17 May 2021 / Accepted: 12 July 2021

Abstract In five dimensions we consider a general shift symmetric and parity preserving scalar tensor action that contains up to second order covariant derivatives of the scalar field. A rotating stealth black hole solution is constructed where the metric is given by the Myers–Perry spacetime with equal momenta and the scalar field is identified with the Hamilton–Jacobi potential. This nontrivial scalar field has an extra hair associated with the rest mass of the test particle, and the solution does not require any fine tuning of the coupling functions of the theory. Interestingly enough, we show that the disformal transformation, generated by this scalar field, and with a constant degree of disformality, leaves invariant (up to diffeomorphisms) the Myers–Perry metric with equal momenta. This means that the hair of the scalar field, along with the constant disformality parameter, can be consistently absorbed into further redefinitions of the mass and of the single angular parameter of the disformed metric. These results are extended in higher odd dimensions with a Myers–Perry metric for which all the momenta are equal. The key of the invariance under disformal transformation of the metric is mainly the cohomogeneity−1 character of the Myers–Perry metric with equal momenta. Starting from this observation, we consider a general class of cohomogeneity−1 metrics in arbitrary dimension, and we list the conditions ensuring that this class of metrics remain invariant (up to diffeomorphisms) under a disformal transformation with a constant degree of disformality and with a scalar field with constant kinetic term. The extension to the Kerr+-de Sitter case is also considered where it is shown that rotating stealth solutions may exist provided some fine tuning of the coupling functions of the scalar tensor theory.

1 Introduction

Even though the detection of gravitational waves [1] has raised Einstein's four-dimensional General Relativity (GR) to an exceptional position, this should not slow down our desire of exploring the theories of gravity in higher dimensions, as well as to study its black hole solutions. The interests in these studies are numerous and diverse. For example one can mention the gauge/gravity duality [2] which allows to relate the properties of black holes in some dimension to the properties of strongly coupled quantum field theories defined in some lower dimension. For example, by applying the AdS/CFT machinery with a five-dimensional AdS black hole, the authors of Ref. [3] were able to show that the ratio of the shear viscosity and the volume density of entropy was close to a certain universal constant, which was further confirmed at the Relativistic Heavy Ion Collider. From a different point of view, it is undeniable that, using a purely mathematical approach, the study of higher-dimensional black holes has required the development of new mathematical tools with significant benefit for the scientific community. For a nice review on higher-dimensional black holes see Ref. [4]. In the same spirit we deem it important to accompany the studies of higher-dimensional black holes with modifying GR in order to explore new promising theoretical possibilities in the realm of gravity. From this angle, scalar tensor theories have attracted a great deal of attention over the past two decades. These theories can be considered as one of the simplest modifications to the theory of gravity, since one only needs to introduce a single scalar field in addition to the metric. One of the pioneering works in this context was provided by Horndeski in the seventies where he presented the most general scalar-tensor theory with second order equations of motion in four dimensions [5]. This requirement of not having more than two derivatives is a sufficient condition that prevents the theory to have a Hamiltonian that is unbounded from below. More recently, it was shown that this virtue of the Horndeski theories can also be extended to scalar tensor theories which

[a] e-mail: olaf.baake@inst-mat.utalca.cl (corresponding author)
[b] e-mail: hassaine@inst-mat.utalca.cl

Published online: 22 July 2021

have higher than second order equations of motion. These latter are now known as the Degenerate Higher Order Scalar Tensor Theories (DHOST theories) [6–11]. Many interesting solutions have been found concerning these DHOST theories, see e.g. [12–23]. While the above-mentioned relevant features are inherent in the four-dimensional DHOST theories, we find it interesting to also consider the extension of general scalar tensor theories in higher dimensions in order to explore, among other things, their possible rotating black hole solutions.

This is precisely the aim of the present paper where we will be investigating the possibility of constructing rotating black hole solutions in these modified gravity theories. As is known, this task is highly non-trivial, especially for scalar-tensor theories, where in general the complexity and the high degree of non-linearity of the field equations almost make it impossible to find spinning solutions. Starting from this observation, we will approach the problem from a very particular perspective, which can be summarized as follows. We will start by fixing the metric background to be a vacuum rotating black hole spacetime, and we will investigate whether this spacetime can be endowed with a non-trivial scalar field such that the full equations of motion are satisfied. Such solutions may be identified with the so-called stealth solutions, see Refs. [24,25] for the original works on stealth configurations. In the present case, we will show that the vacuum Myers–Perry metric [26] in higher odd dimensions, with equal angular momenta, can accommodate a non-trivial scalar field in such a way that the resulting scalar tensor configuration will satisfy the complete field equations of some general scalar tensor theories containing, in particular, the DHOST branch, and also including the sector with unitary speed of gravitational waves. Regarding the scalar field solution, we will see that its expression merges with that of the Hamilton–Jacobi action [27] in which the azimuthal conserved quantities are zero, and the energy is equal to the test particle mass. More precisely, the scalar field is shown to be linearly time-dependent with a radial dependence in such way that its kinetic term is a constant given by minus the square of the test particle mass. These conditions are similar to those found for the disformed Kerr metric in [19,28,29], where it was shown that the restriction on the energy ensures the scalar hair to be well defined from the event horizon up to asymptotic infinity, while the vanishing of the azimuthal conserved quantities guarantees the regularity at the poles. Note that such an ansatz for the scalar field has been proven to be fruitful for finding solutions of Horndeski/DHOST scalar tensor theories, see Ref. [30] and also [31–33].

Following Refs. [28,29], we will make use of the stealth scalar field solution for constructing disformal versions of the metric solution. Surprisingly, we will establish that the odd-dimensional Myers–Perry spacetimes with equal momenta remain invariant (up to diffeomorphisms and redefinitions of the constants) under a disformal transformation generated by the stealth scalar field with a constant degree of disformality. This result is in itself intriguing since the disformal transformations are supposed to map solutions of some classes of scalar tensor theories to other (different) classes, and the resulting disformed metrics are in general quite different from the original ones, [17,18,34]. To illustrate this assertion, we mention the works of Refs. [28,29] where a disformal version of the Kerr spacetime with a regular scalar field was constructed, and where the disformed Kerr metric turned out to be neither Ricci flat nor circular [28]. Note that disformal transformations in the case of static metrics were previously considered in [35].

It is clear that the possibility of constructing rotating stealth black hole configurations, and that the corresponding scalar field leaving the metric invariant under a constant disformal transformation, are highly correlated with the particular symmetries of the metric. Indeed, it is well-known that the Myers–Perry line element in D dimensions can be shown to have an isometry group identified with $\mathbb{R} \times U(1)^n$ where $\mathbb{R}$ corresponds to time translations and $n = [(D-1)/2]$. Nevertheless, this symmetry group in odd dimension, $D = 2N+3$, with all its angular momenta equal, $a_i = a$, is extended to a bigger symmetry group given by $\mathbb{R} \times U(N+1)$. As a direct consequence of this symmetry enhancement, the odd-dimensional Myers–Perry spacetime with equal momenta is cohomogeneity-1, which is to say that it only non-trivially depends on a single coordinate. We may note that the cohomogeneity-1 character of the Myers–Perry metrics has been proven to be of great importance in order to study the stability of theses particular odd-dimensional Myers–Perry spacetimes, see [36], where it was concluded that there was no evidence of instability in five and seven dimensions in contrast with the nine-dimensional case where an instability was found. In the present work, one can claim with sincerity that the cohomogeneity-1 property of the Myers–Perry metrics with equal angular momenta is clearly responsible for its disformal invariance. We will go further in this direction by considering a general class of cohomogeneity-1 metrics in arbitrary dimension, and by establishing a list of conditions for the metric, ensuring its invariance (up to diffeomorphisms) under a disformal transformation with a constant degree of disformality, and with a scalar field whose kinetic term is constant.

The extension to the Kerr-de Sitter case is also considered where it is shown that rotating stealth solutions may exist provided some fine tuning of the coupling functions of the scalar tensor theory.

The plan of the paper is organized as follows. In Sect. 2, we will introduce the scalar tensor theories under consideration and present their field equations in the particular case where the scalar field is taken to have a constant kinetic term. We will establish that for a vacuum black hole metric the complete set of field equations can be satisfied, provided that the scalar field solves a unique constraint equation which, as we will see, can be solved in odd dimensions. In Sect. 3, we will start by considering the five-dimensional case where the rotating stealth black hole will be constructed on the Myers–Perry spacetime with equal angular momenta. We will also see that the scalar field leaves the metric invariant under a constant disformal transformation. These results will then be generalized in higher odd dimensions where the Myers–Perry spacetime with equal momenta is cohomogeneity−1. In Sect. 4, starting from this observation, we consider a general class of cohomogeneity−1 metrics in arbitrary dimension, and we list the conditions ensuring this class of metrics to remain invariant (up to diffeomorphisms) under a constant disformal transformation. The last section is concerned with our conclusions while an Appendix is devoted to extend our results in presence of a cosmological constant.

2 Set up of the theory

In the present work, we will be concerned with the following shift symmetric, and parity preserving scalar tensor action that contains up to second order covariant derivatives of the scalar field

$$\begin{aligned}S[g,\phi]=\int d^Dx\sqrt{-g}\Big[&G(X)R+A_1(X)\\&\times\left[\phi_{\mu\nu}\phi^{\mu\nu}-(\Box\phi)^2\right]+A_3(X)\Box\phi\,\phi^{\mu}\phi_{\mu\nu}\phi^{\nu}\\&+A_4(X)\phi^{\mu}\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho}+A_5(X)\left(\phi^{\mu}\phi_{\mu\nu}\phi^{\nu}\right)^2\Big],\end{aligned}\tag{2.1}$$

Here the coupling functions G, A_1, A_3, A_4 and A_5 are functions of the kinetic term, $X=g^{\mu\nu}\phi_\mu\phi_\nu$, in order to ensure the shift symmetry $\phi\to\phi+$cst. Also, for simplicity we have defined $\phi_\mu=\nabla_\mu\phi$ and $\phi_{\mu\nu}=\nabla_\mu\nabla_\nu\phi$. As mentioned in the introduction, the action (2.1) can propagate healthy degrees of freedom in four dimensions, provided the functions A_4 and A_5 are constrained by

$$\begin{aligned}A_4=&\frac{1}{8(G-XA_1)^2}\Big\{4G\left[3(-A_1+2G')^2-2A_3G\right]\\&-A_3X^2(16A_1G'+A_3G)\\&+4X\left[3A_1A_3G+16A_1^2G'-16A_1(G')^2-4A_1^3+2A_3GG'\right]\Big\},\\A_5=&\frac{1}{8(G-XA_1)^2}(2A_1-XA_3-4G')\\&\times\left(A_1(2A_1+3XA_3-4G')-4A_3G\right),\end{aligned}$$

but for our specific task, we will not *a priori* consider such restrictions on the coupling functions.

Neither will we write down the equations of motion in all of their generality, but instead restrict their expressions in the case of a constant kinetic term $X=$ cst. In doing so, the equations arising from the variation of the general scalar tensor field action (2.1) with respect to the metric reduce to

$$\begin{aligned}&G(X)G_{\mu\nu}+G'(X)R\phi_\mu\phi_\nu\\&-\frac{1}{2}A_3(X)\Big[(\Box\phi)^2-(\phi_{\alpha\beta})(\phi^{\alpha\beta})\Big]\phi_\mu\phi_\nu\\&+\frac{1}{2}A_3(X)\Big[R_{\alpha\beta}\phi^\alpha\phi^\beta\Big]\phi_\mu\phi_\nu\\&+A_1(X)\Big[-R_{\nu\lambda}\phi_\mu\phi^\lambda-R_{\mu\lambda}\phi_\nu\phi^\lambda\\&-\frac{1}{2}g_{\mu\nu}\left[(\Box\phi)^2-(\phi_{\alpha\beta})(\phi^{\alpha\beta})\right]\\&+g_{\mu\nu}\left[R_{\lambda\rho}\phi^\lambda\phi^\rho\right]+\phi_{\mu\nu}\Box\phi+\phi^\lambda\phi_{\lambda\mu\nu}\Big]\\&-A_1'(X)\left[(\Box\phi)^2-(\phi_{\alpha\beta})(\phi^{\alpha\beta})\right]\phi_\mu\phi_\nu=0,\end{aligned}\tag{2.2}$$

while its variation with respect to the scalar field is a conserved current equation given by $\nabla_\mu J^\mu=0$, with

$$\begin{aligned}J^\mu=2\Big(&G'(X)R-\left[A_1'(X)+\frac{1}{2}A_3(X)\right]\\&\times\Big[(\Box\phi)^2-(\phi_{\alpha\beta})(\phi^{\alpha\beta})\Big]\\&+\frac{1}{2}A_3(X)\Big[R_{\alpha\beta}\phi^\alpha\phi^\beta\Big]\Big)\phi^\mu\\&-2A_1(X)R^{\mu\nu}\phi_\nu.\end{aligned}\tag{2.3}$$

Note that this conservation equation is a direct consequence of the shift symmetry of the action.

Further, it is remarkable to note that for any any vacuum metric solution $R_{\mu\nu}=0$, the field equations (2.2–2.3) will be automatically fulfilled, provided that the scalar field satisfies (in addition of having a constant kinetic term) the following two conditions

$$(\Box\phi)^2-\left(\phi_{\mu\nu}\right)\left(\phi^{\mu\nu}\right)=0,\tag{2.4a}$$

$$\phi_{\mu\nu}\,\Box\phi+\phi^\lambda\phi_{\lambda\mu\nu}=0,\tag{2.4b}$$

and this without imposing any restrictions on the coupling functions, $G(X)$, $A_1(X)$, $A_3(X)$, $A_4(X)$ and $A_5(X)$. On the other hand, since the kinetic term of the scalar field is constant, $\phi_\mu\phi^\mu=$ cst, it is easy to see that the trace of the second equation (2.4b) yields the first condition (2.4a). As a direct consequence, for a vacuum metric it will be sufficient for the scalar field to have a constant kinetic term and to satisfy the following tensorial equation

$$\phi_{\mu\nu}\,\Box\phi+\phi^\lambda\phi_{\lambda\mu\nu}=0,\tag{2.5}$$

to ensure the field equations (2.2–2.3) to be satisfied for any coupling functions. This observation will be our guiding principle in order to construct stealth rotating black hole solutions of the general scalar tensor theory defined by (2.1). In presence of matter sources, conditions ensuring the construction of solutions in general quadratic higher order scalar-tensor theories with and without cosmological constant have been derived in [37]. However, in contrast to the present case, these conditions put constraints on the coupling functions of the theories instead of the scalar field (for example requiring that A_1 and A_2 vanish at the constant value of X).

Now, in order to construct the corresponding stealth scalar field, ϕ, we must first make sure that its kinetic term is constant. A simple option would be to identify the scalar field with the Hamilton–Jacobi potential, S, as is done in [19], i.e.

$$\phi \equiv \mathcal{S}, \tag{2.6}$$

where $\mathcal{S}$ satisfies the Hamilton–Jacobi equation of a free particle of mass m,

$$g^{\mu\nu}\,\partial_\mu \mathcal{S}\,\partial_\nu \mathcal{S} = -m^2. \tag{2.7}$$

This hypothesis on the scalar field (2.6) is also useful to take advantage on the known results on the integrability of the Hamilton–Jacobi equations. It is also clear that, in this representation, the constant value of the kinetic term would be given by minus the square of the mass of the particle, i.e. $\phi_\mu \phi^\mu = -m^2$.

In what follows, we will consider vacuum spacetime metrics (representing rotating black holes) with a scalar field identified with the corresponding Hamilton–Jacobi potential (2.6), and we will discuss under which conditions the tensorial equations (2.5) can be fulfilled.

3 Rotating stealth black holes and their disformal transformations

In this section, we construct a concrete example of a rotating stealth black hole solution for the scalar theory defined by the action (2.1) and whose field equations for a constant kinetic term reduce to (2.2–2.3). As shown below, this will be possible in odd dimensions, and for a Myers–Perry vacuum metric [26] where all the angular momenta a_i take a single value, $a_i = a$, together with a scalar field identified with the Hamilton–Jacobi potential (2.6–2.7). Having in hand this scalar tensor solution, namely a metric, g, and a nontrivial scalar field, ϕ, it will be interesting to study the disformal transformation of the metric

$$\bar{g}_{\mu\nu} = g_{\mu\nu} - P(\phi, X)\,\phi_\mu\,\phi_\nu,$$

where P may be an arbitrary function of the scalar field and its kinetic term. The interest on such consideration is mainly due to the fact that such disformal transformations are known to be internal maps of the scalar tensor theories considered here (2.1). The special ingredient in our construction will be the fact that the the scalar field responsible for the disformed metric is related to the geodesics of Myers–Perry spacetimes (2.6–2.7). In Refs. [28,29], this construction was done for a stealth solution defined on the four-dimensional Kerr metric, and it was shown that the deviation of the disformed metric with respect to the Kerr metric is considerable. Indeed, the disformed Kerr metric was shown to be neither Ricci flat, nor circular, and obviously no longer a vacuum metric. In the present case, we will see that the disformal transformation of the Myers–Perry metric with equal angular momenta, denoted by $g_{\mu\nu}^{\mathrm{MP},a_i=a}$, and with a constant disformality parameter, P, i.e.

$$\bar{g}_{\mu\nu} = g_{\mu\nu}^{\mathrm{MP,a_i=a}} - P\,\phi_\mu\,\phi_\nu, \tag{3.1}$$

is diffeomorphic to itself, that is $\bar{g}_{\mu\nu} \underset{\text{diffeo}}{\sim} g_{\mu\nu}^{\mathrm{MP,a_i=a}}$.

We first analyze in detail the five-dimensional case before considering the extension to higher odd dimensions.

3.1 Stealth on the five-dimensional Myers–Perry spacetime and its (invariant) disformed transformation

As we have seen in the previous section, any vacuum metric together with a scalar field satisfying the tensorial equation (2.5) will be a solution of the full field equations (2.2–2.3) without any conditions on the coupling functions. In four dimensions, this problem was considered in Ref. [19], where the authors constructed a stealth solution defined on the Kerr(-de Sitter) metric. Nevertheless, in this case we would like to underline that, as the stealth scalar field does not fulfill the conditions (2.5), restrictions on the coupling functions are necessary. We now turn to the five-dimensional situation where we will notice that this kind of restrictions can be circumvented thanks to the symmetries of the vacuum metric.

The five-dimensional Myers–Perry solution [26] of the vacuum Einstein equations, $R_{\mu\nu} = 0$, in Boyer–Lindquist coordinates, $(t, r, \theta, \varphi, \psi)$, reads

$$\begin{aligned} ds^2_{\mathrm{MP}} = &-\left(1-\frac{2M}{\rho^2}\right)dt^2 + \frac{r^2\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \\ &+\frac{4aM\sin^2\theta}{\rho^2}dt d\varphi + \frac{4bM\cos^2\theta}{\rho^2}dt d\psi \\ &+\frac{4abM\sin^2\theta\cos^2\theta}{\rho^2}d\varphi d\psi \\ &+\sin^2\theta\left(r^2+a^2+\frac{2Ma^2\sin^2\theta}{\rho^2}\right)d\varphi^2 \\ &+\cos^2\theta\left(r^2+b^2+\frac{2Mb^2\cos^2\theta}{\rho^2}\right)d\psi^2, \end{aligned} \tag{3.2}$$

where we have defined

$$\Delta = (r^2+a^2)(r^2+b^2) - 2Mr^2,$$
$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta. \tag{3.3}$$

The metric (3.2) is characterized by its mass, M, and two angular momenta denoted by a and b. Once the vacuum metric is fixed, following the strategy outlined in the previous section, we now identify the scalar field with the Hamilton–Jacobi potential associated to the five-dimensional Myers–Perry solution. Fortunately, it was shown in Ref. [27] that the Hamilton–Jacobi equation of the Myers–Perry metric can be separated into the form

$$\mathcal{S} = \frac{1}{2}m^2\lambda - Et + S_r(r) + S_\theta(\theta) + L_1\varphi + L_2\psi, \tag{3.4}$$

where E is the energy, the L_i's are the conserved quantities associated with each rotation, and

$$S_r(r) = \int^r \frac{\sqrt{X(r)}}{\Delta}\, r\, dr,\; S_\theta(\theta) = \int^\theta \sqrt{\Theta(\theta)}\, d\theta,$$

with

$$X(r) = \Delta\Bigg[r^2(E^2-m^2) + (a^2-b^2)\times\left(\frac{L_1^2}{r^2+a^2} - \frac{L_2^2}{r^2+b^2}\right) - K\Bigg] + 2M(r^2+a^2)(r^2+b^2)\left[E + \frac{aL_1}{r^2+a^2} + \frac{bL_2}{r^2+b^2}\right]^2,$$
$$\Theta(\theta) = (E^2-m^2)\left(a^2\cos(\theta)^2 + b^2\sin(\theta)^2\right) - \frac{L_1^2}{\sin(\theta)^2} - \frac{L_2^2}{\cos(\theta)^2} + K.$$

Here K is a constant allowing to separate the equations of motion, and which plays a role analogous to Carter's constant, [38].

