Torus and additive group actions on affine ind-schemes

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A thesis submitted in partial fulfillment of the requirements
for the degree of Doctor of Mathematics

Institute of Mathematics
University of Talca

November 2021

To Catherine

Acknowledgments

I'm deeply grateful to Universidad de Talca for giving me this opportunity of following my doctoral studies. I wish to extend my special thanks to the Director of this thesis, Profesor Alvaro Liendo who constantly provided me support and advice in mathematical and non mathematical issues, it was fundamental to finalize this thesis project. I would also express my gratitude to my colleagues Gonzalo Manzano and Luis Cid who were active members in sharing ideas and motivation in this opportunity of studying algebraic geometry. I also want to show my appreciation to professor Maximiliano Leyton whose lessons and guidance gave me the inspiration to create the idea of this thesis.Finally a special word of gratitude to the professor Adrien Dubolouz for his great contribution in the development and completion of some research presented to this thesis.

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Introduction

Let $\mathbf{k}$ be a closed field of characteristic 0 and let X be an affine variety on $\mathbf{k}$. By $\mathrm{Aut}(X)$ we mean to the automorphism group of X.

An important aspect in the study of $\mathrm{Aut}(X)$ is its geometric structure, it was proved that there exists a filtration $V_1 \hookrightarrow V_2 \hookrightarrow \dots$ such that, as set, $\mathrm{Aut}(X) \simeq \varinjlim V_i$, where V_i are affine varieties on $\mathbf{k}$ and $V_i \to V_{i+1}$ are closed embeddings, thus we can endow the final topology on $\mathrm{Aut}(X)$ induced by $V_1 \hookrightarrow V_2 \hookrightarrow \dots$. Roughly, we can see to $\mathrm{Aut}(X)$ like a "infinite dimensional variety" (see [**9**]).

Affine ind-variety and the **affine ind-schemes** are natural generalization of finite dimensional affine algebraic variety and affine scheme to infinite dimension. There exist two definition of this structure, one defined by Shafarevich [**22, 23**] and developed by himself, as well as other authors [**9, 16, 17, 18, 27**], see section 0.3.1, a ind-variety is a set $\mathcal{V}$ together a filtration $V_1 \hookrightarrow V_2 \hookrightarrow \dots$ where:

(1) $\mathcal{V} \simeq \varinjlim V_i$,
(2) V_i are finite dimensional varieties on $\mathbf{k}$,
(3) $V_i \to V_{i+1}$.

Ind-varieties are said to be affine (resp. projective) if V_i are affine (resp. projective). Morphisms in the category of ind-varieties ara maps $\psi\colon \varinjlim V_i \to \varinjlim W_j$ such that for all i there exists $j := j(i)$ such that $\psi|_{V_i}\colon V_i \to W_j$ is a morphism of varieties. Ind-groups $\mathcal{G}$ are ind-varieties such that the inverse and the binary operation are morphisms of ind-varieties. In Chapter 1 we present the article **"On toric ind-varieties and pro-affine semigroups"**, this article is devoted to obtain a generalization of affine toric variety to the context of affine ind-variety. More explicitly, the set

$$(\mathbb{C}^*)^\infty = \{(a_1, a_2, \dots) \mid a_i \in \mathbb{C}^* \text{ and } a_i \neq 1 \text{ for finitely many } i\}$$

with the canonical structure of ind-variety given by the filtration $\mathbb{C}^* \overset{\varphi_1}{\hookrightarrow} (\mathbb{C}^*)^2 \overset{\varphi_2}{\hookrightarrow} \dots$, where $\varphi_i(a_1, \dots, a_i) = (a_1, \dots, a_i, 1)$ for all integer $i > 0$, has a natural structure of ind-group where the group law is given by component-wise multiplication. An **ind-torus** $\mathcal{T}$ is an ind-group isomorphic to either an algebraic torus or $(\mathbb{C}^*)^\infty$ so the definition proposed generalizes affine toric variety:

Definition A. An affine toric ind-variety is a curve-connected affine ind-variety $\mathcal{V}$ having an ind-torus $\mathcal{T}$ as an open subset such that the action of $\mathcal{T}$ on itself by translations extends to a regular action of $\mathcal{T}$ on $\mathcal{V}$.

In general an irreducible ind-variety $\mathcal{V}$ does not necessarily admit an equivalent filtration $V_1 \hookrightarrow V_2 \hookrightarrow \dots$ by irreducible varieties as shown in [**3**, Remark 4.3], see also [**9**, Example 1.6.5]. But an ind-variety $\mathcal{V}$ is curve-connected if and only if there exists an equivalent filtration $V_1 \hookrightarrow V_2 \hookrightarrow \dots$ by irreducible varieties [**9**, Proposition 1.6.3]. So the property of curve-connected in Definition A instead of irreducible as the finite dimensional definition of affine toric variety allow us to obtain an equivalent description of affine toric ind-variety expressed in the follow theorem:

THEOREM B. *Let $\mathcal{V} = \varinjlim V_i$ be an ind-variety endowed with a regular action of the ind-torus $\mathcal{T}$. Then $\mathcal{V}$ is an affine toric ind-variety with respect to $\mathcal{T}$ if and only if $\mathcal{V} \simeq \varinjlim W_j$ where W_j are affine toric varieties with acting torus T_j, the closed embedding $\varphi_j \colon W_j \hookrightarrow W_{j+1}$ are toric morphisms and the ind-torus $\mathcal{T}$ is the inductive limit $\varinjlim T_j$.*

We introduce the natural dual objects to affine toric ind-varieties that we call **pro-affine semigroups**. Let $\mathcal{S}$ be a commutative semigroup with a unity and a descending filtration $R_1 \supset R_2 \supset \dots$ of $\mathcal{S}\times\mathcal{S}$ of equivalence relations on $\mathcal{S}$ that satisfy certain compatibility condition with respect to the semigroup operation generating a semigroup operation in the set of equivalence classes $\mathcal{S}/R_i$, see Section 1.3 for details. We call a semigroup $\mathcal{S}$ endowed with such a filtration a **filtered semigroup** to obtain the dual object to afine toric ind-variety presented in the follow definition

DEFINITION C. A pro-affine semigroup $\mathcal{S}$ is a filtered semigroup with filtration $R_1 \supset R_2 \supset \dots$ of compatible equivalence relations in $\mathcal{S}$ that is complete and such that $\mathcal{S}/R_i$ is an affine semigroup, for all integer $i > 0$.

Our main result respect to pro-affine semigroups is:

COROLLARY D. *An abstract semigroup $\mathcal{S}$ admits a filtration by compatible equivalence relations on $\mathcal{S}$ making $\mathcal{S}$ a pro-affine semigroup if and only if there exists an embedding $\iota \colon \mathcal{S} \hookrightarrow \mathbb{Z}^\omega$ where $\iota(\mathcal{S})$ is closed and $(\pi_i \circ \iota)(\mathcal{S})$ is finitely generated for every $i > 0$. Moreover, if such a filtration exits, then it is unique (up to equivalence).*

Finally, our principal result in chapter 1, is the equivalence of categories between pro-affine semigroups and affine toric ind-varieties, announced as follow:

THEOREM E.

(1) The assignment $\mathcal{V}(\bullet)$ is a contravariant functor from the category of pro-affine semigroups with homomorphisms of semigroups to the category of affine toric ind-varieties with toric morphisms.
(2) The assignment $\mathcal{S}(\bullet)$ is a contravariant functor from the category of affine toric ind-varieties with toric morphisms to the category of pro-affine semigroups with homomorphisms of semigroups.
(3) The pair $(\mathcal{V}(\bullet), \mathcal{S}(\bullet))$ is a duality between the categories of affine toric ind-varieties and pro-affine semigroups.

In Chapter 2 we present the submitted article **"Topologically integrable derivations and additive group action on affine ind-schemes"**. In this article we develop a generalization of the correspondence between locally nilpotent derivation and $\mathbb{G}_a$-action to the context of affine ind-scheme, we will see in Section 2.2, it will be necessary to add topological condition on the derivation for to obtain the desired generalization of the correspondence. In Chapter 2, Unlike Chapter 1, we work with a different notion of affine ind-variety, more in relation with that proposed by Kambayashi [**14**, **15**] presented in this thesis in section 0.3.2. This approximation is closer to the Grothendieck theory of ind-representable functors and formal schemes studied in [**1**, **10**].

For explanatory and expository purposes of this introduction, we will restrict ourselves to $\mathbf{k}$ a closed field of zero characteristic and to affine ind-variety, but a broader definition and affine ind-scheme will be developed in Section 2.3.

A commutative topological $\mathbf{k}$-algebra $\mathcal{A}$ is said to be an **pro-affine** $\mathbf{k}$**-algebra** if there exists a fundamental system $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \dots$ of neighborhood of 0 consisting in ideals such that $\mathcal{A} \simeq \varprojlim \mathcal{A}/\mathfrak{a}_i$ and $\{0\} = \cap_{i=0}^{\infty}\mathfrak{a}_i$. Particularly topological $\mathbf{k}$-algebra $\mathcal{A}$ is said to be **linearly topologized** if has a fundamental system $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \dots$ of neighborhood of 0 consisting in ideals. We call a continuous k-derivation ∂ of $\mathcal{A}$ **topologically integrable** if the sequence of $\mathbf{k}$-linear endomorphisms $(\partial^n)_{i\in\mathbb{N}}$ of $\mathcal{A}$ converges continuously to the zero homomorphism, that is, if for every $f \in \mathcal{A}$ and every $i \in \mathbb{N}$, there exists an indices $n_0, j \in \mathbb{N}$ such that $\partial^n(f + \mathfrak{a}_j) \subset \mathfrak{a}_i$ for every integer $n \geq n_0$. This definition coincide with the definition of locally nilpotent derivation when $\mathcal{A}$ is endowed with the discrete topology.

The geometric object proposed by Kambayashi, as alternative definition of affine ind-variety, is the set $\mathrm{Spf}(\mathcal{A})$ of open prime ideals of $\mathcal{A}$, endowed with the subspace topology inherited from the Zariski topology on the usual prime spectrum $\mathrm{Spec}(\mathcal{A})$. Morphisms between such affine ind-varieties are determined by continuous homomorphisms between the corresponding topological algebras, see Section 2.3.

With the previous definition, a simplified version of the main result of Chapter 2 is the following:

THEOREM F. *Let $\mathfrak{X} = \mathrm{Spf}(\mathcal{A})$ be the affine ind-k-scheme associated to a linearly topologized complete $\mathbf{k}$-algebra $\mathcal{A}$ which admits a fundamental system of open neighborhoods of 0 consisting of a countable family of ideals. Then there exists a one-to-one correspondence between $\mathbb{G}_{a,\mathbf{k}}$-actions on $\mathfrak{X}$ and topologically integrable $\mathbf{k}$-derivations of $\mathcal{A}$.*

An importan result of the algebraic theory of locally nilpotent derivations is the existence for every nonzero such derivation ∂ of a $\mathbf{k}$-algebra A of a so-called **local slice**, that is, an element $s \in A$ such that $\partial(s) \in \ker(\partial)$ but $\partial(s) \neq 0$. Not every nonzero topologically integrable derivation $\mathbf{k}$-derivation ∂ of a pro-affine k-algebra $\mathcal{A}$ has a local slice. But, when the topologically integrable derivations admits a local slices, the theory follow closely the finite-dimensional case. Respect to the previous coincidence with the classical case we prove the next theorem:

THEOREM G. *Let $\mathcal{A}$ be linearly topologized complete $\mathbf{k}$-algebra and let $\partial\colon \mathcal{A} \to \mathcal{A}$ be a topologically integrable derivation admitting a slice s such that $\partial(s) = 1$. Then $\mathcal{A} \cong (\ker\partial)\{s\}$ and $\exp(T\partial)$ coincides with the homomorphism of topological $(\ker\partial)$-algebras*

$$(\ker\partial)\{s\} \to (\ker\partial)\{s\}\{T\} \cong (\ker\partial)\{s, T\}, \quad s \mapsto s + T.$$

Preliminar

Set theory

Let I be a set, and $\preceq$ a **preorder relation** on I that is $\preceq$ satisfies the transitivity and the reflexivity, $\preceq$ is said to be **right directed** if for all α and β elements in I there exists γ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. Analogously is said to be **left directed** if for all α and β elements in I there exists γ such that $\gamma \preceq \alpha$ and $\gamma \preceq \beta$. Naturally, we can define a preorder on a subset $J \subset I$, J is said to be a **cofinal subset** of I if for all $\alpha \in I$ there exists an element $\beta \in J$ such that $\alpha \preceq \beta$. J is said to be a **coinitial subset** of I if for all $\alpha \in I$ there exists $\beta \in J$ such that $\beta \preceq \alpha$. Let Γ be a family of sets (or sets with additional structure) and a bijective map $E\colon I \to \Gamma$ of I a preorder set into Γ, if $\alpha \in I$ we will use E_α instead of $E(\alpha)$ for the image of α and $(E_\alpha)_{\alpha\in I}$ instead Γ when we will want to express a family of set is indexed by a preorder set.

Categories

By a **category** $\mathcal{C}$ we will refer to all the next information:

(1) A collection of object $\mathrm{obj}(\mathcal{C})$.
(2) Sets of morphisms $\mathrm{Mor}(A, B)$ defined for each pair of elements $A, B \in \mathrm{obj}(\mathcal{C})$. The elements in $\mathrm{Mor}(A, B)$ are denoted by $f\colon A \to B$
(3) A composition operator $\circ\colon \mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$. $\circ(f, g)$ is often written $g \circ f$.
Additionally it is necessary:
  (a) For any f, g, h is verified $h \circ (g \circ f) = (h \circ g) \circ f$ when the composition is defined.
  (b) For all A in $\mathrm{obj}(\mathcal{C})$ there exists a morphism I_A in $\mathrm{Mor}(A, A)$ such that for all f in $\mathrm{Mor}(A, B)$, $f = \mathrm{I}_B \circ f$ and $f = f \circ \mathrm{I}_A$.

For some authors, this definition is of a locally small category. If $\mathcal{C}$ is a category we can match the **oposite category** to $\mathcal{C}$ denoted $\mathcal{C}^{opp}$ and defined by $\mathrm{obj}(\mathcal{C}^{opp}) := \mathrm{obj}(\mathcal{C})$ but for all $A, A' \in \mathrm{obj}(\mathcal{C}^{opp})$ the set of morphism is defined in the other direction i.e. $\mathrm{Mor}_{\mathcal{C}^{opp}}(A, A') := \mathrm{Mor}_{\mathcal{C}}(A', A)$. A **covariant functor** F from a category $\mathcal{C}$ to a category $\mathcal{C}'$, written $F\colon \mathcal{C} \to \mathcal{C}'$, is the next data: A map $F\colon \mathrm{obj}(\mathcal{C}) \to \mathrm{obj}(\mathcal{C}')$, also for each pair A_1, A_2 in $\mathrm{obj}(\mathcal{C})$ a map $F\colon \mathrm{Mor}(A_1, A_2) \to \mathrm{Mor}(F(A_1), F(A_2))$ such that $F(\mathrm{I}_A) = \mathrm{I}_{F(A)}$ for all $A \in \mathrm{obj}(\mathcal{C})$ and F preserve the composition, i.e. $F(f' \circ f) = F(f') \circ F(f)$. **Contravariant functor** $\mathcal{C} \to \mathcal{C}'$ are defined as covariant functor $\mathcal{C}^{opp} \to \mathcal{C}'$. Also there exists a definition of composition of covariant functor, if $F\colon \mathcal{C} \to \mathcal{C}'$ and $F'\colon \mathcal{C}' \to \mathcal{C}''$ are covariant functors, by $F' \circ F\colon \mathcal{C} \to \mathcal{C}''$ we will refer to the covariant functor defined naturally. A **natural transformation** of covariant functors $F\colon \mathcal{C} \to \mathcal{C}'$ and $F'\colon \mathcal{C} \to \mathcal{C}'$ is a family of morphisms

$\{m(A)\colon F(A) \to F'(A)\}_{A\in \mathrm{obj}(\mathcal{C})}$

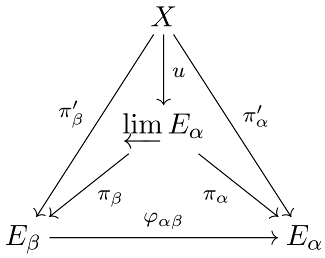
$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle m(A)}\downarrow & & \downarrow{\scriptstyle m(A')} \\ F'(A) & \xrightarrow{F'(f)} & F'(A') \end{array}$$

is commutative for all $f \in \mathrm{Mor}(A, A')$. A natural transformation of covariant functors is said to be **natural isomorphism** of covariant functors if the elements in $\{m(A)\colon F(A) \to F'(A)\}_{A\in \mathrm{obj}(\mathcal{C})}$ are isomorphisms. We will say just F and F' are naturally isomorphic if there exists a natural isomorphism of covariant functors between F and F'. Natural transformation and natural isomorphism on contravariant functor is defined analogously but with some changes in arrow directions.

Finally, two categories $\mathcal{C}$ and $\mathcal{C}'$ are **equivalent** if there exist covariant functors $F\colon \mathcal{C} \to \mathcal{C}'$ and $F'\colon \mathcal{C}' \to \mathcal{C}$ such that $F \circ F'$ is naturally isomorphic to the functor $\mathrm{I}_{\mathcal{C}'}$ and $F' \circ F$ is naturally isomorphic to the functor $\mathrm{I}_{\mathcal{C}}$. Also, two categories $\mathcal{C}$ and $\mathcal{C}'$ are **coequivalent** if $\mathcal{C}^{opp}$ and $\mathcal{C}'$ are equivalent.

0.0.1. Representable functor. Let $\mathcal{C}$ be a category and Set the category of sets. For each object $A \in \mathrm{obj}(\mathcal{C})$ we have the contravariant functor Mor_A from the category $\mathcal{C}$ to the category Set. Mor_A verifies that for all X an object in $\mathrm{obj}(\mathcal{C})$ is sent to $\mathrm{Mor}_A(X) := \mathrm{Mor}(A, X)$ and the morphism $f\colon X \to Y$ in $\mathrm{Mor}(X, Y)$ is sent to $\mathrm{Mor}_A(f) : \mathrm{Mor}_A(Y) \to \mathrm{Mor}_A(X) \in \mathrm{Mor}(\mathrm{Mor}_A(Y), \mathrm{Mor}_A(X))$ where $\mathrm{Mor}_A(f)(\phi\colon Y \to A) = \phi \circ f\colon X \to A$. A contravariant functor F from a category $\mathcal{C}$ to the category of sets Set is said to be a **representable functor** if F is naturally isomorphic to Mor_A for some $A \in \mathrm{obj}(\mathcal{C})$.

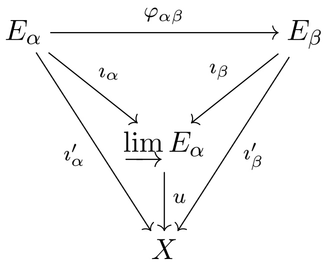
0.0.2. Projective limits. Let I be a preorder set. A **projective system** of object in a category $\mathcal{C}$ relative to the index set I is a pair $((E_\alpha)_{\alpha\in I}, (\varphi_{\alpha\beta}))$ where $(E)_{\alpha\in I}$ is a family of objects in the category $\mathcal{C}$ indexed by I and $(\varphi_{\alpha\beta})$ a collection of morphism $\varphi_{\alpha\beta}\colon E_\beta \to E_\alpha$ defined if $\alpha \preceq \beta$, such that $\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}$ when $\alpha \preceq \beta \preceq \gamma$ and $\varphi_{\alpha\alpha}$ is the identity morphism of E_α. The projective system $((E_\alpha)_{\alpha\in I}, (\varphi_{\alpha\beta}))$ will be denoted by $(E_\alpha, \varphi_{\alpha\beta})$ when the additional information are clear by the context. The **projective limit** (some authors prefer inverse limit) of a projective system $(E_\alpha, \varphi_{\alpha\beta})$ in a category $\mathcal{C}$, is an object $\varprojlim_{\alpha\in I}(E_\alpha, \varphi_{\alpha\beta})$ in the category, again when the context and the additional information are clear, the object $\varprojlim_{\alpha\in I}(E_\alpha, \varphi_{\alpha\beta})$ could be expressed by $\varprojlim_{\alpha\in I} E_\alpha$ or simply $\varprojlim E_\alpha$, together a collection of morphism $(\pi_\beta\colon \varprojlim E_\alpha \to E_\beta)_{\beta\in I}$ called the canonical mappings, satisfying $\pi_\alpha = \varphi_{\alpha\beta} \circ \pi_\beta$ when $\alpha \preceq \beta$. In addition this object and the morphism must be an universal object in the sense if there exists another object X in the category and a collection of morphism $(\pi'_\beta\colon X \to E_\beta)_{\beta\in I}$ there exists a unique morphism $u\colon X \to \varprojlim E_\alpha$ such that the diagram



is commutative.

The projective limit does not necessarily exist but, when it exists and the objects are sets (possibly with more structure) we can consider $\Pi_{\alpha\in I}E_\alpha$ the product of the family $(E_\alpha)_{\alpha\in I}$, with the respective additional structure, and the morphism $(pr_\beta\colon \Pi_{\alpha\in I}E_\alpha \to E_\beta)_{\beta\in I}$ the natural projections so we can see $\varprojlim E_\alpha$ as the subset of $x \in \Pi_{\alpha\in I}E_\alpha$ such that $pr_\alpha(x) = \varphi_{\alpha\beta}\circ pr_\beta(x)$ for all $\alpha,\beta \in I$ such that $\alpha \preceq \beta$ and $(\pi_\beta\colon \varprojlim E_\alpha \to E_\beta)_{\beta\in I}$ as the restriction of $(pr_\beta\colon \Pi_{\alpha\in I}E_\alpha \to E_\beta)_{\beta\in I}$.