For the identification of the scalar field with the Hamilton–Jacobi potential (2.6–3.4), one can show after some calculations (which are somewhat too cumbersome to report them here) that the tensorial equations (2.5) will be satisfied if: (i) the azimuthal conserved quantities are zero, $L_1 = L_2 = 0$, (ii) the Carter constant is zero, $K = 0$, (iii) the values of the the two angular momenta of the Myers–Perry metric are equal, $b = a$, and (iv) the energy, E, is equal to the test particle mass, $E = m$. In other words, the non-trivial stealth scalar field is linear in time and has a radial dependence,

$$\phi(t,r) = -mt + S_r(r) \Longrightarrow \phi_\mu\phi^\mu = -m^2. \tag{3.5}$$

These restrictions are similar to those obtained for the disformed Kerr(-de Sitter) spacetimes [19,28,29] in order to deal with a regular scalar field. We then conclude that a rotating stealth black hole solution of the scalar tensor theory (2.1) with arbitrary coupling functions, $G(X)$, $A_1(X)$, $A_3(X)$, $A_4(X)$ and $A_5(X)$, can be given by

$$\begin{aligned} ds^2_{\mathrm{MP},b=a} =& -\left(1-\frac{2M}{r^2+a^2}\right)dt^2 \\ &+\frac{r^2(r^2+a^2)}{(r^2+a^2)^2-2Mr^2}dr^2 + (r^2+a^2)d\theta^2 \\ &+\frac{4aM\sin^2\theta}{r^2+a^2}dtd\varphi + \frac{4aM\cos^2\theta}{r^2+a^2}dtd\psi \\ &+\frac{4a^2M\sin^2\theta\cos^2\theta}{r^2+a^2}d\varphi d\psi \\ &+\sin^2\theta\left(r^2+a^2+\frac{2Ma^2\sin^2\theta}{r^2+a^2}\right)d\varphi^2 \\ &+\cos^2\theta\left(r^2+a^2+\frac{2Ma^2\cos^2\theta}{r^2+a^2}\right)d\psi^2, \end{aligned} \tag{3.6}$$

together with a scalar field defined by

$$\phi(t,r) = -mt - \sqrt{2M}m\int \frac{r(r^2+a^2)}{(r^2+a^2)^2-2Mr^2}dr. \tag{3.7}$$

Many comments can be made concerning this solution. Firstly, it is important to stress the importance of the linear time dependence of the scalar field (3.7) which ensures the existence of a non-trivial stealth configuration. Indeed, eliminating the time dependency of the scalar field would amount to considering an identically-to-zero scalar field. Leaving aside the physical interpretation of the constant m in (3.7), it can be set to one by exploiting the fact that the tensorial equation (2.5) is quadratic in the scalar field, and hence invariant under a rescaling by a constant, i.e. $\phi \to \frac{1}{m}\phi$. Also, the Myers–Perry metric with equal momenta enjoys an extension of its symmetry given by $U(1)\times SU(2) = U(2)$, which gives rise to its cohomogeneity-1 characteristic. Finally, we shall mention that in the non-rotating limit, $a = 0$, the solution reduces to a stealth configuration defined on the Schwarzschild spacetime.

Surprisingly, the non-trivial scalar field (3.7) will be shown to leave the cohomogeneity-1 metric (3.6) invariant (up to diffeomorphisms) through a disformal transformation (3.1) with a constant degree of disformality P. Indeed, in order to show this result explicitly, the disformed Myers–Perry metric with equal momenta, as defined in (3.1), becomes

$$
\begin{aligned}
d\bar{s}^2_{\text{disf. MP}} = & -\left(1-\frac{2M}{r^2+a^2}+Pm^2\right)dT^2 \\
& +\frac{r^2(r^2+a^2)}{(r^2+a^2)^2-\frac{2M}{Pm^2+1}r^2+\frac{2MPa^2m^2}{Pm^2+1}}dr^2 \\
& +(r^2+a^2)d\theta^2 \\
& +\frac{4aM\sin^2\theta}{r^2+a^2}dTd\Phi+\frac{4aM\cos^2\theta}{r^2+a^2}dTd\Psi \\
& +\frac{4a^2M\sin^2\theta\cos^2\theta}{r^2+a^2}d\Phi d\Psi \\
& +\sin^2\theta\left(r^2+a^2+\frac{2Ma^2\sin^2\theta}{r^2+a^2}\right)d\Phi^2 \\
& +\cos^2\theta\left(r^2+a^2+\frac{2Ma^2\cos^2\theta}{r^2+a^2}\right)d\Psi^2,
\end{aligned} \tag{3.8}
$$

where, in order to eliminate the undesirable cross-terms, we have introduced the new variables T, Φ and Ψ by means of

$$
dt = dT + k_0(r)dr, \qquad d\varphi = d\Phi + k_1(r)dr,
$$
$$
d\psi = d\Psi + k_1(r)dr,
$$

with

$$
\begin{aligned}
k_0(r) &= \frac{Pm^2\sqrt{2M}r(r^2+a^2)\left[(r^2+a^2)^2+2Ma^2\right]}{\left[(r^2+a^2)^2-2Mr^2\right]\left[(r^2+a^2)^2(Pm^2+1)-2M(r^2-Pm^2a^2)\right]}, \\
k_1(r) &= \frac{-(2M)^{3/2}Pm^2r^2(r^2+a^2)}{\left[(r^2+a^2)^2-2Mr^2\right]\left[(r^2+a^2)^2(Pm^2+1)-2M(r^2-Pm^2a^2)\right]}.
\end{aligned}
$$

Finally, under the following redefinitions

$$
\begin{aligned}
&\bar{t} = T\sqrt{1+Pm^2}, \quad \bar{r} = \sqrt{r^2-Pm^2a^2}, \\
&\bar{M} = \frac{M}{1+Pm^2}, \quad \bar{a}^2 = \frac{a^2}{1+Pm^2},
\end{aligned} \tag{3.9}
$$

it is easy to see that the disformed metric (3.8) is nothing but the Myers–Perry metric (3.6) with equal angular momenta $\bar{a}$ and with mass $\bar{M}$. This result is surprising by itself, since one would expect that the hair of the scalar field, m, and the constant disformality factor, P, would have a certain impact in the disformed metric, but this is not the case, since both parameters can be consistently absorbed into the redefinitions of the coordinates, and into the physical constants (3.9). This is clearly in contrast with the four-dimensional disformed Kerr metric [28], where the deviations from General Relativity are strongly codified by the disformality coefficient P, which cannot be absorbed.

Before extending these results to higher odd-dimensions, in which the Myers–Perry spacetime can be as well a cohomogeneity−1 metric, we would like to propose a geometrical explanation of the disformal invariance. Generically, for a disformal transformation of the form

$$
\bar{g}_{\mu\nu} = g_{\mu\nu} - P\phi_\mu\phi_\nu,
$$

with P being constant, and with a scalar field ϕ such that $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \text{cst}$, the Riemann and Ricci tensors of the disformed metric $\bar{g}_{\mu\nu}$ can be expressed as

$$
\begin{aligned}
\bar{R}^\alpha_{\ \beta\mu\nu} &= R^\alpha_{\ \beta\mu\nu} + 2\nabla_{[\mu}\mathcal{K}^\alpha_{\ \nu]\beta} + 2\mathcal{K}^\alpha_{\ \gamma[\mu}\mathcal{K}^\gamma_{\ \nu]\beta} \Longrightarrow \bar{R}_{\mu\nu} \\
&= R_{\mu\nu} + 2\nabla_{[\alpha}\mathcal{K}^\alpha_{\ \mu]\nu} + 2\mathcal{K}^\alpha_{\ \gamma[\alpha}\mathcal{K}^\gamma_{\ \mu]\nu},
\end{aligned} \tag{3.10}
$$

where we have defined

$$
\mathcal{K}^\alpha_{\ \mu\nu} := \bar{\Gamma}^\alpha_{\mu\nu} - \Gamma^\alpha_{\mu\nu} = \bar{g}^{\alpha\lambda}\left(\nabla_{(\mu}\bar{g}_{\nu)\lambda} - \frac{1}{2}\nabla_\lambda\bar{g}_{\mu\nu}\right). \tag{3.11}
$$

We know that for a constant degree of disformality, P, the expression of the tensor $\mathcal{K}^\alpha_{\ \mu\nu}$ reduces to

$$
\mathcal{K}^\alpha_{\ \mu\nu} = -\frac{P}{1-PX}\phi^\alpha\phi_{\mu\nu}. \tag{3.12}
$$

On the other hand, for $X = \text{cst}$, one can easily establish that each term of $\mathcal{K}^\alpha_{\ \gamma[\alpha}\mathcal{K}^\gamma_{\ \mu]\nu}$ vanishes, as well as $\nabla_\mu\mathcal{K}^\alpha_{\ \alpha\nu}$, and hence the expression of the Ricci tensor of the disformed metric (3.10) reduces to

$$
\bar{R}_{\mu\nu} = R_{\mu\nu} + \nabla_\alpha\mathcal{K}^\alpha_{\ \mu\nu} = R_{\mu\nu} - \frac{P}{1-PX}\nabla_\alpha\left(\phi^\alpha\phi_{\mu\nu}\right). \tag{3.13}
$$

This last expression is noting but the Ricci tensor of any disformed metric constructed by means of a constant degree of disformality, P, and with a scalar field whose kinetic term, X, is constant. In our particular situation, since the Myers–Perry metric is a vacuum metric solution, the expression (3.13) becomes

$$
\begin{aligned}
\bar{R}_{\mu\nu} &= -\frac{P}{1-PX}\nabla_\alpha\left(\phi^\alpha\phi_{\mu\nu}\right) \\
&= -\frac{P}{1-PX}\left(\phi_{\mu\nu}\Box\phi + \phi^\lambda\phi_{\lambda\mu\nu}\right),
\end{aligned} \tag{3.14}
$$

and hence, we can conclude that the disformed metric is also a vacuum metric if and only if the following current

$$
j^\alpha_{\mu\nu} := \phi^\alpha\phi_{\mu\nu}, \tag{3.15}
$$

is conserved, i.e. $\nabla_\alpha j^\alpha_{\mu\nu} = 0$. It is easy to see that this divergence is exactly the tensorial equation (2.5) that ensures our construction of the stealth solution for any vacuum metric, and for any coupling functions. Hence, we recover that, for a scalar field satisfying the conditions (2.5) on a vacuum metric, its disformal transformation will be as well a vacuum metric.

3.2 Stealth on the higher odd-dimensional Myers–Perry solution with equal angular momenta and its disformed transformation

In five dimensions we have seen that a stealth solution defined on the Myers–Perry spacetime was possible only if the symmetry of this spacetime is enhanced to a $SU(2)$, and that this can be done by equating the two angular momenta. As a direct consequence, this Myers–Perry spacetime turns out to be a cohomogeneity-1 metric. This particular feature is inherent to the Myers–Perry spacetimes only in odd dimensions, $D = 2N+3$, and in the particular case of equal angular momenta, $a_i = a$. This will be our starting point, in order to generalize the previously found rotating stealth black hole solution in higher odd dimensions ($D > 5$).

In odd dimensions, $D = 2N + 3$, the cohomogeneity-1 Myers–Perry metric can be conveniently represented as

$$\begin{aligned} ds^2_{\text{MP}} = &-(1 - f^2(r, r_M, a))dt^2 + g^2(r, r_M, a)dr^2 \\ &+h^2(r, r_M, a)[d\psi + A_j dx^j \\ &-\Omega(r, r_M, a)dt]^2 + r^2\hat{g}_{ij}dx^i dx^j, \end{aligned} \tag{3.16}$$

with the metric functions given by

$$\begin{aligned} g^2(r, r_M, a) &= \left(1 - \frac{r_M^{2N}}{r^{2N}} + \frac{r_M^{2N}a^2}{r^{2N+2}}\right)^{-1} \\ h^2(r, r_M, a) &= r^2\left(1 + \frac{r_M^{2N}a^2}{r^{2N+2}}\right), \\ f(r, r_M, a) &= \sqrt{1 - \frac{r^2}{g^2(r, r_M, a)\, h^2(r, r_M, a)}} \\ \Omega(r, r_M, a) &= \frac{r_M^{2N}a}{r^{2N}h^2(r, r_M, a)}. \end{aligned}$$

Here, r_M is the mass radius parameter, $\hat{g}_{ij}$ is the Fubini-Study metric on CP^N with Ricci tensor $\hat{R}_{ij} = 2(N+1)\hat{g}_{ij}$, and $A = A_j dx^j$ is related to the Kähler form J by $dA = 2J$. Following closely the five-dimensional case, we look for a scalar field depending on the radial coordinate, r, and linearly in time, satisfying the constraint (2.5), and with a kinetic term $X = -m^2$. Such a scalar field for the metric representation (3.16) is given by

$$\phi(t, r) = -mt + m\int \frac{g(r, r_M, a) f(r, r_M, a)}{\sqrt{1 - f(r, r_M, a)^2}}dr, \tag{3.17}$$

and hence, one can easily conclude that in odd dimensions, $D = 2N + 3$, the cohomogeneity-1 metric (3.16) together with the non-trivial scalar field (3.17) will represent a rotating stealth solution of the field equations associated to the scalar tensor theory (2.1) without imposing any constraints on the coupling functions of the theory.

Continuing the analogy with the five-dimensional case, we now consider the disformed transformation of the cohomogeneity-1 metric (3.16) using the stealth scalar field (3.17). In order to bring the disformed metric in the same form as (3.16), we redefine the time, t, and the angular coordinate, ψ, as

$$\begin{aligned} dt \to\ & \frac{dt}{\sqrt{1 + Pm^2}} \\ &- \frac{Pm^2 f(r, r_M, a) g(r, r_M, a)}{\sqrt{1 - f^2(r, r_M, a)}(1 + Pm^2 - f^2(r, r_M, a))}dr, \\ d\psi \to\ & d\psi \\ &- \frac{Pm^2 f(r, r_M, a) g(r, r_M, a)\Omega(r, r_M, a)}{\sqrt{1 - f^2(r, r_M, a)}(1 + Pm^2 - f^2(r, r_M, a))}dr, \end{aligned}$$

and, after some algebraic manipulations, the disformed Myers–Perry metric reads

$$\begin{aligned} d\bar{s}^2_{\text{disf. MP}} = &-\left(1 - \frac{f^2(r, r_M, a)}{1 + Pm^2}\right)dt^2 \\ &+ \frac{g^2(r, r_M, a)\left(1 - f^2(r, r_M, a)\right)}{1 - \frac{f^2(r, r_M, a)}{(1+Pm^2)}}dr^2 \\ &+h^2(r, r_M, a)\left(d\psi + A_j dx^j - \frac{\Omega(r, r_M, a)}{\sqrt{1+Pm^2}}dt\right)^2 \\ &+r^2\hat{g}_{ij}dx^i dx^j. \end{aligned} \tag{3.18}$$

Now it remains to proof that this disformed metric is diffeomorphic to the original cohomogeneity-1 metric (3.16). It is easy to see that under the following redefinitions of the constants of integration

$$\bar{a} = \left(1 + Pm^2\right)^{\frac{1}{2}} a, \qquad \bar{r}_M = \left(1 + Pm^2\right)^{-\frac{1}{2N}} r_M, \tag{3.19}$$

the metric functions change as

$$\begin{aligned} &\frac{f^2(r, r_M, a)}{1 + Pm^2} = f^2(r, \bar{r}_M, \bar{a}), \quad \frac{\Omega(r, r_M, a)}{\sqrt{1 + Pm^2}} = \Omega(r, \bar{r}_M, \bar{a}) \\ &h(r, r_M, a) = h(r, \bar{r}_M, \bar{a}), \quad g^2(r, r_M, a)^2\left(1 - f^2(r, r_M, a)\right) \\ &\qquad = g^2(r, \bar{r}_M, \bar{a})\left(1 - f^2(r, \bar{r}_M, \bar{a})\right), \end{aligned} \tag{3.20}$$

and hence the disformed metric (3.18) is nothing but the original Myers–Perry spacetime (3.16) with the constants of integration given by $\bar{a}$ and $\bar{r}_M$, as defined in (3.19). As in the five-dimensional case, the scalar field's hair, m, along with the constant disformality parameter, P, can be reasonably absorbed into the redefinitions of the constants of integration (3.19). As a last remark, one can note that by redefining the constants of integration as

$$x_M := r_M^N, \quad y := \frac{1}{a}, \tag{3.21}$$

the conditions (3.20) can be interpreted as requiring f and Ω to be homogeneous functions of degree one with respect to theses "new" integration constants, while the function h and the determinant of the sector (t, r) of the metric, namely $\det_{[tr]} := -g^2(1-f^2)$, are of degree zero, i. e.

$$\begin{aligned} f(r, \alpha x_M, \alpha y) &= \alpha f(r, x_M, y), \\ \Omega(r, \alpha x_M, \alpha y) &= \alpha \Omega(r, x_M, y), \\ h(r, \alpha x_M, \alpha y) &= h(r, x_M, y), \\ \det_{[tr]}(r, \alpha x_M, \alpha y) &= \det_{[tr]}(r, x_M, y), \end{aligned} \tag{3.22}$$

for any $\alpha \in \mathbb{R}\backslash\{0\}$.

4 Conditions for the disformal invariance of cohomogeneity−1 metrics

In the previous section we have seen that the odd-dimensional Myers–Perry metric with equal momenta remains invariant by means of a disformal transformation with a constant degree of disformality and with a scalar field given by (3.5). This result is strongly correlated to the cohomogeneity−1 character of the Myers–Perry metric. We will go further in this direction by considering a general class of cohomogeneity−1 metrics (not necessarily a vacuum metric), and by identifying the conditions which ensure that its disformed transformation remains invariant (up to some diffeomorphisms and some redefinitions of the constants of integration). In order to achieve this task, we consider a class of cohomogeneity−1 metrics in arbitrary, D, dimensions parametrized as follows for latter convenience

$$\begin{aligned} ds^2 &= (-1+g_{tt}(r,a_\alpha))\,dt^2 + g_{rr}(r,a_\alpha)dr^2 \\ &\quad +2\sum_{i=1}^{D-2} g_{(it)}(r,a_\alpha)\,dt\,dx^i \\ &\quad +\sum_{i\neq j=1}^{D-2} g_{ij}(r,a_\alpha)\,dx^i\,dx^j \\ &\quad +\sum_{i=1}^{D-2} g_{ii}(r,a_\alpha)(dx^i)^2, \end{aligned} \tag{4.1}$$

where the a_α are some constants (like the mass, angular momenta, electromagnetic charges, $\cdots$). Note that we do not consider off-diagonal terms of the form $g_{tr}(r,a_\alpha)dtdr$ in the ansatz (4.1), since these can always be eliminated due to the cohomogeneity−1 property of the metric.

Following the same steps as before, a solution of $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -m^2$ will be given by

$$\phi(t,r) = -mt - m\int\sqrt{-g_{rr}(r,a_\alpha)\Big(1+\frac{|\Delta_n(r,a_\alpha)|}{|\Delta_{n+1}(r,a_\alpha)|}\Big)}\,dr, \tag{4.2}$$

where $|\Delta_{n+1}|$ and $|\Delta_n|$ are the determinants of the following reduced metrics

$$|\Delta_{n+1}| = \begin{vmatrix} -1+g_{tt}(r,a_\alpha) & g_{tx_1}(r,a_\alpha) & \dots & g_{tx_n}(r,a_\alpha) \\ g_{tx_1}(r,a_\alpha) & g_{x_1x_1}(r,a_\alpha) & \dots & g_{x_1x_n}(r,a_\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ g_{tx_n}(r,a_\alpha) & g_{x_1x_n}(r,a_\alpha) & \dots & g_{x_nx_n}(r,a_\alpha) \end{vmatrix},$$

$$|\Delta_n| = \begin{vmatrix} g_{x_1x_1}(r,a_\alpha) & \dots & g_{x_1x_n}(r,a_\alpha) \\ \vdots & \ddots & \vdots \\ g_{x_1x_n}(r,a_\alpha) & \dots & g_{x_nx_n}(r,a_\alpha) \end{vmatrix}.$$

With these definitions, it is clear that the determinant of the cohomogeneity−1 metric (4.1) denoted by $\det(g)$ is given by

$$\det(g)(r,a_\alpha) = |\Delta_{n+1}(r,a_\alpha)|\,g_{rr}(r,a_\alpha). \tag{4.3}$$

Using the scalar field defined by (4.2), the disformed metric,

$$d\bar{s}^2_{\text{disf}} = ds^2 - P\,(d\phi)^2\,, \tag{4.4}$$

yields, after eliminating the undesired off-diagonal term $dt - dr$ (which is possible since we have a cohomogeneity−1 metric),

$$\begin{aligned} d\bar{s}^2_{\text{disf}} &= \Big(-1-Pm^2+g_{tt}(r,a_\alpha)\Big)\,dt^2 + \bar{g}_{rr}(r,a_\alpha)dr^2 \\ &\quad +2\sum_{i=1}^{D-2} g_{(it)}(r,a_\alpha)\,dt\,dx^i \qquad (4.5)\\ &\quad +2\sum_{i\neq j=1}^{D-2} g_{ij}(r,a_\alpha)\,dx^i\,dx^j + \sum_{i=1}^{D-2} g_{ii}(r,a_\alpha)(dx^i)^2, \end{aligned} \tag{4.6}$$

where, for simplicity we have defined

$$\begin{aligned} \bar{g}_{rr}(r,a_\alpha) &:= \frac{(1+Pm^2)\,|\Delta_{n+1}(r,a_\alpha)|\,g_{rr}(r,a_\alpha)}{|\Delta_{n+1}(r,a_\alpha)| - Pm^2|\Delta_n(r,a_\alpha)|} \\ &= \frac{(1+Pm^2)\det(g)(r,a_\alpha)}{|\Delta_{n+1}(r,a_\alpha)| - Pm^2|\Delta_n(r,a_\alpha)|}. \end{aligned} \tag{4.7}$$

Hence, although the form of the metric is preserved by the disformed metric, there is a priori no reason for the latter to be diffeomorphic to the original metric (4.1), unless, as we will see now, the metric functions satisfy certain conditions. First of all, by noting that the dt^2−term can be rewritten as

$$\Big(-1-Pm^2+g_{tt}(r,a_\alpha)\Big)\,dt^2 = \Big(-1+\frac{g_{tt}(r,a_\alpha)}{1+Pm^2}\Big)\,d\bar{t}^2$$

with $d\bar{t} = dt\,\sqrt{1+Pm^2}$, it is easy to see that the disformal factor $1+Pm^2$ can be absorbed if the metric function g_{tt} satisfies the following homogeneity condition

$$\frac{g_{tt}(r,a_\alpha)}{1+Pm^2} = g_{tt}(r,\bar{a}_\alpha), \tag{4.8}$$

where $\bar{a}_\alpha$ are some redefinitions of the constants a_α, i.e.

$$\bar{a}_\alpha = f_\alpha\left(a_\alpha, P, m^2\right). \tag{4.9}$$

Now, because of the time redefinition, we also need to require that the off-diagonal terms on the $t-$row must satisfy as well a homogeneity condition given by

$$\frac{g_{it}(r, a_\alpha)}{\sqrt{1+Pm^2}} = g_{it}(r, \bar{a}_\alpha). \tag{4.10}$$

In addition, even if the $g_{ij}-$ terms are not affected by the disformal transformation, nor by the time redefinition, we still have to demand that they remain invariant under the redefinitions (4.9), that is

$$g_{ij}(r, x_k, a_\alpha) = g_{ij}(r, x_k, \bar{a}_\alpha). \tag{4.11}$$

Finally, the condition on the dr^2-disformed term (4.7) that will ensure the full disformed metric to be diffeomorphic to itself reads

$$\frac{|\Delta_{n+1}(r, a_\alpha)|\, g_{rr}(r, a_\alpha)}{|\Delta_{n+1}(r, a_\alpha)| - Pm^2|\Delta_n(r, a_\alpha)|} = \frac{1}{1+Pm^2} g_{rr}(r, \bar{a}_\alpha). \tag{4.12}$$

Nevertheless, it is simple to see that if the conditions (4.8–4.11) are fulfilled, we will automatically have

$$|\Delta_{n+1}(r, a_\alpha)| - Pm^2|\Delta_n(r, a_\alpha)| = (1+Pm^2)|\Delta_{n+1}(r, \bar{a}_\alpha)|, \tag{4.13}$$

and as a direct consequence, the condition (4.12) will be achieved if

$$|\Delta_{n+1}(r, a_\alpha)|\, g_{rr}(r, a_\alpha) = |\Delta_{n+1}(r, \bar{a}_\alpha)|\, g_{rr}(r, \bar{a}_\alpha). \tag{4.14}$$

From the definition (4.3), this last equation is equivalent to requiring the determinant of the metric to remain invariant under the redefinition of the constants of integration (4.9), i.e.

$$\det(g)(r, a_\alpha) = \det(g)(r, \bar{a}_\alpha). \tag{4.15}$$

To summarize, we have shown that, in order for the cohomogeneity-1 metric (4.1) to remain invariant under a disformal transformation generated by a scalar field with constant kinetic term, and with a constant factor of disformality, P, the metric functions have to satisfy the listed conditions (4.8), (4.10), (4.11) and (4.15). In particular, these conditions ensure that the hair of the scalar field, m, and the constant, P, can be absorbed into the redefinitions of the constants (4.9). It is also easy to check that in the case of the odd-dimensional Myers–Perry metric with equal momenta these conditions reduce to the homogeneous conditions listed previously (3.22).

5 Conclusions

The main objective of the present work is to look for non-trivial rotating black hole solutions of some general extended scalar tensor theories. Here, we restrict our study to a general shift symmetric and parity preserving scalar tensor action that contains up to second order covariant derivatives of the scalar field. In order to tackle the problem of finding non-trivial rotating configurations of the field equations, we fix the metric spacetime to be a vacuum rotating black hole spacetime, and we investigate whether this metric can accommodate a non-trivial scalar field. Since the metric is fixed and corresponds to a vacuum metric, it is reasonable to identify these solutions with stealth configurations. In order to ensure that the solution will not require any fine-tuning of the coupling functions, our ansatz for the scalar field is such that its kinetic term is assumed to be constant. In doing so, we prove that such a scalar field has to satisfy a single tensorial equation (2.5), and this solution would exist for any coupling functions of the theory. Our hypothesis on the scalar field is also useful to identify it with the Hamilton–Jacobi potential, and to take advantage on the known results on the integrability of the Hamilton–Jacobi equations.

In four dimensions, it was known that the condition (2.5) cannot be satisfied in the Kerr metric. Starting from this observation, we consider the problem of finding rotating stealth black hole solutions defined on the five-dimensional Myers–Perry metric. In doing so, we have noticed that such an ansatz is not appropriate, unless we impose some restrictions on the metric, and on the scalar field. In particular, the angular momentum parameters of the Myers–Perry metric must be chosen to be equal. This restriction is known to enhance the symmetry group of the metric to a $U(2)$ symmetry, and allows to express the metric components entirely as functions of a single (radial) coordinate. In addition, we have shown that the scalar field solution must depend linearly on time and on the radial coordinate (3.5). The non-trivial scalar field solution corresponds to the Hamilton–Jacobi potential in which the energy must be given by the particle mass, the conserved quantities associated to each rotation must be taken to be zero, and where the equivalent of the Carter's constant is taken to be zero, $K = 0$.

Interestingly enough, we have also shown that the disformed cohomogeneity-1 Myers–Perry spacetime obtained using this stealth scalar field is diffeomorphic to itself. This means that the hair of the scalar field identified with the particle mass and the constant disformality parameter can be consistently absorbed into further redefinitions of the mass and of the single angular parameter of the disformed metric. In other words, the invariance of the disformal transformation can be viewed as a map that brings a rotating black hole configuration with mass M and angular momentum a to another rotating configuration with rescaled mass and angular momentum, and where this rescaling is quantified by the hair and the disformality parameter. The invariance of the cohomogeneity−1 Myers–Perry spacetime under a disformal transformation can also be explained from the transformation of the Ricci tensor (3.14) together with the fact that the condition (2.5) rescales with an overall factor of $\frac{1}{(1-PX)^2}$ under a disformal a transformation, i.e.

$$\begin{aligned}&\bar{\nabla}_\mu\bar{\nabla}_\nu\phi\,\bar{\Box}\phi+\bar{\nabla}^\lambda\phi\left(\bar{\nabla}_\lambda\bar{\nabla}_\mu\bar{\nabla}_\nu\phi\right)\\&\quad=\frac{1}{(1-PX)^2}\left(\nabla_\mu\nabla_\nu\phi\,\Box\phi+\nabla^\lambda\phi\left(\nabla_\lambda\nabla_\mu\nabla_\nu\phi\right)\right).\end{aligned} \tag{5.1}$$

This would imply that for any vacuum metric with a scalar field satisfying the tensorial equations (2.5), its disformal transformation generated by the scalar field would as well be a solution of the field equations (2.2). Also, since the kinetic term, X, of the non-trivial stealth scalar field is constant, one could easily consider more general disformal and conformal transformations that respect the symmetry $\phi\to\phi+\text{cst}$, i.e.

$$\bar{g}_{\mu\nu}=A(X)\,g_{\mu\nu}-P(X)\,\phi_\mu\phi_\nu,$$

and this will not affect our results. All these results are shown to hold in higher-odd dimensions, where the Myers–Perry metric with equal momenta is known to be of cohomogeneity−1 class. Starting from this observation, we have listed the conditions on a general class of cohomogeneity−1 metrics, ensuring its invariance (up to diffeomorphisms) under a disformal transformation with a constant degree of disformality and with a scalar field with constant kinetic term. Finally, in the appendix, we consider the extension to the five-dimensional Kerr–de Sitter metric, where it is shown that rotating stealth solutions exist, provided some fine tuning of the coupling functions of the extended scalar tensor theory.

Acknowledgements We would like to thank Timothy Anson, Eloy Ayón–Beato, Eugeny Babichev and Christos Charmousis for very useful discussions. OB is funded by the Ph.D. scholarship of the University of Talca. The work of MH has been partially supported by FONDECYT grant 1210889.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: Since the paper is not experimental there is no data.]

Funded by SCOAP[3].

Appendix: Rotating stealth solution with an Einstein space

The rotating stealth black hole solutions defined on the odd-dimensional cohomogeneity-1 Myers–Perry metric can also be extended in the presence of a cosmological constant with some subtleties as we shall see. In this appendix, we present in detail the five-dimensional case, but its extension to higher odd-dimensions is straightforward.

In order to achieve this task, we consider the following scalar tensor theory

$$\begin{aligned}S[g,\phi]=\int d^5x\sqrt{-g}\Big[&K(X)+G(X)R+A_1(X)\\&\times\left[\phi_{\mu\nu}\phi^{\mu\nu}-(\Box\phi)^2\right]\\&+A_3(X)\Box\phi\,\phi^\mu\phi_{\mu\nu}\phi^\nu\\&+A_4(X)\phi^\mu\phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho+A_5(X)\left(\phi^\mu\phi_{\mu\nu}\phi^\nu\right)^2\Big],\end{aligned} \tag{5.1}$$

whose field equations for a constant kinetic scalar field reduce to

$$\begin{aligned}&-\frac{1}{2}K(X)g_{\mu\nu}+K'(X)\phi_\mu\phi_\nu\\&\quad+G(X)G_{\mu\nu}+G'(X)R\phi_\mu\phi_\nu\\&\quad-\frac{1}{2}A_3(X)\Big[(\Box\phi)^2-(\phi_{\alpha\beta})(\phi^{\alpha\beta})-R_{\alpha\beta}\phi^\alpha\phi^\beta\Big]\phi_\mu\phi_\nu\\&\quad+A_1(X)\Big[-R_{\nu\lambda}\phi_\mu\phi^\lambda-R_{\mu\lambda}\phi_\nu\phi^\lambda\\&\quad-\frac{1}{2}g_{\mu\nu}\left[(\Box\phi)^2-(\phi_{\alpha\beta})(\phi^{\alpha\beta})\right]\\&\quad+g_{\mu\nu}\left[R_{\lambda\rho}\phi^\lambda\phi^\rho\right]+\phi_{\mu\nu}\Box\phi+\phi^\lambda\phi_{\lambda\mu\nu}\Big]\\&\quad-A_1'(X)\left[(\Box\phi)^2-(\phi_{\alpha\beta})(\phi^{\alpha\beta})\right]\phi_\mu\phi_\nu=0,,\end{aligned} \tag{5.2}$$

and the conserved scalar field current, $\nabla_\mu J^\mu=0$, with

$$\begin{aligned}J^\mu=2\Big(&G'(X)R-\left[A_1'(X)+\frac{1}{2}A_3(X)\right]\\&\times\left[(\Box\phi)^2-(\phi_{\alpha\beta})(\phi^{\alpha\beta})\right]\\&+\frac{1}{2}A_3(X)\left[R_{\alpha\beta}\phi^\alpha\phi^\beta\right]+K'(X)\Big)\phi^\mu\\&-2A_1(X)R^{\mu\nu}\phi_\nu.\end{aligned} \tag{5.3}$$

In analogy with the Myers–Perry case, we consider the cohomogeneity−1 five-dimensional Kerr-de Sitter metric [39] satisfying $R_{\mu\nu}=4\lambda g_{\mu\nu}$,

$$ds^2 = -\frac{\Delta}{\rho^2}\left[dt - \frac{a}{\Xi_a}\left(\sin^2\theta d\varphi + \cos^2\theta d\psi\right)\right]^2 + \frac{\Xi_a \sin^2\theta}{\rho^2}\left[adt - \frac{\rho^2}{\Xi_a}d\varphi\right]^2 + \frac{\Xi_a \cos^2\theta}{\rho^2}\left[adt - \frac{\rho^2}{\Xi_a}d\psi\right]^2 + \frac{(1-r^2\lambda)}{r^2\rho^2}\left[a^2 dt - \frac{a\rho^2}{\Xi_a}\left(\sin^2\theta d\varphi + \cos^2\theta d\psi\right)\right]^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\Xi_a}d\theta^2, \tag{5.4}$$

where we have defined

$$\Delta = \frac{1}{r^2}(r^2+a^2)^2(1-r^2\lambda) - 2M,$$
$$\Xi_a = 1 + a^2\lambda, \qquad \rho^2 = r^2 + a^2.$$

For an Einstein metric satisfying $R_{\mu\nu} = 4\lambda g_{\mu\nu}$, it is easy to see that the field equations (5.2) reduce to

$$\Big[-\frac{1}{2}K(X) - 6\lambda G(X) + 4\lambda A_1(X)X - \frac{1}{2}A_1(X)\Big((\Box\phi)^2 - (\phi_{\alpha\beta})(\phi^{\alpha\beta})\Big)\Big]g_{\mu\nu} + \Big[K'(X) + 20\lambda G'(X) - 8\lambda A_1(X) - A_1'(X) \times\Big((\Box\phi)^2 - (\phi_{\alpha\beta})(\phi^{\alpha\beta})\Big) - \frac{1}{2}A_3(X)\Big((\Box\phi)^2 - (\phi_{\alpha\beta})(\phi^{\alpha\beta}) - 4\lambda X\Big)\Big]\phi_\mu\phi_\nu + A_1(X)\left(\phi_{\mu\nu}\Box\phi + \phi^\lambda\phi_{\lambda\mu\nu}\right) = 0. \tag{5.5}$$

We do not pretend to solve these equations in full generality but instead opt for a strategy similar to the asymptotically flat case. Indeed, we will consider a scalar field whose kinetic term is a constant, $X = -m^2$, that is

$$\phi(t,r) = -mt - m\int \frac{r(r^2+a^2)\sqrt{\lambda(r^2+a^2)^2 + 2M}}{(r^2+a^2)^2(1-\lambda r^2) - 2Mr^2} dr \Longrightarrow \phi_\mu\phi^\mu = -m^2. \tag{5.6}$$

Since we are considering the de Sitter case $\lambda > 0$, the scalar field is well defined. Nevertheless, one can see that in contrast with the asymptotically flat case, the scalar field, as defined by (5.6), does not satisfy the tensorial conditions (2.5), but instead

$$\phi_{\mu\nu}\Box\phi + \phi^\lambda\phi_{\lambda\mu\nu} = 4\lambda\phi_\mu\phi_\nu + 4m^2\lambda g_{\mu\nu}. \tag{5.7}$$

This might seem like an obstruction, but given the structure of the Eq. (5.5), these can be recast using the relation (5.7) into

$$\left[-\frac{1}{2}K - 6\lambda G - 6m^2\lambda A_1\right]g_{\mu\nu} + \left[K' + 20\lambda G' - 4\lambda A_1 - 12m^2\lambda A_1' - 8m^2\lambda A_3\right]\phi_\mu\phi_\nu = 0, \tag{5.8}$$

where we have explicitly used that $X = -m^2$, as well as the trace of Eq. (5.7) which yields $(\Box\phi)^2 - (\phi_{\alpha\beta})(\phi^{\alpha\beta}) = 12m^2\lambda$. Now, since $g_{tr} = 0$ while $\phi_t\phi_r \neq 0$, each bracket of (5.8) must vanish independently which in turn implies the conditions

$$K(X) = -12\lambda\Big(G(X) - XA_1(X)\Big),$$
$$G'(X) + A_1(X) + 3XA_1' + XA_3(X) = 0. \tag{5.9}$$

In [37] different conditions, for example requiring that A_1 and A_2 vanish at the constant value of X, have been implemented to look for solutions of more general quadratic theories.

It is also straightforward to see that under these restrictions, the current J^μ as defined in (5.3) vanishes identically, and hence the equations of motion for the scalar field are well verified. Hence, we conclude that the Kerr–de Sitter metric (5.4) together with the scalar field (5.6) will be a solution of the field equations (5.2–5.3) provided that the coupling functions are tied as (5.9).

Finally, as in the asymptotically flat case, the disformed metric generated by the scalar field (5.6) with a constant degree of disformality, $d\bar{s}^2 = ds^2 - P(d\phi)^2$, is as well an Einstein metric. Indeed, combining the equations (3.13) and (5.7), one gets

$$\bar{R}_{\mu\nu} = R_{\mu\nu} - \frac{P}{1-PX}\left[4\lambda\phi_\mu\phi_\nu + 4m^2\lambda g_{\mu\nu}\right]$$
$$= 4\lambda g_{\mu\nu} - \frac{P}{1+Pm^2}\left[4\lambda\phi_\mu\phi_\nu + 4m^2\lambda g_{\mu\nu}\right]$$
$$= \frac{4\lambda}{1+Pm^2}\bar{g}_{\mu\nu}.$$

Generalizations to higher odd dimensions can be done with the Kerr–de Sitter metric with equal momenta given in [40].

References

1. B.P. Abbott et al. (LIGO Scientific and Virgo), Phys. Rev. Lett. **116**(6), 061102 (2016). https://doi.org/10.1103/PhysRevLett.116.061102. arXiv:1602.03837 [gr-qc]
2. J.M. Maldacena, Adv. Theor. Math. Phys. **2**, 231–252 (1998). https://doi.org/10.1023/A:1026654312961. arXiv:hep-th/9711200
3. P. Kovtun, D.T. Son, A.O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005). https://doi.org/10.1103/PhysRevLett.94.111601. arXiv:hep-th/0405231
4. R. Emparan, H.S. Reall, Living Rev. Relativ. **11**, 6 (2008). https://doi.org/10.12942/lrr-2008-6. arXiv:0801.3471 [hep-th]

5. G.W. Horndeski, Int. J. Theor. Phys. **10**, 363–384 (1974). https://doi.org/10.1007/BF01807638
6. D. Langlois, K. Noui, JCAP **1602**, 034 (2016). https://doi.org/10.1088/1475-7516/2016/02/034
7. D. Langlois, K. Noui, JCAP **1607**, 016 (2016). https://doi.org/10.1088/1475-7516/2016/07/016
8. M. Crisostomi, M. Hull, K. Koyama, G. Tasinato, JCAP **03**, 038 (2016). https://doi.org/10.1088/1475-7516/2016/03/038
9. M. Crisostomi, K. Koyama, G. Tasinato, JCAP **1604**, 044 (2016). https://doi.org/10.1088/1475-7516/2016/04/044
10. H. Motohashi, K. Noui, T. Suyama, M. Yamaguchi, D. Langlois, JCAP **07**, 033 (2016). https://doi.org/10.1088/1475-7516/2016/07/033
11. J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui, G. Tasinato, JHEP **1612**, 100 (2016). https://doi.org/10.1007/JHEP12(2016)100
12. E. Babichev, C. Charmousis, A. Lehébel, JCAP **1704**, 027 (2017). https://doi.org/10.1088/1475-7516/2017/04/027
13. E. Babichev, A. Lehébel, JCAP **12**, 027 (2018). https://doi.org/10.1088/1475-7516/2018/12/027
14. J. Chagoya, G. Tasinato, JCAP **1808**, 006 (2018). https://doi.org/10.1088/1475-7516/2018/08/006
15. T. Kobayashi, T. Hiramatsu, Phys. Rev. D **97**(10), 104012 (2018). https://doi.org/10.1103/PhysRevD.97.104012
16. M. Minamitsuji, J. Edholm, Phys. Rev. D **101**(4), 044034 (2020). https://doi.org/10.1103/PhysRevD.101.044034
17. E. Babichev, C. Charmousis, G. Esposito-Farèse, A. Lehébel, Phys. Rev. Lett. **120**(24), 241101 (2018). https://doi.org/10.1103/PhysRevLett.120.241101
18. E. Babichev, C. Charmousis, G. Esposito-Farèse, A. Lehébel, Phys. Rev. D **98**(10), 104050 (2018). https://doi.org/10.1103/PhysRevD.98.104050
19. C. Charmousis, M. Crisostomi, R. Gregory, N. Stergioulas, Phys. Rev. D **100**(8), 084020 (2019). https://doi.org/10.1103/PhysRevD.100.084020
20. E. Babichev, G. Esposito-Farese, Phys. Rev. D **95**(2), 024020 (2017). https://doi.org/10.1103/PhysRevD.95.024020. arXiv:1609.09798 [gr-qc]
21. T. Kobayashi, Rept. Prog. Phys. **82**(8), 086901 (2019). https://doi.org/10.1088/1361-6633/ab2429
22. E. Babichev, C. Charmousis, A. Cisterna, M. Hassaine, JCAP **06**, 049 (2020). https://doi.org/10.1088/1475-7516/2020/06/049. arXiv:2004.00597 [hep-th]
23. O. Baake, C. Charmousis, M. Hassaine, M. San Juan, Regular black holes and gravitational particle-like solutions in generic DHOST theories (2021). arXiv:2104.08221 [hep-th]
24. E. Ayon-Beato, C. Martinez, J. Zanelli, Gen. Relativ. Gravity **38**, 145–152 (2006). https://doi.org/10.1007/s10714-005-0213-x. arXiv:hep-th/0403228
25. E. Ayon-Beato, C. Martinez, R. Troncoso, J. Zanelli, Phys. Rev. D **71**, 104037 (2005). https://doi.org/10.1103/PhysRevD.71.104037. arXiv:hep-th/0505086
26. R.C. Myers, M.J. Perry, Ann. Phys. **172**, 304 (1986). https://doi.org/10.1016/0003-4916(86)90186-7
27. V.P. Frolov, D. Stojkovic, Phys. Rev. D **67**, 084004 (2003). https://doi.org/10.1103/PhysRevD.67.084004. arXiv:gr-qc/0211055
28. T. Anson, E. Babichev, C. Charmousis, M. Hassaine, JHEP **01**, 018 (2021). https://doi.org/10.1007/JHEP01(2021)018. arXiv:2006.06461 [gr-qc]
29. J. Ben Achour, H. Liu, H. Motohashi, S. Mukohyama, K. Noui, JCAP **11**, 001 (2020). https://doi.org/10.1088/1475-7516/2020/11/001. arXiv:2006.07245 [gr-qc]
30. E. Babichev, C. Charmousis, JHEP **08**, 106 (2014). https://doi.org/10.1007/JHEP08(2014)106. arXiv:1312.3204 [gr-qc]
31. T. Kobayashi, N. Tanahashi, PTEP **2014**, 073E02 (2014). https://doi.org/10.1093/ptep/ptu096. arXiv:1403.4364 [gr-qc]
32. M. Bravo-Gaete, M. Hassaine, Phys. Rev. D **89**, 104028 (2014). https://doi.org/10.1103/PhysRevD.89.104028. arXiv:1312.7736 [hep-th]
33. M. Minamitsuji, J. Edholm, Phys. Rev. D **100**(4), 044053 (2019). https://doi.org/10.1103/PhysRevD.100.044053. arXiv:1907.02072 [gr-qc]
34. M. Zumalacárregui, J. García-Bellido, Phys. Rev. D **89**, 064046 (2014). https://doi.org/10.1103/PhysRevD.89.064046. arXiv:1308.4685 [gr-qc]
35. J. Ben Achour, H. Liu, S. Mukohyama, JCAP **02**, 023 (2020). https://doi.org/10.1088/1475-7516/2020/02/023. arXiv:1910.11017 [gr-qc]
36. O.J.C. Dias, P. Figueras, R. Monteiro, H.S. Reall, J.E. Santos, JHEP **05**, 076 (2010). https://doi.org/10.1007/JHEP05(2010)076. arXiv:1001.4527 [hep-th]
37. K. Takahashi, H. Motohashi, JCAP **06**, 034 (2020). https://doi.org/10.1088/1475-7516/2020/06/034. arXiv:2004.03883 [gr-qc]
38. B. Carter, Phys. Rev. **174**, 1559– 1571 (1968). https://doi.org/10.1103/PhysRev.174.1559
39. S.W. Hawking, C.J. Hunter, M. Taylor, Phys. Rev. D **59**, 064005 (1999). https://doi.org/10.1103/PhysRevD.59.064005. arXiv:hep-th/9811056
40. W. Chen, H. Lu, C.N. Pope, Class. Quantum Gravity **23**, 5323-5340 (2006). https://doi.org/10.1088/0264-9381/23/17/013. arXiv:hep-th/0604125 [hep-th]

Chapter III

Quantum effects in curved space-times

This chapter discusses possible quantum effects around black holes. The starting point will be a little overview of the field of quantum gravity, followed by a motivation as to why it is interesting to consider semiclassical gravity. Then a work about the backreaction of a quantum scalar field on an overspinning BTZ geometry is presented. This is a work in progress in collaboration with Jorge Zanelli.