0.0.3. Inductive limits. Let I be a right directed preorder set. A **direct system** of object in a category $\mathcal{C}$ relative to the index set I is a pair $((E_\alpha)_{\alpha\in I}, (\varphi_{\alpha\beta}))$ where $(E)_{\alpha\in I}$ is a family of object in the category $\mathcal{C}$ indexed by I and $(\varphi_{\alpha\beta})$ a collection of morphism $\varphi_{\alpha\beta}\colon E_\alpha \to E_\beta$ defined if $\alpha \preceq \beta$, such that $\varphi_{\alpha\gamma} = \varphi_{\beta\gamma}\circ\varphi_{\alpha\beta}$ when $\alpha \preceq \beta \preceq \gamma$ and $\varphi_{\alpha\alpha}$ is the identity morphism of E_α. The direct system $((E_\alpha)_{\alpha\in I}, (\varphi_{\alpha\beta}))$ will be denoted by $(E_\alpha, \varphi_{\alpha\beta})$ when the additional information is clear by the context. The **inductive limit** (some authors prefer direct limit) of a direct system $(E_\alpha, \varphi_{\alpha\beta})$ in a category $\mathcal{C}$, is an object $\varinjlim_{\alpha\in I}(E_\alpha, \varphi_{\alpha\beta})$ in the category, again when the context and the additional information is clear, the object $\varinjlim_{\alpha\in I}(E_\alpha, \varphi_{\alpha\beta})$ could be expressed by $\varinjlim_{\alpha\in I} E_\alpha$ or simply $\varinjlim E_\alpha$. Together a collection of morphism $(\imath_\beta\colon E_\beta \to \varinjlim E_\alpha)_{\beta\in I}$, called the canonical mappings, satisfying $\imath_\alpha = \imath_\beta\circ\varphi_{\alpha\beta}$ when $\alpha \preceq \beta$. In addition this object and the morphism must be an universal object in the sense if there exists another object X in the category and a collection of morphism $(\imath'_\beta\colon E_\beta \to X)_{\beta\in I}$ there exists a unique morphism $u\colon \varinjlim E_\alpha \to X$ such that the diagram



is commutative. The inductive limit does not necessarily exist but, when it exists and the objects are sets (possibly with more structure) we can see $\varprojlim E_\alpha$ as $\sqcup_{\alpha\in I}E_\alpha/\sim$ the disjoint union of the $(E_\alpha)_{\alpha\in I}$, with the respective additional structure, and the equivalence relation: $e_\alpha \in E_\alpha$ and $e_\beta \in E_\beta$ then $e_\alpha \sim e_\beta$ if and only if there is some $\gamma \in I$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$ and such that $\varphi_{\alpha\gamma}(e_\alpha) = \varphi_{\beta\gamma}(e_\beta)$. the morphisms are naturally defined as to send an element to its equivalence class $(\imath_\beta\colon E_\beta \to \sqcup_{\alpha\in I}E_\alpha/\sim)_{\beta\in I}$.

Example H. Two particular examples of the above construction will appear several time in the chapter 1 of this thesis. Recall that $\mathbb{Z}^\omega$ is the group of arbitrary sequences of integer numbers. This group is also called the Baer-Specker group [**26**]. A sequence a in $\mathbb{Z}^\omega$ is denoted by $a = (a_1, a_2, \dots)$. Equivalently, $\mathbb{Z}^\omega$ is the projective limit of the system $\mathbb{Z}^1 \leftarrow \mathbb{Z}^2 \leftarrow \dots$, where the morphisms $\varphi_i \colon \mathbb{Z}^{i+1} \to \mathbb{Z}^i$ are the projections forgetting the last coordinate and the projection $\pi_i \colon \mathbb{Z}^\omega \to \mathbb{Z}^i$ is forget are coordinates after of coordinate i. Furthermore, the subgroup of $\mathbb{Z}^\omega$ of eventually zero sequences is denoted by $\mathbb{Z}^\infty$, so $a \in \mathbb{Z}^\infty$ is such that $a_i = 0$ except for finitely many positive integers i. Equivalently, $\mathbb{Z}^\infty$ is the inductive limit of the system $\mathbb{Z}^1 \to \mathbb{Z}^2 \to \dots$, where the maps are the injections setting the last coordinate to 0. If we take any inductive or projective subsystem of the system defining $\mathbb{Z}^\infty$ or $\mathbb{Z}^\omega$, respectively with the obvious morphisms given by compositions, then the limits are canonically isomorphic to $\mathbb{Z}^\infty$ or $\mathbb{Z}^\omega$, respectively.

More generally, a projective or inductive system is called split if every morphism in the system admits a section. It is a straightforward computation to show that for any split projective system $\mathbb{Z}^{n_1} \leftarrow \mathbb{Z}^{n_2} \leftarrow \dots$, with a strictly increasing sequence $n_1 < n_2 < \dots$ of positive integers, the limit is isomorphic to $\mathbb{Z}^\omega$. Similarly, for any split inductive system $\mathbb{Z}^{n_1} \to \mathbb{Z}^{n_2} \to \dots$, with a strictly increasing sequence $n_1 < n_2 < \dots$ of positive integers, the limit is isomorphic to $\mathbb{Z}^\infty$

0.1. Algebraic geometry summary

The main object of study in this thesis was introduced by Shafarevich, so in this section we summarize the main definitions of algebraic geometry in terms of the ideas proposed by the same author [**24, 25**] but other definition here can be studied in [**11**].

By a **ringed space** we mean to a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ a sheave of ring on X. A morphism between ringed space is a pair $(\varphi, \varphi^\#) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ verifying $\varphi \colon X \to Y$ is a continuous map and $\varphi^\# : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ the map of sheaves on Y induced by $\varphi \colon X \to Y$. A ringed space $(X, \mathcal{O}_X)$ is a **locally ringed space** if for all $x \in X$ the rings $\mathcal{O}_{X,x}$ are local rings. A morphism of locally ringed spaces is a morphism of ringed spaces $(\varphi, \varphi^\#) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that for each $x \in X$ the map induced $\varphi^\#_x \colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local ring i.e $\varphi^\#_x(\mathfrak{m}_{Y,f(x)}) \subset \mathfrak{m}_{X,x}$.

By a **scheme** we mean to a ringed space $(X, \mathcal{O}_X)$ such that X has a covering of open $\{U_i\}_{i \in I}$ such that $(U, \mathcal{O}_{X|U})$ is isomorphic with $(\mathrm{Spec}(A), \mathcal{O})$ where $\mathcal{O}(\mathrm{Spec}(A)) \simeq A$ for some ring A. Morphisms between schemes $(\varphi, \varphi^\#) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ are the morphisms corresponding to the local ringed space. Commonly we will only refer to scheme X instead of $(X, \mathcal{O}_X)$ and a morphism of schemes by $\varphi \colon X \to Y$ instead $(\varphi, \varphi^\#) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. By affine scheme we mean to X such that $X = \mathrm{Spec}(A)$ for some ring $A \simeq \mathcal{O}_X(X)$. If we fix an scheme S, by a **Scheme on** S we mean to a scheme X together a morphism of scheme $\varphi \colon X \to S$, commonly if $S = \mathrm{Spec}(A)$ for some ring A we say a scheme on A instead a scheme on $\mathrm{Spec}(A)$. Particularly all the rings are $\mathbb{Z}$-algebras so we can see any scheme X as a scheme on $\mathbb{Z}$.

0.1.0.1. *Some properties of schemes.* Let X be a scheme, X is said to be **reduced** if for all open $U \subset X$ the rings $\mathcal{O}_X(U)$ are reduced i.e $\mathcal{O}_X(U)$ has no nilpotent elements.

Definition A. A morphism of schemes $\varphi\colon Y \to X$ is a **closed embedding** if every point $x \in X$ has an affine neighbourhood U such that $\varphi^{-1}(U) \subset Y$ is an affine subscheme and the homomorphism $\varphi_U^{\#}\colon \mathcal{O}_X(U) \to \mathcal{O}_Y(\varphi^{-1}(U))$ is surjective.

Recall a scheme X is said to be **separated** if the diagonal morphism $\Delta := (\mathrm{id},\mathrm{id})\colon X \to X \times_{\mathrm{Spec}(\mathbb{Z})} X$ is a closed embedding. Finally if X is a scheme on a ring B, X is said of **finite type** on B if there exists a finite covering $\{U_i\}_{i\in I}$ of X such that $U_i = \mathrm{Spec}(A_i)$ and A_i are of finite type over B, equivalently, A_i are B-algebras finitely generated as B-algebra.

Definition B. A **variety** over an algebraically closed field $\mathbf{k}$ is a reduced separated scheme of finite type over $\mathbf{k}$,

Morphisms of varieties are morphisms of schemes.
other authors define the varieties similarly but with the additional property of irreducibility [**11**].

0.2. Linear topology on Groups, Rings and Modules

In this section I make a summary about the principal definitions and results in the literature relative to linear topology on groups, rings and modules. Principally the reference to these contents is [**21**, Chapter 9]

Definition A. A **topological group** is a group G with a topology such that the mappings $\mathrm{M}_1\colon G \times G \to G$ defined as $(g, g') \mapsto g + g'$ and $\mathrm{M}_2\colon G \to G$ defined as $g \mapsto -g$ are continuous, where $G \times G$ is endowed with the product topology.

The continuity of M_1 and M_2 is equivalent with the continuity of the map $\mathrm{M}_3\colon G \times G \to G$ defined as $(g, g') \mapsto g - g'$. In addition, we can conclude, for all $g' \in G$ the continuity of the mappings $\mathrm{M}_{g'+}\colon G \to G$ defined $g \mapsto g' + g$ and $\mathrm{M}_{g'-}\colon G \to G$ defined $g \mapsto g' - g$. Naturally the morphisms between topological groups are defined as continuous homomorphisms.
If we endow of group structure and a topology to a set G, the group structure and a the topology are said to be **compatible** if the mappings in the previous definition are continuous.

Example B. The discrete topology and the group structure of any group G are compatible.

Example C. The additive group structure and the Zariski topology on $\mathbb{Z}$, are not compatible since that the preimage of $\{0\}$ in $\mathbb{Z} \times \mathbb{Z}$, the set $\{(g, -g) \in \mathbb{Z} \times \mathbb{Z}\}$, is not a closed set, because any (h, h') in the complement of $\{(g, -g) \in \mathbb{Z} \times \mathbb{Z}\}$ and neighborhood U of (h, h') intersected with $\{(g, -g) \in \mathbb{Z} \times \mathbb{Z}\}$ is a set no empty.

Through this thesis, we will be interested in a special class of topological group G with a necessary property when we study geometric objects matched with G.

Definition D. Let $X := (X, \mathfrak{T})$ be a topological space, a **fundamental system** of neighborhoods of a set $A \subset X$ is any $\mathfrak{S} \subset \mathfrak{T}$ such that for each neighborhood V of A there is a neighborhood $W \in \mathfrak{S}$ such that $W \subset V$.

When $A = \{x\} \subset X$ we are going to refer to a fundamental system of a point x. So, from now on,

ours topological group always will be assumed with a fundamental system of neighborhoods $\mathfrak{S}$, of the neutral element 0 of G, consisting in a countable family of subgroups of G.

This assumption gives us the possibility to obtain a fundamental system $(H_n)_{n\in\mathbb{N}}$ indexed by the set $\mathbb{N}$ of non-negative integers with the property $H_n \subset H_m$ when $m \leq n$ and $H_0 = G$. The topologies verifying the previous condition will be called **linear topology** and we will refer to $\mathfrak{S}$ briefly as the fundamental system of G.

Example E. Any group G endowed with the discrete topology is a topological group with linear topology, where the fundamental system is the family $\mathfrak{S} = \{\{0\}\}$.

Example F. $(\mathbb{R}, +)$ with de euclidean topology and the fundamental system the family of intervals $\mathfrak{S} = \{(-\frac{1}{n}, \frac{1}{n}) \mid n \in \mathbb{N}\}$.

Proposition G. *Let G be a topological group with fundamental system $\mathfrak{S}$ then the topology is the family of set V verifying the condition "If an element $g \in G$ is an element in V, there exists $H \in \mathfrak{S}$ such that the set $g + H$ is contained in V"*

Proof. Let V be an open of G and g an element of V, 0 is an element in $-g+V$ so there exists $H \in \mathfrak{S}$ such that $H \subset -g+V$ then $g+H \subset V$. Also the sets $g+H$ are open because is the preimage of H under continuous map $\mathrm{M}_{\text{(-g)+}}$ thus the set V with the condition "If an element $g \in G$ is an element in V, there exists $H \in \mathfrak{S}$ such that the set $g+H$ is contained in V" are open. □

Easily with the proof in the previous proposition we can conclude the family $\mathfrak{B} = \{g + H \mid g \in G \text{ and } H \in \mathfrak{S}\}$ is a basis of the topology of G and another fundamental system gives us the same topology. Also is posible to conclude that our topological group are first-countable topological space.

Remark H. If $H < G$ is an open subgroup the group G/H endowed with the discrete topology is isomorphic as topological group to G/H with the quotient topology defined by the relation $g \sim g'$ if $g - g' \in H$.

More general, for all subgroup H_α of G the subspace topology on H_α is compatible with the operation preserved of G, so the subgroup will be consider as topological group with subspace topology also the quotient G/H_α will be endowed with the quotient topology and the quotient map will be denoted $P_\alpha\colon G \to G/H_\alpha$.

Lemma I. *Let G be a topological group and $H_\alpha < H_\beta < G$ subgroups. The natural map $P_{\alpha\beta}\colon G/H_\beta \to G/H_\alpha$ is continuous.*

Proof. The continuity of $P_{\alpha\beta}$ is deduced of the equality $P_\alpha = P_{\alpha\beta} \circ P_\beta$ and the definition of the quotient map. □

0.2.1. Complete topological group. Let G be a topological group with fundamental system $(H_\alpha)_{\alpha\in J}$, the property of being separated as topological space is equivalent with the intersection of all open groups consist of the set $\{0\}$ and equivalent with the intersection of open groups in the fundamental system consist of the set $\{0\}$. If $(H_\alpha)_{\alpha\in I}$ is the family of open subgroups of G where I is a preordered set and the preorder is given by $\alpha \preceq \beta$ if $H_\beta \subset H_\alpha$. We can define the projective system $(G/H_\alpha, P_{\alpha\beta})$ of topological group and the respective projective limit the separated topological group $\widehat{G} := \varprojlim_{\alpha\in I} G/H_\alpha$ with the structure induced by $\Pi_{\alpha\in I} G/H_\alpha$

together the continuous homomorphism $\pi_\alpha\colon \widehat{G} \to H_\alpha$. In our particular case, with the assumption of a countable fundamental system, $\widehat{G}$ is a topological group in our sense because for all $\alpha \in I$ the topological groups G/H_α have the discrete topology then $(\operatorname{Ker}\pi_\alpha)_{\alpha\in I}$ is a fundamental system of $\widehat{G}$. By $c\colon G \to \varprojlim_{\alpha\in I} G/H_\alpha$ we will refer to the map induced by the morphisms $P_\alpha\colon G \to G/H_\alpha$, $\alpha \in I$ this map is a homomorphism of topological groups whose image is a dense subgroup of $\widehat{G}$ and whose kernel is equal to the closure of $\{0\}$ in G. Furthermore, the induced morphism of topological groups $c\colon G \to c(G)$ is open [4, III.7.3 Proposition 2].

Proposition J. *Let G be a topological group with fundamental system $(H_\alpha)_{\alpha\in J}$ and $(H_\alpha)_{\alpha\in I}$ the family of open subgroups of G then it is verified that $\widehat{G}$ is a separated topological group isomorphic as topological group to $\varprojlim_{\alpha\in J} G/H_\alpha$ endowed with the inverse topology.*

Proof. $\widehat{G}$ is separated because $\cap_{\alpha\in I}\operatorname{Ker}\pi_\alpha = \{0\}$. By the definition of fundamental system of neighborhood, the family J is a cofinal subset of direct set I so the canonical mapping $\widehat{G} \to \varprojlim_{\alpha\in J} G/H_\alpha$ defined by $x \mapsto (\pi_\alpha(x))_{\alpha\in J}$ is a bijective map by [4, III.7.2, Proposition 3] and by [5, III.7.2] is a continuous homomorphism, easily we can see the inverse of a bijective homomorphism is homomorphism. In addition I is a right directed set so by [4, I.4.4] the map is an homeomorphism. □

Proposition K. *With the notation in this section, the continuous homomorphism $\pi_\alpha\colon \widehat{G} \to G/H_\alpha$ are surjective.*

Proof. By the existence of a countable fundamental system and the morphisms $P_{\alpha\beta}$ are surjective, we can use directly the corollary I of Mittag-Leffler theorem [4, II.3.5 Corollary I]. □

The topological group $\widehat{G}$ will be called the **separated completion** of the topological group G and we will say that G is complete if the continuous homomorphism $c\colon G \to \widehat{G}$ is an isomorphism of topological group.

Proposition L. *Let G and G' be topological groups with respective separated completions $c\colon G \to \widehat{G}$ and $c'\colon G' \to \widehat{G}'$. Then for every homomorphism of topological groups $h\colon G \to G'$ there exists a unique homomorphism of topological groups $\widehat{h}\colon \widehat{G} \to \widehat{G}'$ such that $c' \circ h = \widehat{h} \circ c$.*

Conversely, every homomorphism of topological groups $\widehat{h}\colon \widehat{G} \to \widehat{G}'$ is uniquely determined by its "restriction" $\widehat{h} \circ c\colon G \to \widehat{G}'$ to G.

Proof. The first assertion is an immediate consequence of the universal property of the separated completion homomorphism $c\colon G \to \widehat{G}$ [4, III.3.4 Proposition 8], which says that given a homomorphism of topological groups $f\colon G \to G''$ where G'' is complete, there exists a unique homomorphism of topological groups $\widehat{f}\colon \widehat{G} \to G''$ such that $f = \widehat{f} \circ c$. The second assertion follows from the fact that the image of the separated completion homomorphism $c\colon G \to \widehat{G}$ is dense. □

Lemma M. *Let $(G_n)_{n\in\mathbb{N}}$ be an inverse system of complete topological groups with surjective transition homomorphisms $p_{m,n}\colon G_m \to G_n$ for every $m \geq n \geq 0$ and $\mathfrak{S}_n$ the respective fundamental systems. Then the inverse limit $\mathcal{G} = \varprojlim_{n\in\mathbb{N}} G_n$ endowed* *with the inverse limit topology is a complete topological group and each canonical projection* $\widehat{p}_n\colon \mathcal{G} \to G_n$ *is a surjective homomorphism of topological groups.*

Proof. The fact that $\mathcal{G}$ endowed with the inverse limit topology is a linearly topologized abelian group with fundamental system $\cup_{n\in\mathbb{N}}\{\widehat{P}_n^{-1}(H) \mid H \in \mathfrak{S}_n\}$ and the fact that the canonical projections $\widehat{p}_n\colon \mathcal{G} \to G_n$ are continous homomorphisms are clear. The surjectivity of $\widehat{p}_n$ follows again from Mittag-Leffler theorem [4, II.3.5 Corollary I]. Finally, since each G_n is complete, it follows from [4, II.3.5 Corollary to Proposition 10] that $\mathcal{G}$ is complete. □

0.2.2. Recollection on topological rings. A topological ring $\mathcal{A}$ is a topological group with a multiplicative structure such that multiplication map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous. Like the topological group, from now, we will assume the topological ring $\mathcal{A}$ endowed with a linear topology where the fundamental system of neighborhoods of 0 consist in a countable family $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ of ideals of $\mathcal{A}$. Again, we will be able to assume a fundamental system such that $\mathfrak{a}_n \subset \mathfrak{a}_m$ when $m \leq n$ and $\mathfrak{a}_0 = \mathcal{A}$ the continuous homomorphism between topological ring will be refer as homomorphism of topological ring. A topological ring is said to be complete if is complete in the sense of topological group. Below, I present a summary of basic results and definitions contained in [15] relating to $\mathcal{A}$ a separated and complete topological ring.

Proposition N. *The group of units* $U(\mathcal{A})$ *of* $\mathcal{A}$ *is topologically closed.*

Proposition O. *Let* $\mathfrak{b}$ *be a closed ideal in* $\mathcal{A}$ *and* $A_i := \mathcal{A}/\mathfrak{a_i}$*, then*

$$\mathcal{A}/\mathfrak{b} \simeq \varprojlim (A_i)/\pi_i(\mathfrak{b})$$

Proposition P. *For any maximal ideal* $\mathfrak{m} \subset \mathcal{A}$*, the following conditions are equivalent:*

(1) $\mathfrak{m}$ *is closed;*
(2) For some i*,* $\pi_i(\mathfrak{m}) \subsetneq A_i$*;*
(3) For some i*,* $\mathfrak{a}_i \subset \mathfrak{m}$*;*
(4) For some i*,* $\mathfrak{m} = \pi_i^{-1}(m)$ *for some maximal ideal in* $\mathcal{A}$*;*
(5) $\mathfrak{m}$ *is open.*

Proposition Q. *For any prime ideal* $\mathfrak{p} \subset \mathcal{A}$*, the following conditions are equivalent:*

(1) $\mathfrak{p}$ *is open;*
(2) For some i*,* $\mathfrak{p} = \pi_i^{-1}(\pi_i(\mathfrak{p}))$*;*
(3) For some j *and a prime ideal* $\mathfrak{q}$*,* $\mathfrak{p} = \pi_j^{-1}(\pi_j(\mathfrak{q}))$*.*

In accordance with the definitions of Kambayashi [15], let $\mathcal{A}$ be a topological ring, the **formal nilradical** of $\mathcal{A}$ is the ideal $\mathfrak{N}(\mathcal{A}) := \bigcap_{\mathfrak{p}\subset\mathcal{A}} \mathfrak{p}$, with the $\mathfrak{p}$'s range the open prime ideal of $\mathcal{A}$. The **formal radical** of $\mathcal{A}$ is the ideal $\mathfrak{R}(\mathcal{A}) := \bigcap_{\mathfrak{m}\subset\mathcal{A}} \mathfrak{m}$, with the $\mathfrak{m}$'s ranges the open maximal ideal of $\mathcal{A}$. Finally for all $\mathfrak{a}$ ideal of $\mathcal{A}$, the formal radical of $\mathfrak{a}$ is defined by $\sqrt{\mathfrak{a}} := \bigcap_{\mathfrak{a}\subset\mathfrak{p}} \mathfrak{p}$ with $\mathfrak{p}$'s again ranging the open prime ideal of $\mathcal{A}$. The previous definition let us make two type of reduction of $\mathcal{A}$, basically $\mathcal{A}_{red} := \mathcal{A}/\mathfrak{N}(\mathcal{A})$ and $\mathcal{A}_{RED} := \varprojlim((A_i)_{red})$[14, 15] so $\mathcal{A}$ is said to be **reduced** if $\mathcal{A} = \mathcal{A}_{red}$ and **strongly reduced** if $\mathcal{A} = \mathcal{A}_{RED}$.

0.2.2.1. *Localization.* Let f be an element in $\mathcal{A}$ a topological ring, f is said to be **topologically nilpotent** if $\lim_{N\to\infty} f^N = 0$. By a multiplicative set S in $\mathcal{A}$ a topological ring we mean to a $S \subset \mathcal{A}$ such that $1 \in S$ and $0 \notin \overline{S}$ so $S^{-1}\mathcal{A}$ is the typical normalization and by $\mathcal{A}_S$ we mean to $\varprojlim S_i^{-1} A_i$ endowed with the topology co-induced by the map $\mathfrak{i}_S \colon \mathcal{A} \to \mathcal{A}_S$ i.e. the linear system of $\mathcal{A}_S$ is defined by $\{\langle\varphi(\mathfrak{a}_i)\rangle\}_{i\in\mathbb{N}}$ where $\langle\varphi(\mathfrak{a}_i)\rangle$ is the ideal generated by $\varphi(\mathfrak{a}_i)$. Since $\varprojlim S_i^{-1} A_i \simeq \varprojlim \overline{S}_i^{-1} A_i$ we will assume S is a closed set. Principal localizations are to consider $S = \mathcal{A} - \mathfrak{p}$ for some open prime ideal $\mathfrak{p}$ or $S = \{f^n \mid n \text{ non negative integer}\}$ for some $f \in \mathcal{A}$ non topologically nilpotent. These localization are denoted $\mathcal{A}_\mathfrak{p}$ and $\mathcal{A}_f$ respectivelly.

Proposition R. *Let $\varphi \colon \mathcal{A} \to \mathcal{B}$ be a homomorphism of topological ring, if $\varphi(f) = g$ then the ring homomorphism $\varphi_f : \mathcal{A}_f \to \mathcal{B}_g$ induced by φ is continuous.*

Proof. Let $\langle\mathfrak{i}_g(\mathfrak{b})\rangle$ basic open neighborhood of 0 in $\mathcal{B}_g$ where $\mathfrak{i}_g \colon \mathcal{B} \to \mathcal{B}_g$. There exists $\mathfrak{a}$ basic open neighborhood of 0 in $\mathcal{A}$ such that $\varphi(\mathfrak{a}) \subset \mathfrak{b}$. We use the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\ {\scriptstyle \mathfrak{i}_f}\downarrow & & \downarrow{\scriptstyle \mathfrak{i}_g} \\ \mathcal{A}_f & \xrightarrow[\varphi_f]{} & \mathcal{B}_g \end{array}$$

to conclude $\varphi_f(\langle\mathfrak{i}_f(\mathfrak{a})\rangle) \subset \langle\mathfrak{i}_g(\mathfrak{b})\rangle$

□

0.2.2.2. *Topological modules.* An $\mathcal{A}$-module M over a topological ring $\mathcal{A}$ is called **topological $\mathcal{A}$-module** if M is a topological group and the scalar multiplication $\mathcal{A} \times M \to M$ is continuous, obviously $\mathcal{A} \times M$ is endowed with product topology. the topological $\mathcal{A}$-modules M will be assumed with a linear topology with fundamental system $(M_n)_{n\in\mathbb{N}}$ of topological sub $\mathcal{A}$-modules with the property $M_n \subset M_m$ when $m \leq n$ and $M_0 = M$. Continuous homomorphisms between topological $\mathcal{A}$-modules will be called homomorphism of topological $\mathcal{A}$-modules.

0.2.2.3. *Topological Algebras.* A R-algebra $\mathcal{A}$ over a topological ring R, is called a **topological R-algebra** if $\mathcal{A}$ is topological ring and the scalar multiplication $R \times \mathcal{A} \to \mathcal{A}$ is continuous. When the context is clear we will omite the ring R and just we will say a topological algebra instead of topological R-algebra. The continuos R-homomorphism between topological algebras will be called homomorphisms of topological algebras. By the continuity of homomorphisms $\varphi \colon \mathcal{A} \to \mathcal{B}$ between the topological algebras $\mathcal{A}$ and $\mathcal{B}$, For all $\mathfrak{b}_m$ there exists $\mathfrak{a}_n$ such that $\varphi(\mathfrak{a}_n) \subset \mathfrak{b}_m$ and the next diagram is commutative:

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{A}/\mathfrak{a}_n & \longrightarrow & \mathcal{B}/\mathfrak{b}_m \end{array}$$

Special case is when R is a Field $\mathbf{k}$ endowed with the discrete topology. A topological $\mathbf{k}$-algebras $\mathcal{A}$ is said to be **pro-affine** if is complete and separated.

If $\mathcal{A}$ and $\mathcal{B}$ are pro-affine **k**-algebras with fundamental system of neighborhoods of 0 $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ and $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ respectively, the product $\mathcal{A}\times\mathcal{B}$ is pro-affine **k**-algebra with fundamental system of neighborhoods of 0 $\{\mathfrak{a}_n\times\mathfrak{b}_n \mid n\in\mathbb{N}\}$ and the complete tensor product $\mathcal{A}\widehat{\otimes}_{\mathbf{k}}\mathcal{B} := \widehat{\mathcal{A}\otimes_{\mathbf{k}}\mathcal{B}}$ also is pro-affine with fundamental system $\{\mathfrak{a}_n\otimes_{\mathbf{k}}\mathcal{B} + A\otimes_{\mathbf{k}}\mathfrak{b}_n \mid n\in\mathbb{N}\}$ of $\mathcal{A}\otimes_{\mathbf{k}}\mathcal{B}$ [15].

Theorem S. *Let $\mathcal{A}$ be a pro-affine* **k**-*algebra. For the canonical map $\rho\colon\mathcal{A}\to\mathcal{A}_{RED}$ the following are verified:*

(1) $\ker(\rho)=\mathfrak{N}(\mathcal{A})$;
(2) The sequence $0\to\mathfrak{N}(\mathcal{A})\to\mathcal{A}\to\mathcal{A}_{RED}$ is exact and $\mathrm{im}(\rho)$ *is dense in $\mathcal{A}_{RED}$;*
(3) $\mathfrak{N}(\mathcal{A})=\{f\in\mathcal{A}\mid \lim_{N\to\infty}f^N=0\}$ *the set of topologically nilpotent elements of $\mathcal{A}$.*

A proof of previous theorem is proposed in [14, Proposition 1.2]. Pro-affine **k**-algebra is said to be **algebraic** over **k** or **k**-algebraic if we can find a fundamental system of neighborhoods of 0 consist in a countable family $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ of ideals of $\mathcal{A}$ such that $\mathcal{A}/\alpha_i$ are finitely generated over **k**.

Theorem T (Nullstellensatz.). *If a pro-affine* **k**-*algebra $\mathcal{A}$ is algebraic over* **k**, *then $\mathfrak{R}(\mathcal{A})=\mathfrak{N}(\mathcal{A})$.*

0.3. Ind-Geometry

A motivation to study the Ind-geometry is the Jacobian conjeture. Basically the Jacobian conjeture proposes:

Let **k** *be a field, if the characteristic of* **k** *is zero, then every algebraic endomorphism P of an affine space $\mathbb{A}^n$ over* **k**, *whose Jacobian is a non-zero constant, is an automorphism.*

So, a way to understand this conjeture, it is to study $\mathrm{Aut}(\mathbb{A}^n)$ the automorphisms group of affine space n dimensional. Shafarevich [22, 23] noticed that it is possible to give a geometry structure to $\mathrm{Aut}(\mathbb{A}^n)$ and this structure could be interpreted as a natural generalization of algebraic group to infinite dimensional algebraic group. Kambayashi [13] noted some details in the theory proposed by Shafarevich and gives a different way more general to approximate infinite dimensional algebraic varieties [14, 15]. In this section I make a brief summary of both theories and the main results in the literature used in the articles in this thesis presented.

0.3.1. Shafarevich's Ind-varieties.

For the study of the structure proposed by Shafarevich I propose the lecture of [17, Chapter 4] also [9, 27], those works have a detailed review of the concepts in current terminology also important results and examples used in the development of this thesis.

Definition A. An **ind-variety** over a field **k** is the inductive limit $\mathcal{V}=\varinjlim V_i$ of the direct system induced by a filtration $V_1\hookrightarrow V_2\hookrightarrow\dots$ such that each V_n is a finite-dimensional algebraic variety over **k** and each inclusion $V_i\hookrightarrow V_{i+1}$ are closed embeddings.

And ind-variety is said to be **affine** if each V_i is affine and **projective** if each V_i is projective. The ind-variety has a structure of topological space with the inductive topology, that is: For all $i\in\mathbb{N}$ $\imath_i\colon V_i\to\mathcal{V}$ are the canonical mappings, a subset $U\subset\mathcal{V}$ is an open if $\imath_i(U)^{-1}$ is an open in V_i for all $i\in\mathbb{N}$. Naturally we can identify each V_i as a subset of $\mathcal{V}$ so we can see $V_1 \subset V_2 \subset \cdots \subset \mathcal{V}$ and for the above we will prefer to use $\mathcal{V} = \cup_{i\in\mathbb{N}} V_i$ and a subset $U \subset \mathcal{V}$ an open in $\mathcal{V}$ if $U \cap V_i$ is open in each V_i, this topology will be called the **Ind-topology**. A morphism between ind-varieties $\mathcal{V}$ and $\mathcal{V}'$ with filtrations $\{V_i\}_{i\in\mathbb{N}}$ and $\{V'_j\}_{j\in\mathbb{N}}$ respectively, is a map $\varphi\colon \mathcal{V} \to \mathcal{V}'$ satisfying that for every $i \in \mathbb{N}$ there exists a $j \in \mathbb{N}$ such that $\varphi(V_i) \subset V'_j$ and $\varphi|_{V_i}\colon V_i \to V'_j$ is a morphism of algebraic varieties. A morphism φ of ind-varieties is an isomorphism if φ is bijective and φ^{-1} is a morphism of ind-varieties. Furthermore, two filtrations $V_1 \hookrightarrow V_2 \hookrightarrow \ldots$ and $W_1 \hookrightarrow W_2 \hookrightarrow \ldots$ on the same underlying set $\mathcal{V}$ are equivalent if the identity map is a isomorphism of ind-varieties. If we take any subfiltration of the filtration $V_1 \hookrightarrow V_2 \hookrightarrow \ldots$, the ind-varieties obtained by both filtrations are equivalent.

Definition B. An ind-group is an ind-variety $\mathcal{G}$ endowed with a group structure such that the inversion and multiplication maps are morphisms of ind-varieties.

Example C. If V is an algebraic variety, V has natural structure of Ind-variety with filtration $V_i = V$ for all $i \in \mathbb{N}$ and closed embedding the identity morphism.

Example D. Let $\mathcal{V}$ and $\mathcal{W}$ be ind-varieties with filtration $V_1 \subset V_2 \subset \ldots$ and $W_1 \subset W_2 \subset \ldots$ respectively $\mathcal{V} \times \mathcal{W}$ has structure of ind-variety with filtration $V_1 \times W_1 \subset V_2 \times W_2 \subset \ldots$ where $V_i \times W_i$ is the product variety induced by V_i and W_i.

Example E.

(1) The infinite-dimensional vector space

$$\mathbb{C}^\infty := \{(a_1, \dots) \mid a_i \in \mathbb{C} \text{ and } a_i \neq 0 \text{ for finitely many } i\}$$

has a canonical structure of ind-variety given by the filtration $\mathbb{C} \overset{\varphi_1}{\hookrightarrow} \mathbb{C}^2 \overset{\varphi_2}{\hookrightarrow} \mathbb{C}^3 \overset{\varphi_3}{\hookrightarrow} \ldots$ where $\varphi_n(a_1, \dots, a_i) = (a_1, \dots, a_i, 0)$, for all $i > 0$. This ind-variety is called the infinite-dimensional affine space. Remark that we can change the complex number 0 in the filtration definition of $\mathbb{C}^\infty$ and in $(i+1)$-th coordinate of φ_i by any other number. The ind-variety obtained this way is easily seen to be isomorphic to $\mathbb{C}^\infty$. For instance, we denote by $\mathbb{C}^\infty_1$ the ind-variety isomorphic to the infinite-dimensional affine space given by $\mathbb{C}^\infty_1 := \{(a_1, \dots) \mid a_i \in \mathbb{C} \text{ and } a_i \neq 1 \text{ for finitely many } i\}$.

(2) The set

$$(\mathbb{C}^*)^\infty = \{(a_1, a_2, \dots) \mid a_i \in \mathbb{C}^* \text{ and } a_i \neq 1 \text{ for finitely many } i\}$$

has a canonical structure of ind-group given by the filtration $\mathbb{C}^* \overset{\varphi_1}{\hookrightarrow} (\mathbb{C}^*)^2 \overset{\varphi_2}{\hookrightarrow} (\mathbb{C}^*)^3 \overset{\varphi_3}{\hookrightarrow} \ldots$, where $\varphi_i(a_1, \dots, a_i) = (a_1, \dots, a_i, 1)$ for all $i > 0$. This ind-variety is an open set in the infinite-dimensional affine space. This follows straightforward from the isomorphism $\mathbb{C}^\infty \simeq \mathbb{C}^\infty_1$ above. Remark that $(\mathbb{C}^*)^\infty$ has a natural structure of ind-group given by component-wise multiplication.

Example F. Let V be an affine algebraic variety then $\operatorname{Aut}(V)$ has structure of affine ind-group [9, Theorem 5.1.1]

If we endow $\mathbb{A}^1_{\mathbf{k}}$ with structure of ind-variety, the morphisms $\mathcal{V} \to \mathbb{A}^1_{\mathbf{k}}$ are the elements in $\mathbf{k}[\mathcal{V}] := \varprojlim \mathbf{k}[V_i]$. This ring is called the ring of regular function on $\mathcal{V}$ and the element in $\mathbf{k}[\mathcal{V}]$ are the regular map on $\mathcal{V}$. Additionally $\mathbf{k}[\mathcal{V}]$ is endowed with the projective topology induced by $\mathbf{k}[V_i]$ with the discrete topology. Easily we can see, since the definition of morphism between ind-varieties, a morphism $\varphi\colon \mathcal{V} \to \mathcal{W}$ induces a continuous $\mathbf{k}$-homomorphism $\varphi^*\colon \mathbf{k}[\mathcal{W}] \to \mathbf{k}[\mathcal{V}]$ so we can identify each point v in an ind-variety $\mathcal{V}$ with the continuous $\mathbf{k}$-homomorphism $\mathbf{k}[\mathcal{V}] \to \mathbf{k}$ defined by $f \mapsto f(v)$.

0.3.2. Kambayashi's affine ind-scheme. The study of Kambayashi gives us a notion of Ind-variety more in relation with the modern idea of affine scheme, Kambayashi defined the affine ind-scheme also the affine ind-variety through of a special ring and making a special ringed space. Kambayashi in [**15**] showed a detailed construction of this object and interesting example about this theory. In this section I make a summary and a generalization about the principal concept and results.

Definition G. Let $\mathcal{A}$ be a topological ring, the **formal spectrum** and the **formal maximal spectrum** of $\mathcal{A}$ are the topological space $\mathrm{Spf}(\mathcal{A}) = \{\mathfrak{p} \in \mathcal{A} \mid \mathfrak{p} \text{ open prime ideal}\}$ and $\mathrm{Spfm}(\mathcal{A}) = \{\mathfrak{m} \in \mathcal{A} \mid \mathfrak{m} \text{ open maximal ideal}\}$ with the topology defined by declaring that the closed sets are the subsets of the form

$$V(E) = \{\mathfrak{p} \in \mathrm{Spf}(\mathcal{A}) \mid \mathfrak{p} \supseteq E\}$$

and

$$V(E) = \{\mathfrak{m} \in \mathrm{Spfm}(\mathcal{A}) \mid \mathfrak{m} \supseteq E\}$$

respectively, where E ranges through all subsets of $\mathcal{A}$. This topology we will called it the **Zariski topology** on $\mathrm{Spf}(\mathcal{A})$ and $\mathrm{Spfm}(\mathcal{A})$.

The previous sets define a topology this is possible to verify as a consequence of the below proposition [**14**, Proposition 2.1.1]

Proposition H. *Let E be a subset of $\mathcal{A}$, the following sentences are verifying:*

(1) Let $\mathfrak{a} := \langle E\rangle$ the ideal generated in $\mathcal{A}$ by E, and let $\sqrt{\mathfrak{a}}$. Then, $V(\mathfrak{a}) = V(E) = V(\sqrt{\mathfrak{a}})$;
(2) $V(\{0\}) = \mathrm{Spf}(\mathcal{A})$, $V(\{1\}) = \phi$;
(3) Given a family $\{E_i\}_{i\in I}$ of subset of $\mathcal{A}$. then

$$\mathcal{V}\left(\bigcup_{i\in I} E_i\right) = \bigcap_{i\in I} V(E_i)$$

;
(4) For ideals $\mathfrak{b}$ and $\mathfrak{c}$, then $V(\mathfrak{b}\mathfrak{c}) = V(\mathfrak{b}) \cup V(\mathfrak{c})$.

Like the Zariski topology on affine scheme the sets $D(f) := \mathrm{Spf}(\mathcal{A})\backslash V(\{f\})$ where f ranges through all elements of $\mathcal{A}$ are the principal open and give us a basis of the Zariski topology of $\mathrm{Spf}(\mathcal{A})$ (analogous with $\mathrm{Spfm}(\mathcal{A})$) as can be concluded from the following proposition.

Proposition I. *Let f, g and a family $\{f_i\}_{i\in I}$ be elements of $\mathcal{A}$. Then,*

(1) $D(f) \cap D(g) = D(f \cdot g)$;
(2) $\bigcup_{\alpha\in I} D(f_i) = V(\langle\{f_i\}_{i\in I}\rangle)^c$;
(3) $D(f) = \phi \Leftrightarrow f \in \mathfrak{N}(\mathcal{A}) \Leftrightarrow f$ is topologically nilpotent;
(4) If $\mathcal{A}$ is a separated and complete ring, then $D(f) = \mathrm{Spf}(\mathcal{A}) \Leftrightarrow f$ is a unit;
(5) $D(g) \subset D(f) \Leftrightarrow g \in \sqrt{\langle\{f\}\rangle}$.

The next proposition was proposed by Kambayashi for make the ringed space structure on $\mathrm{Spf}(\mathcal{A})$ when $\mathcal{A}$ is a separated and complete topological ring.

PROPOSITION J. *Let $\mathcal{A}$ be a separated and complete topological ring. If $U = D(f)$, $V = D(g)$, $\mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(U) := \mathcal{A}_f$ and $\mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(V) := \mathcal{A}_g$*

(1) *If $U = V$, then $\mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(U) \simeq \mathcal{O}_{\mathrm{Spf}(A)}(V)$*
(2) *If $V \subset U$, then there exit a canonical homomorphism of topological ring $\rho_{UV} \colon \mathcal{O}_{\mathrm{Spf}(A)}(U) \to \mathcal{O}_{\mathrm{Spf}(A)}(V)$*
(3) *Let U, V be as above and $W = D(h)$ for some $h \in \mathcal{A}$. If $W \subset V \subset U$, we have $\rho_{UU} = \mathrm{Id}_{\mathcal{O}_{\mathrm{Spf}(A)}(U)}$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$*

Finally the structural sheave on $\mathrm{Spf}(\mathcal{A})$ a separated complete topological ring $\mathcal{A}$ is defined matched to an open $U \subset \mathrm{Spf}(\mathcal{A})$ the inverse limit $\mathcal{O}_{\mathrm{Spf}(A)}(U) := \varprojlim \mathcal{O}_{\mathrm{Spf}(A)}(V)$ where V ranges through all the standard open $V \subset U$.

With the previous information [**15**, Theorem 2.2.3] defines a sheaf on $\mathrm{Spf}(\mathcal{A})$ of topological ring for some separated complete topological ring so the *affine ind-scheme* matched to a separated complete topological ring $\mathcal{A}$ is denoted an defined by $\mathcal{X}_{\mathcal{A}} := (\mathrm{Spf}(\mathcal{A}), \mathcal{O}_{\mathrm{Spf}(\mathcal{A})})$ Particularly if $\mathcal{A}$ is an pro-affine $\mathbf{k}$-algebra similarly the *affine ind-variety* matched to $\mathcal{A}$ is denoted and defined by $\mathcal{V}_{\mathcal{A}} := (\mathrm{Spfm}(\mathcal{A}), \mathcal{O}_{\mathrm{Spfm}(\mathcal{A})})$.

We can see if $\mathcal{V} \simeq \varinjlim V_i$ is an affine ind-variety on a closed field $\mathbf{k}$ in the Shafarevich sense, $\mathbf{k}[V_i]$ are reduced $\mathbf{k}$-algebras finitely generated, additionally the closed embedding $V_i \hookrightarrow V_{i+1}$ the respective map $\mathbf{k}[V_{i+1}] \to \mathbf{k}[V_i]$ are surjective $\mathbf{k}$-homomorphisms and if we endow all $\mathbf{k}[V_i]$ with the discrete topology the topological $\mathbf{k}$-algebra $\mathbf{k}[\mathcal{V}] := \varprojlim \mathbf{k}[V_i]$ is a strongly reduced algebraic pro-affine $\mathbf{k}$-algebra with fundamental system $(\ker(\mathbf{k}[\mathcal{V}] \to \mathbf{k}[V_i])_{i\in\mathbb{N}}$ and we can obtain a affine ind-variety in the Kambayashi sense $\mathcal{V}_{\mathbf{k}[\mathcal{V}]} := (\mathrm{Spfm}(\mathbf{k}[\mathcal{V}]), \mathcal{O}_{\mathbf{k}[\mathcal{V}]})$. In the other direction, If $\mathcal{A}$ is a strongly reduced algebraic pro-affine $\mathbf{k}$-algebra, with fundamental system $\{\mathfrak{a}_i\}_{i\in\mathbb{N}}$ such that $A_i := \mathcal{A}/\mathfrak{a}_i$ is reduced we take the affine varieties $V_i := (\mathrm{Spec}(A_i), \mathcal{O}_{A_i})$ naturally the surjective morphism $A_{i+1} \to A_i$ defines an affine ind-variety in Shafarevich sense.

0.3.2.1. *Morphism.* As usual in the algebraic geometry we defined the affine ind-scheme step to step. First we defined a set, then we endow a topology on the set and finally we gave it a topological ring structure on the topology space. A first approximation to the morphism in this try of category was proposed in [**14**]. A homomorphism of topological ring $\varphi \colon \mathcal{A} \to \mathcal{B}$ induces a continuos map $\varphi^* \colon \mathrm{Spf}(\mathcal{B}) \to \mathrm{Spf}(\mathcal{A})$ So the morphisms between affine ind-schemes are the induced by the corresponding continuous homomorphism between the respective topological rings

0.4. Locally nilpotent derivation

Locally nilpotent derivation on rings are closely related with the geometric structure of the corresponding scheme. A complete study of this special derivation will be study in [**8**].

Let B be a k-algebra, a **derivation** on B is a map $\delta \colon B \to B$ such that for $a, b \in B$ verifies $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = a\delta(b) + b\delta(a)$.

EXAMPLE A. If $f \in \mathbf{k}[x_1, \dots x_n]$,$f\dfrac{d}{dx_i} \colon \mathbf{k}[x_1, \dots x_n] \to \mathbf{k}[x_1, \dots x_n]$ is a derivation for all i.

A derivation on B is said to be **locally nilpotent** (lnd henceforth) if for all $a \in B$ there exists a non negative integer n depending of a such that $\delta^{(n)}(a) = 0$.

EXAMPLE B. $x^n \dfrac{d}{dy}\colon \mathbf{k}[x,y] \to \mathbf{k}[x,y]$ is a lnd. Analogously $y^n \dfrac{d}{dx}$ also is lnd.