1 How to (not) quantize gravity

In the previous chapters, we were solely concerned with theories of gravity in the absence of quantum effects. But why not go beyond classical effects? Let us consider regions where the effects of gravity are strong, for example in the vicinity of black holes or other compact astrophysical objects like neutron stars. It would be foolish to assume that these regions are devoid of microscopic particles and therefore, we need to understand how gravity interacts with these particles on a quantum level. Even worse, black hole solutions in general relativity always contain a singularity close to which classical physics breaks down, and so it could be an indication for the need of including quantum effects into the theory. In the previous chapter we have shown that classical modifications can avoid these singularities and therefore one may believe that the observational effects in strong gravitational fields are governed by classical physics and a quantum description is not necessary. But even then, there is still the question of what happened close to the big bang. There is no way that this primordial, densely packed *soup of particles* can be entirely described within the framework of classical physics. In these extreme regions, one may even assume that space-time itself undergoes quantum fluctuations. The treatment of gravity differs drastically from the other three fundamental interactions of physics, so how do we combine gravity and quantum field theory? Naively, the answer is straightforward: since general relativity can be written as a classical field theory, we just need to quantize it, following the usual procedure when going from classical to quantum field theory. It is no secret that we will eventually run into trouble due to the non-renormalizability of GR. Any textbook on QFT will explain that whether a theory is renormalizable or not depends on the mass dimension of its coupling constant. If it is positive, as is the case of GR, then the theory is not renormalizable. Let us sketch how this happens for gravity through an example. Consider the variation of flat space-time[1]

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}h_{\mu\nu}, \tag{1.1}$$

with $G = 16\pi G_N$ and G_N being the Newton constant. The choice of this normalization factor will become clear below. Remember that the stress energy tensor in GR is defined through the variation of the matter action with respect to the metric:

$$T^{\mu\nu}(x) = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}(x)}. \tag{1.2}$$

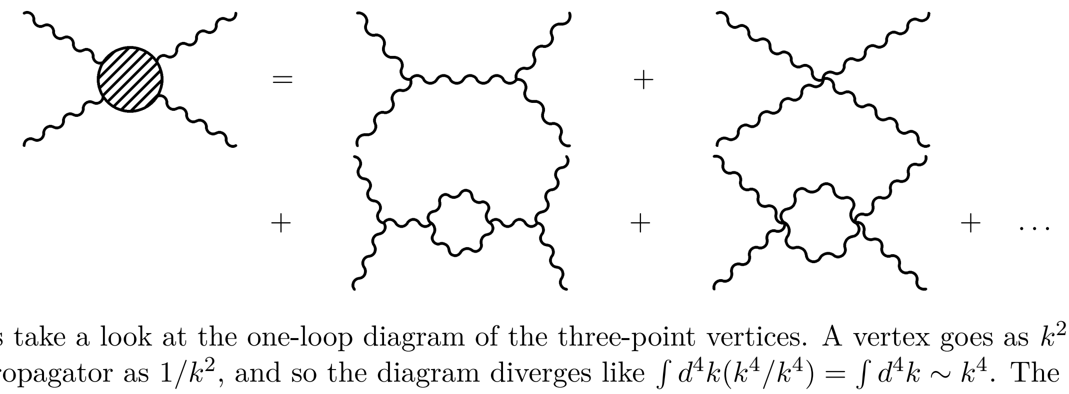
Then, expanding the matter action S_M to first order in h, we get something of the form

$$S_M[h] = S_M[h=0] - \int d^4x \left[\frac{1}{2}h_{\mu\nu}T^{\mu\nu} + O(h^2)\right]. \tag{1.3}$$

In analogy to the photon field coupling to the electromagnetic current, we can interpret $h_{\mu\nu}$ as the graviton field that couples to the matter (or actually anything with energy and momentum) through the stress energy tensor. Expanding the scalar curvature, the perturbative action reads schematically

$$S \sim \int d^4x \left(\partial h \partial h + \sqrt{G}h\partial h \partial h + Gh^2 \partial h \partial h + \dots + \sqrt{G}hT + \dots\right). \tag{1.4}$$

Note that the perturbative action of gravity is an infinite series, but this is all we need here. The first term corresponds to the graviton propagator, which is properly normalized due to the factor we introduced in equation (1.1). The second and third terms represent the self interaction terms of the graviton. We can see that there are three- and four-point vertices. According to this, the first order Feynman diagrams of a graviton-graviton scattering process are:



Let us take a look at the one-loop diagram of the three-point vertices. A vertex goes as k^2 and the propagator as $1/k^2$, and so the diagram diverges like $\int d^4k(k^4/k^4) = \int d^4k \sim k^4$. The next loop order adds three more propagators, two more vertices and another integral, increasing

[1] We could as well expand around a curved metric.

the degree of divergence by two to $\sim k^6$. This goes on and on indefinitely, with every order being worse than the previous. To remedy this, we would have to introduce infinitely many counterterms, which clearly is not feasible. Thus, a modification to GR is needed. One option is to modify it classically in such a way that it becomes renormalizable. However, by now, GR is enjoying a tremendous amount of experimental evidence with an impressive accuracy, which puts strong constraints on these theories. Another option is a fundamentally different theory that contains general relativity as an effective theory. Nowadays, there are numerous such attempts. One of the difficulties of this approach is that quantum gravitational effects only appear at length scales near the Planck scale that is only accessible with enormous energies. Even with modern technology, these energies cannot be reached, and so it is not possible to experimentally rule out most of these new theories. So, how do we choose from this *swamp* of possibilities? We may want to get a better understanding of what these theories have to be capable of before we can have a well-informed opinion on this. Therefore, we need to have a better understanding of the regimes close to the breakdown of the established theories. We can achieve this by studying the behavior of quantum fields in curved space-time, that is, we keep gravity as it is and only quantize the matter fields in an underlying space-time.

1.1 Semiclassical gravity

Due to the difficulty of finding a theory that unifies general relativity with quantum field theory, a good way of developing a better understanding of the issue is to give an approximate description of quantum gravity. This is done in the model of semiclassical gravity, where we treat the matter fields as being quantum while keeping the gravitational field classical. Since the background metric is not flat anymore, the behavior of the matter fields is now governed by the theory of quantum fields in curved space-time. We cannot derive the equations of motion directly from an action principle, but rather replace the stress energy tensor in the classical Einstein equations with the renormalized expectation value of its quantum operator:

$$G_{\mu\nu} = \kappa \langle \psi | \hat{T}_{\mu\nu} | \psi \rangle \,, \tag{1.5}$$

where ψ indicates the quantum state of the matter fields. From now on, we will omit the hat and abbreviate the renormalized stress energy tensor (RSET) as $\langle T_{\mu\nu} \rangle$. Solving these semiclassical Einstein equations is a highly difficult task in (3+1)-dimensions, and to the best of our knowledge it has not been achieved for black hole space-times yet (at least not with an explicitly calculated RSET without further approximations). It is therefore natural to study a simpler model that still shares the important conceptual features of GR, for example by going to (2+1)-dimensions. Even though the dynamics of this model is quite different from standard GR, it still allows us to learn more about the general analysis of many problems, for instance, the construction of states and observables, or the different approaches to quantization. The classical solution we have at hand is the BTZ metric (see section 1 of the first chapter), so we will have to include a cosmological constant:

$$G_{\mu\nu} - l^{-2} g_{\mu\nu} = \kappa \langle T_{\mu\nu} \rangle \,. \tag{1.6}$$

In the next section, a work in progress in collaboration with Jorge Zanelli is presented, where we are calculating the quantum back reactions of a massless, conformally coupled scalar field on the BTZ metric. We will outline some preliminaries here that will not appear in the publication. Before we can calculate the quantum operator, we have to take a look at the classical equations.

The action of the model reads

$$S = \int d^3x\sqrt{-g}\left[\frac{R+2l^{-2}}{2\kappa} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{16}R\phi^2\right], \tag{1.7}$$

and results in the field equations

$$\begin{aligned} G_{\mu\nu} - l^{-2}g_{\mu\nu} &= \kappa T_{\mu\nu}, \\ \left(\Box - \frac{1}{8}R\right)\phi &= 0. \end{aligned} \tag{1.8}$$

with the classical stress energy tensor

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\lambda\rho}\nabla_\lambda\phi\nabla_\rho\phi + \frac{1}{8}\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}\right)\phi^2. \tag{1.9}$$

Using the equation of motion for ϕ, together with the fact that $G_{\mu\nu} = l^{-2}g_{\mu\nu}$ holds for AdS_3, it is straightforward to show that it is conserved and traceless. From this expression, we can obtain the quantum stress tensor $\langle T_{\mu\nu}\rangle$ by taking the coincidence limit of the corresponding *point-split* expectation value (see, for example, [101–103]):

$$\kappa\left\langle T_{\mu\nu}(x)\right\rangle = \pi l_P \lim_{y\to x}\left(3\nabla^x_\mu\nabla^y_\nu - g_{\mu\nu}g^{\lambda\rho}\nabla^x_\lambda\nabla^y_\rho - \nabla^x_\mu\nabla^x_\nu - \frac{1}{4l^2}g_{\mu\nu}\right)G(x,y), \tag{1.10}$$

where $G(x,y)$ is the Green's function of the differential operator acting on the scalar field in equation (1.8). To renormalize the quantum stress energy tensor, we follow the Hadamard-regularization method. It is implemented at the level of the Green's function and consists of subtracting a locally determined elementary solution with the same singularity structure in the coincidence limit $y \to x$. Therefore, instead of using the divergent propagator of the scalar field in (1.10), we replace it with the regularized expression $G(x,y)_{\text{reg}}$:

$$G(x,y)_{\text{reg}} = G(x,y) - G_{\text{div}}(x,y). \tag{1.11}$$

But what does the scalar field propagator look like? Consider the embedding of AdS_3 into four dimensional flat space through

$$\eta_{ab}x^a x^b = -(x^0)^2 + (x^1)^2 + (x^2)^2 - (x^3)^2 = -l^2 \tag{1.12}$$

Then the two-point function of the scalar field shown to be [101, 104–108]

$$G(x,y) = \frac{1}{|x-y|}. \tag{1.13}$$

The derivation is not that trivial, but it is easy to show that it satisfies the same equation of motion as the scalar field:

$$\left(\nabla^\mu\nabla_\mu + \frac{3}{4l^2}\right)G(x,y) = 0, \quad x \neq y. \tag{1.14}$$

To achieve this we define the projection operator for AdS_3 as

$$P^{ab} = \eta^{ab} + \frac{x^a x^b}{l^2}, \tag{1.15}$$

which satisfies $P^{ab}x_b = 0$ and $P^{ab}\eta_{ab} = 3$. This allows us to re-write the three-dimensional d'Alembert operator in terms of the embedding coordinates:

$$\begin{aligned}
\nabla^\mu\nabla_\mu =& g^{\mu\nu}(\partial_\mu x^c)(\partial_\nu x^d)P^a_{\ c}\partial_a\left(P^b_{\ d}\partial_b\right)\\
=&\eta^{cd}P^a_{\ c}\partial_a\left(P^b_{\ d}\partial_b\right) = P^{ad}\partial_a\left(P^b_{\ d}\partial_b\right)\\
=&P^{ab}\partial_a\partial_b + \frac{1}{l^2}P^{ab}\left(\delta^c_a x_b + \eta_{ab}x^c\right)\partial_c\\
=&P^{ab}\partial_a\partial_b + \frac{1}{l^2}P^{ab}\eta_{ab}x^c\partial_c\\
=&P^{ab}\partial_a\partial_b + \frac{3}{l^2}x^a\partial_a.
\end{aligned} \tag{1.16}$$

With this at hand, it is straightforward to insert the propagator into the scalar field equation of motion. We shall compute the relevant terms here, starting with

$$x^a\partial_a|x-y|^{-1} = -x^a\frac{x_c - y_c}{|x-y|^3} = \frac{l^2 + \eta_{ab}x^a y^b}{|x-y|^3}, \tag{1.17}$$

and

$$P^{ab}\partial_a\partial_b|x-y|^{-1} = P^{ab}\left[\frac{3\left(x_a - y_a\right)\left(x_b - y_b\right)}{|x-y|^5} - \frac{\eta_{ab}}{|x-y|^3}\right] = \frac{3P_{ab}y^a y^b}{|x-y|^5} - \frac{3}{|x-y|^3}. \tag{1.18}$$

Inserting these into the equation, then yields

$$\begin{aligned}
\nabla^\mu\nabla_\mu G(x,y) =& \left(P^{ab}\partial_a\partial_b + \frac{3}{l^2}x^a\partial_a\right)|x-y|^{-1}\\
=&\frac{3P_{ab}y^a y^b}{|x-y|^5} - \frac{3}{|x-y|^3} + \frac{3}{l^2}\frac{l^2 + \eta_{ab}x^a y^b}{|x-y|^3}\\
=&\frac{3P_{ab}y^a y^b}{|x-y|^5} + \frac{3}{l^2}\frac{\eta_{ab}x^a y^b}{|x-y|^3}\\
=&\frac{3}{l^2}\frac{\left(-l^4 + x_a x_b y^a y^b\right) - 2x_a y^a\left(l^2 + x_b y^b\right)}{|x-y|^5}\\
=&-\frac{3}{l^2}\frac{\left(l^2 + x_a y^a\right)^2}{|x-y|^5} = -\frac{3}{4l^2}G(x,y)
\end{aligned} \tag{1.19}$$

and hence the field equation is satisfied.

These are the principal concepts that are used during our calculations. We will present it in the following section.

2 Quantum backreaction for overspinning BTZ geometries

Since the dawn of general relativity, many black hole solutions to Einstein's field equations have been found. And all these black holes are hiding something, namely a space-time singularity. Furthermore, one can even tune the parameters of such a solution to remove its event horizon and obtain naked singularities, which is an exact solution to the classical equations of general relativity as well. And to top it off in the vicinity of these singularities causality is greatly violated, which is why Roger Penrose suggested the existence of the (weak) cosmic censorship hypothesis [22]. The cosmic censorship hypothesis states that singularities must be hidden behind an event horizon. However, classically these solutions cannot be rules out on theoretical grounds, and it is difficult to prove whether all possible processes forming a black hole lead to the occurrence of an event horizon, but at least so far no naked singularity has been found in the universe. One could argue that in such a strong gravity regime general relativity breaks down and there is the need for a theory that combines gravity with quantum effects, which hopefully gets rid of these singularities altogether or at least makes sure that every one of them is hidden behind a horizon. In the latter case, at least one would not need to worry anymore about them, since there would be no causal connection from the singularity going outside to an observer.

Now, with every year passing by, we have more and more experimental and observational confirmation of the predictions of general relativity. This puts very tight constraints on any possible theory incorporating both, general relativity and quantum field theory. Since both theories are so well established, it is therefore sensible to start looking at common regimes where one can use a semi-classical approach to obtain a better understanding of the issues at hand. Calculating quantum effects on a curved background space-time is notoriously difficult, but there is a wonderful testing ground where these problems become a lot easier yet still provide meaningful information to learn from: The (2+1)-dimensional Bañados-Teitelboim-Zanelli (BTZ) space-time. In previous works (cf. [108] and references therein) this has been done for several cases of the BTZ geometry, the static black hole, the rotating and extremal black holes, and the conical singularity. Here we are concerned with the so-called overspinning geometry of the BTZ space-time, which is essentially when the angular momentum is greater than a certain threshold (cf. next section). For this type of geometry, we include the quantum effects in the stress-energy tensor of a conformally coupled massless scalar field and solve the resulting semi-classical Einstein equations. The quantized stress-energy tensor for matter fields contains ultraviolet divergences and therefore has to be renormalized. With this at hand, we can perturbatively solve the equations to obtain the quantum-backreacted metric. Our results show that the quantum effects indeed cause the singularity to be hidden behind an event horizon, thus supporting the (weak) cosmic censorship conjecture.

2.1 Overspinning BTZ space-time

We will give a brief review of the classical background that is being used, that is the rotating BTZ metric [26, 109], which is given by:

$$\mathrm{d}s^2 = -\left(\frac{r^2}{l^2} - M\right)\mathrm{d}t^2 - J\mathrm{d}t\mathrm{d}\theta + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2, \tag{2.1}$$

where the coordinate ranges are: $-\infty < t < \infty$, $0 < r < \infty$ and $0 \leq \theta < 2\pi$, $\Lambda = -l^{-2}$ is the cosmological constant, and M and J are mass and angular momentum respectively.

To classify the space-times that are given for different values of M and J, one has to study the four roots of the equation $g^{rr}=0$, which take the form

$$\lambda_\pm = \frac{l}{2}\left[\sqrt{M+\frac{J}{l}} \pm \sqrt{M-\frac{J}{l}}\right], \tag{2.2a}$$

$$\tilde{\lambda}_\pm = -\lambda_\pm. \tag{2.2b}$$

The different classifications are explained in detail in [109], however, in our case we will only concern ourselves with the parameter relation $|J|>Ml$, the so-called overspinning case. Its classical properties are studied in [28], so here we will be looking into its quantum corrections, as has been done for other parameter ranges in [108, 110–112]. To achieve this, note that in general the BTZ space-time can be obtained as a quotient space of the universal covering of anti-De Sitter space-time (CAdS_3), using an appropriate Killing field to identify certain points [109]. Remember that the AdS_3 pseudosphere can be embedded in $(\mathbb{R}^{(2,2)},\eta)$ through

$$\eta_{AB}X^AX^B = -\left(X^0\right)^2+\left(X^1\right)^2+\left(X^2\right)^2-\left(X^3\right)^2=-l^2. \tag{2.3}$$

Then, in the overspinning geometry, we can parametrize the embedding coordinates for $J>0$ as [28]:

$$\begin{aligned} X^0 =& \frac{l}{2}\sqrt{A+1}\cosh\left[a\left(t/l-\theta\right)\right]\left\{\cos\left[b\left(\theta+t/l\right)\right]-\sin\left[b\left(\theta+t/l\right)\right]\right\} \\ &+\epsilon\frac{l}{2}\sqrt{A-1}\sinh\left[a\left(t/l-\theta\right)\right]\left\{\sin\left[b\left(\theta+t/l\right)\right]+\cos\left[b\left(\theta+t/l\right)\right]\right\}, \end{aligned} \tag{2.4a}$$

$$\begin{aligned} X^1 =& \frac{l}{2}\sqrt{A+1}\sinh\left[a\left(t/l-\theta\right)\right]\left\{\cos\left[b\left(\theta+t/l\right)\right]-\sin\left[b\left(\theta+t/l\right)\right]\right\} \\ &+\epsilon\frac{l}{2}\sqrt{A-1}\cosh\left[a\left(t/l-\theta\right)\right]\left\{\sin\left[b\left(\theta+t/l\right)\right]+\cos\left[b\left(\theta+t/l\right)\right]\right\}, \end{aligned} \tag{2.4b}$$

$$\begin{aligned} X^2 =& \frac{l}{2}\sqrt{A+1}\sinh\left[a\left(t/l-\theta\right)\right]\left\{\sin\left[b\left(\theta+t/l\right)\right]+\cos\left[b\left(\theta+t/l\right)\right]\right\} \\ &-\epsilon\frac{l}{2}\sqrt{A-1}\cosh\left[a\left(t/l-\theta\right)\right]\left\{\cos\left[b\left(\theta+t/l\right)\right]-\sin\left[b\left(\theta+t/l\right)\right]\right\}, \end{aligned} \tag{2.4c}$$

$$\begin{aligned} X^3 =& \frac{l}{2}\sqrt{A+1}\cosh\left[a\left(t/l-\theta\right)\right]\left\{\sin\left[b\left(\theta+t/l\right)\right]+\cos\left[b\left(\theta+t/l\right)\right]\right\} \\ &-\epsilon\frac{l}{2}\sqrt{A-1}\sinh\left[a\left(t/l-\theta\right)\right]\left\{\cos\left[b\left(\theta+t/l\right)\right]-\sin\left[b\left(\theta+t/l\right)\right]\right\}, \end{aligned} \tag{2.4d}$$

with

$$a=\frac{\sqrt{|J|/l+M}}{2}, \qquad b=\frac{\sqrt{|J|/l-M}}{2}, \tag{2.5}$$

$$A=\frac{2\sqrt{\frac{J^2}{4}+\frac{r^4}{l^2}-Mr^2}}{\sqrt{J^2-l^2M^2}}, \tag{2.6}$$

and where $\epsilon=1$ for $2r^2\leq l^2M$ and $\epsilon=-1$ for $2r^2\geq l^2M$. Note that both cases will lead to the same RSET in the end, and hence to the same results. Without loss of generality, we will assume $J>0$ for the rest of this work.