If $\mathrm{Der}(B)$ is the set of derivations on B, by $\mathrm{LND}(B)$ we will refer to the set of $\delta \in \mathrm{Der}(B)$ such that δ is lnd and by $\mathrm{Der}_{\mathbf{k}}(B)$ the set of $\delta \in \mathrm{Der}(B)$ such that $\delta(k) = 0$ for all $k \in \mathbf{k}$. If M is an abelian monoid a M-graded $\mathbf{k}$-algebra is an associative, commutative $\mathbf{k}$-algebra with unity and a direct sum decomposition $B = \bigoplus_{m\in M} B_m$ where B_m are $\mathbf{k}$-submodules and $B_m B_{m'} \subset B_{m+m'}$. The B_m are the M-homogenous component and we will say that the elements a in B_m for some $m \in M$ are M-homogeneous elements of degree m. Special derivation are defined on M-graded $\mathbf{k}$-algebra B, a derivation $\delta \in \mathrm{Der}(B)$ is said to be **homogenous** if we send homogeneous elements into homogeneous elements and by the **degree** of a homogenous derivation δ we mean to the element $e \in M$ such that $\delta B_m \subset B_{m+e}$ for all $m \in M$. The degree of δ is denoted by $\deg \delta$. The study of locally nilpotent derivation has importance since the geometric perspective in particular if X is an affine variety and $\mathcal{O}(X)$ the rings of regular function, there exists a correspondence between $\mathrm{LND}(\mathcal{O}(X))$ and $\{\lambda\colon \mathbb{G}_a \times X \to X \mid \lambda \text{ is a regular action}\}$ where the correspondence is defined as follows: Let $\lambda\colon \mathbb{G}_a \times X \to X$ be a regular $\mathbb{G}_a$-action on X we can match it a locally nilpotent derivation $\delta_\lambda\colon \mathcal{O}(X) \to \mathcal{O}(X)$ defined by $f \mapsto [\frac{d}{dt} \circ \lambda^*(f)]_{t=0}$ furthermore, every regular $\mathbb{G}_a$-action on X arises from such a locally nilpotent derivation δ_λ. The regular $\mathbb{G}_a$-action $\lambda_\delta\colon \mathbb{G}_a \times X \to X$ is the comorphism of $\lambda_\delta^*\colon \mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathbf{k}[t] \simeq \mathcal{O}(X)[t]$ defined by $f \mapsto \exp(t\delta)(f) := \sum_{i=0}^{\infty} \dfrac{t^i \delta^{(i)}(f)}{i!}$. Commonly we will refer to a regular action $\lambda\colon G \times X \to X$ of an algebraic group G as a homomorphism of group $\lambda\colon G \to \mathrm{Aut}(X)$, naturally by $\mathrm{Aut}(X)$ we will refer to the automorphism group of X. Finally we will say that two LNDs δ and δ' on B are **equivalent** if $\ker \delta = \ker \delta'$ Geometrically this means that the generic orbits of the associated $\mathbb{G}_a$-actions coincide.

0.4.1. Locally finite iterative higher derivation. Sometimes we work with rings B such that $X = \mathrm{Spec}(B)$ is an affine variety over an arbitrary field $\mathbf{k}$, previous correspondence fails when $\mathbf{k}$ has characteristic different of 0 because in the proof we need to divide for, possibly, some integers equal to 0 module the characteristic of $\mathbf{k}$.

DEFINITION C. Let $\partial = \{\partial^{(i)}\}_{i\in\mathbb{Z}_{\geq 0}}$ be a sequence of $\mathbf{k}$-linear operators on B. We say that ∂ is a **locally finite iterative higher derivation** (LFIHD henceforth) if it satisfies the following conditions:

(1) The operator $\partial^{(0)}$ is the identity map.
(2) For any $i \in \mathbb{Z}_{\geq 0}$ anf for all $f_1, f_2 \in B$ we have the Leiniz rule

$$\partial^{(i)}(f_1 f_2) = \sum_{j=0}^{i} \partial^{(j)}(f_1)\partial^{(i-j)}(f_2)$$

.

(3) The sequence ∂ is locally finite, i.2. for any $f \in B$ there exists a positive integer n such that for any $i \geq n$, $\partial^{(f)} = 0$.
(4) For all $i, j \in \mathbb{Z}_{\geq 0}$ and for any regular function $f \in B$ we have

$$\left(\partial^{(i)} \circ \partial^{(j)}\right)(f) = \binom{i+j}{i} \partial^{(i+j)}(f)$$

.

When ∂ verifies only (1) and (2), ∂ is said to be **Hasse-Schmidt derivation** [**12**] and if ∂ verifies only (1), (2) and (4) ∂ is said to be **iterative higher derivation.**

With the same idea in the previous section we have a correspondence between $\mathbb{G}_a$-action on X and LFIHD on B [**20**]. An easy computation proves that when the characteristic of $\mathbf{k}$ is 0 then ∂ is uniquelly determinated by $\partial^{(1)}$.

0.5. Affine toric variety

An **algebraic torus** $\mathbb{T}$ is an algebraic group isomorphic to $\mathbb{G}_m^n$ where n is a non negative integer. The algebraic torus is matched with free abelian group of rank n

$$M := \{m\colon \mathbb{T} \to \mathbb{G}_m \mid m \text{ is a homomorphism group}\}$$

$$N := \{u\colon \mathbb{G}_m \to \mathbb{T} \mid u \text{ is a homomorphism group}\}$$

and a bilinear map $\langle\ ,\ \rangle\colon M \times N \to \mathbb{Z}$ defined by $(m, u) \mapsto \langle m, u\rangle := z$ where z is the integer such that $m \circ u(t) = t^z$. Naturally we can identify $M = \mathrm{Hom}(N, \mathbb{Z})$ $N = \mathrm{Hom}(M, \mathbb{Z})$ and the bilinear map is expressed as $\langle m, u\rangle = m(u)$. The previous definition gives us the identification $\mathbf{k}[\mathbb{T}]$ the ring of regular maps of $\mathbb{T}$ with $\mathbf{k}[M] := \bigoplus_{m\in M} \mathbf{k}\chi^m$ the M-graduated $\mathbf{k}$-algebra where the multiplication rule is given by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$ and $\chi^0 = 1$.

By an **affine toric variety** we will refer to an irreducible affine algebraic variety X containing a torus $\mathbb{T}$ as a Zariski open subset such that the action of $\mathbb{T}$ on itself extends to an algebraic action of $\mathbb{T}$ on X. This definition is the proposed in [**2**] so we don't assume the property of normality on X unlike other authors [**28**, **7**]. The category of affine toric varieties is dual with the opposite category of affine semigroups. An **affine semigroup** is a finitely generated monoid S embeddable in a free abelian group M of rank n, S is said to be **saturated** if for all $m \in M$ such that there exists a positive integers l verifying $l \cdot m$ is an element in S then $m \in S$. To obtain the correspondence between objects, if S is an affine semigroup, the algebraic affine variety $X_S = \mathrm{Spec}(\mathbf{k}[S])$ matched is affine toric variety where $\mathbf{k}[S] := \bigoplus_{m\in S} \chi^m$ and the torus $\mathbb{T} = \mathrm{Spec}(\mathbf{k}[M])$ corresponding to the free abelian group $M = \mathbb{Z}S := \{m_1 - m_2 \mid m_1, m_2 \in S\}$. In the other direction if X is an affine toric variety with torus $\mathbb{T}$ we can use the dominant inclusion $\mathbb{T} \subset X$ so $\mathbf{k}[X] \subset \mathbf{k}[\mathbb{T}] \simeq \mathbf{k}[M]$ thus the affine semigroup S_X matched to X is the semigroup $\{m \in M \mid \chi^m \in \mathbf{k}[X]\}$.

Although our interest is in non necessarily normal affine toric variety but the property of normality on an affine toric variety will be so used in this paper, normal affine toric varieties are in correspondence with saturated affine semigroups and object in the convex geometry. These objects give us several tools to advance the study of this type of algebraic varieties.

General Bibliography

[1] M. Artin, A. Grothendieck, J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas, Tome 1: Théorie des topos*, Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4). Lecture Notes in Mathematics 269, Springer 1972.

[2] D. Cox, J. Little, H. Schenck, *Toric varieties,* Graduate Studies in Mathematics, volume 124. American Mathematical Society, Providence, RI, 2011.

[3] Jérémy Blanc and Jean-Philippe Furter, *Topologies and structures of the Cremona groups.* Ann. of Math. (2) 2013, 178 (2013), no. 3, 1173–1198.

[4] N. Bourbaki, *Elements of mathematics. General topology,* Chapters 1-4. Translated from the French. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966.

[5] N. Bourbaki, *Elements of mathematics. Theory of sets,* Hermann, 1968.

[6] M. Demazure, *Sous-groupes algèbriques de rang maximum du groupe de Cremona,* Ann. Sci. Ecole Norm. Sup. (4) 3, 1970, 507-588.

[7] W. Fulton, *Introduction to toric varieties,* Annals of Mathematics Studies, volume 131. Princeton University Press, Princeton, NJ, 1993.

[8] G. Freudenburg, *Algebraic theory of locally nilpotent derivations.* Encyclopaedia of Mathematical Sciences 136, Invariant Theory and Algebraic Transformation Groups 7, Springer, 2017.

[9] J.-P. Furter, H. Kraft, *On the geometry of the automorphism groups of affine varieties,* arXiv:1809.04175 [math.AG] (2018).

[10] A. Grothendieck, J. Dieudonné, *Éléments de géométrie algébrique: I. Le langage des schémas,* Grundlehren der Mathematischen Wissenschaften (in French). 166 (2nd ed.). Berlin; New York: Springer-Verlag. ISBN 978-3-540-05113-8.

[11] R. Hartshome, *Algebraic Geometry,* Arcata 1974 (Vol. 29). American Mathematical Soc..

[12] H. Hasse, F.K. Schmidt, *Noch eine Bergründung der Theorie der höheren Differential quotienten in einemalgebraischen Functionenkörper einer Unbestimmten,*J. Reine Angew. Math. 177 (1937) 215–237

[13] T. Kambayashi, *Review of* [**22**],, Math. Reviews 58 (1979), #5697.

[14] T. Kambayashi, *Pro-affine algebras, ind-affine groups and the Jacobian Problem,* Journal of Algebra, 185 (1996), 481-501.

[15] T. Kambayashi, *Some basic results on pro-afine algebras and ind-affine schemes,* Osaka J. Math. 40 (2003), no. 3, 621-638.

[16] T. Kambayashi, M. Miyanishi, *On two recent views of the Jacobian conjecture*, Affine algebraic geometry, 113-138, Contemp. Math., 369, Amer. Math. Soc., Providence, RI, 2005.

[17] S. Kumar, *Kac-Moody groups, their ag varieties and representation theory, Progress in Mathematics*, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.

[18] H. Kraft, A. Regeta, *Automorphisms of the Lie algebra of vector fields on affine n-space*, J. Eur. Math. Soc. (JEMS) 19 (2017), no. 5, 1577-1588.

[19] A. Liendo, *Affine $\mathbb{T}$-varieties of complexity one and locally nilpotent derivations.* Transform. Groups 15 (2010), no. 2, 389-425.

[20] M. Miyanishi, *A remark on an iterative infinite higher derivation*, J. Math. Kyoto Univ. 8 (1968) 411-415.

[21] D. Northcott, *Lessons on rings, modules and multiplicities*, Cambridge University Press Bentley House, 1968.

[22] I. Shafarevich, *On some infinite-dimensional groups*, Rend. Mat. Appl. (5) 25 (1-2) (1966) 208-212.

[23] I. Shafarevich, *On some infinite-dimensional groups. II*, Math. USSR Izvestija 18 (1982), 185-194.

[24] I. Shafarevich, *Basic algebraic geometry, varieties in projective space,* Berlin: Springer-verlag, 2013.

[25] I. Shafarevich, *Basic algebraic geometry 2: Schemes and complex manifolds,* Berlin: Springer-verlag, 2013.

[26] E. Specker, *Additive Gruppen von Folgen ganzer Zahlen,* Portugaliae Mathematica, 9 (1950): 131–140.

[27] I. Stampfli, *On the topologies on ind-varieties and related irreducibility questions*, J. Algebra 372 (2012), 531-541.

[28] T. Oda, *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties,* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), volume 15. Springer-Verlag, Berlin, 1988.

CHAPTER 1

On toric ind-varieties and pro-affine semigroups

In this chapter, I present the research developed with Alvaro Liendo.

ON TORIC IND-VARIETIES AND PRO-AFFINE SEMIGROUPS

ROBERTO DÍAZ AND ALVARO LIENDO

ABSTRACT. An ind-variety is an inductive limit of closed embeddings of algebraic varieties and an ind-group is a group object in the category of ind-varieties. These notions were first introduced by Shafarevich in the study of the automorphism group of affine spaces and have been studied by many authors afterwards. An ind-torus is an ind-group obtained as an inductive limit of closed embeddings of algebraic tori that are also algebraic group homomorphisms. In this paper, we introduce the natural definition of toric ind-varieties as ind-varieties having an ind-torus as an open set and such that the action of the ind-torus on itself by translations extends to a regular action on the whole ind-variety. We are brought to introduce and study pro-affine semigroups, which turn out to be unital semigroups isomorphic to closed subsemigroups of the group of arbitrary integer sequences with the product topology such that their projection to the first i coordinates is finitely generated for all positive integers i. Our main result is a duality between the categories of affine toric ind-varieties and the the category of pro-affine semigroups.

INTRODUCTION

Shafarevich first introduced in [14, 15] the notion of infinite-dimensional algebraic varieties and infinite-dimensional algebraic groups, the so called ind-varieties and ind-groups, respectively. These notions were later expanded and revisited by several authors, see for instance [10, 9, 17] and the recent preprint [8] that includes a detailed exposition of generalities on ind-varieties and ind-groups. In this paper, we generalize the notion of toric varieties to the category of ind-varieties.

We work over the field of complex numbers $\mathbb{C}$. An *ind-variety* is a set $\mathcal{V}$ together with a filtration $V_1 \hookrightarrow V_2 \hookrightarrow \ldots$ such that $\mathcal{V} = \bigcup V_i$, where each V_i is a finite-dimensional algebraic variety and the inclusions $\varphi_i\colon V_i \hookrightarrow V_{i+1}$ are closed embeddings. Morphisms in the category of ind-varieties are defined in the natural way, see Section 1.3 for details. An *ind-group* is a group object in the category of ind-varieties, i.e., it is an ind-variety endowed with a group structure such that the inversion and multiplication maps are morphisms of ind-varieties. The set

$$(\mathbb{C}^*)^\infty = \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{C}^* \text{ and } a_i \neq 1 \text{ for finitely many } i\}$$

with the canonical structure of ind-variety given by the filtration $\mathbb{C}^* \overset{\varphi_1}{\hookrightarrow} (\mathbb{C}^*)^2 \overset{\varphi_2}{\hookrightarrow} \ldots$, where $\varphi_i(a_1, \ldots, a_i) = (a_1, \ldots, a_i, 1)$ for all integer $i > 0$, has a natural structure of ind-group where the group law is given by component-wise multiplication. An *algebraic torus* T is an algebraic group isomorphic to $(\mathbb{C}^*)^k$ for some integer $k \geq 0$. An *ind-torus* $\mathcal{T}$ is an ind-group isomorphic to either an algebraic torus or $(\mathbb{C}^*)^\infty$.

A toric variety V is an irreducible algebraic variety having an algebraic torus T as an open set and such that the action of T on itself by translations extends to a regular action on V. Toric varieties can be classified by certain combinatorial devices, see [12, 6, 4]. This classification allows to translate many algebro-geometric properties of a toric variety into combinatorial terms that may then be computed algorithmically. Hence, toric varieties represent a fertile testing ground for theories in algebraic geometry. Toric morphisms between toric varieties are characterized by the property that they restrict to a morphism of algebraic groups between the corresponding algebraic tori. For affine toric varieties their combinatorial nature is represented by the fact that the category of affine toric varieties is dual to the category of affine semigroups, i.e., finitely generated semigroups that can be embedded in $\mathbb{Z}^k$ for some integer $k \geq 0$. By convention, all our semigroups will be commutative and unital. A unital semigroup is usually called a monoid.

In this paper we introduce the natural notion of toric ind-variety. A *toric ind-variety* $\mathcal{V}$ is an ind-variety having an ind-torus $\mathcal{T}$ as an open set and such that the action of $\mathcal{T}$ on itself by translations extends to a regular action on $\mathcal{V}$, see Definition 2.1. Furthermore, toric morphisms between toric ind-varieties are morphisms that restrict to morphisms of ind-groups between the corresponding ind-tori, see Definition 2.5. Our first result in this paper, contained in Theorem 2.3, shows that every toric ind-variety can be obtained as an inductive limit of toric varieties. This result allows us to investigate toric ind-varieties applying usual methods from toric geometry.

In Section 3 we introduce the natural dual objects to affine toric ind-varieties that we call pro-affine semigroups. We need to develop the theory of pro-affine semigroups from scratch since, to our knowledge, only the case of pro-finite semigroups has been previously studied in the literature in detail, see for instance [3]. Let $\mathcal{S}$ be a commutative unital semigroup. In analogy with the case of topological algebras [13, Section 9.2] taking into account the lack of the notion of ideal of a semigroup, the natural way to endow the semigroup $\mathcal{S}$ with a topology is with a descending filtration $R_1 \supset R_2 \supset \ldots$ of $\mathcal{S} \times \mathcal{S}$ of equivalence relations on $\mathcal{S}$ that satisfy certain compatibility condition with respect to the semigroup operation allowing to define a semigroup operation in the set of equivalence classes $\mathcal{S}/R_i$, see Section 3 for details. We call a semigroup $\mathcal{S}$ endowed with such a filtration a *filtered semigroup*. A *pro-affine semigroup* $\mathcal{S}$ is a filtered semigroup with filtration $R_1 \supset R_2 \supset \ldots$ of compatible equivalence relations in $\mathcal{S}$ that is complete and such that $\mathcal{S}/R_i$ is an affine semigroup, for all integer $i > 0$. Our main result concerning pro-affine semigroups is contained in Corollary 3.11 and is a classification of pro-affine semigroups as semigroups isomorphic to subsemigroups $\mathcal{S}$ of $\mathbb{Z}^\omega$, the group of arbitrary sequences of integers, that are closed in the product topology and such that $\pi_i(\mathcal{S})$ is finitely generated for all integer $i > 0$, where $\pi_i \colon \mathbb{Z}^\omega \to \mathbb{Z}^i$ is the projection to the first i-th coordinates.

Finally, our main result in this paper is Theorem 4.5 where we show that the category of affine toric ind-varieties with toric morphisms is dual to the category of pro-affine semigroups with homomorphisms of semigroups.

The contents of the paper are as follows. In Section 1 we collect the preliminary notions of toric varieties, inductive and projective limits and ind-varieties required in this paper. In Section 2 we introduce toric ind-varieties. In Section 3 we define pro-affine semigroups. In Section 4 we prove the duality of categories that is our main result. Finally, in Section 5 we provide some examples to ilustrate our results.

Acknowledgements. The authors would like to thank the anonymous referee of this manuscript for useful comments and for spotting a gap in a proof. Part of this work was done during a stay of both authors at IMPAN in Warsaw. We would like to thank IMPAN and the organizers of the Simons semester "Varieties: Arithmetic and Transformations" for the hospitality.

1. Preliminaries

In this section we recall the notions of toric geometry, injective and projective limits and ind-varieties needed for this paper.

1.1. **Toric varieties.** To fix notation we recall the basics of toric geometry. For details, see [12, 6, 4]. An algebraic torus T is a linear algebraic group isomorphic to $(\mathbb{C}^*)^k$ for some integer $k \geq 0$. A toric variety on $\mathbb{C}$ is an irreducible algebraic variety V having an algebraic torus as a dense open set such that the action of T on itself by translations extends to a regular action of T on V. Similarly to [4], we will not assume that a toric variety is necessarily normal. It is well known that affine toric varieties are in correspondence with affine semigroups S, i.e., with finitely generated semigroups that admit an embedding in $\mathbb{Z}^k$ for some integer $k \geq 0$. By convention, all our semigroups are commutative and unital.

Indeed, given an affine semigroup S, the corresponding affine toric variety is given by $\mathcal{V}(S) = \operatorname{Spec} \mathbb{C}[S]$, where $\mathbb{C}[S]$ is the semigroup algebra given by $\mathbb{C}[S] = \bigoplus_{m \in S} \mathbb{C} \cdot \chi^m$. Here, χ^m are new symbols and the multiplication rule is defined by $\chi^0 = 1$ and $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. On the other hand, the character lattice M of the torus T is a finitely generated free abelian group $M \simeq \mathbb{Z}^k$ of rank $k = \dim T$. Let V be an affine toric variety with acting torus T. We define the semigroup $\mathcal{S}(V)$ of the toric variety V as the semigroup of characters of T in M that extend to regular functions on V.

A toric morphism between toric varieties is a regular map that restricts to a morphism of algebraic groups between the corresponding algebraic tori acting on each toric variety. It is well known that the assignments $\mathcal{V}(\bullet)$ and $\mathcal{S}(\bullet)$ extend to functors from the category of affine varieties with toric morphisms to the category of affine semigroups and vice versa, respectively. Furthermore, the functors $\mathcal{V}(\bullet)$ and $\mathcal{S}(\bullet)$ together form a duality between the categories of affine toric varieties with toric morphisms and affine semigroups with homomorphisms of semigroups.

1.2. **Inductive and projective limits.** In this paper we will require several instances of inductive and projective limits of algebraic and geometric objects. We give here a brief account to fix notation. For details, see any reference on category theory such as [11, Chapter III]. All the systems of morphisms required in this paper will be indexed by the positive integers with the usual order. Hence we restrict the exposition to this setting.

An inductive system indexed by the positive integers in a category $\mathcal{C}$ is a sequence

$$X_1 \overset{\varphi_1}{\to} X_2 \overset{\varphi_2}{\to} X_3 \overset{\varphi_3}{\to} \dots ,$$

where the X_i are objects in $\mathcal{C}$ and $\varphi_i\colon X_i \to X_{i+1}$ are morphisms in $\mathcal{C}$. We denote such an inductive system by (X_i, φ_i). For every $i, j > 0$ with $i \leq j$, we define $\varphi_{ij}\colon X_i \to X_j$ as $\varphi_{ij} = \varphi_j \circ \varphi_{j-1} \circ \cdots \circ \varphi_i$, where by definition $\varphi_{ii} = \mathrm{id}\colon X_i \to X_i$. The inductive limit of an inductive system (X_i, φ_i) is an object $\varinjlim X_i$ in $\mathcal{C}$ and morphisms $\psi_i\colon X_i \to \varinjlim X_i$ verifying $\psi_i = \psi_j \circ \varphi_{ij}$ and satisfying the following universal property: if there exist another object Y and morphisms $\psi'_i\colon X_i \to Y$ verifying $\psi'_i = \psi'_j \circ \varphi_{ij}$, then there exists a unique morphism $u\colon \varinjlim X_i \to Y$ such that $\psi'_i = u \circ \psi_i$ for all $i > 0$.