The rotating BTZ space-time is now obtained through identifications generated by the Killing field, ξ, represented as [28, 109]

$$\xi = b(J_{03}-J_{12})-a(J_{01}-J_{23}), \tag{2.7}$$

which can be written as $\xi = \frac{1}{2}\omega^{AB} J_{AB}$, where the antisymmetric matrix ω^{AB} characterizes the identification. The Killing field in matrix form reads

$$\xi = \begin{pmatrix} 0 & -a & 0 & -b \\ -a & 0 & -b & 0 \\ 0 & b & 0 & -a \\ b & 0 & -a & 0 \end{pmatrix}. \tag{2.8}$$

The action under the Killing field in the embedding space, $\mathbb{R}^{(2,2)}$, is a matrix, $H(\xi) = e^{2\pi\xi}$, which takes the form

$$H = \begin{pmatrix} C(a)c(b) & -S(a)c(b) & S(a)s(b) & -C(a)s(b) \\ -S(a)c(b) & C(a)c(b) & -C(a)s(b) & S(a)s(b) \\ -S(a)s(b) & C(a)s(b) & C(a)c(b) & -S(a)c(b) \\ C(a)s(b) & -S(a)s(b) & -S(a)c(b) & C(a)c(b) \end{pmatrix}, \tag{2.9}$$

where $C(a) \equiv \cosh(2\pi a)$, $S(a) \equiv \sinh(2\pi a)$ $c(b) \equiv \cos(2\pi b)$, and $s(b) \equiv \sin(2\pi b)$. Iterating the identification by H is equivalent to acting with

$$H^n = \begin{pmatrix} C(na)c(nb) & -S(na)c(nb) & S(na)s(nb) & -C(na)s(nb) \\ -S(na)c(nb) & C(na)c(nb) & -C(na)s(nb) & S(na)s(nb) \\ -S(na)s(nb) & C(na)s(nb) & C(na)c(nb) & -S(na)c(nb) \\ C(na)s(nb) & -S(na)s(nb) & -S(na)c(nb) & C(na)c(nb) \end{pmatrix}. \tag{2.10}$$

Before we continue with the calculations, we will take a closer look at the identification. The Killing vector, (2.7), is a linear combination of a rotation and a boost. We can use this fact to treat the rotational plane and the boosted plane separately by splitting the identification matrix in the following way:
Consider writing $n = qm + p$, where $p \in \{0, 1, \ldots, m-1\}$, $q \in \{0, 1, \ldots, \infty\}$ and m is some positive integer. Now we have the following structure:

$$\begin{array}{cccccc} \mathbb{1} & H & H^2 & H^3 & \ldots & H^{m-1} \\ H^m & H^{m+1} & H^{m+2} & H^{m+3} & \ldots & H^{2m-1} \\ H^{2m} & H^{2m+1} & H^{2m+2} & H^{2m+3} & \ldots & H^{3m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \tag{2.11}$$

Purely looking at the general structure, each column can be associated with a black hole. To see this, first note that in the rotational plane, we need the parameter to be a rational number [108]. This is because the quotient space of a manifold by a rotation Killing vector requires the identification angle to be a rational fraction of 2π. Otherwise, each point is identified with infinitely many images, densely covering a circle, and the resulting image set would not be a smooth manifold. Therefore, the coefficient b must be a rational number, $b = k/m$, where k and m are relative primes. No restrictions are necessary for a, since boosts act transitively in a non-compact manner. Now define

$$H_a = H|_{b=0}\,, \tag{2.12a}$$
$$H_b = H|_{a=0}\,, \tag{2.12b}$$

to obtain matrices having the following beautiful properties:

$$H = H_a \cdot H_b = H_b \cdot H_a. \tag{2.13}$$

There is a deeper reason behind this: The boost and rotation generators, that is, $K \equiv J_{01} - J_{23}$ and $J \equiv J_{03} - J_{12}$, respectively, commute $[K, J] = 0$. Thus, the identification matrix H can be factored in this way. And so, using the well-known properties of the trigonometric functions, we can write

$$H^{qm+p} == H_a^{qm} H_a^p H_b^p = H_{a\cdot m}^q H_a^p H_b^p. \tag{2.14}$$

So that the p-th column reads

$$H_a^p H_b^p \left\{ \mathbb{1}, H_{a\cdot m}^1, H_{a\cdot m}^2, H_{a\cdot m}^3, \cdots \right\}. \tag{2.15}$$

Then, remembering that H_a is the identification matrix for the black hole [108], the previous statement follows. Finally, we can go on with the calculations.

2.2 Renormalized stress tensor

We consider the semi-classical Einstein equations

$$G_{\mu\nu} - l^{-2} g_{\mu\nu} = \kappa \left\langle T_{\mu\nu} \right\rangle, \tag{2.16}$$

where $\langle T_{\mu\nu} \rangle$ is the so-called renormalized expectation value of the quantum stress-energy tensor (RSET):

$$\kappa \left\langle T_{\mu\nu}(x) \right\rangle = \pi l_P \lim_{x' \to x} \left(3 \nabla_\mu^x \nabla_\nu^{x'} - g_{\mu\nu} g^{\lambda\rho} \nabla_\lambda^x \nabla_\rho^{x'} - \nabla_\mu^x \nabla_\nu^x - \frac{1}{4l^2} g_{\mu\nu} \right) G(x, x'). \tag{2.17}$$

Here, the propagator, $G(x, x') = \{\phi(x), \phi(x')\}$ is the anti-commutator of the scalar field, which takes the form (using the method of images) [101, 104–108]

$$G(x, x') = \frac{1}{2\sqrt{2}\pi} \sum_{n \in I} \frac{\Theta(\sigma(x, H^n x'))}{\sqrt{\sigma(x, H^n x')}}, \tag{2.18}$$

where

$$\sigma(x, x') = \frac{1}{2} \left[-\left(X^0 - X'^0 \right)^2 + \left(X^1 - X'^1 \right)^2 + \left(X^2 - X'^2 \right)^2 - \left(X^3 - X'^3 \right)^2 \right] \tag{2.19}$$

and Θ is the Heaviside step function, that was introduced in [108] since, in the rotating case, $\sigma(x, H^n x)$ can be negative. Writing

$$d_n = 2\sigma(x, H^n x), \tag{2.20}$$

the RSET takes the form [101, 108]

$$\kappa \left\langle T_{\mu\nu} \right\rangle = \frac{3 l_{\mathrm{P}}}{2} \sum_{n \in I \setminus \{0\}}{}' \left(S_{\mu\nu}^n - \frac{1}{3} g_{\mu\nu} g^{\lambda\rho} S_{\lambda\rho}^n \right), \tag{2.21}$$

with

$$S_{ab}^n = \frac{H_{ab}^n}{d_n^{3/2}} + \frac{3 H_{ac}^n X^c H_{bd}^{-n} X^d - H_{ac}^n X^c H_{bd}^n X^d}{d_n^{5/2}}, \tag{2.22}$$

and the prime attached to the sum symbol denotes the appearance of the Heaviside function:

$$\sum_n{}' s_n \equiv \sum_n \Theta(d_n) s_n. \tag{2.23}$$

Now is the time to clarify what set I we are summing over. We need to sum over all distinct images. However, we will treat the boosted and the rotational plane differently: Recall the splitting that was shown in the previous subsection. In this spirit, we will have to sum over q and p, and hence one cannot expect to have the same summation ranges for both parameters. In fact, remembering that each column in (2.11) had the structure of a black hole, we deduce that $q \in \{0, 1, \ldots, \infty\}$. Further, for $p = 0$ we essentially have copies of the $n = 0$ image, and therefore this has to be considered for the renormalization of the stress energy tensor. We conclude that $p \in \{1, \ldots m-1\}$. In summary, going back to the sum over n, one excludes $n = 0$ and additionally all n that are multiples of m. Then the non-vanishing components of the stress-energy tensor can be written in the overspinning case as

$$\begin{aligned}\kappa \langle T^t{}_t \rangle =& \frac{l_P l^2}{8ab} \sum_{\substack{n=1 \\ m \nmid n}}^{\infty}{}' \frac{1}{d_n^{5/2}} \Big\{ 6 \left(a^2 + b^2\right) B b_n - 4ab\bar{b}_n + 12 B \bar{a}_n \\ & + \Big[3\left(a^2 - b^2\right) B - 2ab\Big] (\bar{c}_n - 8) + \Big[3(a^2 - b^2) + 2abB\Big] c_n e_n \Big\},\end{aligned} \tag{2.24a}$$

$$\kappa \langle T^t{}_\theta \rangle = \frac{3 l_P l^3}{8ab} \sum_{\substack{n=1 \\ m \nmid n}}^{\infty}{}' \frac{2\left[\left(b^2 - a^2\right) B - 4ab\right] b_n - 4 B a_n - \left(a^2 + b^2\right)\left[B\left(\bar{c}_n - 8\right) + e_n c_n\right]}{d_n^{5/2}}, \tag{2.24b}$$

$$\kappa \langle T^r{}_r \rangle = l_P \sum_{\substack{n=1 \\ m \nmid n}}^{\infty}{}' \frac{c_n}{d_n^{3/2}} \tag{2.24c}$$

$$\kappa \langle T^\theta{}_t \rangle = \frac{3 l_P l}{8ab} \sum_{\substack{n=1 \\ m \nmid n}}^{\infty}{}' \frac{2\left[\left(a^2 - b^2\right) B - 4ab\right] b_n + 4 B a_n + \left(a^2 + b^2\right)\left[B\left(\bar{c}_n - 8\right) + c_n e_n\right]}{d_n^{5/2}}, \tag{2.24d}$$

$$\kappa \langle T^\theta{}_\theta \rangle = -\kappa \left[\langle T^t{}_t \rangle + \langle T^r{}_r \rangle\right], \tag{2.24e}$$

where we have defined the auxiliary functions

$$B = B(r) = \frac{l^2 M - 2r^2}{4abl^2}, \tag{2.25a}$$

$$d_n = 2l^2 \left[-1 + \cosh(2\pi a n)\cos(2\pi b n) - B \sinh(2\pi a n)\sin(2\pi b n)\right], \tag{2.25b}$$

$$e_n = 4 \sinh(2\pi a n)\sin(2\pi b n), \tag{2.25c}$$

and

$$a_n = a^2 \cos(4\pi b n) + b^2 \cosh(4\pi a n), \tag{2.25d}$$

$$\bar{a}_n = a^2 \cos(4\pi b n) - b^2 \cosh(4\pi a n), \tag{2.25e}$$

$$b_n = \cos(4\pi b n) - \cosh(4\pi a n), \tag{2.25f}$$

$$\bar{b}_n = \cos(4\pi b n) + \cosh(4\pi a n), \tag{2.25g}$$

$$c_n = 2\cosh(2\pi a n)\cos(2\pi b n) + 2, \tag{2.25h}$$

$$\bar{c}_n = 2\cosh(4\pi a n)\cos(4\pi b n) + 2. \tag{2.25i}$$

Note that, if $b = k/m$, with m being an even number, the sum contains infinitely many values $n = qm + p = qm + m/2$ for which d_n is independent of r. For these values, the RSET diverges at radial infinity so that the perturbative approximation breaks down. We will therefore exclude these values for b in the analysis and assume that m is an odd number. For if m is odd, the sine function is only zero if $kp = mz$, for some positive integer z. But since k and m are relatively prime, this implies that z is a multiple of k. This is a contraction since p has to be smaller than m, and hence we can safely work with m being an odd number. In the next subsection, we will insert this into the field equations (2.16) to look for a quantum corrected (up to first order in l_P) solution.

2.3 Backreacted metric

We use the ansatz

$$\begin{aligned} \mathrm{d}s^2 =& -N(r)^2 f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta + k(r)\mathrm{d}t\right)^2, \\ N(r) =& N_0(r) + l_P N_1(r) + O(l_P^2), \\ f(r) =& f_0(r) + l_P f_1(r) + O(l_P^2), \\ k(r) =& k_0(r) + l_P k_1(r) + O(l_P^2). \end{aligned} \tag{2.26}$$

The zeroth order equations are obviously the vacuum equations that yield the BTZ metric as a solution, or in other words

$$N_0(r) = 1, \qquad f_0(r) = \frac{r^2}{l^2} - M + \frac{J^2}{4r^2}, \qquad k_0(r) = -\frac{J}{2r^2}. \tag{2.27}$$

Now, the field equations (up to first order in l_P) can be written as

$$N_1' = -\frac{J\kappa\left\langle T^t{}_\theta\right\rangle - 2r^2\kappa\left\langle T^r{}_r\right\rangle + 2r^2\kappa\left\langle T^t{}_t\right\rangle}{2rl_P f_0}, \tag{2.28a}$$

$$\left(r^3 f_1'\right)' = -2r^3 f_0 N_1'' + 2r^2\left(f_0 + 3M - \frac{6r^2}{l^2}\right)N_1' + \frac{2r^3}{l_P}\kappa\left\langle T^r{}_r\right\rangle, \tag{2.28b}$$

$$Jk_1' = -f_1' + \frac{J^2 N_1}{r^3} + \frac{J}{rl_P}\kappa\left\langle T^t{}_\theta\right\rangle + \frac{2r}{l_P}\kappa\left\langle T^t{}_t\right\rangle. \tag{2.28c}$$

And we obtain

$$N_1(r) = \frac{\kappa}{l_P}\int \mathrm{d}r\frac{1}{f_0(r)}\left(r\left\langle T^r{}_r\right\rangle - r\left\langle T^t{}_t\right\rangle - \frac{J}{2r}\left\langle T^t{}_\theta\right\rangle\right) + K_1, \tag{2.29a}$$

$$\begin{aligned} f_1(r) =& \int \mathrm{d}r\left[-2f_0(r)N_1'(r) + \left(\frac{J^2}{r^3} - \frac{2M}{r}\right)N_1(r)\right. \\ & \left. + \frac{2}{r^3}\int \mathrm{d}r\left(2MrN_1(r) + \frac{\kappa}{l_P}r^3\left\langle T^r{}_r\right\rangle\right)\right] + \frac{K_2}{r^2} + K_3, \end{aligned} \tag{2.29b}$$

$$Jk_1(r) = -f_1(r) - 2f_0(r)N_1(r) + 2\int r\mathrm{d}r\left(\frac{2}{l^2}N_1(r) + \frac{\kappa}{l_P}\left\langle T^r{}_r\right\rangle\right) + K_4, \tag{2.29c}$$

where the integration constants will be chosen to be zero in order for the metric corrections to vanish for a vanishing RSET.

This in turn leads to

$$N_1(r) = -\frac{l^2}{2ab}\sum_{\substack{n=1\\ m\nmid n}}^{\infty}{}' \frac{\mathcal{A}_n}{e_n d_n^{3/2}}, \tag{2.30a}$$

$$f_1(r) = \sum_{\substack{n=1\\ m\nmid n}}^{\infty}{}' \frac{l^2\mathcal{A}_n\left(r^2 e_n f_0 - 16a^2b^2 B d_n\right) - 32a^3b^3 c_n d_n^2}{abr^2 e_n^2 d_n^{3/2}}, \tag{2.30b}$$

$$k_1(r) = \frac{4}{r^2}\sum_{\substack{n=1\\ m\nmid n}}^{\infty}{}' \frac{\mathcal{B}_n}{e_n^2 d_n^{1/2}}, \tag{2.30c}$$

where we have defined

$$\mathcal{A}_n = 2\left(a^2+b^2\right)b_n + 4\left(a^2-b^2\right)\left(\cosh^2(\pi an)\cos^2(\pi bn)-1\right) + abc_n e_n, \tag{2.31a}$$

$$\mathcal{B}_n = 2l\left[\left(a^2-b^2\right)b_n + 2\left(a^2+b^2\right)\left(\cosh^2(\pi an)\cos^2(\pi bn)-1\right)\right]. \tag{2.31b}$$

These functions solve the field equations (up to first order in l_P) and in the following subsections we will analyze them to study their physical properties. But before note that at large r the functions behave like $N_1 \sim \frac{1}{r^3}$, $f_1 \sim \frac{1}{r}$ and $k_1 \sim \frac{1}{r^3}$, and so they do not modify the asymptotic behavior of the solution, since the zeroth order functions are dominating at infinity.

2.4 Quantum field acting as cosmic censor

In this subsection, we will determine whether the quantum backreaction yields to the formation of an event horizon. We will closely follow the steps that were done in [108] for the case of the rotating naked singularity.

First, note that the zeroes of d_n are determined by

$$\frac{r_n^2}{l^2} = a^2 - b^2 + 2ab\frac{1-\cosh(2\pi an)\cos(2\pi bn)}{\sin(2\pi bn)\sinh(2\pi an)}. \tag{2.32}$$

We remind the reader that $b = k/m$ is a rational number such that k and m are relatively prime, and m is odd. Then it is easy to see that this expression is bounded from above for discrete values of $n = qm + p$. In fact, the upper bound is an accumulation point of the radii as $q \to \infty$ and $n = qm + (m+1)/2$. We will denote the upper bound by r_* and approximate the functions in its vicinity:

$$d_{n_*}(r) \approx \frac{2r_*\left(r-r_*\right)}{ab}\sinh(2an_*\pi)\sin(2bn_*\pi), \tag{2.33}$$

so that the correction (2.30b) becomes

$$f_1(r) = \frac{\Xi f_0(r_*)}{\left(r-r_*\right)^{3/2}} + C, \quad r \to r_*, \tag{2.34}$$

with

$$\Xi = \frac{\sqrt{8ab}\,l^2\mathcal{A}_{n_*}}{e_{n_*}\left(r_* e_{n_*}\right)^{3/2}}. \tag{2.35}$$

The constant C includes all other terms of the sum in (2.30b) that are being dominated by the n_*-term. Note that this has the same form as the case of the rotating naked singularity in [108]. We further remind the reader about the introduction of the Heaviside step function in (2.18), which, together with (2.33), ensures the positivity of the sine factor in the denominator. This is also why the n_* at the beginning of this subsection were written with the factors of two and a half respectively. Then, using (2.34), the defining equation for the quantum corrected horizon, $g^{rr}(r_+^{(q)}) = 0$, reads

$$0 = \left[f_0(r_+^{(q)}) + l_P C\right] \left(r_+^{(q)} - r_*\right)^{3/2} + l_P \Xi f_0(r_*), \tag{2.36}$$

and expanding $f_0(r_+^{(q)})$ around r_* it follows that $r_+^{(q)} - r_*$ is of order $l_P^{2/3}$ and that we can safely ignore the constant C (cf. [108]). Thus, the defining equation for the horizon can be written as

$$r_+^{(q)} = r_* + (-\Xi\, l_P)^{2/3} + O\left(l_P^{7/3}\right). \tag{2.37}$$

With $\mathcal{A}_{n_*}$, and therefore also Ξ, being manifestly negative it follows that the horizon is indeed hiding all the singularities occurring in the metric.

Having established the geometry to be that of a black hole, it is now interesting to look at another feature of such a solution, the static limit surface. Its defining equation is

$$g_{tt} = -N(r)^2 f(r) + r^2 k(r)^2 = 0, \tag{2.38}$$

which can be written up to first order in l_P as [108]

$$0 = -\left(\frac{r^2}{l^2} - M\right) - l_P \left(2 f_0 N_1 + f_1 + J k_1\right). \tag{2.39}$$

In the case of the overspinning geometry we have

$$2 f_0 N_1 + f_1 + J k_1 = \sum_{\substack{n=1 \\ m \nmid n}}^{\infty}{}' \frac{16 \left(\left[a^2 + b^2\right] b_n + 2\left[a^2 - b^2\right]\left[\cos^2(2\pi b n)\cosh^2(2\pi a n) - 1\right]\right)}{e_n^2 d_n^{1/2}}. \tag{2.40}$$

This term is negative, as can be easily seen:

$$\begin{aligned} 2\left[a^2 - b^2\right]\left[\cos^2(2\pi b n)\cosh^2(2\pi a n) - 1\right] \leq& 2\left[a^2 + b^2\right]\left[\cos^2(2\pi b n)\cosh^2(2\pi a n) - 1\right] \\ <& 2\left[a^2 + b^2\right]\left[\cosh^2(2\pi a n) - \cos^2(2\pi b n)\right] \\ =& -\left[a^2 + b^2\right] b_n, \end{aligned} \tag{2.41}$$

and the result follows.
This in turn can be approximately solved near the singularity by

$$r_{SL}^{(q)} = r_* + \mu l_P^2, \tag{2.42}$$

with,

$$\mu = \frac{128 a b \left(\left[a^2 + b^2\right] b_{n_*} + 2\left[a^2 - b^2\right]\left[\cos^2(2\pi b n_*)\cosh^2(2\pi a n_*) - 1\right]\right)^2}{r_* e_{n_*}^5 \left[\left(\frac{r_*}{2l}\right)^2 - (a^2 - b^2)\right]^2}, \tag{2.43}$$

which is manifestly positive (remember (2.33) and the fact that it is positive). Note the similarities with the naked singularity studied in [108]. In particular, in the overspinning case the ergoregion is non-existent for the same reason: Its distance $r_{SL}^{(q)} - r_*$ is of order $O(l_P^2)$, while $r_+^{(q)} - r_*$ is of order $O(l_P^{2/3})$, and so the static limit surface is hidden behind the event horizon. We can therefore conclude that for the overspinning black hole, we do not have an ergoregion.

2.5 Thermodynamics

In the previous section we have shown the existence of an event horizon, and consequently ascertained the solution to be that of a black hole. It is now natural to go one step further and explore its thermodynamic properties. Due to the semi-classical approach that we are using, it should be evident that there is no straightforward approach to this. However, it is known that $O(r^{-1})$ corrections to the metric functions leave the ADM mass ([113]) unchanged, and because of this, it is reasonable to work with the mass of the unperturbed space-time, M. Starting there, we can assume the validity of the first law of thermodynamics to calculate the entropy. To do so, we will have to find the temperature first, and so we begin by transforming the metric ansatz (2.26) to Eddington-Finkelstein-like coordinates:

$$dv = dt + \frac{dr}{Nf}, \qquad d\tilde{\phi} = d\phi - \frac{k}{Nf}dr, \tag{2.44}$$

which yields

$$ds^2 = -N^2 f dv^2 + 2N dv dr + r^2 \left(d\tilde{\phi} + k dv\right)^2. \tag{2.45}$$

Note that f vanishes at the horizon, making it a null hypersurface. In fact, the horizon is a Killing horizon generated by the Killing vectorfield

$$\chi = \partial_v - k(r_+)\partial_{\tilde{\phi}}, \tag{2.46}$$

and writing r_+ instead of $r_+^{(q)}$ from now on. Then, the surface gravity can be calculated, using:

$$\kappa^2 = -\frac{1}{2}\nabla^\mu \chi^\nu \nabla_\mu \chi_\nu, \tag{2.47}$$

which is directly related to the temperature:

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi}|N(r_+)f'(r_+)|, \tag{2.48}$$

with the functions being approximated at the horizon as

$$f'(r_+) = f_0'(r_+) + \frac{3f_0(r_*)}{2(-l_P\Xi)^{2/3}}, \qquad N(r_+) = \frac{3}{2}. \tag{2.49}$$

Now, assuming the validity of the first law of thermodynamics, one may calculate the entropy via

$$S = \int \frac{dM}{T}. \tag{2.50}$$

The integrand is a very complex function of the mass and is quite lengthy. Also note that the integration constant of this integral would possibly depend on J! We can not solve it analytically, but we can still use it to get some information about the entropy. Remembering that a and b depend on M, we can write the temperature as

$$\beta = \frac{1}{T} = \frac{2}{3}\left(\frac{3\left(\frac{l^2\left(a^2+b^2\right)^2}{r_*^2} - 2a^2 + 2b^2 + \frac{r_*^2}{l^2}\right)}{2(-l_P\Xi)^{2/3}} - \frac{2l^2\left(a^2+b^2\right)^2}{\left[(-l_P\Xi)^{2/3}+r_*\right]^3} + \frac{2\left[(-l_P\Xi)^{2/3}+r_*\right]}{l^2}\right)^{-1}, \tag{2.51}$$

where a, b, Ξ and r_* depend non-trivially on M. Clearly, the first term dominates the others. Taking the limit $a \to b$, or essentially $M \approx 0$, the first order term, and therefore the entropy, is positive:

$$S = \int \left[\beta(0) + O(M)\right] dM. \tag{2.52}$$

For arbitrary M it is not clear since the integral of the inverse temperature cannot be calculated analytically. However, interestingly, even without calculating the integral, one can immediately see that the classical limit of the entropy, $l_P \to 0$, is zero. This is as expected, since its classical counterpart is not a black hole.