The notion of projective limit is dual to the notion of inductive limit and is defined as follows. A projective system indexed by the positive integers in a category $\mathcal{C}$ is a sequence

$$X_1 \overset{\varphi_1}{\leftarrow} X_2 \overset{\varphi_2}{\leftarrow} X_3 \overset{\varphi_3}{\leftarrow} \dots ,$$

where the X_i are objects in $\mathcal{C}$ and $\varphi_i\colon X_{i+1} \to X_i$ are morphisms in $\mathcal{C}$. We denote such a projective system by (X_i, φ_i). For every $i, j > 0$ with $i \leq j$, we define $\varphi_{ij}\colon X_j \to X_i$ as $\varphi_{ij} = \varphi_i \circ \varphi_{i+1} \circ \cdots \circ \varphi_j$, where by definition $\varphi_{ii} = \mathrm{id}\colon X_i \to X_i$. The projective limit of a projective system (X_i, φ_i) is an object $\varprojlim X_i$ in $\mathcal{C}$ and morphisms $\pi_i\colon \varprojlim X_i \to X_i$ verifying $\pi_i = \varphi_{ij} \circ \pi_j$ and satisfying the following universal property: if there exist another object Y and morphisms $\pi_i'\colon Y \to X_i$ verifying $\pi_i' = \varphi_{ij}' \circ \pi_j'$, then there exists a unique morphism $u\colon Y \to \varprojlim X_i$ such that $\pi_i' = \pi_i \circ u$ for all $i > 0$.

Both limits may not exist in arbitrary categories but in the categories of our interest (sets, groups, rings, algebras, semigroups, topological spaces) both limits can be realized by explicit constructions. Indeed, the inductive limit $\varinjlim X_i$ of an inductive system (X_i, φ_i) can be constructed as $\varinjlim X_i = \bigsqcup_{i>0} X_i / \sim$, where $\sim$ is the equivalence relation given by $x_i \sim x_j$, where $x_i \in X_i$ and $x_j \in X_j$, if there exist k verifying $i \leq k$ and $j \leq k$ such that $\varphi_{ik}(x_i) = \varphi_{jk}(x_j)$. The morphisms $\psi\colon X_i \to \varinjlim X_i$ are induced by the natural injections $X_i \to \bigsqcup_{i>0} X_i$. Furthermore, if the morphisms φ_i are injective, then we can naturally regard each X_i as a subobject of the inductive limit $\varinjlim X_i$. On the other hand, the projective limit $\varprojlim X_i$ of the projective system (X_i, φ_i) can be constructed as

$$\varprojlim X_i = \left\{ (x_1, x_2, \dots) \in \prod_{i>0} X_i \mid x_i \in X_i \text{ and } \varphi_{ij}(x_j) = x_i \right\},$$

and the morphisms $\pi_i\colon \varprojlim X_i \to X_i$ are induced by the natural projections $\prod_{i>0} X_i \to X_i$. Furthermore, if the morphisms φ_i are surjective, then we can naturally regard each X_i as a quotient of the projective limit $\varprojlim X_i$. Finally, in the case where the X_i are topological spaces, the topology on the projective limit $\varprojlim X_i$ coincides with the subspace topology induced by $\prod_{i>0} X_i$ with the product topology.

Example 1.1. Two particular instances of the above construction will appear very often in this paper. Recall that $\mathbb{Z}^\omega$ is the group of arbitrary sequences of integer numbers. This group is also called the Baer-Specker group. A sequence in $a \in \mathbb{Z}^\omega$ is denoted by $a = (a_1, a_2, \dots)$. Equivalently, $\mathbb{Z}^\omega$ is the projective limit of the system $\mathbb{Z}^1 \leftarrow \mathbb{Z}^2 \leftarrow \dots$, where the morphisms $\varphi_i\colon \mathbb{Z}^{i+1} \to \mathbb{Z}^i$ are the projections forgetting the last coordinate. Furthermore, the subgroup of $\mathbb{Z}^\omega$ of eventually zero sequences is denoted by $\mathbb{Z}^\infty$, so $a \in \mathbb{Z}^\infty$ is such that $a_i = 0$ except for finitely many positive integers i. Equivalently, $\mathbb{Z}^\infty$ is the inductive limit of the system $\mathbb{Z}^1 \to \mathbb{Z}^2 \to \dots$, where the maps are the injections setting the last coordinate to 0.

If we take any inductive or projective subsystem of the system defining $\mathbb{Z}^\infty$ or $\mathbb{Z}^\omega$, respectively with the obvious morphisms given by compositions, then the limits are canonically isomorphic to $\mathbb{Z}^\infty$ or $\mathbb{Z}^\omega$, respectively. More generally, a projective or inductive system is called split if every morphism in the system admits a section. It is a straightforward computation to show that for any split projective system $\mathbb{Z}^{n_1} \leftarrow \mathbb{Z}^{n_2} \leftarrow \dots$, with a strictly increasing sequence $n_1 < n_2 < \dots$ of positive integers, the limit is isomorphic to $\mathbb{Z}^\omega$. Similarly, for any split inductive system $\mathbb{Z}^{n_1} \to \mathbb{Z}^{n_2} \to \dots$, with a strictly increasing sequence $n_1 < n_2 < \dots$ of positive integers, the limit is isomorphic to $\mathbb{Z}^\infty$.

In the sequel we will need the following lemma showing that $\mathbb{Z}^\omega$ and $\mathbb{Z}^\infty$ are mutually dual. Showing that $\mathrm{Hom}(\mathbb{Z}^\infty, \mathbb{Z}) \simeq \mathbb{Z}^\omega$ is a straightforward exercise, but showing $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \simeq \mathbb{Z}^\infty$ is more involved, see [16] for the original proof or [5, Example 3.22] for a modern proof.

Lemma 1.2. *The groups $\mathbb{Z}^\omega$ and $\mathbb{Z}^\infty$ are mutually dual and this duality is realized by the usual dot product*

$$\langle\ ,\ \rangle\colon \mathbb{Z}^\omega \times \mathbb{Z}^\infty \to \mathbb{Z}, \qquad (m, p) \mapsto \sum_{i>0} (m_i \cdot p_i)\,.$$

1.3. **General ind-varieties.** In this section we introduce the necessary notions and results regarding ind-varieties. The definitions are borrowed from [10], [9] and [8].

Recall that an ind-variety is a set $\mathcal{V}$ together with a filtration $V_1 \hookrightarrow V_2 \hookrightarrow \dots$ such that $\mathcal{V} = \varinjlim V_i := \bigcup V_i$, each V_i is a finite-dimensional variety over $\mathbb{C}$, and the inclusion $\varphi_i \colon V_i \hookrightarrow V_{i+1}$ is a closed embedding. An ind-variety $\mathcal{V}$ is affine if each V_i is affine. We also define the ind-topology on an ind-variety $\mathcal{V}$ as the topology where a set $U \subset \mathcal{V}$ is open if and only if $U \cap V_i$ is open in V_i for all $i > 0$. In particular, the filtration $V_1 \hookrightarrow V_2 \hookrightarrow \dots$ is an inductive system and the set $\mathcal{V}$ is the inductive limit. The topology defined on $\mathcal{V}$ corresponds to the inductive topology given by this inductive system. The dimension of $\mathcal{V}$ is $\lim \dim(V_i)$ as i tends to infinity.

A morphism between ind-varieties $\mathcal{V}$ and $\mathcal{V}'$ with filtrations V_i and V_j' respectively, is a map $\varphi \colon \mathcal{V} \to \mathcal{V}'$ satisfying that for every $i > 0$ there exists a positive integer $j > 0$ such that $\varphi(V_i) \subset V_j'$ and $\varphi|_{V_i} \colon V_i \to V_j'$ is a morphism of varieties. A morphism φ of ind-varieties is an isomorphism if φ is bijective and φ^{-1} is a morphism of ind-varieties. Furthermore, two filtrations $V_1 \hookrightarrow V_2 \hookrightarrow \dots$ and $W_1 \hookrightarrow W_2 \hookrightarrow \dots$ on the same underlying set $\mathcal{V}$ are equivalent if the identity map is an isomorphism of ind-varieties. In analogy with the similar Example 1.1, if we take any subfiltration of the filtration $V_1 \hookrightarrow V_2 \hookrightarrow \dots$, the ind-varieties obtained by both filtrations are isomorphic. The Cartesian product of two ind-varieties is again an ind-variety with the product filtration. Moreover, an ind-group is an ind-variety $\mathcal{G}$ endowed with a group structure such that the inversion and multiplication maps are morphisms of ind-varieties.

Recall that a topological space is irreducible if it is not equal to the union of two proper closed sets. An irreducible ind-variety $\mathcal{V}$ does not necessarily admit an equivalent filtration $V_1 \hookrightarrow V_2 \hookrightarrow \dots$ by irreducible varieties as shown in [1, Remark 4.3], see also [8, Example 1.6.5]. An ind-variety $\mathcal{V}$ is called curve-connected if for any two points $a, b \in \mathcal{V}$ there exists an irreducible algebraic curve C and a morphism $C \to \mathcal{V}$ whose image contains a and b. An ind-variety $\mathcal{V}$ is curve-connected if and only if there exists an equivalent filtration $V_1 \hookrightarrow V_2 \hookrightarrow \dots$ by irreducible varieties [8, Proposition 1.6.3].

Recall that a set in a topological space is locally closed if it is the intersection of an open set and a closed set. Let $\mathcal{V} = \varinjlim V_i$ be an ind-variety. A subset $A \subset \mathcal{V}$ is called algebraic if it is locally closed and contained in V_i for some $i > 0$, so A has a natural structure of an algebraic variety. A morphism $\alpha \colon \mathcal{V} \to \mathcal{V}'$ is called an embedding if the image $\alpha(\mathcal{V}) \subset \mathcal{V}'$ is locally closed and induces an isomorphism of ind-varieties between $\mathcal{V}$ and $\alpha(\mathcal{V})$. An embedding is called a closed embedding (resp. an open embedding) if $\alpha(\mathcal{V}) \subset \mathcal{V}'$ is closed (resp. open). Finally, recall that a constructible set is a finite union of locally closed subsets.

Example 1.3. (1) The infinite-dimensional vector space

$$\mathbb{C}^\infty := \{(a_1, \dots) \mid a_i \in \mathbb{C} \text{ and } a_i \neq 0 \text{ for finitely many } i\}$$

has a canonical structure of ind-variety given by the filtration $\mathbb{C} \overset{\varphi_1}{\hookrightarrow} \mathbb{C}^2 \overset{\varphi_2}{\hookrightarrow} \mathbb{C}^3 \overset{\varphi_3}{\hookrightarrow} \dots$ where $\varphi_n(a_1, \dots, a_i) = (a_1, \dots, a_i, 0)$, for all $i > 0$. This ind-variety is called the infinite-dimensional affine space. Remark that we can change the complex number 0 in the $(i+1)$-th coordinate of φ_i by any other number. The ind-variety obtained this way is easily seen to be isomorphic to $\mathbb{C}^\infty$. For instance, we denote by $\mathbb{C}_1^\infty$ the ind-variety isomorphic to the infinite-dimensional affine space given by $\mathbb{C}_1^\infty := \{(a_1, \dots) \mid a_i \in \mathbb{C} \text{ and } a_i \neq 1 \text{ for finitely many } i\}$.

(2) The set

$$(\mathbb{C}^*)^\infty = \{(a_1, a_2, \dots) \mid a_i \in \mathbb{C}^* \text{ and } a_i \neq 1 \text{ for finitely many } i\}$$

has a canonical structure of ind-variety given by the filtration $\mathbb{C}^* \overset{\varphi_1}{\hookrightarrow} (\mathbb{C}^*)^2 \overset{\varphi_2}{\hookrightarrow} (\mathbb{C}^*)^3 \overset{\varphi_3}{\hookrightarrow} \ldots$, where $\varphi_i(a_1, \ldots, a_i) = (a_1, \ldots, a_i, 1)$ for all $i > 0$. This ind-variety is an open set in the infinite-dimensional affine space. This follows straightforward from the isomorphism $\mathbb{C}^\infty \simeq \mathbb{C}_1^\infty$ above. Remark that $(\mathbb{C}^*)^\infty$ has a natural structure of ind-group given by component-wise multiplication.

A commutative topological $\mathbb{C}$-algebra $\mathcal{A}$ is pro-affine if it is Hausdorff, complete and admits a base $\{I_i\}_{i>0}$ of open neighborhoods of 0, where $I_i \subset \mathcal{A}$ is an ideal for all $i > 0$. Furthermore, we can assume that the I_i form a descending filtration $I_1 \supset I_2 \supset \ldots$ of ideals of $\mathcal{A}$. Recall that the Hausdorff property is equivalent to $\bigcap I_i = \{0\}$ and completeness is equivalent to $\mathcal{A} = \varprojlim \mathcal{A}_i$ where the algebra $\mathcal{A}_i := \mathcal{A}/I_i$ is taken with the discrete topology, see [13, Section 9.2] for details. A pro-affine algebra $\mathcal{A}$ is algebraic if $\mathcal{A}_i$ is finitely generated over $\mathbb{C}$ for all $i > 0$. Every finitely generated algebra over $\mathbb{C}$ is pro-affine algebraic with $I_i = \{0\}$ for all $i > 0$. In the sequel all pro-affine algebras are assumed to be algebraic, so we will drop algebraic from the notation.

For an ind-variety $\mathcal{V}$ with filtration $V_1 \overset{\varphi_1}{\hookrightarrow} V_2 \overset{\varphi_2}{\hookrightarrow}, \ldots$ the ring of regular functions $\mathbb{C}[\mathcal{V}]$ is defined as $\varprojlim \mathbb{C}[V_i]$ with respect to the projective system $\mathbb{C}[V_1] \overset{\varphi_1^*}{\leftarrow} \mathbb{C}[V_2] \overset{\varphi_2^*}{\leftarrow} \ldots$ where each $\mathbb{C}[V_i]$ is taken with the discrete topology and $\varprojlim \mathbb{C}[V_i]$ has the projective limit topology i.e.,

$$\mathbb{C}[\mathcal{V}] = \varprojlim \mathbb{C}[V_i] = \Big\{ (f_1, f_2, \ldots) \mid f_i \in \mathbb{C}[V_i] \text{ and } \varphi_i^*(f_{i+1}) = f_i \Big\} \subset \prod_{i>0} \mathbb{C}[V_i],$$

with the subspace topology. The projective limit comes equipped with natural projections $\pi_i \colon \mathbb{C}[\mathcal{V}] \to \mathbb{C}[V_i]$.

Let $\alpha \colon \mathcal{V} \to \mathcal{V}'$ be a morphism of ind-varieties. Then for every $i > 0$ there exists $j > 0$ such that α induces a homomorphism $\mathbb{C}[V_j'] \to \mathbb{C}[V_i]$ and so α induces a continuous homomorphism of pro-affine algebras $\alpha^* \colon \mathbb{C}[\mathcal{V}'] \to \mathbb{C}[\mathcal{V}]$. Conversely, every continuous homomorphism $\beta \colon \mathbb{C}[V'] \to \mathbb{C}[V]$ of pro-affine algebras induces for every $i > 0$ a homomorphism $\mathbb{C}[V_j'] \to \mathbb{C}[V_i]$ for some $j > 0$ and so it induces a morphism $V_i \to V_j$ which in turns gives a morphism $\beta^* \colon \mathcal{V} \to \mathcal{V}'$ [10, 9]. This yields an equivalence of categories between pro-affine algebras and affine ind-varieties.

2. Toric ind-varieties

An algebraic torus T is an algebraic group isomorphic to $(\mathbb{C}^*)^i$ for some $i \geq 0$. An ind-torus $\mathcal{T}$ is an ind-group isomorphic to either an algebraic torus or $(\mathbb{C}^*)^\infty$. A regular action of an ind-torus $\mathcal{T}$ on an ind-variety $\mathcal{V}$ is a group action $\alpha \colon \mathcal{T} \times \mathcal{V} \to \mathcal{V}$ by automorphisms of $\mathcal{V}$ such that α is also a morphism of ind-varieties.

Definition 2.1. A toric ind-variety is a curve-connected ind-variety $\mathcal{V}$ having an ind-torus $\mathcal{T}$ as an open subset such that the action of $\mathcal{T}$ on itself by translations extends to a regular action of $\mathcal{T}$ on $\mathcal{V}$.

If $\mathcal{V}$ is finite dimensional, then this definition coincides with the usual notion of toric variety since curve-connectedness is equivalent to irreducibility in the finite-dimensional case, see for instance [4, Definition 1.1.3]. Asking for an ind-toric variety $\mathcal{V}$ to be curve-connected is equivalent to asking for $\mathcal{V}$ to be presented as the inductive limit of irreducible varieties [8, Proposition 1.6.3]. Remark that similarly to [4] and unlike other references [12, 6], we do not require toric varieties to be normal.

Example 2.2. Recall that $\mathbb{Z}^\infty$ is defined as the inductive limit of the inductive system $\mathbb{Z} \to \mathbb{Z}^2 \to \dots$ where the maps are the injections setting the last coordinate to 0. Taking tensor product of this system with $\mathbb{C}^*$ we obtain the inductive system defining $(\mathbb{C}^*)^\infty$. In analogy with the finite-dimensional case, we denote this by by $(\mathbb{C}^*)^\infty = \mathbb{Z}^\infty \otimes_{\mathbb{Z}} \mathbb{C}^*$. Now, it follows directly from Example 1.1 that for every sequence $\mathbb{C}^* \stackrel{\varphi_1}{\hookrightarrow} (\mathbb{C}^*)^2 \stackrel{\varphi_2}{\hookrightarrow} (\mathbb{C}^*)^3 \stackrel{\varphi_3}{\hookrightarrow} \dots$ with φ_i injective homomorphisms of algebraic groups, the corresponding ind-variety is an ind-group isomorphic to $(\mathbb{C}^*)^\infty$.

In the next theorem we show that for every toric ind-variety, we can find an equivalent filtration composed of toric varieties and toric morphisms.

Theorem 2.3. *Let $\mathcal{V} = \varinjlim V_i$ be an ind-variety endowed with a regular action of the ind-torus $\mathcal{T}$. Then $\mathcal{V}$ is an affine toric ind-variety with respect to $\mathcal{T}$ if and only if $\mathcal{V} \simeq \varinjlim W_j$ where W_j are affine toric varieties with acting torus T_j, the closed embedding $\varphi_j\colon W_j \hookrightarrow W_{j+1}$ are toric morphisms and the ind-torus $\mathcal{T}$ is the inductive limit $\varinjlim T_j$.*

Proof. The finite dimensional case is trivial since we can take $W_j = \mathcal{V}$ and $T_j = \mathcal{T}$, for all $j > 0$. Hence, we only deal with the case where $\mathcal{T} = (\mathbb{C}^*)^\infty$. To prove the "only if" part we may assume that each V_i is irreducible since $\mathcal{V}$ is curve-connected. Let W_j be the closure of $(\mathbb{C}^*)^j$ in $\mathcal{V}$. The acting torus in W_j is $T_j = (\mathbb{C}^*)^j$ and so it follows that $\mathcal{T} = \varinjlim T_j$. Fix an integer $j > 0$. Let A be a closed set in $\mathcal{V}$. Then $A \cap (\mathbb{C}^*)^\infty$ is closed in $(\mathbb{C}^*)^\infty$ so $A \cap (\mathbb{C}^*)^{j+1}$ is closed in $(\mathbb{C}^*)^{j+1}$. Hence, the inclusion $(\mathbb{C}^*)^{j+1} \hookrightarrow \mathcal{V}$ is continuous and so by [8, Lemma 1.1.5], there exist $i > 0$ such that $W_{j+1} \subset V_i$. Furthermore, the inclusion $(\mathbb{C}^*)^j \hookrightarrow (\mathbb{C}^*)^{j+1}$ induces an inclusion $\varphi_j\colon W_j \hookrightarrow W_{j+1}$. Since V_i is closed in $\mathcal{V}$ we have that W_j and W_{j+1} are closed in V_i and so φ_j is a closed embedding.

We claim that the varieties W_j are toric with respect to the algebraic tori $T_j = (\mathbb{C}^*)^j$ and that the morphisms $\varphi_j\colon W_j \hookrightarrow W_{j+1}$ are toric. Indeed, since $(\mathbb{C}^*)^j$ is irreducible, W_j is also irreducible, for all $j > 0$. Furthermore, the T_j-action on T_j by translations extends to a T_j-action in W_j since for every $t \in T_j$, we have $t.W_j$ is contained in the closure of $t.(\mathbb{C}^*)^j = (\mathbb{C}^*)^j$ and so W_j is stabilized by T_j. Finally, by [2, Proposition 1.11], the T_j-orbit $(\mathbb{C}^*)^j$ is locally closed in W_j and so we conclude that $(\mathbb{C}^*)^j$ is an open set in W_j. Hence W_j is a toric variety. Furthermore, the morphism $\varphi_j\colon W_j \hookrightarrow W_{j+1}$ is toric since its restriction to the acting torus is a group homomorphism by definition.

Finally, we prove that $\mathcal{V} \simeq \varinjlim W_j$ by proving that the filtrations given by V_i and W_j respectively, are equivalent. We already proved above that for every $j > 0$ there exists $i > 0$ such that $W_j \subset V_i$ is a closed embedding. To prove the other direction, we need to prove that for every V_i there exists W_k with $V_i \subset W_k$. Without loss of generality, we may and will assume that $W_1 \subset V_i$. Observe that the set $X = V_i \cap (\mathbb{C}^*)^\infty$ is a non-empty algebraic subset of $\mathcal{V}$. Furthermore, since $(\mathbb{C}^*)^\infty \subset \bigcup_{j>0} W_j$ and $X \subset (\mathbb{C}^*)^\infty$ we have $X = \bigcup_{j>0} X \cap W_j$. By [8, Lemma 1.3.1], there exists a positive integer k such that $X = X \cap W_k$ and so $X \subset W_k$. Moreover, the closure of X in $\mathcal{V}$ is V_i since V_i is irreducible by our assumption above. Since W_k is closed, we conclude that $V_i \subset W_k$ is a closed embedding. This concludes the proof of the "only if" part of the theorem.

We now prove the "if" direction of the theorem. The ind-variety $\mathcal{V} \simeq \varinjlim W_j$ is curve-connected since each W_j is irreducible. Furthermore, by Example 2.2 the limit $\mathcal{T} = \varinjlim T_j$ is an ind-torus. Moreover, $\mathcal{T}$ is an open set in $\varinjlim W_j$ by the definition of the ind-topology. Moreover, the action of $\mathcal{T}$ on itself by multiplication extends to $\varinjlim W_j$ since the same holds in all the strata for T_j acting on W_j. This concludes the proof. $\square$

Remark 2.4. The above theorem can be generalized to the case of ind-varieties endowed with an action of a nested ind-group, i.e., an ind-group admitting an equivalent filtration by algebraic groups [8, Section 9.4]. We restrict to the case of the ind-torus for simplicity.

Let $\mathcal{V} = \varinjlim V_i$ be a toric ind-variety. We say that $V_1 \hookrightarrow V_2 \hookrightarrow \ldots$ is a toric filtration if for every $i > 0$ the variety V_i is toric with acting torus T_i, the closed embedding $\varphi_i\colon V_i \hookrightarrow V_{i+1}$ is a toric morphism and the acting ind-torus $\mathcal{T}$ is the inductive limit $\varinjlim T_i$. Theorem 2.3 above ensures the every toric ind-variety admits a toric filtration.

We define toric morphisms in direct analogy with the case of classical toric varieties.

Definition 2.5. Let $\mathcal{T}_\mathcal{V}$ and $\mathcal{T}_{\mathcal{V}'}$ be ind-tori acting on toric ind-varieties $\mathcal{V} = \varinjlim V_i$ and $\mathcal{V}' = \varinjlim V'_j$, respectively. A morphism $\alpha\colon \mathcal{V} \to \mathcal{V}'$ of ind-varieties is toric if the image of $\mathcal{T}_\mathcal{V}$ by α is contained in $\mathcal{T}_{\mathcal{V}'}$ and $\alpha|_{\mathcal{T}_\mathcal{V}}\colon \mathcal{T}_\mathcal{V} \to \mathcal{T}_{\mathcal{V}'}$ is a morphism of ind-groups.

Proposition 2.6. *Let $\alpha\colon \mathcal{V} \to \mathcal{V}'$ be a morphism of toric ind-varieties $\mathcal{V}$ and $\mathcal{V}'$. Then α is a toric morphism if and only if for every pair of toric filtrations $V_1 \hookrightarrow V_2 \hookrightarrow \ldots$ and $V'_1 \hookrightarrow V'_2 \hookrightarrow \ldots$ of $\mathcal{V}$ and $\mathcal{V}'$, respectively, and every $i > 0$, there exists an integer $j > 0$ such that $\alpha|_{V_i}\colon V_i \to V'_j$ is a toric morphism.*

Proof. To prove the "only if" direction of the proposition, we assume that α is toric and by Theorem 2.3 we let $V_1 \hookrightarrow V_2 \hookrightarrow \ldots$ and $V'_1 \hookrightarrow V'_2 \hookrightarrow \ldots$ be toric filtrations of $\mathcal{V}$ and $\mathcal{V}'$, respectively. By definition of morphism of ind-varieties, for every $i > 0$ there exists $j > 0$ such that α restricts to a morphism of varieties $\alpha|_{V_i}\colon V_i \to V'_j$. Let $\mathcal{T}_\mathcal{V} = \varinjlim T_i$ and $\mathcal{T}_{\mathcal{V}'} = \varinjlim H_j$ are the acting tori with the filtration coming from the toric filtration of $\mathcal{V}$ and $\mathcal{V}'$, respectively. By the definition of toric morphism, we have $\alpha(T_i) \subset \mathcal{T}_{\mathcal{V}'}$ and so $\alpha(T_i) \subset H_j = V'_j \cap \mathcal{T}_{\mathcal{V}'}$. Since $\alpha\colon \mathcal{T}_\mathcal{V} \to \mathcal{T}_{\mathcal{V}'}$ is a group homomorphism, the same holds for $\alpha|_{T_i}\colon T_i \to H_j$. This proves this direction of the proposition.

To prove the "if" part, we let $V_1 \hookrightarrow V_2 \hookrightarrow \ldots$ and $V'_1 \hookrightarrow V'_2 \hookrightarrow \ldots$ be toric filtrations of $\mathcal{V}$ and $\mathcal{V}'$, respectively. We further assume that for every $i > 0$, there exists an integer $j > 0$ such that $\alpha|_{V_i}\colon V_i \to V'_j$ is a toric morphism. Furthermore, replacing the toric filtration of $\mathcal{V}'$ by a renumbered subfiltration we may and will assume $\alpha|_{V_i}\colon V_i \to V'_i$ is a toric morphism. It follows that $\alpha(T_i) \subset H_i$, where $\mathcal{T}_\mathcal{V} = \varinjlim T_i$ and $\mathcal{T}_{\mathcal{V}'} = \varinjlim H_j$ be the acting tori with the filtration coming from the toric filtration of $\mathcal{V}$ and $\mathcal{V}'$, respectively. Hence, we conclude $\alpha(\mathcal{T}_\mathcal{V}) \subset \mathcal{T}_{\mathcal{V}'}$. Similarly, the fact that $\alpha|_{T_i}\colon T_i \to H_i$ is a homomorphism of groups implies that $\alpha|_{\mathcal{T}_\mathcal{V}}\colon \mathcal{T}_\mathcal{V} \to \mathcal{T}_{\mathcal{V}'}$ is a homomorphism of ind-groups, proving the proposition. □

Remark 2.7. It is straightforward to show that a toric morphism $\alpha\colon \mathcal{V} \to \mathcal{V}'$ of toric ind-varieties is equivariant, i.e., $\alpha(t.x) = \alpha(t).\alpha(x)$, for all $t \in \mathcal{T}_\mathcal{V}$ and all $x \in \mathcal{V}$.

A character of an ind-torus $\mathcal{T}$ is a morphism $\chi\colon \mathcal{T} \to \mathbb{C}^*$ of ind-varieties that is also a group homomorphism. The set of characters of $\mathcal{T}$ forms a group denoted by $\mathcal{M}$. If $\dim \mathcal{T} < \infty$ it is well known that $\mathcal{M}$ is a finitely generated free abelian group of rank $\dim \mathcal{T}$. Similarly, a one-parameter subgroup of $\mathcal{T}$ is a morphism $\lambda\colon \mathbb{C}^* \to \mathcal{T}$ of ind-varieties that is also a group homomorphism. The set of one-parameter subgroups of $\mathcal{T}$ forms a group denoted by $\mathcal{N}$. If $\dim \mathcal{T} < \infty$ it is well known that $\mathcal{N}$ is also a finitely generated free abelian group of rank $\dim \mathcal{T}$. Furthermore, if $\dim \mathcal{T} < \infty$, then the groups $\mathcal{M}$ and $\mathcal{N}$ are dual with duality $\mathcal{M} \times \mathcal{N} \to \mathbb{Z}$ given by $\langle \chi, \lambda \rangle = k$ where k is the unique integer such that $\chi \circ \lambda\colon \mathbb{C} \to \mathbb{C}$ maps t to t^k.

We now compute the groups of characters and one-parameter subgroups of the infinite-dimensional ind-torus and prove the analogous duality result. Let $\mathcal{T}$ be the infinite-dimensional ind-torus with toric filtration $T_1 \hookrightarrow T_2 \hookrightarrow \ldots$. Letting M_i and N_i be the character lattice and the one-parameter subgroup lattice of T_i, respectively, the filtration induces naturally a projective system $M_1 \leftarrow M_2 \leftarrow \dots$ and an inductive system $N_1 \to N_2 \to \dots$.

Proposition 2.8. *Let $\mathcal{T}$ be the infinite-dimensional ind-torus with toric filtration $T_1 \hookrightarrow T_2 \hookrightarrow \dots$. Then*

1. *The group of characters $\mathcal{M}$ of $\mathcal{T}$ is $\varprojlim M_i$ and is isomorphic to $\mathbb{Z}^\omega$.*
2. *The group of one-parameter subgroups $\mathcal{N}$ of $\mathcal{T}$ is $\varinjlim N_i$ and is isomorphic to $\mathbb{Z}^\infty$.*
3. *The groups $\mathcal{M}$ and $\mathcal{N}$ are naturallly dual to each other and the duality is realized by the pairing $\langle\ ,\ \rangle\colon \mathcal{M} \times \mathcal{N} \to \mathbb{Z}$ given by $\langle \chi, \lambda\rangle = k$, where $\lambda \circ \chi\colon \mathbb{C}^* \to \mathbb{C}^*$ maps $t \mapsto t^k$.*

Proof. To prove (1), we let $\chi\colon \mathcal{T} \to \mathbb{C}^*$ be a character of $\mathcal{T}$. By the definition of morphism of ind-varieties, we have that $\chi|_{T_i}\colon T_i \to \mathbb{C}^*$ is a character of T_i for all $i > 0$. This produces homomorphisms $\pi_i\colon \mathcal{M} \to M_i$ satisfying $\pi_i = \varphi_i^* \circ \pi_{i+1}$, where $\varphi_i^*\colon M_{i+1} \to M_i$ is the map induced by $\varphi\colon T_i \to T_{i+1}$. By the universal property of the projective limit we have a homomorphism $\mathcal{M} \to \varprojlim M_i$. On the other hand, we define the inverse homomorphism $\varprojlim M_i \to \mathcal{M}$ in the following way. Let $(\chi_1, \chi_2, \dots)$ be an element in the projective limit $\varprojlim M_i$. We associate a character $\chi \in \mathcal{M}$ given by $\chi\colon \mathcal{T} \to \mathbb{C}^*$ via $t \mapsto \chi_k(t)$ for any $k > 0$ such that $t \in T_k$. By the definition of projective limit this map is well defined. It is a straightforward verification that it is a homomorphism. This proves that $\mathcal{M}$ is the projective limit $\varprojlim M_i$. Finally, $\mathcal{M}$ is isomorphic to $\mathbb{Z}^\omega$ by Example 1.1.

To prove (2), let $\lambda_i\colon \mathbb{C}^* \to T_i$ be a one-parameter subgroup in N_i. Composing with the injection $T_i \hookrightarrow \mathcal{T}$ we obtain a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to \mathcal{T}$ of the ind-torus. This yields homomorphisms $\psi_i\colon N_i \to \mathcal{N}$. By the universal property of the inductive limit we have a homomorphism $\varinjlim N_i \to \mathcal{N}$. On the other hand, we define the inverse homomorphism in the following way. Let $\lambda\colon \mathbb{C}^* \to \mathcal{T}$ be a one-parameter subgroup of $\mathcal{T}$. By the definition of morphism of ind-varieties, we have that there exists $k > 0$ such that the one-parameter subgroup λ restricts to $\lambda_k\colon \mathbb{C}^* \to T_k$ is a one-parameter subgroup of T_k. Hence, $\lambda_k \in N_k$ and composing with $\psi_k\colon N_k \to \varinjlim N_i$ we obtain a homomorphism $\mathcal{N} \to \varinjlim N_i$. By the definition of inductive limit this map is well defined. It is a straightforward verification that it is a homomorphism. Finally, $\mathcal{N}$ is isomorphic to $\mathbb{Z}^\infty$ by Example 1.1.

To prove (3), a routine computation shows that $\langle\ ,\ \rangle$ is bilinear and under the isomorphisms in (1) and (2) corresponds to the usual dot product defined in Lemma 1.2. This proves the proposition. □

In the proof of our main result, we will need the following lemma whose proof is straightforward.

Lemma 2.9. *Let $\mathcal{T}$ and $\mathcal{T}'$ be ind-tori and let $\alpha\colon \mathcal{T} \to \mathcal{T}'$ be an ind-group homomorphism with character group $\mathcal{M}_\mathcal{T}$ and $\mathcal{M}_{\mathcal{T}'}$ and one-parameter subgroup group $\mathcal{N}_\mathcal{T}$ and $\mathcal{N}_{\mathcal{T}'}$. Then α induces homomorphisms $\alpha^*\colon \mathcal{M}_{\mathcal{T}'} \to \mathcal{M}_\mathcal{T}$ and $\alpha_*\colon \mathcal{N}_\mathcal{T} \to \mathcal{N}_{\mathcal{T}'}$.*

3. PRO-AFFINE SEMIGROUPS

A semigroup is a set $(\mathcal{S}, +)$ with an associative binary operation. All our semigroups will be commutative and unital. A semigroup S is called affine if it is finitely generated and can be embedded in a $\mathbb{Z}^k$ for some $k \geq 0$. It is well known that the category of affine toric varieties with toric morphisms is dual to the category of affine semigroups with homomorphisms of semigroups. The main result of this paper is a generalization of this result to the case of affine toric ind-varieties. In this section, we define and study the semigroups $\mathcal{S}$ that will appear as the semigroup of an affine toric ind-variety $\mathcal{V}$.

Recall that the ring of regular functions $\mathbb{C}[\mathcal{V}]$ of an ind-variety is a pro-affine algebra and so it is endowed with a topology holding the information of the filtration of $\mathcal{V}$ [9]. We will first transport the notion of pro-affine algebra into the context of semigroups. A pro-affine algebra $\mathcal{A}$ is defined using a filtration of ideals on $\mathcal{A}$ and the projective limit topology induced by the quotients of $\mathcal{A}$ by the ideals in this filtration. In the case of semigroups, there exists an analog notion of ideal, but there is no bijection between ideals and quotient semigroups. For this reason, in the context of semigroups, we need the more general notion of compatible equivalence relations to keep track of all the possible quotients.

An equivalence relation on a set $\mathcal{S}$ is a subset $R \subset \mathcal{S} \times \mathcal{S}$ satisfying the usual properties of being reflexive, symmetric and transitive. An equivalence relation on a semigroup $\mathcal{S}$ is called compatible if for every (m, n) and (m', n') in R we have that $(m + m', n + n')$ also belongs to R. In this case, the set of equivalence classes $\mathcal{S}/R$ inherits a natural structure of semigroup with binary operation given by $[m] + [m'] = [m + m']$, where $[m]$ denotes the class of m in $\mathcal{S}/R$.

A filtered semigroup is a couple $(\mathcal{S}, F)$, where $\mathcal{S}$ is a semigroup and F is a descending filtration $R_1 \supset R_2 \supset \dots$ of $\mathcal{S} \times \mathcal{S}$ of compatible equivalence relations on $\mathcal{S}$. We denote a filtered semigroup simply by $\mathcal{S}$ if F is clear from the context. In close analogy with [13, Section 9.2], the filtration of compatible equivalence relations on $\mathcal{S}$ defines a topology on $\mathcal{S}$ having basis $\{E_{m,k} \mid m \in \mathcal{S}, k > 0\}$, where $E_{m,k} = \{m' \in \mathcal{S} \mid (m, m') \in R_k\}$ is the equivalence class of m under the equivalence relation R_k. It is straightforward to verify that this topology coincides with the finest topology making all the quotient morphisms $\mathcal{S} \to \mathcal{S}/R_k$ continuous where $\mathcal{S}/R_k$ is taken with the discrete topology. The trivial equivalence relation on $\mathcal{S}$ corresponds to the diagonal in $\mathcal{S} \times \mathcal{S}$. The trivial filtration on a semigroup $\mathcal{S}$ is given by setting each equivalence relation R_i to be trivial. In this case the induced topology on $\mathcal{S}$ is the discrete topology.

Let $\mathcal{S}$ be filtered semigroup with filtration $R_1 \supset R_2 \supset \dots$ of compatible equivalence relations in $\mathcal{S}$. It is straightforward to verify that the topology on $\mathcal{S}$ is Hausdorff if and only if $\bigcap_{k>0} R_k$ equals the diagonal in $\mathcal{S} \times \mathcal{S}$. Additionally, we can generalize the notion of Cauchy sequence to this context of semigroups. Indeed, a sequence $\{a^{(i)}\}_{i>0} \subset \mathcal{S}$ in the semigroup is said to be Cauchy sequence if given any $k > 0$ there exists an integer N such that $(a^{(i)}, a^{(j)}) \in R_k$ for all $i, j > N$. A direct computation shows that a convergent sequence is always Cauchy. We say that a filtered semigroup $\mathcal{S}$ is complete if every Cauchy sequence converges.

Given a projective system $S_1 \leftarrow S_2 \leftarrow \dots$ of semigroups we define a filtration $R_1 \supset R_2 \supset \dots$ of compatible equivalence relations on the projective limit $\mathcal{S} = \varprojlim S_i$ by $R_i = \{(m, m') \in \mathcal{S} \times \mathcal{S} \mid \pi_i(m) = \pi_i(m')\}$. The topology induced on $\mathcal{S}$ by this filtration coincides with the projective limit topology.

Proposition 3.1. *Let $S_1 \leftarrow S_2 \leftarrow \dots$ be a projective system of semigroups where each S_i carries the discrete topology. Then the projective limit semigroup $\mathcal{S} = \varprojlim S_i$ is Hausdorff and complete.*

Proof. A couple $(m, m') \subset \mathcal{S} \times \mathcal{S}$ belongs to R_k if and only if $m_i = m'_i$ for all $i \leq k$. Hence, the couple (m, m') belongs to $\bigcap_{k>0} R_k$ if and only if $m = m'$. We conclude that $\bigcap_{k>0} R_k$ equals the diagonal of $\mathcal{S} \times \mathcal{S}$ and so $\mathcal{S}$ is Hausdorff. To prove that $\mathcal{S}$ is complete, let $\{m^{(i)}\}_{i>0} \subset \mathcal{S}$ be a Cauchy sequence in $\mathcal{S}$. Recall that, by the definition of projective limit, each $m^{(i)}$ equals $\left(m_1^{(i)}, m_2^{(i)}, \dots\right) \in \prod_{i>0} S_i$. For every $k > 0$ there exist N such that $(m^{(i)}, m^{(i+1)}) \in R_k$ for all $i > N$. Hence, for every k there exist N such that

$m_k^{(i)} = m_k^{(i+1)} = m_k^{(i+2)} = \cdots$ when $i > N$. Letting $m_k = m_k^{(i)} \in S_k$ for any $i > N$, we let $m = (m_1, m_2 \dots) \in \mathcal{S}$. Now, for every $k > 0$ there exist N such that $\big(m, m^{(i)}\big) \in R_k$ for all $i > N$ and so the Cauchy sequence $\big\{m^{(i)}\big\}_{i>0} \subset \mathcal{S}$ converges to m. □

Remark 3.2. If a filtered semigroup $\mathcal{S}$ with filtration $R_1 \supset R_2 \supset \dots$ of compatible equivalence relations in $\mathcal{S}$ is Hausdorff and complete, then $\varprojlim S_i$, where $S_i = \mathcal{S}/R_i$ with the morphism induced from $R_i \supset R_{i+1}$, is canonically isomorphic to $\mathcal{S}$. Indeed, the canonical map $\mathcal{S} \to \varprojlim S_i$ into the projective limit given by $m \mapsto (\pi_1(m), \pi_2(m), \dots)$ has inverse given by $([m_1], [m_2], \dots) \mapsto \lim m_i$, where $\{m_i\}_{i>0}$ is the Cauchy sequence given in $\mathcal{S}$ by $\{m_1, m_2, \dots\}$.

We now define the natural notion of morphism of filtered semigroups.

Definition 3.3. (1) Let S and S' be filtered semigroups with filtrations $R_1 \supset R_2 \supset \dots$ and $R'_1 \supset R'_2 \supset \dots$, respectively. A map $\beta\colon \mathcal{S} \to \mathcal{S}'$ is called a morphism of filtered semigroups if β is a semigroup homomorphism and for every $i > 0$ there exists $j > 0$ such that $(\beta \times \beta)(R_j) \subset R'_i$. In particular, every morphism $\beta\colon \mathcal{S} \to \mathcal{S}'$ of filtered semigroups is continuous since the condition $(\beta \times \beta)(R_j) \subset R'_i$ implies point-wise continuity at every $m \in \mathcal{S}$. As usual, an isomorphism $\beta\colon \mathcal{S} \to \mathcal{S}'$ of filtered semigroups is a bijective morphism whose inverse is also a morphism. We also say that two filtrations $R_1 \supset R_2 \supset \dots$ and $R'_1 \supset R'_2 \supset \dots$ on the same semigroup $\mathcal{S}$ are equivalent if the identity map is an isomorphism of filtered semigroups.

(2) Let $\mathcal{S}$ be a filtered semigroup with filtration $R_1 \supset R_2 \supset \dots$. A filtered subsemigroup is a semigroup $\mathcal{S}' \subset \mathcal{S}$ endowed with the filtration of compatible equivalence relations $R_i \cap (\mathcal{S}' \times \mathcal{S}')$ on $\mathcal{S}'$.

Lemma 3.4. *With the notation in Definition 3.3, the morphism $\beta\colon \mathcal{S} \to \mathcal{S}'$ of filtered semigroups induces a natural homomorphism of semigroups $\beta_{ij}\colon S_j \to S'_i$ where $S_j = \mathcal{S}/R_j$ and $S'_i = \mathcal{S}'/R'_i$ such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\beta} & \mathcal{S}' \\ {\scriptstyle \pi_j}\big\downarrow & & \big\downarrow{\scriptstyle \pi'_i} \\ S_j & \xrightarrow{\beta_{ij}} & S'_i \end{array}$$

Proof. The map $\beta_{ij}\colon S_j \to S_i$ defined naturally by $[m] \mapsto [\beta(m)]$ is well defined due to the condition $(\varphi \times \varphi)(R_j) \subset R'_i$. The rest of the proof is straightforward. □

We now define pro-affine semigroups, which are the generalization of the affine semigroups that are the objects dual to classical affine toric varieties.

Definition 3.5.
A pro-affine semigroup $\mathcal{S}$ is a filtered semigroup with filtration $R_1 \supset R_2 \supset \dots$ of compatible equivalence relations in $\mathcal{S}$ that is complete, Hausdorff and such that every $\mathcal{S}/R_i$ is an affine semigroup.

Example 3.6.

(1) We define the canonical filtration $\widetilde{R}_1 \supset \widetilde{R}_2 \supset \dots$ of equivalence relations on the semigroup $\mathbb{Z}^\omega$ by $\widetilde{R}_k = \{(m, m') \in \mathbb{Z}^\omega \times \mathbb{Z}^\omega \mid m_i = m'_i, \text{ for all } i \leq k\}$. By Proposition 3.1, we conclude that $\mathbb{Z}^\omega$ is complete. Furthermore, $\mathbb{Z}^\omega/R_i$ is naturally

isomorphic to $\mathbb{Z}^i$ with quotient morphism $\pi_i\colon \mathbb{Z}^\omega \to \mathbb{Z}^i$ the projection to the first i-th coordinates. Hence, $\mathbb{Z}^\omega/R_i$ is an affine semigroup and so the filtered semigroup $\mathbb{Z}^\omega$ is a pro-affine semigroup.

(2) The filtered subsemigroups $\mathbb{Z}^\omega_{\geq 0}$ of $\mathbb{Z}^\omega$ of arbitrary sequences of non-negative integers is also pro-affine by a similar argument as in (1).

(3) Any affine semigroup $S \subset \mathbb{Z}^i$ with the constant filtration given by the trivial equivalence relation is pro-affine.

(4) Let $e_i = (0, \ldots, 0, 1, 0, \ldots) \in \mathbb{Z}^\omega$, where the non-zero coefficient is located at the position $i > 0$. The subsemigroup $\mathcal{S} = \mathbb{Z}^\omega_{\geq 0} \setminus \{e_1\}$ of $\mathbb{Z}^\omega$ is not complete and so is not pro-affine. Indeed, the sequence $\{a_i = e_1 + e_i\}_{i>0}$ is Cauchy but not convergent in $\mathcal{S}$.