2.6 Summary

We have shown that the presence of a conformally coupled quantum scalar field on an overspinning BTZ geometry gives rise to infinitely many curvature singularities which are all hidden behind an event horizon, thus supporting the cosmic censorship conjecture in three dimensions. Additionally, we have established that the static limit surface is also hidden behind the horizon. It is interesting to point out that behind the horizon there is an infinite number of curvature singularities hidden. This is very different from the classical black hole and an exclusive feature of the overspinning solution. In the derivation of the horizon radius we only used the outmost singularity, but one can do the same procedure using any of those, which hints at each singularity being covered by a horizon, giving the space-time an onion-like structure. It should be noted though that at a curvature singularity the linear approximation breaks down, so it is questionable how reliable the results are between the singularities, in particular those that are fairly close to one another. Even though the singularities itself appear in the RSET and hence are not just perturbative artifacts but rather exact results of the theory, in their vicinity one would still have to consider strong quantum gravitational effects to study its geometry. However, from a more pragmatic perspective, one would be more concerned with the geometry outside the event horizon, for which the linear approximation is quite reasonable. It is therefore interesting that when one takes quantum effects into account, using a semi-classical approach, the cosmic censorship conjecture is satisfied naturally. Even though this has been established for conical and (mildly) rotating singularities before (cf. [108] and references therein), this is not obvious for such an extreme scenario that is the overspinning geometry.

Conclusions and outlook

In this thesis, we examined several methods of sourcing black holes through scalar fields, each allowing for a different interpretation of the scalar field's physical meaning. We presented projects carried out in different theories, naturally splitting the thesis into three parts. The first chapter discusses the work we have done in standard general relativity, while the second considers classical modifications to GR by introducing changes to the Einstein-Hilbert action by means of a scalar field. In the last part of the thesis, we covered the semiclassical approach, where the matter field was quantized but the metric not. We will summarize our results in the following.

In chapter I we study the possibility of constructing black holes where the torsion of the underlying geometry does not vanish. Usually, general relativity considers a torsion-free connection; however, we can turn on the torsion part, by including the scalar field like a Lagrange multiplier of the Gauß-Bonnet term. This approach was used in [40] to study possible cosmological solutions with torsion. Through this trick, instead of being a topological invariant, we have an additional contribution to the field equations. In particular, using the first order formalism of gravity, these can cause a non-zero torsion. We argue that the model has some flaws by showing that, making a spherically symmetric and static ansatz, the system of equations is underdetermined, leaving a function unspecified. One possible reason for this is that, due to the field equations, the Gauß-Bonnet term needs to vanish. This gives a strong constraint on the geometry, yet it adds only one equation to the system. It is therefore reasonable to assume that one may have to modify the model such that this constraint gets slightly relaxed.

In the second part of this chapter, we consider a scalar field as matter in the field equations of GR and study its critical behavior during a gravitational collapse. To achieve this, we chose affine-null coordinates, like in [49], and present a novel regularization scheme to tackle possible singularities of the equations, which allowed us to solve the equations through pseudo-spectral methods. With a consumer-grade computer, and relatively little computation time, our code can not only reproduce the characteristic features like mass scaling and echoing of the Choptuik critical solution [19], but also the exponential decay and oscillations of the Bondi mass first discovered in [48]. As an additional feature, the collapsing time of our initial data is linearly proportional to one of its parameters, allowing for a manual adjustment of the computation time. Given the high potential of the code, it would be interesting to generalize it in the future, for example to axial symmetry, or a different kind of matter field.

Chapter II discusses different aspects of modifying the Einstein's theory of general relativity. Here, the scalar field modifies the underlying theory of gravity under consideration. In particular, we study a series of problems in DHOST theories that lead to three publications [23–25], and which we will summarize here in chronological order. In all our projects, the coupling functions of the DHOST action solely depend on the standard kinetic term of the scalar field, thus ensuring the invariance under the constant shift of the scalar field.

The first one considers the DHOST action in three dimensions together with a stationary metric ansatz together with a purely radial scalar field, for which we show that the equations of motion can be fully integrated. The kinetic term of the scalar field must then be constant, and the only possible solution is shown to be a BTZ-like metric with an effective cosmological constant expressed in terms of the coupling functions of the theory. This is very different from four dimensions, where DHOST theories that are much more restrictive admit various solutions with different asymptotic behavior, see for example [88, 90–93, 95, 97, 114, 115]. What is particularly interesting is, that in three dimensions there is an algebraic relation which forces the kinetic term to be constant, and that the general structure of the thermodynamics remains as in the standard BTZ case, even though we coupled a scalar field to the equations. It would be interesting to investigate whether the uniqueness of BTZ-like solutions is a more general feature of three-dimensional theories, given that in [97] the uniqueness was shown for the quadratic Horndeski action.

In our second work, we constructed regular (without curvature singularities), asymptotically flat black holes by making use of a generalized Kerr-Schild solution generating method in spherical symmetry. The solutions that we found depend on a mass integration constant, admit a smooth core of chosen regularity (depending on a parameter that we can vary), and are asymptotically flat. Interestingly, even though we use a spherically symmetric ansatz, they generally have an inner and outer event horizon. Fine-tuning the mass below a certain threshold, the solutions even become horizonless, yet still massive, hence particle-like. We examined a few observational consequences of our solutions, scanning through different strengths of the gravitational field. These results look very promising, which makes the solutions great models to compare their predictions with observational data. Therefore, it would be interesting to calculate further details allowing for more predictions. We also calculate the thermodynamic properties and show that the entropy does not comply with the area law. An interesting extension would be to find such solutions using an axial symmetric ansatz and compare them to the Kerr solution.

In the third project, we constructed a rotating stealth black hole solution in five dimensions. Its metric is given by the Myers–Perry spacetime with equal angular momenta, and the scalar field is identified with the Hamilton–Jacobi potential. Interestingly, the coupling functions of the theory remain arbitrary, we did not need to fine-tune them. We further showed that this scalar field generates a disformal transformation that, with a constant degree of disformality, leaves the metric invariant[2]. We were able to extend these results to higher order odd dimensions, for which the Myers-Perry metric needs to have all angular momenta to be equal. We concluded that this is mainly due to the Myers-Perry metric being of cohomogeneity-1. This allowed us to find conditions under which a general cohomogeneity-1 remains invariant under a disformal transformation, generated by such a scalar field, with a constant degree of disformality.

In the last part of the thesis, chapter III, we take quantum effects under consideration. Considering the Einstein equations in three dimensions together with a conformally coupled scalar field as matter source, we then studied the semiclassical backreaction when one quantizes the scalar field. As a background geometry, we use an overspinning BTZ metric, that is, where the relation between angular momentum and mass reads $|J| > lM$, which classically leads to a naked singularity. Using the Hadamard regularization, we computed the renormalized quantum stress-energy tensor of the scalar field to obtain the semiclassical Einstein equations and solved

[2] By invariant we mean up to diffeomorphisms.

them perturbatively to first order in l_P, the Planck length. Our results show that the quantum backreaction causes infinitely many curvature singularities to arise, which are all hidden behind an event horizon. Similar results have been shown for different BTZ geometries in [108, 110–112], though in these cases, they have not observed infinitely many curvature singularities, which is an exclusive feature of the overspinning geometry. The arising of an event horizon due to quantum effects is a strong indication in favor of the cosmic censorship conjecture, and it would be intriguing to study whether this holds true in four dimensions as well, though this would be a highly complicated task.

Appendix A

Appendix

1 Regular Black Hole solutions

Our starting point will be the action:

$$S[g,\phi] = \int d^4x\sqrt{-g} \quad \Big[K(X) + G(X)R + A_1(X)\left[\phi_{\mu\nu}\phi^{\mu\nu} - (\Box\phi)^2\right] + A_3(X)\Box\phi\,\phi^{\mu}\phi_{\mu\nu}\phi^{\nu} \\ +A_4(X)\phi^{\mu}\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho} + A_5(X)\left(\phi^{\mu}\phi_{\mu\nu}\phi^{\nu}\right)^2\Big] \tag{1.1}$$

The coupling functions, K, G, A_1, A_3, A_4, A_5, depend solely on the kinetic term of the scalar field, defined by $X = \phi^{\mu}\phi_{\mu}$, where we have used $\phi_\mu = \partial_\mu\phi$ and $\phi_{\mu\nu} = \partial_\mu\partial_\nu\phi$. Further, we impose the following condition on the functions:

$$\begin{aligned} A_4 &= \frac{1}{8(G - XA_1)^2}\Big\{4G\Big[3(-A_1 + 2G_X)^2 - 2A_3G\Big] - A_3X^2(16A_1G_X + A_3G) \\ &\quad +4X\Big[-3A_2A_3G + 16A_1^2G_X - 16A_1G_X^2 - 4A_1^3 + 2A_3GG_X\Big]\Big\}, \\ A_5 &= \frac{1}{8(G - XA_1)^2}(2A_1 - XA_3 - 4G_X)\left(A_1(2A_1 + 3XA_3 - 4G_X) - 4A_3G\right). \end{aligned} \tag{1.2}$$

This is to avoid the so-called Ostrogradski ghosts. For later convenience we further define the auxiliary functions

$$\begin{aligned} &\mathcal{H}(X) = A_1(X)\,X - G(X), \qquad \mathcal{B}(X) = A_3(X)\,X + 4G_X(X) - 2A_1(X), \\ &\mathcal{Z}(X) = A_3(X) + A_4(X) + X\,A_5(X), \end{aligned} \tag{1.3}$$

and impose a simplifying hypothesis:

$$\frac{3\mathcal{B}}{8\mathcal{H}} = \frac{1}{X}, \tag{1.4}$$

Thus we can express the functions as

$$\mathcal{H} = \frac{3}{(rX' - 3X)}, \qquad G = \frac{1}{X}\left(1 - \frac{rX'}{X}\right) - \frac{Kr^2}{2}. \tag{1.5}$$

Writing

$$A_3 = -\frac{4G_X}{X} + \frac{2A_1}{X} + \frac{8\mathcal{H}}{3X^2}, \qquad A_1 = \frac{\mathcal{H}+G}{X} \tag{1.6}$$

leads to

$$K = -\frac{2\left[3X\left(rX'' + 2X'\right) + r^2X^{-1}X'^3 - 7rX'^2\right]}{rX\left(rX' - 3X\right)^2}. \tag{1.7}$$

A more general form, where $f(r) = 1 - \frac{\mu X^\lambda}{r}$, is:

$$\mathcal{H} = \frac{H_0}{X^\lambda\left(\frac{\lambda rX'}{3X} - 1\right)}, \tag{1.8}$$

$$K = -\frac{\mathcal{A}_{,r}}{rX^{\lambda/3}}, \tag{1.9}$$

$$\mathcal{A} = \frac{H_0\left(1 - \frac{\lambda rX'}{X}\right)}{X^{2\lambda/3}\left(\frac{\lambda rX'}{3X} - 1\right)}, \tag{1.10}$$

$$G = \frac{H_0}{X^\lambda}\left(1 - \frac{\lambda rX'}{X}\right) - \frac{1}{2}Kr^2, \tag{1.11}$$

where H_0 is an integration constant.
Following the same procedure that is outlined in [97], one can obtain an even more general form, where the metric function, $f(r)$, is arbitrary:

$$\mathcal{H} = \frac{H_0}{X^\lambda\left(\frac{\lambda rX'}{3X} - 1\right)}, \tag{1.12}$$

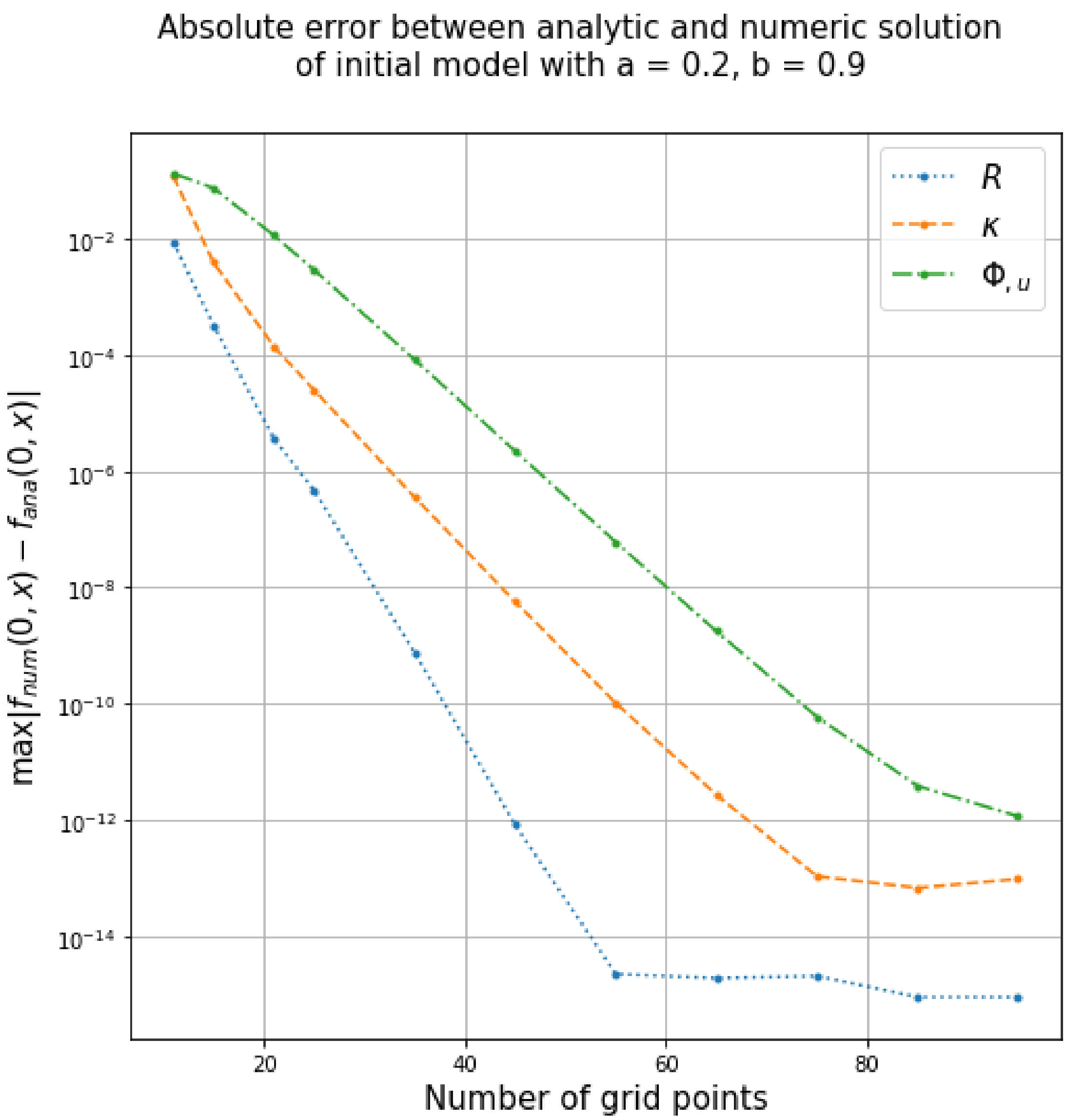
$$G = \frac{H_0}{X^\lambda}\left[rf' + f\left(1 - \frac{\lambda rX'}{X}\right)\right] - \frac{1}{2}Kr^2, \tag{1.13}$$

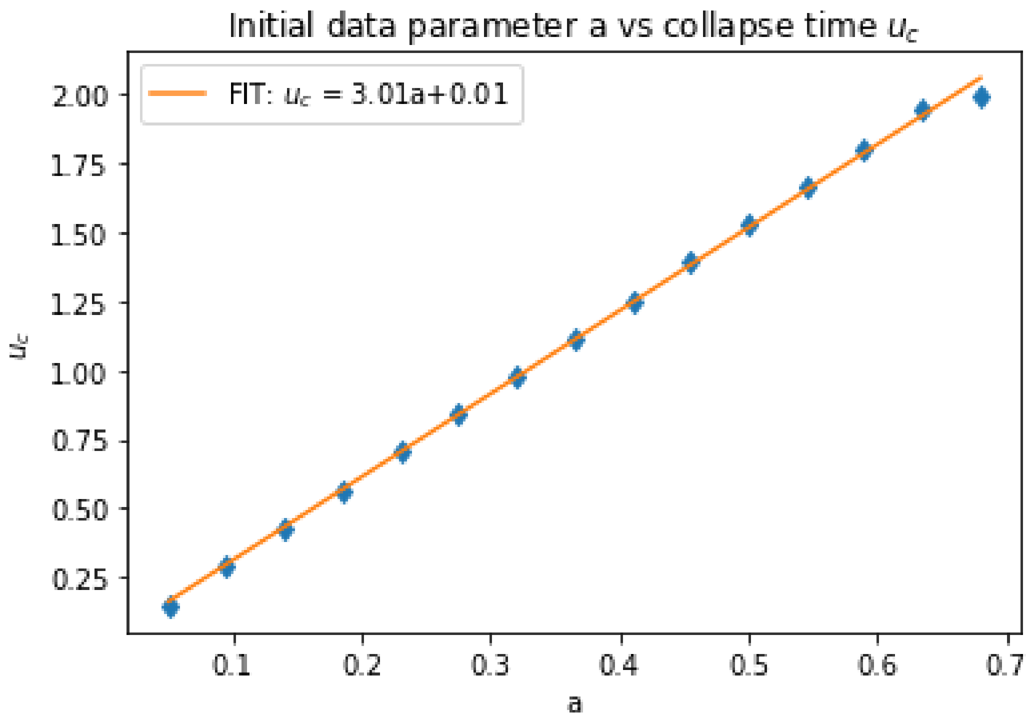
$$K = \frac{K_{\mathrm{N}}}{rX^{1+\lambda}(r\lambda X' - 3X)^2}, \tag{1.14}$$

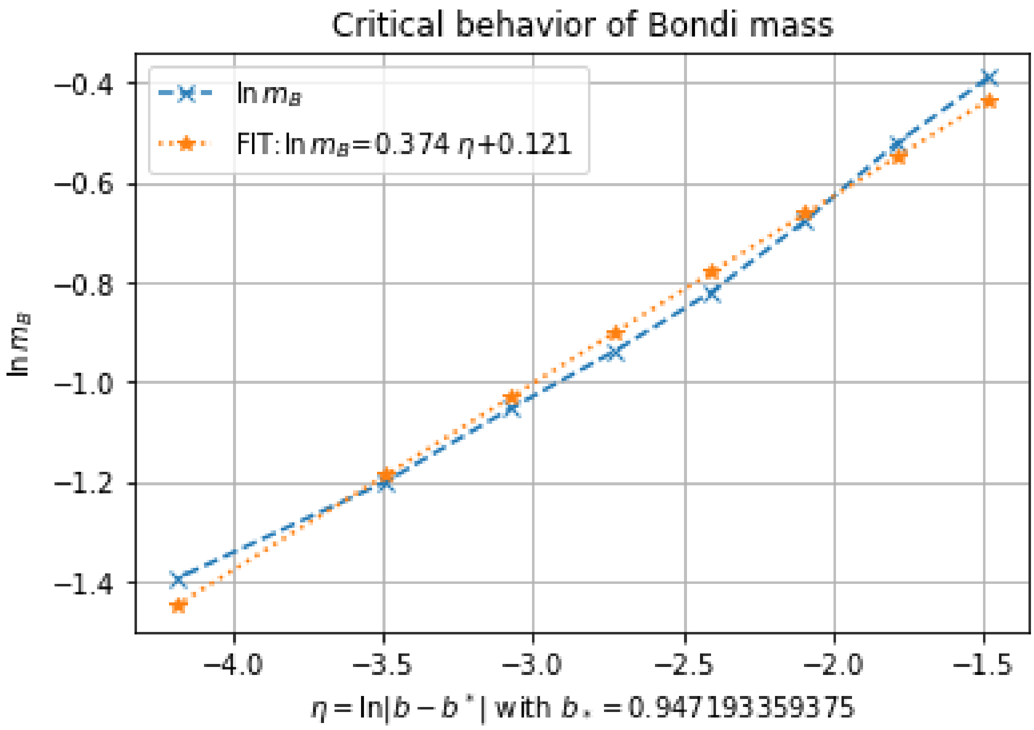
with the numerator of K being

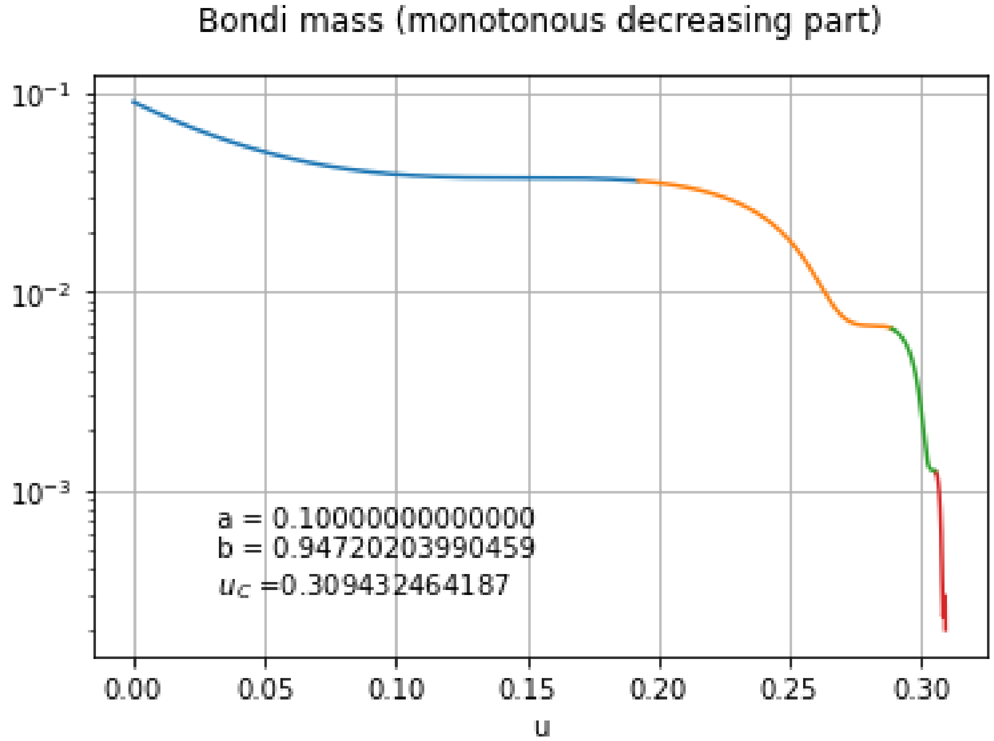
$$\begin{aligned} K_{\mathrm{N}} = & -3H_0(-r\lambda X^2(rf' - 2f)X'' + X^2r(r\lambda X' - 3X)f'' + \frac{2}{3}r^2\lambda^3 fX'^3 \\ & -\frac{5}{3}(r(\lambda - \frac{3}{5})f' + \frac{8}{5}(\frac{3}{4} + \lambda)f)X\lambda rX'^2 + 6X^2(\frac{2}{3}f + rf')\lambda X' - 6X^3f') \end{aligned} \tag{1.15}$$

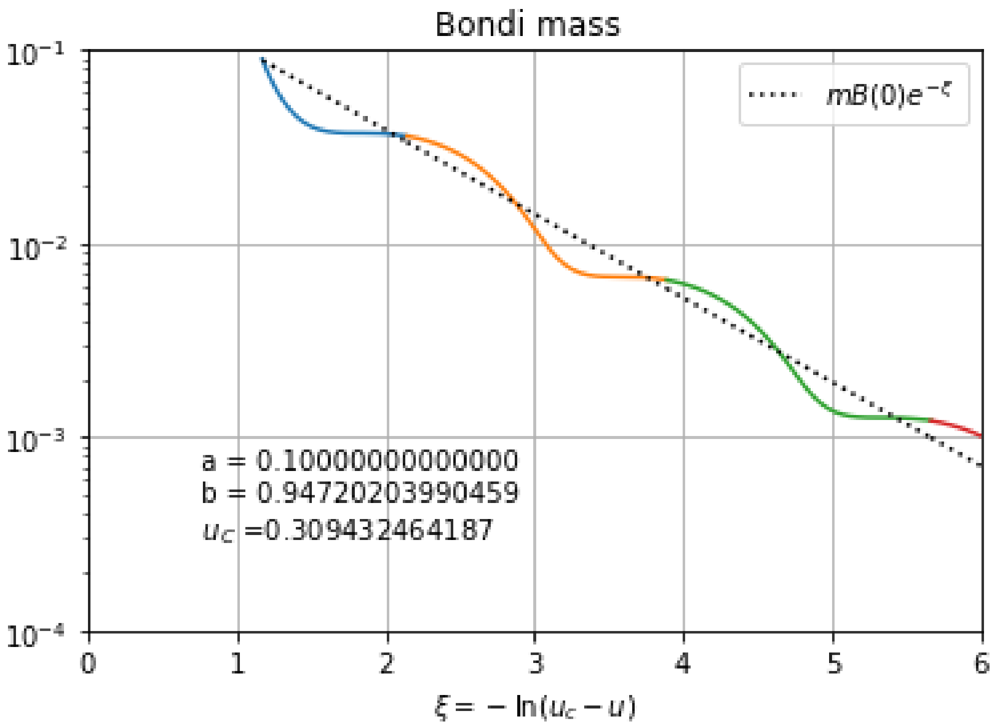
2 Figures of our critical collapse results

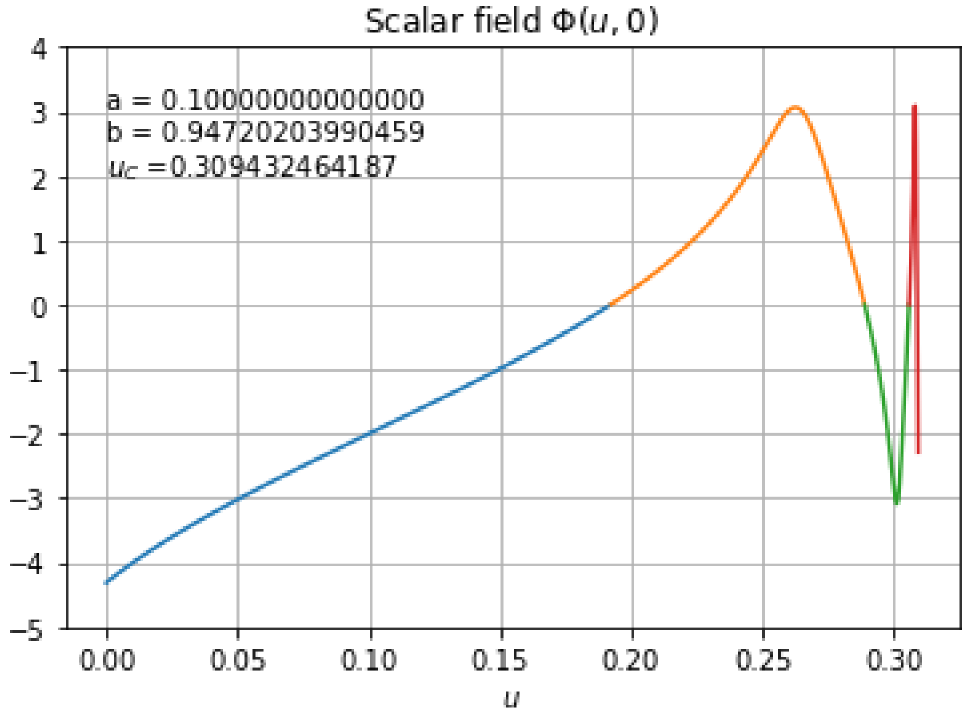


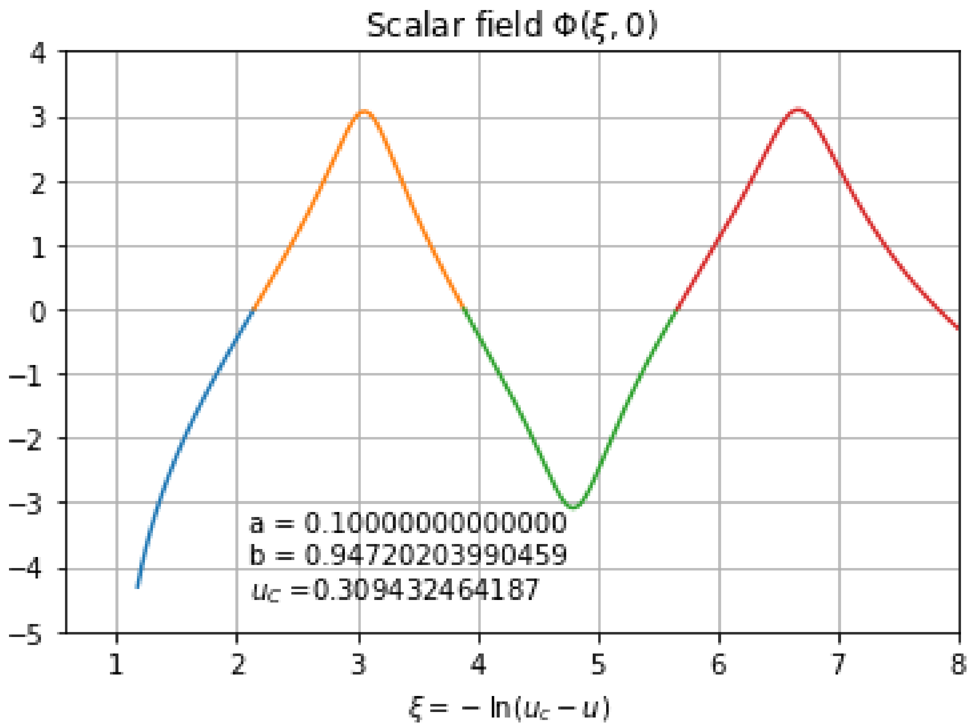

Initial data parameter a vs collapse time u_c
FIT: u_c = 3.01a+0.01
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
u_c
0.1
0.2
0.3
0.4
0.5
0.6
0.7
a

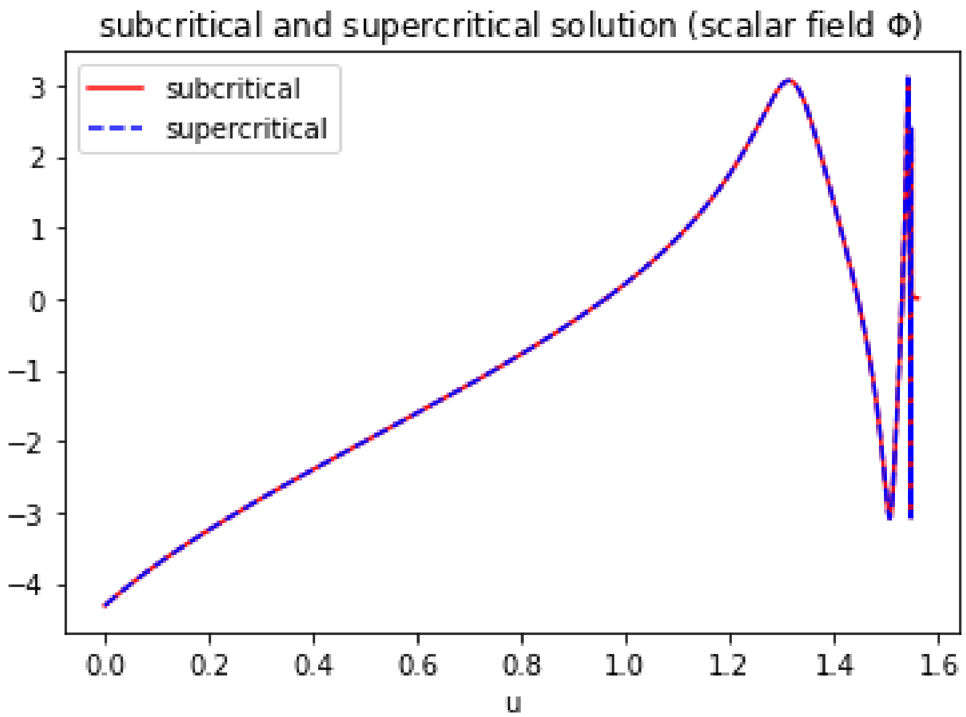

Critical behavior of Bondi mass
$\ln m_B$
FIT: $\ln m_B$=0.374 η+0.121
−0.4
−0.6
−0.8
−1.0
−1.2
−1.4
$\ln m_B$
−4.0
−3.5
−3.0
−2.5
−2.0
−1.5
$\eta = \ln|b - b^*|$ with $b_* = 0.947193359375$

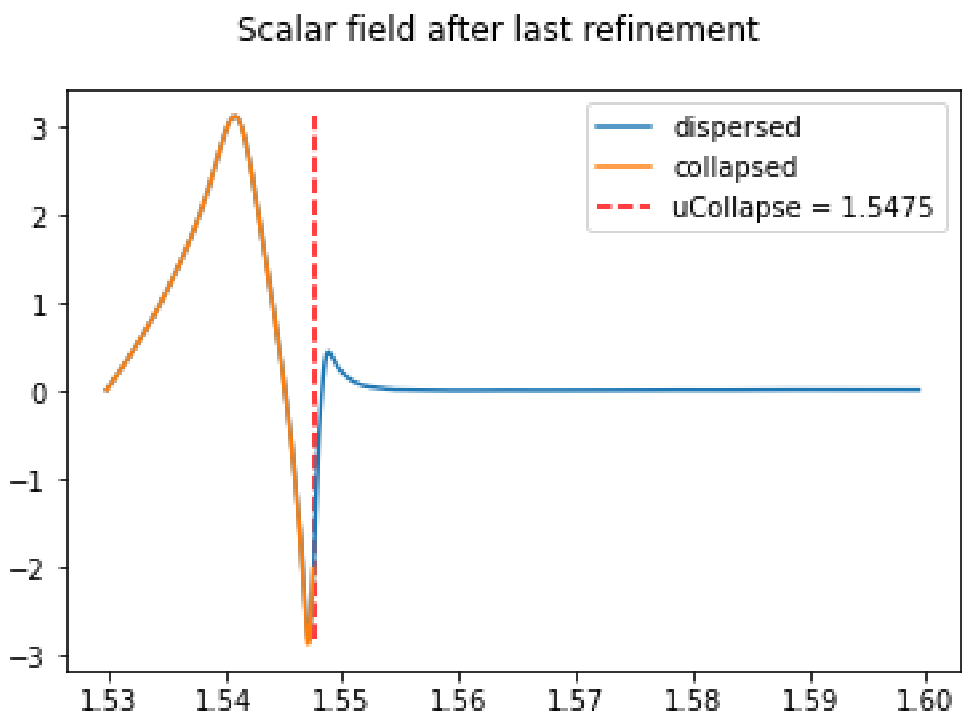

Bondi mass (monotonous decreasing part)
10^{-1}
10^{-2}
10^{-3}
a = 0.10000000000000
b = 0.94720203990459
u_C =0.309432464187
0.00
0.05
0.10
0.15
0.20
0.25
0.30
u


Bondi mass
$mB(0)e^{-\xi}$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
a = 0.10000000000000
b = 0.94720203990459
u_C =0.309432464187
0
1
2
3
4
5
6
$\xi = -\ln(u_c - u)$


Scalar field Φ(u, 0)
a = 0.10000000000000
b = 0.94720203990459
u_c =0.309432464187
u


Scalar field Φ(ξ, 0)
a = 0.10000000000000
b = 0.94720203990459
u_c =0.309432464187
$\xi = -\ln(u_c - u)$


subcritical and supercritical solution (scalar field Φ)
subcritical
supercritical
3
2
1
0
−1
−2
−3
−4
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
u


Scalar field after last refinement
dispersed
collapsed
uCollapse = 1.5475
3
2
1
0
−1
−2
−3
1.53
1.54
1.55
1.56
1.57
1.58
1.59
1.60

Bibliography

[1] B. P. Abbott et al. “Observation of Gravitational Waves from a Binary Black Hole Merger”. In: *Phys. Rev. Lett.* 116.6 (2016), p. 061102. DOI: 10.1103/PhysRevLett.116.061102. arXiv: 1602.03837 [gr-qc].

[2] Kazunori Akiyama et al. “First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole”. In: *Astrophys. J. Lett.* 875 (2019), p. L1. DOI: 10.3847/2041-8213/ab0ec7. arXiv: 1906.11238 [astro-ph.GA].

[3] Kazunori Akiyama et al. “First M87 Event Horizon Telescope Results. II. Array and Instrumentation”. In: *Astrophys. J. Lett.* 875.1 (2019), p. L2. DOI: 10.3847/2041-8213/ab0c96. arXiv: 1906.11239 [astro-ph.IM].

[4] Kazunori Akiyama et al. “First M87 Event Horizon Telescope Results. III. Data Processing and Calibration”. In: *Astrophys. J. Lett.* 875.1 (2019), p. L3. DOI: 10.3847/2041-8213/ab0c57. arXiv: 1906.11240 [astro-ph.GA].

[5] Kazunori Akiyama et al. “First M87 Event Horizon Telescope Results. IV. Imaging the Central Supermassive Black Hole”. In: *Astrophys. J. Lett.* 875.1 (2019), p. L4. DOI: 10.3847/2041-8213/ab0e85. arXiv: 1906.11241 [astro-ph.GA].

[6] Kazunori Akiyama et al. “First M87 Event Horizon Telescope Results. V. Physical Origin of the Asymmetric Ring”. In: *Astrophys. J. Lett.* 875.1 (2019), p. L5. DOI: 10.3847/2041-8213/ab0f43. arXiv: 1906.11242 [astro-ph.GA].

[7] Kazunori Akiyama et al. “First M87 Event Horizon Telescope Results. VI. The Shadow and Mass of the Central Black Hole”. In: *Astrophys. J. Lett.* 875.1 (2019), p. L6. DOI: 10.3847/2041-8213/ab1141. arXiv: 1906.11243 [astro-ph.GA].

[8] Kazunori Akiyama et al. “First M87 Event Horizon Telescope Results. VII. Polarization of the Ring”. In: *Astrophys. J. Lett.* 910.1 (2021), p. L12. DOI: 10.3847/2041-8213/abe71d. arXiv: 2105.01169 [astro-ph.HE].

[9] Kazunori Akiyama et al. “First M87 Event Horizon Telescope Results. VIII. Magnetic Field Structure near The Event Horizon”. In: *Astrophys. J. Lett.* 910.1 (2021), p. L13. DOI: 10.3847/2041-8213/abe4de. arXiv: 2105.01173 [astro-ph.HE].

[10] Albert Einstein. “The Field Equations of Gravitation”. In: *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* 1915 (1915), pp. 844–847.

[11] J. A. Wheeler and K. Ford. *Geons, black holes, and quantum foam: A life in physics*. 1998.

[12] Karl Schwarzschild. “On the gravitational field of a mass point according to Einstein’s theory”. In: *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* 1916 (1916), pp. 189–196. arXiv: physics/9905030.

[13] H. Reissner. "Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie". In: *Annalen der Physik* 355.9 (1916), pp. 106–120. DOI: https://doi.org/10.1002/andp.19163550905. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/andp.19163550905. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/andp.19163550905.

[14] G. Nordström. "On the Energy of the Gravitation field in Einstein's Theory". In: *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences* 20 (Jan. 1918), pp. 1238–1245.

[15] Roy P. Kerr. "Gravitational field of a spinning mass as an example of algebraically special metrics". In: *Phys. Rev. Lett.* 11 (1963), pp. 237–238. DOI: 10.1103/PhysRevLett.11.237.

[16] Werner Israel. "Event horizons in static vacuum space-times". In: *Phys. Rev.* 164 (1967), pp. 1776–1779. DOI: 10.1103/PhysRev.164.1776.

[17] Werner Israel. "Event horizons in static electrovac space-times". In: *Commun. Math. Phys.* 8 (1968), pp. 245–260. DOI: 10.1007/BF01645859.

[18] B. Carter. "Axisymmetric Black Hole Has Only Two Degrees of Freedom". In: *Phys. Rev. Lett.* 26 (1971), pp. 331–333. DOI: 10.1103/PhysRevLett.26.331.

[19] Matthew W. Choptuik. "Universality and scaling in gravitational collapse of a massless scalar field". In: *Phys. Rev. Lett.* 70 (1993), pp. 9–12. DOI: 10.1103/PhysRevLett.70.9.

[20] Roger Penrose. "Gravitational collapse and space-time singularities". In: *Phys. Rev. Lett.* 14 (1965), pp. 57–59. DOI: 10.1103/PhysRevLett.14.57.

[21] S. W. Hawking and R. Penrose. "The Singularities of gravitational collapse and cosmology". In: *Proc. Roy. Soc. Lond. A* 314 (1970), pp. 529–548. DOI: 10.1098/rspa.1970.0021.

[22] Roger Penrose. "Gravitational Collapse: the Role of General Relativity". In: *Nuovo Cimento Rivista Serie* 1 (Jan. 1969), p. 252.

[23] Olaf Baake, Moises F. Bravo Gaete, and Mokhtar Hassaine. "Spinning black holes for generalized scalar tensor theories in three dimensions". In: *Phys. Rev. D* 102.2 (2020), p. 024088. DOI: 10.1103/PhysRevD.102.024088. arXiv: 2005.10869 [hep-th].

[24] Olaf Baake, Christos Charmousis, Mokhtar Hassaine, and Miguel San. "Regular black holes and gravitational particle-like solutions in generic DHOST theories". In: (Apr. 2021). arXiv: 2104.08221 [hep-th].

[25] Olaf Baake and Mokhtar Hassaine. "Rotating stealth black holes with a cohomogeneity-1 metric". In: *Eur. Phys. J. C* 81.7 (2021), p. 642. DOI: 10.1140/epjc/s10052-021-09449-2. arXiv: 2104.13834 [hep-th].

[26] Maximo Banados, Claudio Teitelboim, and Jorge Zanelli. "The Black hole in three-dimensional space-time". In: *Phys. Rev. Lett.* 69 (1992), pp. 1849–1851. DOI: 10.1103/PhysRevLett.69.1849. arXiv: hep-th/9204099.

[27] Steven Carlip. "The (2+1)-Dimensional black hole". In: *Class. Quant. Grav.* 12 (1995), pp. 2853–2880. DOI: 10.1088/0264-9381/12/12/005. arXiv: gr-qc/9506079.

[28] Matías Briceño, Cristián Martínez, and Jorge Zanelli. "Overspinning naked singularities in AdS_3 spacetime". In: (May 2021). arXiv: 2105.06488 [gr-qc].

[29] S. W. Hawking. “Black holes in general relativity”. In: *Commun. Math. Phys.* 25 (1972), pp. 152–166. DOI: 10.1007/BF01877517.

[30] J. D. Bekenstein. “Black holes and the second law”. In: *Lett. Nuovo Cim.* 4 (1972), pp. 737–740. DOI: 10.1007/BF02757029.

[31] Jacob D. Bekenstein. “Black holes and entropy”. In: *Phys. Rev. D* 7 (1973), pp. 2333–2346. DOI: 10.1103/PhysRevD.7.2333.

[32] S. W. Hawking. “Particle Creation by Black Holes”. In: *Commun. Math. Phys.* 43 (1975). Ed. by G. W. Gibbons and S. W. Hawking. [Erratum: Commun.Math.Phys. 46, 206 (1976)], pp. 199–220. DOI: 10.1007/BF02345020.

[33] S. W. Hawking. “Black hole explosions”. In: *Nature* 248 (1974), pp. 30–31. DOI: 10.1038/248030a0.