Theorem 3.7. *Let $\mathcal{S}$ be a pro-affine semigroup, then $\mathcal{S}$ is isomorphic to a filtered subsemigroup of $\mathbb{Z}^\omega$. Moreover, we can assume that $\mathcal{S}$ is embedded in $\mathcal{M}$ with $\mathbb{Z}\mathcal{S} = \mathcal{M}$, where $\mathcal{M} \simeq \mathbb{Z}^\omega$ or $\mathcal{M} \simeq \mathbb{Z}^k$ for some $k > 0$.*

Proof. Letting $R_1 \supset R_2 \supset \ldots$ be the filtration of compatible equivalence relations in $\mathcal{S}$ we let $S_i = \mathcal{S}/R_i$ and $\varphi_i\colon S_{i+1} \to S_i$ be the homomorphisms given by the inclusions $R_i \supset R_{i+i}$. Hence, we have a commutative diagram

$$\begin{array}{ccccccc}
S_1 & \longleftarrow & S_2 & \longleftarrow & S_3 & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Z}S_1 & \longleftarrow & \mathbb{Z}S_2 & \longleftarrow & \mathbb{Z}S_3 & \longleftarrow & \cdots
\end{array}$$

where $\mathbb{Z}S_i$ is the group generated by S_i for any embedding $S_i \hookrightarrow \mathbb{Z}^k$ and the homomorphisms $\mathbb{Z}S_{i+1} \to \mathbb{Z}S_i$ are induced by $S_{i+1} \to S_i$, for all $i > 0$. Since the homomorphisms in the upper system are surjective, the same holds for the lower system. Hence, the lower projective system is split. If the homomorphisms in the lower system become also injective for i large enough, then the projective limit of the lower system is isomorphic to $\mathbb{Z}^k$ for some $k \geq 0$. Furthermore, since $\mathbb{Z}^k$ is embedded in $\mathbb{Z}^\omega$ the first statement follows in this case. Assume now that there is no integer $i > 0$ such that the homomorphisms in the lower system become injective for all integer $j > i$. In this case, by Example 1.1 we have that the lower projective limit is isomorphic to $\mathbb{Z}^\omega$ and under this isomorphism we have that $\varprojlim S_i \subset \mathbb{Z}^\omega$ is an embedding of filtered semigroups. Since $\mathcal{S}$ is Hausdorff and complete, by Remark 3.2 we have $\mathcal{S} = \varprojlim S_i$. The second statement follows directly from the construction above in this proof. □

In the following example we show the surprising consequence of the Specker Theorem (Lemma 1.2) that every group homomorphism $\beta\colon \mathbb{Z}^\omega \to \mathbb{Z}^\omega$ is a morphism of filtered semigroups for the canonical filtration $\widetilde{R}_i$.

Example 3.8. (1) Every homomorphism $\beta\colon \mathbb{Z}^\omega \to \mathbb{Z}^\omega$ is a morphism of filtered semigroups with respect to the canonical filtration. Indeed, since $\mathbb{Z}^\omega$ is a group, we have that $E_{0,k}$ is a subgroup of $\mathbb{Z}^\omega$ and

$$\widetilde{R}_k = \bigcup_{m \in \mathbb{Z}^\omega} (m + E_{0,k}) \times (m + E_{0,k})$$

Hence, it is enough to show that for every $i > 0$ there exists $j > 0$ such that $\beta(E_{0,j}) \subset E_{0,i}$. By Lemma 1.2, the composition $\pi_i \circ \beta\colon \mathbb{Z}^\omega \to \mathbb{Z}^i$ corresponds to an element in $(p_1, \ldots, p_i) \in (\mathbb{Z}^\infty)^i$ under the isomorphism $\operatorname{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \simeq \mathbb{Z}^\infty$ given by the duality map, see also [7, Theorem 94.3 and Corollary 94.5]. By definition of inductive limit, each $p_i \in \mathbb{Z}^{j_i}$ for some $j_i > 0$. Taking j to be the maximum of $\{j_1, \dots, j_i\}$ we obtain that $\beta(E_{0,j}) \subset E_{0,i}$.

(2) A similar argument shows that every homomorphism $\beta\colon \mathbb{Z}^\omega \to \mathbb{Z}^k$ is a morphism of filtered semigroups with respect to the canonical filtration in $\mathbb{Z}^\omega$ and trivial filtration in $\mathbb{Z}^k$, for every $k \geq 0$.

The above example allows us to prove that every homomorphism between pro-affine semigroups is a morphism.

Proposition 3.9. *Let $\mathcal{S}$ and $\mathcal{S}'$ be pro-affine semigroups. If $\beta\colon \mathcal{S} \to \mathcal{S}'$ is any homomorphism of semigroups then β is a morphism of filtered semigroups.*

Proof. By Theorem 3.7, we can assume that $\mathcal{S}$ is a subsemigroup of $\mathcal{M} = \mathbb{Z}^\omega$ or $\mathcal{M} = \mathbb{Z}^k$ for some $k \geq 0$ with $\mathbb{Z}S = \mathcal{M}$. Similarly, we can assume that $\mathcal{S}'$ is a subsemigroup of $\mathcal{M}' = \mathbb{Z}^\omega$ or $\mathcal{M}' = \mathbb{Z}^\ell$ for some $\ell \geq 0$ with $\mathbb{Z}\mathcal{S}' = \mathcal{M}'$. The homomorphism β can be extended to a homomorphism $\widehat{\beta}\colon \mathcal{M} \to \mathcal{M}'$ via $m - m' \mapsto \beta(m) - \beta(m')$. If $\mathcal{M} = \mathbb{Z}^k$, then $\widehat{\beta}$ is trivially a morphism of filtered semigroups since the filtration by equivalence relation on $\mathbb{Z}^k$ is trivial. Furthermore, if $\mathcal{M} = \mathbb{Z}^\omega$ the homomorphism $\widehat{\beta}$ is also a morphism of filtered semigroups by Example 3.8. Now, the proposition follows since $\mathcal{S}$ and $\mathcal{S}'$ are filtered subsemigroups of $\mathcal{M}$ and $\mathcal{M}'$, respectively. $\square$

Remark 3.10. It follows from Proposition 3.9 above that two different filtrations $R_1 \supset R_2 \supset \dots$ and $R_1' \supset R_2' \supset \dots$ of compatible equivalence relations in a pro-affine semigroup $\mathcal{S}$ are always equivalent since the identity is an isomorphism of semigroups and so it is also an isomorphism of filtered semigroups.

It is straightforward to prove, mimicking the classical argument for metric spaces, that a subsemigroup in a complete filtered semigroup is complete if and only if it is closed. This allows us to derive the following corollary that acts as alternative definition of pro-affine semigroups. Recall that $\mathbb{Z}^\omega/\widetilde{R}_i$ is naturally isomorphic $\mathbb{Z}^i$ with quotient morphism $\pi_i\colon \mathbb{Z}^\omega \to \mathbb{Z}^i$ the projection to the first i-th coordinates.

Corollary 3.11. *An abstract semigroup $\mathcal{S}$ admits a filtration by compatible equivalence relations on $\mathcal{S}$ making $\mathcal{S}$ a pro-affine semigroup if and only if there exists an embedding $\iota\colon \mathcal{S} \hookrightarrow \mathbb{Z}^\omega$ where $\iota(\mathcal{S})$ is closed and $(\pi_i \circ \iota)(\mathcal{S})$ is finitely generated for every $i > 0$. Moreover, if such a filtration exists, then it is unique (up to equivalence).*

Proof. If $\mathcal{S}$ admits a structure of pro-affine semigroup, then the "only if" part follows from Theorem 3.7. On the other hand, if $\mathcal{S}$ is embedded in $\mathbb{Z}^\omega$, then it inherits a filtration $R_1 \supset R_2 \supset$ from this embedding. By definition $\mathcal{S}/R_i \simeq (\pi_i \circ \iota)(\mathcal{S})$ which is assumed to be finitely generated. Furthermore, $\mathcal{S}$ is complete with the induced filtration since $\iota(S)$ is closed in $\mathbb{Z}^\omega$ and $\mathcal{S}$ is Hausdorff since $\mathbb{Z}^\omega$ is. This yields that $\mathcal{S}$ is a pro-affine semigroup with this filtration. Finally, the uniqueness statement follows from Proposition 3.9 and Remark 3.10. $\square$

4. AFFINE TORIC IND-VARIETIES AND PRO-AFFINE SEMIGROUPS

In this section we prove that the category of affine toric ind-varieties with toric morphisms is dual to the category of pro-affine semigroups with homomorphisms of semigroups.

Given an affine toric ind-variety $\mathcal{V}$ with toric filtration $V_1 \hookrightarrow V_2 \hookrightarrow \dots$, applying the functor $\mathcal{S}(\bullet)$ defined in Section 1.1, we obtain a projective system

$$\begin{array}{ccccccc} V_1 & \xrightarrow{\varphi_1} & V_2 & \xrightarrow{\varphi_2} & V_3 & \xrightarrow{\varphi_3} & \cdots \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ S_1 & \xleftarrow{\mathcal{S}(\varphi_1)} & S_2 & \xleftarrow{\mathcal{S}(\varphi_2)} & S_3 & \xleftarrow{\mathcal{S}(\varphi_3)} & \cdots \end{array}$$

where each semigroup $S_i = \mathcal{S}(V_i)$ is the affine semigroup associated to the toric variety V_i, i.e, $\mathbb{C}[V_i] = \mathbb{C}[S_i]$ and $\mathcal{S}(\varphi_i)\colon S_{i+1} \to S_i$ is the semigroup homomorphism corresponding to the toric morphism $\varphi_i\colon V_i \to V_{i+1}$ [4, Proposition 1.3.14]. We define the semigroup $\mathcal{S}(\mathcal{V})$ associated to $\mathcal{V}$ as the projective limit $\varprojlim S_i$ of this projective system. By Proposition 3.1 and the paragraph preceding it, we have that $\mathcal{S}(\mathcal{V})$ is a pro-affine semigroup.

On the other hand, given a pro-affine semigroup $\mathcal{S}$ with the filtration $R_1 \supset R_2 \supset \dots$ of compatible equivalence relations on $\mathcal{S}$, we let $S_1 \leftarrow S_2 \leftarrow \dots$ be the associated projective system of semigroups where each $S_i = \mathcal{S}/R_i$ is an affine semigroup and the homomorphisms $\varphi_i\colon S_{i+1} \to S_i$ are given by $[m]_{i+1} \mapsto [m]_i$, where $[m]_i$ is the class of $m \in \mathcal{S}$ inside the quotient S_i. The homomorphisms φ_i are surjetive. Hence, applying the functor $\mathcal{V}(\bullet)$ defined in Section 1.1 for toric varieties, we obtain an inductive system of closed embeddings

$$\begin{array}{ccccccc} S_1 & \xleftarrow{\varphi_1} & S_2 & \xleftarrow{\varphi_2} & S_3 & \xleftarrow{\varphi_3} & \cdots \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ V_1 & \xrightarrow{\mathcal{V}(\varphi_1)} & V_2 & \xrightarrow{\mathcal{V}(\varphi_2)} & V_3 & \xrightarrow{\mathcal{V}(\varphi_3)} & \cdots \end{array}$$

where each $V_i = \mathcal{V}(S_i)$ is the toric variety associated to the semigroup S_i and $\mathcal{V}(\varphi_i)\colon V_i \to V_{i+1}$ is the toric morphism corresponding to the semigroup homomorphism $\varphi_i\colon S_{i+1} \to S_i$. The corresponding inductive limit $\varinjlim V_i$ of this system is an affine toric ind-variety by Theorem 2.3 that we denote by $\mathcal{V}(\mathcal{S})$. The ind-torus acting on $\mathcal{V}(\mathcal{S})$ is $\mathcal{T} = \varinjlim T_i$, where T_i is the algebraic torus acting on V_i. It is clear that these constructions provide a bijection between affine toric varieties and pro-affine semigroups up to isomorphisms.

Let now $\mathcal{V}$ be an affine toric ind-variety and let $\mathcal{S} = \mathcal{S}(\mathcal{V})$. In general, projective limits do not commute with direct sums, hence we cannot expect to have, as in the classical case, an isomorphism between the ring of regular functions $\mathbb{C}[\mathcal{V}]$ on $\mathcal{V}$ and the semigroup algebra $\mathbb{C}[\mathcal{S}]$, see Example 5.2 below. Nevertheless, the semigroup algebra carries a natural descending filtration of ideals $I_1 \supset I_2 \supset \dots$, where $I_i = \ker \pi_i$ and π_i is the natural projection $\pi_i\colon \mathbb{C}[\mathcal{S}] \to \mathbb{C}[V_i]$, for all $i > 0$ induced by the projections $\overline{\pi}_i\colon \mathcal{S} \to S_i$ coming from the projective limit. It follows directly from [13, Chapter 9, Theorem 10] that the algebra $\mathbb{C}[\mathcal{V}]$ is the completion of $\mathbb{C}[\mathcal{S}]$ with respect to $I_1 \supset I_2 \supset \dots$.

In the following proposition, we summarize the considerations above.

Proposition 4.1. *The assignments $\mathcal{V} \mapsto \mathcal{S}(\mathcal{V})$ for every affine toric ind-variety and $\mathcal{S} \mapsto \mathcal{V}(\mathcal{S})$ for every pro-affine semigroup are inverses up to isomorphism, i.e., $\mathcal{V}(\mathcal{S}(\mathcal{V}))$ is isomorphic to $\mathcal{V}$ for every affine toric ind-variety and $\mathcal{S}(\mathcal{V}(\mathcal{S}))$ is isomorphic to $\mathcal{S}$ for every pro-affine semigroup $\mathcal{S}$. Furthermore, for every affine toric ind-variety $\mathcal{V}$, the ring of regular functions $\mathbb{C}[\mathcal{V}]$ is isomorphic as filtered algebra to the completion of $\mathbb{C}[\mathcal{S}]$.*

We will also need the following lemma generalizing the usual equivalent statement in the classical case.

Lemma 4.2. *Let $\mathcal{V}$ be an affine toric ind-variety with acting ind-torus $\mathcal{T}$ whose character lattice is $\mathcal{M}$. Then $\mathcal{S}(\mathcal{V})$ is naturally embedded in $\mathcal{M}$ with $\mathbb{Z}\mathcal{S}(\mathcal{V}) = \mathcal{M}$. On the other hand, let $\mathcal{S}$ be a pro-affine semigroup embedded in $\mathcal{M} \simeq \mathbb{Z}^\omega$ or $\mathcal{M} \simeq \mathbb{Z}^k$ for some $k \geq 0$ as filtered semigroup with $\mathbb{Z}\mathcal{S} = \mathcal{M}$. Then the character lattice of the ind-torus $\mathcal{T}$ acting on the affine toric ind-variety $\mathcal{V}(\mathcal{S})$ is naturally isomorphic to $\mathcal{M}$.*

Proof. The case where $\mathcal{M} \simeq \mathbb{Z}^k$ corresponds to the classical case of affine toric varieties. Hence, we will only deal with the case where $\mathcal{M} \simeq \mathbb{Z}^\omega$. Assume first that $\mathcal{V}$ is an affine toric ind-variety. With the above notation, by the classical case we have that each S_i is naturally embedded in the character lattice M_i of the algebraic torus T_i acting on V_i with $M_i = \mathbb{Z}S_i$. By Theorem 2.3, we have that $\mathcal{T}$ equals the inductive limit $\varinjlim T_i$. Furthermore, by Proposition 2.8 we have that $\mathcal{M}$ equals $\varprojlim M_i$. The first assertion now follows. On the other hand, given $\mathcal{S}$ embedded in $\mathcal{M} \simeq \mathbb{Z}^\omega$, we let M_i be the character lattice of the torus T_i acting on V_i. By the classical finite dimensional case of the lemma, we have $\mathbb{Z}S_i = M_i$. The result now follows again from Proposition 2.8. ☐

We come now to morphisms in both categories. Let first $\mathcal{S}$ and $\mathcal{S}'$ be pro-affine semigroups and let $\beta\colon \mathcal{S} \to \mathcal{S}'$ be a semigroup homomorphism. By Proposition 3.9 the pro-affine semigroups $\mathcal{S}$ and $\mathcal{S}'$ admit filtrations of equivalence relations $R_1 \supset R_2 \supset \ldots$ and $R'_1 \supset R'_2 \supset \ldots$, respectively, such that β is a morphism of filtered semigroups with respect to these filtrations. We let $\mathcal{V} = \mathcal{V}(\mathcal{S})$ and $\mathcal{V}' = \mathcal{V}(\mathcal{S}')$ be the corresponding affine toric ind-varieties defined above with the toric filtrations $V_1 \hookrightarrow V_2 \hookrightarrow \ldots$ and $V'_1 \hookrightarrow V'_2 \hookrightarrow \ldots$, respectively, where $V_i = \mathcal{V}(S_i)$, $V'_i = \mathcal{V}(S'_i)$ and the closed embeddings are $\mathcal{V}(\varphi_i)$ and $\mathcal{V}(\varphi'_i)$ associated to the surjective semigroup homomorphisms $\varphi_i\colon S_{i+1} \to S_i$ and $\varphi'_i\colon S'_{i+1} \to S'_i$ respectively. We define a homomorphism $\mathbb{C}[\mathcal{S}] \to \mathbb{C}[\mathcal{S}']$ of semigroup algebras by $\chi^m \mapsto \chi^{\beta(m)}$, for all $m \in \mathcal{S}$. By abuse of notation, we denote this map also by $\beta\colon \mathbb{C}[\mathcal{S}] \to \mathbb{C}[\mathcal{S}']$.

Lemma 4.3. *The homomorphism $\beta\colon \mathbb{C}[\mathcal{S}] \to \mathbb{C}[\mathcal{S}']$ is a continuous homomorphism of topological algebras and so we can extend β to an unique continuous homomorphism $\mathcal{V}(\beta)^*\colon \mathbb{C}[\mathcal{V}] \to \mathbb{C}[\mathcal{V}']$ whose comorphism defines a toric morphism of affine toric ind-varieties $\mathcal{V}(\beta)\colon \mathcal{V}' \to \mathcal{V}$.*

Proof. To prove that $\beta\colon \mathbb{C}[\mathcal{S}] \to \mathbb{C}[\mathcal{S}']$ is continuous we have to prove that for all $i > 0$ there exists $j > 0$ such that $\beta(I_j) \subset I'_i$. Here $I_j = \ker \pi_j$ and π_j is the projection $\pi_j\colon \mathbb{C}[\mathcal{S}] \to \mathbb{C}[S_j]$ induced by $\mathcal{S} \to S_j$, for all $j > 0$ and similarly $I'_i = \ker \pi'_i$ and π'_i is the projection $\pi'_i\colon \mathbb{C}[\mathcal{S}'] \to \mathbb{C}[S'_i]$ induced by $\mathcal{S}' \to S'_i$, for all $i > 0$.

Let $i > 0$ be an integer. By Proposition 3.9 and the definition of morphism of filtered semigroup, there exists $j > 0$ such that $(\beta \times \beta)(R_j) \subset R'_i$. Let $f = \sum a_m \chi^m$ be an element in I_j where the sum is finite. Belonging to I_j is equivalent to $\pi_j(f) = \sum a_m \chi^{\pi_j(m)} = 0$. On the other hand, $\pi'_i(\beta(f)) = \sum a_m \chi^{(\pi'_i \circ \beta)(m)}$. By Lemma 3.4, the homomorphism β induces a homomorphism $\beta_{ij}\colon S_j \to S'_i$ and we have $\pi'_i \circ \beta = \beta_{ij} \circ \pi_j$ so we have $\pi'_i(\beta(f)) = \sum a_m \chi^{(\beta_{ij} \circ \pi_j)(m)} = \beta_{ij}\big(\sum a_m \chi^{\pi_j(m)}\big) = 0$. We conclude that $\beta(I_j) \subset I'_i$ and so $\beta\colon \mathbb{C}[\mathcal{S}] \to \mathbb{C}[\mathcal{S}']$ is continuous.

Finally, the algebra $\mathbb{C}[\mathcal{S}]$ is dense in $\mathbb{C}[\mathcal{V}]$ by the second statement of Proposition 4.1. Hence, the homomorphism β can be extended to a continuous homomorphism $\mathcal{V}(\beta)^*\colon \mathbb{C}[\mathcal{V}] \to \mathbb{C}[\mathcal{V}']$ as required, see [13, Ch.9, Th. 5]. Moreover, by Proposition 2.6, the morphism $\mathcal{V}(\beta)\colon \mathcal{V}' \to \mathcal{V}$ is toric. ☐

Let now $\alpha\colon \mathcal{V} \to \mathcal{V}'$ be a toric morphism of affine toric ind-varieties and let $\mathcal{S} = \mathcal{S}(\mathcal{V})$ and $\mathcal{S}' = \mathcal{S}(\mathcal{V}')$ be the corresponding pro-affine semigroups. By Lemma 4.2 we have that

$\mathcal{S}$ and $\mathcal{S}'$ are naturally embedded in $\mathcal{M}$ and $\mathcal{M}'$, respectively. In particular, we have that $\alpha|_{\mathcal{T}_\mathcal{V}}\colon \mathcal{T}_\mathcal{V} \to \mathcal{T}_{\mathcal{V}'}$ is a homomorphism of ind-groups and so by Lemma 2.9 the comorphism $(\alpha|_{\mathcal{T}_\mathcal{V}})^*$ induces a semigroup homomorphism $\alpha^\vee\colon \mathcal{M}' \to \mathcal{M}$ on the character lattices via $(\alpha|_{\mathcal{T}_\mathcal{V}})^*(\chi^m) = \chi^{\alpha^\vee(m)}$. Furthermore, given $m \in \mathcal{S}'$, the regular function $\chi^m \in \mathbb{C}[\mathcal{V}']$ is mapped to the regular function $\chi^{\alpha^\vee(m)} \in \mathbb{C}[\mathcal{V}]$. This yields $\alpha^\vee(m) \in \mathcal{S}$, for all $m \in \mathcal{S}'$. Hence $\alpha^\vee$ restricts to a homomorphism $\mathcal{S}' \to \mathcal{S}$. We denote this homomorphism by $\mathcal{S}(\alpha)$.

In the following proposition, we summarize the considerations above.