[34] James M. Bardeen, B. Carter, and S. W. Hawking. “The Four laws of black hole mechanics”. In: *Commun. Math. Phys.* 31 (1973), pp. 161–170. DOI: 10.1007/BF01645742.

[35] S. Carlip. “Black hole thermodynamics”. In: *One Hundred Years of General Relativity: From Genesis and Empirical Foundations to Gravitational Waves, Cosmology and Quantum Gravity (Volume 2)*. Vol. Volume 2. 2017, pp. 415–465. DOI: 10.1142/9789814678506_0010.

[36] Robert M. Wald. *General Relativity*. The University of Chicago Press, 1984.

[37] G. W. Gibbons and S. W. Hawking. “Action Integrals and Partition Functions in Quantum Gravity”. In: *Phys. Rev. D* 15 (1977), pp. 2752–2756. DOI: 10.1103/PhysRevD.15.2752.

[38] James W. York Jr. “Role of conformal three geometry in the dynamics of gravitation”. In: *Phys. Rev. Lett.* 28 (1972), pp. 1082–1085. DOI: 10.1103/PhysRevLett.28.1082.

[39] Tullio Regge and Claudio Teitelboim. “Role of Surface Integrals in the Hamiltonian Formulation of General Relativity”. In: *Annals Phys.* 88 (1974), p. 286. DOI: 10.1016/0003-4916(74)90404-7.

[40] Adolfo Toloza and Jorge Zanelli. “Cosmology with scalar–Euler form coupling”. In: *Class. Quant. Grav.* 30 (2013), p. 135003. DOI: 10.1088/0264-9381/30/13/135003. arXiv: 1301.0821 [gr-qc].

[41] Christos Charmousis, Antoine Lehébel, Evangelos Smyrniotis, and Nikolaos Stergioulas. “Astrophysical constraints on compact objects in 4D Einstein-Gauss-Bonnet gravity”. In: *JCAP* 02.02 (2022), p. 033. DOI: 10.1088/1475-7516/2022/02/033. arXiv: 2109.01149 [gr-qc].

[42] Thomas W. Baumgarte and Stuart L. Shapiro. *Numerical Relativity: Solving Einstein’s Equations on the Computer*. Cambridge University Press, 2010. DOI: 10.1017/CBO9781139193344.

[43] A. M. Abrahams and C. R. Evans. “Critical behavior and scaling in vacuum axisymmetric gravitational collapse”. In: *Phys. Rev. Lett.* 70 (1993), pp. 2980–2983. DOI: 10.1103/PhysRevLett.70.2980.

[44] Carsten Gundlach and Jose M. Martin-Garcia. “Critical phenomena in gravitational collapse”. In: *Living Rev. Rel.* 10 (2007), p. 5. DOI: 10.12942/lrr-2007-5. arXiv: 0711.4620 [gr-qc].

[45] David Garfinkle. "Choptuik scaling in null coordinates". In: *Phys. Rev. D* 51 (1995), pp. 5558–5561. DOI: 10.1103/PhysRevD.51.5558. arXiv: gr-qc/9412008.

[46] Rufus S. Hamade and John M. Stewart. "The Spherically symmetric collapse of a massless scalar field". In: *Class. Quant. Grav.* 13 (1996), pp. 497–512. DOI: 10.1088/0264-9381/13/3/014. arXiv: gr-qc/9506044.

[47] Shahar Hod and Tsvi Piran. "Fine structure of Choptuik's mass scaling relation". In: *Phys. Rev. D* 55 (1997), pp. 440–442. DOI: 10.1103/PhysRevD.55.R440. arXiv: gr-qc/9606087.

[48] Michael Purrer, Sascha Husa, and Peter C. Aichelburg. "News from critical collapse: Bondi mass, tails and quasinormal modes". In: *Phys. Rev. D* 71 (2005), p. 104005. DOI: 10.1103/PhysRevD.71.104005. arXiv: gr-qc/0411078.

[49] J. A. Crespo, H. P. de Oliveira, and J. Winicour. "Affine-null formulation of the gravitational equations: Spherical case". In: *Phys. Rev. D* 100.10 (2019), p. 104017. DOI: 10.1103/PhysRevD.100.104017. arXiv: 1910.03439 [gr-qc].

[50] Leonardo R. Werneck, Zachariah B. Etienne, Elcio Abdalla, Bertha Cuadros-Melgar, and C. E. Pellicer. "NRPyCritCol & SFcollapse1D: an open-source, user-friendly toolkit to study critical phenomena". In: *Class. Quant. Grav.* 38.24 (2021), p. 245005. DOI: 10.1088/1361-6382/ac33bf. arXiv: 2106.06553 [gr-qc].

[51] Erik Jimenez-Vazquez and Miguel Alcubierre. "Critical gravitational collapse of a massive complex scalar field". In: (June 2022). arXiv: 2206.01389 [gr-qc].

[52] Frans Pretorius and Matthew W. Choptuik. "Gravitational collapse in (2+1)-dimensional AdS space-time". In: *Phys. Rev. D* 62 (2000), p. 124012. DOI: 10.1103/PhysRevD.62.124012. arXiv: gr-qc/0007008.

[53] Florencia Benitez, Rodolfo Gambini, Luis Lehner, Steve Liebling, and Jorge Pullin. "Critical collapse of a scalar field in semiclassical loop quantum gravity". In: *Phys. Rev. Lett.* 124.7 (2020), p. 071301. DOI: 10.1103/PhysRevLett.124.071301. arXiv: 2002.04044 [gr-qc].

[54] Thomas Mädler and Jeffrey Winicour. "Bondi-Sachs Formalism". In: *Scholarpedia* 11 (2016), p. 33528. DOI: 10.4249/scholarpedia.33528. arXiv: 1609.01731 [gr-qc].

[55] Pierre Carbonnelle. *PopularitY of Programming Language index*. 2022. URL: https://pypl.github.io/PYPL.html (visited on 06/28/2022).

[56] TIOBE Software BV. *TIOBE Index*. 2022. URL: https://www.tiobe.com/tiobe-index/ (visited on 06/28/2022).

[57] J. Winicour. "Affine-null metric formulation of Einstein's equations". In: *Phys. Rev. D* 87.12 (2013), p. 124027. DOI: 10.1103/PhysRevD.87.124027. arXiv: 1303.6969 [gr-qc].

[58] Thomas Mädler. "Affine-null metric formulation of general relativity at two intersecting null hypersurfaces". In: *Phys. Rev. D* 99.10 (2019), p. 104048. DOI: 10.1103/PhysRevD.99.104048. arXiv: 1810.04743 [gr-qc].

[59] Louis A. Tamburino and Jeffrey H. Winicour. "Gravitational Fields in Finite and Conformal Bondi Frames". In: *Phys. Rev.* 150 (1966), pp. 1039–1053. DOI: 10.1103/PhysRev.150.1039.

[60] Richard Mikael Slevinsky and Sheehan Olver. "A fast and well-conditioned spectral method for singular integral equations". In: *Journal of Computational Physics* 332 (Mar. 2017), pp. 290–315. DOI: 10.1016/j.jcp.2016.12.009.

[61] Jan S. Hesthaven, Sigal Gottlieb, and David Gottlieb. *Spectral Methods for Time-Dependent Problems*. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, 2007. DOI: 10.1017/CBO9780511618352.

[62] Casey J Handmer and Béla Szilágyi. "Spectral characteristic evolution: a new algorithm for gravitational wave propagation". In: *Classical and Quantum Gravity* 32.2 (Dec. 2014), p. 025008. DOI: 10.1088/0264-9381/32/2/025008. URL: https://doi.org/10.1088/0264-9381/32/2/025008.

[63] Jeffrey H. Winicour and Thomas Mädler. Private communication. Feb. 3, 2013.

[64] Jose M. Martin-Garcia and Carsten Gundlach. "Global structure of Choptuik's critical solution in scalar field collapse". In: *Phys. Rev. D* 68 (2003), p. 024011. DOI: 10.1103/PhysRevD.68.024011. arXiv: gr-qc/0304070.

[65] Michael Reiterer and Eugene Trubowitz. "Choptuik's critical spacetime exists". In: *Commun. Math. Phys.* 368.1 (2019), pp. 143–186. DOI: 10.1007/s00220-019-03413-8. arXiv: 1203.3766 [gr-qc].

[66] D. Lovelock. "The Einstein tensor and its generalizations". In: *J. Math. Phys.* 12 (1971), pp. 498–501. DOI: 10.1063/1.1665613.

[67] Bruno Zumino. "Gravity Theories in More Than Four-Dimensions". In: *Phys. Rept.* 137 (1986), p. 109. DOI: 10.1016/0370-1573(86)90076-1.

[68] Mokhtar Hassaine and Jorge Zanelli. *Chern-Simons (super)gravity*. 2016th ed. Vol. 2. 100 years of general relativity. Hackensack: World Scientific, 2016. ISBN: 978-981-4730-93-8. DOI: 10.1142/9863.

[69] Maximo Banados, Claudio Teitelboim, and Jorge Zanelli. "Dimensionally continued black holes". In: *Phys. Rev. D* 49 (1994), pp. 975–986. DOI: 10.1103/PhysRevD.49.975. arXiv: gr-qc/9307033.

[70] Gregory Walter Horndeski. "Second-order scalar-tensor field equations in a four-dimensional space". In: *Int. J. Theor. Phys.* 10 (1974), pp. 363–384. DOI: 10.1007/BF01807638.

[71] David Langlois and Karim Noui. "Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability". In: *JCAP* 02 (2016), p. 034. DOI: 10.1088/1475-7516/2016/02/034. arXiv: 1510.06930 [gr-qc].

[72] David Langlois and Karim Noui. "Hamiltonian analysis of higher derivative scalar-tensor theories". In: *JCAP* 07 (2016), p. 016. DOI: 10.1088/1475-7516/2016/07/016. arXiv: 1512.06820 [gr-qc].

[73] David Langlois. "Degenerate Higher-Order Scalar-Tensor (DHOST) theories". In: *52nd Rencontres de Moriond on Gravitation*. 2017, pp. 221–228. arXiv: 1707.03625 [gr-qc].

[74] Philippe Brax, Clare Burrage, Christoph Englert, and Michael Spannowsky. "LHC Signatures Of Scalar Dark Energy". In: *Phys. Rev. D* 94.8 (2016), p. 084054. DOI: 10.1103/PhysRevD.94.084054. arXiv: 1604.04299 [hep-ph].

[75] Morad Aaboud et al. "Constraints on mediator-based dark matter and scalar dark energy models using $\sqrt{s} = 13$ TeV pp collision data collected by the ATLAS detector". In: *JHEP* 05 (2019), p. 142. DOI: 10.1007/JHEP05(2019)142. arXiv: 1903.01400 [hep-ex].

[76] Lucas Lombriser and Andy Taylor. “Breaking a Dark Degeneracy with Gravitational Waves”. In: *JCAP* 03 (2016), p. 031. DOI: 10.1088/1475-7516/2016/03/031. arXiv: 1509.08458 [astro-ph.CO].

[77] Dario Bettoni, Jose María Ezquiaga, Kurt Hinterbichler, and Miguel Zumalacárregui. “Speed of Gravitational Waves and the Fate of Scalar-Tensor Gravity”. In: *Phys. Rev. D* 95.8 (2017), p. 084029. DOI: 10.1103/PhysRevD.95.084029. arXiv: 1608.01982 [gr-qc].

[78] Paolo Creminelli and Filippo Vernizzi. “Dark Energy after GW170817 and GRB170817A”. In: *Phys. Rev. Lett.* 119.25 (2017), p. 251302. DOI: 10.1103/PhysRevLett.119.251302. arXiv: 1710.05877 [astro-ph.CO].

[79] Jose María Ezquiaga and Miguel Zumalacárregui. “Dark Energy After GW170817: Dead Ends and the Road Ahead”. In: *Phys. Rev. Lett.* 119.25 (2017), p. 251304. DOI: 10.1103/PhysRevLett.119.251304. arXiv: 1710.05901 [astro-ph.CO].

[80] Jeremy Sakstein and Bhuvnesh Jain. “Implications of the Neutron Star Merger GW170817 for Cosmological Scalar-Tensor Theories”. In: *Phys. Rev. Lett.* 119.25 (2017), p. 251303. DOI: 10.1103/PhysRevLett.119.251303. arXiv: 1710.05893 [astro-ph.CO].

[81] M. Ostrogradsky. “Mémoires sur les équations différentielles, relatives au problème des isopérimètres”. In: *Mem. Acad. St. Petersbourg* 6.4 (1850), pp. 385–517.

[82] Richard P. Woodard. “Ostrogradsky’s theorem on Hamiltonian instability”. In: *Scholarpedia* 10.8 (2015), p. 32243. DOI: 10.4249/scholarpedia.32243. arXiv: 1506.02210 [hep-th].

[83] Tsutomu Kobayashi. “Horndeski theory and beyond: a review”. In: *Rept. Prog. Phys.* 82.8 (2019), p. 086901. DOI: 10.1088/1361-6633/ab2429. arXiv: 1901.07183 [gr-qc].

[84] Marco Crisostomi, Matthew Hull, Kazuya Koyama, and Gianmassimo Tasinato. “Horndeski: beyond, or not beyond?” In: *JCAP* 03 (2016), p. 038. DOI: 10.1088/1475-7516/2016/03/038. arXiv: 1601.04658 [hep-th].

[85] Marco Crisostomi, Kazuya Koyama, and Gianmassimo Tasinato. “Extended Scalar-Tensor Theories of Gravity”. In: *JCAP* 04 (2016), p. 044. DOI: 10.1088/1475-7516/2016/04/044. arXiv: 1602.03119 [hep-th].

[86] Hayato Motohashi, Karim Noui, Teruaki Suyama, Masahide Yamaguchi, and David Langlois. “Healthy degenerate theories with higher derivatives”. In: *JCAP* 07 (2016), p. 033. DOI: 10.1088/1475-7516/2016/07/033. arXiv: 1603.09355 [hep-th].

[87] Jibril Ben Achour et al. “Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order”. In: *JHEP* 12 (2016), p. 100. DOI: 10.1007/JHEP12(2016)100. arXiv: 1608.08135 [hep-th].

[88] Eugeny Babichev, Christos Charmousis, and Antoine Lehébel. “Asymptotically flat black holes in Horndeski theory and beyond”. In: *JCAP* 04 (2017), p. 027. DOI: 10.1088/1475-7516/2017/04/027. arXiv: 1702.01938 [gr-qc].

[89] E. Babichev and A. Lehébel. “The sound of DHOST”. In: *JCAP* 12 (2018), p. 027. DOI: 10.1088/1475-7516/2018/12/027. arXiv: 1810.09997 [gr-qc].

[90] Javier Chagoya and Gianmassimo Tasinato. “Compact objects in scalar-tensor theories after GW170817”. In: *JCAP* 08 (2018), p. 006. DOI: 10.1088/1475-7516/2018/08/006. arXiv: 1803.07476 [gr-qc].

[91] Tsutomu Kobayashi and Takashi Hiramatsu. "Relativistic stars in degenerate higher-order scalar-tensor theories after GW170817". In: *Phys. Rev. D* 97.10 (2018), p. 104012. DOI: 10.1103/PhysRevD.97.104012. arXiv: 1803.10510 [gr-qc].

[92] Masato Minamitsuji and James Edholm. "Black holes with a nonconstant kinetic term in degenerate higher-order scalar tensor theories". In: *Phys. Rev. D* 101.4 (2020), p. 044034. DOI: 10.1103/PhysRevD.101.044034. arXiv: 1912.01744 [gr-qc].

[93] Eugeny Babichev, Christos Charmousis, Gilles Esposito-Farèse, and Antoine Lehébel. "Stability of Black Holes and the Speed of Gravitational Waves within Self-Tuning Cosmological Models". In: *Phys. Rev. Lett.* 120.24 (2018), p. 241101. DOI: 10.1103/PhysRevLett.120.241101. arXiv: 1712.04398 [gr-qc].

[94] Eugeny Babichev, Christos Charmousis, Gilles Esposito-Farèse, and Antoine Lehébel. "Hamiltonian unboundedness vs stability with an application to Horndeski theory". In: *Phys. Rev. D* 98.10 (2018), p. 104050. DOI: 10.1103/PhysRevD.98.104050. arXiv: 1803.11444 [gr-qc].

[95] Christos Charmousis, Marco Crisostomi, Ruth Gregory, and Nikolaos Stergioulas. "Rotating Black Holes in Higher Order Gravity". In: *Phys. Rev. D* 100.8 (2019), p. 084020. DOI: 10.1103/PhysRevD.100.084020. arXiv: 1903.05519 [hep-th].

[96] Eugeny Babichev and Gilles Esposito-Farese. "Cosmological self-tuning and local solutions in generalized Horndeski theories". In: *Phys. Rev. D* 95.2 (2017), p. 024020. DOI: 10.1103/PhysRevD.95.024020. arXiv: 1609.09798 [gr-qc].

[97] Eugeny Babichev, Christos Charmousis, Adolfo Cisterna, and Mokhtar Hassaine. "Regular black holes via the Kerr-Schild construction in DHOST theories". In: *JCAP* 06 (2020), p. 049. DOI: 10.1088/1475-7516/2020/06/049. arXiv: 2004.00597 [hep-th].

[98] Jibril Ben Achour, David Langlois, and Karim Noui. "Degenerate higher order scalar-tensor theories beyond Horndeski and disformal transformations". In: *Phys. Rev. D* 93.12 (2016), p. 124005. DOI: 10.1103/PhysRevD.93.124005. arXiv: 1602.08398 [gr-qc].

[99] Jacob D. Bekenstein. "The Relation between physical and gravitational geometry". In: *Phys. Rev. D* 48 (1993), pp. 3641–3647. DOI: 10.1103/PhysRevD.48.3641. arXiv: gr-qc/9211017.

[100] David Langlois, Karim Noui, and Hugo Roussille. "Quadratic degenerate higher-order scalar-tensor theories revisited". In: *Phys. Rev. D* 103.8 (2021), p. 084022. DOI: 10.1103/PhysRevD.103.084022. arXiv: 2012.10218 [gr-qc].

[101] Alan R. Steif. "The Quantum stress tensor in the three-dimensional black hole". In: *Phys. Rev. D* 49 (1994), pp. 585–589. DOI: 10.1103/PhysRevD.49.R585. arXiv: gr-qc/9308032.

[102] N. D. Birrell and P. C. W. Davies. *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1982. DOI: 10.1017/CBO9780511622632.

[103] Leonard Parker and David Toms. *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2009. DOI: 10.1017/CBO9780511813924.

[104] S. J. Avis, C. J. Isham, and D. Storey. "Quantum Field Theory in anti-De Sitter Space-Time". In: *Phys. Rev. D* 18 (1978), p. 3565. DOI: 10.1103/PhysRevD.18.3565.

[105] Kiyoshi Shiraishi and Takuya Maki. “Quantum fluctuation of stress tensor and black holes in three dimensions”. In: *Phys. Rev. D* 49 (1994), pp. 5286–5294. DOI: 10.1103/PhysRevD.49.5286. arXiv: 1804.07872 [gr-qc].

[106] Kiyoshi Shiraishi and Takuya Maki. “Vacuum polarization near asymptotically anti-de Sitter black holes in odd dimensions”. In: *Class. Quant. Grav.* 11 (1994), pp. 1687–1696. DOI: 10.1088/0264-9381/11/7/009. arXiv: 1901.00977 [gr-qc].

[107] Yves Decanini and Antoine Folacci. “Off-diagonal coefficients of the Dewitt-Schwinger and Hadamard representations of the Feynman propagator”. In: *Phys. Rev. D* 73 (2006), p. 044027. DOI: 10.1103/PhysRevD.73.044027. arXiv: gr-qc/0511115.

[108] Marc Casals, Alessandro Fabbri, Cristián Martínez, and Jorge Zanelli. “Quantum-corrected rotating black holes and naked singularities in (2+1) dimensions”. In: *Phys. Rev. D* 99.10 (2019), p. 104023. DOI: 10.1103/PhysRevD.99.104023. arXiv: 1902.01583 [hep-th].

[109] Maximo Banados, Marc Henneaux, Claudio Teitelboim, and Jorge Zanelli. “Geometry of the (2+1) black hole”. In: *Phys. Rev. D* 48 (1993). [Erratum: Phys.Rev.D 88, 069902 (2013)], pp. 1506–1525. DOI: 10.1103/PhysRevD.48.1506. arXiv: gr-qc/9302012.

[110] Marc Casals, Alessandro Fabbri, Cristián Martínez, and Jorge Zanelli. “Quantum dress for a naked singularity”. In: *Phys. Lett. B* 760 (2016), pp. 244–248. DOI: 10.1016/j.physletb.2016.06.044. arXiv: 1605.06078 [hep-th].

[111] Marc Casals, Alessandro Fabbri, Cristián Martínez, and Jorge Zanelli. “Quantum Backreaction on Three-Dimensional Black Holes and Naked Singularities”. In: *Phys. Rev. Lett.* 118.13 (2017), p. 131102. DOI: 10.1103/PhysRevLett.118.131102. arXiv: 1608.05366 [gr-qc].

[112] Marc Casals, Alessandro Fabbri, Cristián Martínez, and Jorge Zanelli. “Quantum fields as Cosmic Censors in $(2+1)$-dimensions”. In: *Int. J. Mod. Phys. D* 27.11 (2018). Ed. by L. C. B. Crispino, S. R. Dolan, C. J. Fewster, G. E. A. Matsas, and D. A. T. Vanzella, p. 1843011. DOI: 10.1142/S0218271818430113.

[113] Richard L. Arnowitt, Stanley Deser, and Charles W. Misner. “The Dynamics of general relativity”. In: *Gen. Rel. Grav.* 40 (2008), pp. 1997–2027. DOI: 10.1007/s10714-008-0661-1. arXiv: gr-qc/0405109.

[114] Kazufumi Takahashi, Hayato Motohashi, and Masato Minamitsuji. “Linear stability analysis of hairy black holes in quadratic degenerate higher-order scalar-tensor theories: Odd-parity perturbations”. In: *Phys. Rev. D* 100.2 (2019), p. 024041. DOI: 10.1103/PhysRevD.100.024041. arXiv: 1904.03554 [gr-qc].

[115] Antoine Lehébel. “Compact astrophysical objects in modified gravity”. Other thesis. Oct. 2018. arXiv: 1810.04434 [gr-qc].