Proposition 4.4. *let $\mathcal{S}$ and $\mathcal{S}'$ be pro-affine semigroups. Then, for every homomorphism $\beta\colon \mathcal{S} \to \mathcal{S}'$ the map $\mathcal{V}(\beta)\colon \mathcal{V}(\mathcal{S}') \to \mathcal{V}(\mathcal{S})$ is a toric morphism of affine toric ind-varieties. Moreover, for every toric morphism $\alpha\colon \mathcal{V}(\mathcal{S}') \to \mathcal{V}(\mathcal{S})$ there exists a unique homomorphism $\beta\colon \mathcal{S} \to \mathcal{S}'$ such that $\alpha = \mathcal{V}(\beta)$. In particular, for every pair of pro-affine semigroups $\mathcal{S}$ and $\mathcal{S}'$ there is a bijection between semigroup homomorphisms $\mathcal{S} \to \mathcal{S}'$ and toric morphisms $\mathcal{V}(\mathcal{S}') \to \mathcal{V}(\mathcal{S})$.*

The assignment $\mathcal{V}(\bullet)$ is a contravariant functor, i.e., $\mathcal{V}(\mathrm{id}) = \mathrm{id}$ and $\mathcal{V}(\beta' \circ \beta) = \mathcal{V}(\beta) \circ \mathcal{V}(\beta')$, for every pair of semigroup homomorphisms $\beta\colon \mathcal{S} \to \mathcal{S}'$ and $\beta'\colon \mathcal{S}' \to \mathcal{S}''$, where $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{S}''$ are pro-affine semigroups. This follows directly from the definition of $\mathcal{V}(\beta)$ as the comorphism of the unique extension of the morphism $\mathbb{C}[\mathcal{S}] \to \mathbb{C}[\mathcal{S}']$ given by $\chi^m \mapsto \chi^{\beta(m)}$.

On the other hand, the assignment $\mathcal{S}(\bullet)$ is also a contravariant functor. Indeed, let $\alpha'\colon \mathcal{V}'' \to \mathcal{V}'$ and $\alpha\colon \mathcal{V}' \to \mathcal{V}$ be morphisms of affine toric ind-varieties $\mathcal{V}$, $\mathcal{V}'$ and $\mathcal{V}''$. By Proposition 4.1 and Proposition 4.4, there exist pro-affine semigroups $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S}''$ such that $\mathcal{V} = \mathcal{V}(\mathcal{S})$, $\mathcal{V}' = \mathcal{V}(\mathcal{S}')$ and $\mathcal{V}'' = \mathcal{V}(\mathcal{S}'')$ with morphisms $\beta\colon \mathcal{S} \to \mathcal{S}'$ and $\beta'\colon \mathcal{S}' \to \mathcal{S}''$ such that $\beta = \mathcal{S}(\alpha)$ and $\beta' = \mathcal{S}(\alpha')$. By Proposition 4.4, we have $\mathcal{V}(\beta' \circ \beta) = \alpha \circ \alpha'$ or, equivalently, $\beta' \circ \beta = \mathcal{S}(\alpha \circ \alpha')$ so that $\mathcal{S}(\alpha') \circ \mathcal{S}(\alpha) = \mathcal{S}(\alpha \circ \alpha')$.

In the following theorem, that is our main result, we summarize the results in this section.

Theorem 4.5.

(1) The assignment $\mathcal{V}(\bullet)$ is a contravariant functor from the category of pro-affine semigroups with homomorphisms of semigroups to the category of affine toric ind-varieties with toric morphisms.

(2) The assignment $\mathcal{S}(\bullet)$ is a contravariant functor from the category of affine toric ind-varieties with toric morphisms to the category of pro-affine semigroups with homomorphisms of semigroups.

(3) The pair $(\mathcal{V}(\bullet), \mathcal{S}(\bullet))$ is a duality between the categories of affine toric ind-varieties and pro-affine semigroups.

A well-known feature of the classical duality between affine toric varieties and affine semigroups is the correspondence between points on the toric variety and semigroup homomorphism to $(\mathbb{C}, \cdot)$. In the following proposition, we generalize this result to the case of affine toric ind-varieties.

Recall that a semigroup $\mathcal{S}$ has the cancellation property if $m + m' = m + m''$ implies $m' = m''$, with $m, m', m'' \in \mathcal{S}$. Let $(\mathbb{C}, \cdot)$ be the semigroup of complex numbers under multiplication. This semigroup is not pro-affine since it does not have the cancellation property and all pro-affine semigroups inherit the cancellation property from the embedding in $\mathbb{Z}^\omega$ shown in Corollary 3.11.

We endow $(\mathbb{C}, \cdot)$ with the trivial descending filtration $R_1' \supset R_2' \supset \ldots$ of compatible equivalence relations $R_i' = \{(t,t) \in \mathbb{C} \times \mathbb{C} \mid t \in \mathbb{C}\}$ so that $\mathbb{C}/R_i \simeq \mathbb{C}$. Unlike the case of pro-affine semigroups, not every semigroup homomorphism $\mathcal{S} \to (\mathbb{C}, \cdot)$ is a filtered morphism. See [7, page 159] and apply the fact that $(\mathbb{C}, \cdot)$ contains an isomorphic copy Q

of the additive group of the rational numbers. For instance, we can take $Q = \{\exp(q) \mid q \in \mathbb{Q}\}$ where $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is the usual exponential map.

Proposition 4.6. *Let $\mathcal{V}$ be an affine toric ind-variety and let $\mathcal{S} = \mathcal{S}(\mathcal{V})$. Then there are bijective correspondences between the following:*

(1) Points v in $\mathcal{V}$.
(2) Closed maximal ideals $\mathfrak{m}$ in $\mathbb{C}[\mathcal{V}]$, which is isomorphic to the completion of $\mathbb{C}[\mathcal{S}]$.
(3) Morphisms of filtered semigroups $\Lambda\colon \mathcal{S} \to (\mathbb{C}, \cdot)$.

Proof. The correspondence between of (1) and (2) is general for ind-varieties and was first proven in [9]. Let $R_1 \supset R_2 \supset \ldots$ be the filtration of compatible equivalence relations in $\mathcal{S}$ and let $\Lambda\colon \mathcal{S} \to \mathbb{C}$ be a filtered semigroup morphism. By the definition of filtered semigroups, there exists $j > 0$ such that $(\Lambda \times \Lambda)(R_j)$ is contained in the diagonal in $\mathbb{C} \times \mathbb{C}$ defining the trivial equivalence relation in $\mathbb{C}$. By Lemma 3.4, the morphism Λ induces a semigroup homomorphisms $\Lambda_j\colon S_j \to \mathbb{C}$, where $S_j = \mathcal{S}/R_j$. The homomorphism $\Lambda_j\colon S_j \to \mathbb{C}$ induces a surjective $\mathbb{C}$-algebra homomorphism $\overline{\Lambda}_j\colon \mathbb{C}[S_j] \to \mathbb{C}$ given by $\chi^m \mapsto \Lambda_j(m)$. Since $\mathbb{C}$ is a field, we have $\overline{\mathfrak{m}} = \ker \overline{\Lambda}_j$ is a maximal ideal. The preimage $\mathfrak{m}$ of $\overline{\mathfrak{m}}$ by the homomorphism $\widehat{\pi}_j\colon \mathbb{C}[\mathcal{V}] \to \mathbb{C}[S_j]$ coming from the projective system $\mathbb{C}[S_1] \leftarrow \mathbb{C}[S_2] \leftarrow \ldots$ is also maximal. By [9, Proposition 1.2.2] we have that $\mathfrak{m}$ is closed since $\widehat{I}_j = \ker \widehat{\pi}_j$ is subset of $\mathfrak{m}$.

On the other hand, let $\mathfrak{m}$ be a closed maximal ideal in $\mathbb{C}[\mathcal{V}]$. By [9, Proposition 1.2.2], there exist $j > 0$ and $\overline{\mathfrak{m}}$ a maximal ideal of $\mathbb{C}[S_j]$ such that $\mathfrak{m}$ is the preimage of $\overline{\mathfrak{m}}$ by $\widehat{\pi}_j$. This maximal ideal $\overline{\mathfrak{m}}$ defines an algebra homomorphism $\overline{\Lambda}_j\colon \mathbb{C}[S_j] \to \mathbb{C} \simeq \mathbb{C}[S_j]/\overline{\mathfrak{m}}$. By [4, proposition 1.3.1], this algebra homomorphism defines a semigroup homomorphism $\Lambda_j\colon S_j \to \mathbb{C}$ given by $\Lambda_j(m) = \overline{\Lambda}_j(\chi^m)$. We define $\Lambda\colon \mathcal{S} \to \mathbb{C}$ by $\Lambda = \Lambda_j \circ \pi_j$, where $\pi_j\colon \mathcal{S} \to S_j$ is the quotient morphism. The semigroup homomorphism Λ is a filtered semigroup morphism since $(\Lambda \times \Lambda)(R_j)$ is contained in the diagonal in $\mathbb{C} \times \mathbb{C}$ defining the trivial equivalence relation in $\mathbb{C}$. It is a straightforward verification that these constructions provide the required bijection. ☐

5. Examples

To conclude the paper, we provide the following three examples of affine toric ind-varieties.

Example 5.1. The ind-torus $\mathcal{T} = (\mathbb{C}^*)^\infty$ is a toric ind-variety. Furthermore, since the algebra of regular functions of $(\mathbb{C}^*)^i$ is $\mathbb{C}[\mathbb{Z}^i]$ we obtain that $\mathcal{S}(\mathcal{T}) = \mathbb{Z}^\omega$ by Example 1.1.

Example 5.2. The infinite dimensional affine space $\mathbb{C}^\infty_1 \simeq \mathbb{C}^\infty$ defined in Example 1.3 is a toric ind-variety. Furthermore, since the algebra of regular functions of $\mathbb{C}^i$ is $\mathbb{C}[\mathbb{Z}^i_{\geq 0}]$ we obtain that $\mathcal{S}(\mathbb{C}^\infty_1) = \mathbb{Z}^\omega_{\geq 0}$, see also Example 3.6.

We take advantage of this example to show that in general $\mathbb{C}[\mathcal{S}] \subsetneq \mathbb{C}[\mathcal{V}]$. To do so, we show that $\mathbb{C}[\mathbb{Z}^\omega_{\geq 0}]$ is not a complete topological ring. Recall that

$$\mathbb{C}[\mathbb{Z}^\omega_{\geq 0}] = \bigoplus_{m \in \mathbb{Z}^\omega_{\geq 0}} \mathbb{C}\chi^m\,.$$

We also let

$$x_i = \chi^m, \text{ with } m = (\underbrace{1, 1, \ldots, 1}_{i\text{-times}}, 0, 0, \ldots), \quad f_1 = x_1, \quad \text{and} \quad f_i = \frac{x_i}{2^{i-1}} + \sum_{k=1}^{i-1} \frac{x_k}{2^k}\,.$$

The sequence $(f_i)_{i>0}$ is Cauchy since

$$f_j - f_i = \frac{x_j}{2^{j-1}} + \sum_{k=i}^{j-1} \frac{x_k}{2^k} - \frac{x_i}{2^{i-1}} \quad \text{for all } j > i\,,$$

and a straightforward computation shows that

$$\pi_\ell(f_j - f_i) = \left(\frac{1}{2^{j-1}} + \sum_{k=i}^{j-1} \frac{1}{2^k} - \frac{1}{2^{i-1}} \right) \chi^{m'} = 0 \quad \text{for all } j > i \text{ and } i, j > \ell\,.$$

Here $m' = (1, 1, \ldots, 1) \in \mathbb{Z}^\ell$ and $\pi_\ell \colon \mathbb{C}[\mathbb{Z}^\omega_{\geq 0}] \to \mathbb{C}[\mathbb{Z}^\ell_{\geq 0}]$ is the natural morphism coming from the projective limit that in this case corresponds to the semigroup homomorphism $\mathbb{Z}^\omega_{\geq 0} \to \mathbb{Z}^\ell_{\geq 0}$ restriction to the ℓ first coordinates. Finally, the sequence is not convergent in $\mathbb{C}[\mathbb{Z}^\omega_{\geq 0}]$ since the limit is an infinite sum that cannot belong to the direct sum in the definition of $\mathbb{C}[\mathbb{Z}^\omega_{\geq 0}]$.

Example 5.3. Let $\mathcal{V}$ be the affine toric ind-variety $V_1 \hookrightarrow V_2 \hookrightarrow \ldots$, where V_i is the i-dimensional affine toric variety given in $\mathbb{C}^{i+1}$ with coordinates $(y, x_1, \ldots, x_i)$ by the equation $y^2 = x_1 \cdots x_i$, where $V_i \hookrightarrow V_{i+1}$ is the closed embedding given by the toric map $(y, x_1, \ldots, x_i) \mapsto (y, x_1, \ldots, x_i, 1)$.

From the equation $y^2 = x_1 \cdots x_i$ we obtain that the embedding of the acting torus of V_i in the acting torus of $\mathbb{C}^{i+1}$ corresponds to the homomorphism of the character lattices $\mathbb{Z}^{i+1} = M(\mathbb{C}^{i+1}) \to M(V_i) = \mathbb{Z}^i$ given by the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Hence, we obtain that $V_i = \operatorname{Spec} \mathbb{C}[S_i]$ where $S_i = \omega_i \cap \mathbb{Z}^i$ and ω_i is the cone spanned in $\mathbb{R}^i$ by the rays

$$\begin{pmatrix} 2 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

The semigroup S_i is thus given in $\mathbb{Z}^i$ by

$$S_i = \left\{ (m_1, \ldots, m_i) \in \mathbb{Z}^i \mid m_1 \geq 0, \text{ and } m_1 + 2m_j \geq 0, \text{ for all } j = 2, 3, \ldots, i \right\}.$$

Furthermore, the closed embedding $V_i \hookrightarrow V_{i+1}$ corresponds to the map $\mathbb{Z}^{i+1} \to \mathbb{Z}^i$ of the respective character lattices given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

Taking the projective limit we obtain that the pro-affine semigroup $\mathcal{S}$ corresponding to the affine toric ind-variety $\mathcal{V}$ is given by

$$\mathcal{S}=\left\{(m_1,m_2,\ldots)\in\mathbb{Z}^\omega \mid m_1\geq 0,\ \text{and}\ m_1+2m_j\geq 0,\ \text{for all}\ j\geq 2\right\}.$$

REFERENCES

[1] Jérémy Blanc and Jean-Philippe Furter, *Topologies and structures of the Cremona groups.* Ann. of Math. (2) 2013, 178 (2013), no. 3, 1173–1198.

[2] Michel Brion, *Introduction to actions of algebraic groups,* Graduate Studies in "Les cours du CIRM", volume 1 pages 1-22, http://eudml.org/doc/116362.

[3] J.H. Carruth, J.A. Hildebrant and R.J. Koch, *The Theory of Topological Semigroups I* (Marcel Dekker, New York, 1983).

[4] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties,* Graduate Studies in Mathematics, volume 124. American Mathematical Society, Providence, RI, 2011.

[5] Radoslav Dimitric, *Slenderness, Volume 1. Abelian Categories*, Cambridge Tracts in Mathematics, volume 215, 2019.

[6] William Fulton, *Introduction to toric varieties,* Annals of Mathematics Studies, volume 131. Princeton University Press, Princeton, NJ, 1993

[7] László Fuchs, *Infinite Abelian Groups* Pure and applied mathematics, volume 36-II. A series of Monographs and Textbook, 1973.

[8] Jean-Philippe Furter and Hanspeter Kraft, *On the geometry of the automorphism groups of affine varieties* arXiv:1809.04175v1.

[9] Tatsuji Kambayashi, *Some basic results on pro-affine algebras and ind-affine schemes,* Osaka J. Math. 40 (2003), no. 3, 621–638.

[10] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.

[11] Saunders Mac Lane, *Categories for the Working Mathematician,* Graduate Texts in Mathematics, volume 5. Springer-Verlag New York, 1971.

[12] Tadao Oda, *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties,* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), volume 15. Springer-Verlag, Berlin, 1988.

[13] Douglas G. Northcott, *Lessons on rings, modules and multiplicities,* Cambridge University Press Bentley House, 1968.

[14] Igor R. Shafarevich, *On some infinite-dimensional groups,* Rend. Mat. e Appl. (5) 25 (1966), no. 1-2, 208–212.

[15] Igor R. Shafarevich, *On some infinite-dimensional groups. II,* Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981), no. 1, 214–226, 240.

[16] Ernst Specker, *Additive Gruppen von Folgen ganzer Zahlen,* Portugaliae Mathematica, 9 (1950): 131–140.

[17] Immanuel Stampfli, *On the topologies on ind-varieties and related irreducibility questions,* J. Algebra 372 (2012), 531–541.

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CHAPTER 2

Topologically integrable derivations and additive group actions on affine ind-schemes

In this chapter, I present the research developed with Adrien Dubouloz and Alvaro Liendo.

TOPOLOGICALLY INTEGRABLE DERIVATIONS AND ADDITIVE GROUP ACTIONS ON AFFINE IND-SCHEMES

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ABSTRACT. Affine ind-varieties are infinite dimensional generalizations of algebraic varieties which appear naturally in many different contexts, in particular in the study of automorphism groups of affine spaces. In this article we introduce and develop the basic algebraic theory of topologically integrable derivations of complete topological rings. We establish a bijective algebro-geometric correspondence between additive group actions on affine ind-varieties and topologically integrable derivations of their coordinate pro-rings which extends the classical fruitful correspondence between additive group actions on affine varieties and locally nilpotent derivations of their coordinate rings.

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INTRODUCTION

Motivated by the study of algebro-geometric properties of some "infinite dimensional" groups which appear naturally in algebraic geometry, such as for instance the group of algebraic automorphisms of the affine n-space $\mathbb{A}^n_k$ over a field k, $n \geq 2$, Shafarevich [20, 21] introduced and developed the notions of infinite-dimensional affine algebraic variety and infinite-dimensional affine algebraic group. In Shafarevich's sense, an affine ind-variety over an algebraically closed field k is a topological space X which is homeomorphic to the colimit $\varinjlim_{n\in\mathbb{N}} X_n$ of a countable inductive system of closed embeddings $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$ of ordinary affine algebraic k-varieties, endowed with the final topology. One declares that a morphism between two such ind-varieties $\varinjlim_{n\in\mathbb{N}} X_n$ and $\varinjlim_{n\in\mathbb{N}} Y_n$ consists of a collection of compatible morphisms of ordinary affine algebraic varieties between the corresponding inductive systems, and a group object in the so-defined category is then called an affine ind-group. Since Shafarevich pioneering work, this notion has been developed further by many authors [16, 15, 22, 17, 9, 6] driven mainly by its numerous applications to the study of algebraic automorphism groups of affine varieties.

A different approach to affine ind-varieties, closer to the Grothendieck theory of ind-representable functors and formal schemes [10, 1], was proposed by Kambayashi [12, 13, 14] in the form of a category of locally ringed spaces anti-equivalent to the category whose objects are linearly topologized complete k-algebras $\mathcal{A}$ which admit fundamental systems of open neighborhoods of 0 consisting of a countable families of ideals $(\mathfrak{a}_n)_{n\in\mathbb{N}}$, with the property that all the quotients $A_n = \mathcal{A}/\mathfrak{a}_n$ are integral finitely generated k-algebras. The underlying topological space of an affine ind-k-variety in Kambayashi's sense is defined as the set $\mathrm{Spf}(\mathcal{A})$ of open prime ideals of $\mathcal{A}$, endowed with the subspace topology inherited from the Zariski topology on the usual prime spectrum $\mathrm{Spec}(\mathcal{A})$. Morphisms between such ind-k-varieties are in turn simply determined by continuous homomorphisms between the corresponding topological algebras, see Section 2.

It is known that these two notions of ind-k-varieties are not equivalent, even already at the topological level (see [22] for an in-depth comparison). Despite its natural definition and its algebraic flavor which allows to easily extend it to more general complete topological algebras, leading to a natural theory of affine ind-schemes, so far, the applications of Kambayashi's notion of affine ind-varieties have not been researched as much as those of Shafarevich's version.

The main goal of this paper is to develop the basic tools to extend the existing rich algebro-geometric theory of additive group actions on affine varieties and schemes to Kambayashi's affine ind-varieties and ind-schemes. To explain our results and put them into context, we restrict ourselves in this introduction to affine schemes and ind-schemes defined over an algebraically closed field k of characteristic zero. Every algebraic action of the additive group $\mathbb{G}_{a,k} = \mathrm{Spec}(k[T])$ on an affine k-scheme is uniquely determined by its comorphism $\mu\colon A \to A \otimes_k k[T] = A[T]$. The fact that μ is the comorphism of a $\mathbb{G}_{a,k}$-action implies that the map which associates to every $f \in A$ the element $\frac{d}{dT}(\mu(f))|_{T=0}$ is a k-derivation ∂ of A, which corresponds geometrically to the velocity vector field along the orbits of the action on X. Conversely, an algebraic vector field ∂ on X determines an algebraic action of $\mathbb{G}_{a,k}$ on X if and only if its formal flow is algebraic, that is, if and only if the formal exponential homomorphism

$$\exp(T\partial)\colon A \to A[[T]], \qquad f \mapsto \sum_n \frac{\partial^n(f)}{n!} T^n$$

factors through the polynomial ring $A[T] \subset A[[T]]$. Clearly, a k-derivation ∂ of A satisfies this polynomial integrability property if and only if for every $f \in A$, there exists $n \in \mathbb{N}$ such that $\partial^n(f) = 0$. Derivations with this property are called *locally nilpotent*, and we obtain the well-known correspondence between $\mathbb{G}_{a,k}$-actions on an affine k-variety $X = \mathrm{Spec}(A)$ and locally nilpotent k-derivations of A.

Let now $\mathcal{A}$ be a linearly topologized complete k-algebra which admits a fundamental system of open neighborhoods of 0 consisting of a countable family $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ of ideals of $\mathcal{A}$. We call a continuous k-derivation ∂ of $\mathcal{A}$ *topologically integrable* if the sequence of k-linear endomorphisms $(\partial^n)_{i\in\mathbb{N}}$ of $\mathcal{A}$ converges continuously to the zero homomorphism, that is, if for every $f \in \mathcal{A}$ and every $i \in \mathbb{N}$, there exist indices $n_0, j \in \mathbb{N}$ such that $\partial^n(f + \mathfrak{a}_j) \subset \mathfrak{a}_i$ for every integer $n \geq n_0$ (see Definition 1.8). Note that in the case where the topology on $\mathcal{A}$ is the discrete one, a k-derivation of $\mathcal{A}$ is topologically integrable precisely when it is locally nilpotent. Our main result is the following extension of the classical correspondence for affine k-varieties to the case of affine ind-k-schemes (see Theorem 3.6 for the general version which applies to arbitrary relative affine ind-schemes over a base affine ind-scheme).

Theorem. *Let $\mathfrak{X} = \mathrm{Spf}(\mathcal{A})$ be the affine ind-k-scheme associated to a linearly topologized complete k-algebra $\mathcal{A}$ which admits a fundamental system of open neighborhoods of 0 consisting of a countable family of ideals. Then there exists a one-to-one correspondence between $\mathbb{G}_{a,k}$-actions on $\mathfrak{X}$ and topologically integrable k-derivations of $\mathcal{A}$.*

This correspondence is made explicit as follows. The topological integrability of a continuous k-derivation ∂ of $\mathcal{A}$ is equivalent to the property that its associated formal exponential homomomorphism $\exp(T\partial)$ factors through a continuous homomorphism with values in the subring $\mathcal{A}\{T\} \subset \mathcal{A}[[T]]$ of *restricted power series*, consisting of formal power series $\sum_{i\in\mathbb{N}} a_i T^i$ whose coefficients a_i tend to 0 for the topology on $\mathcal{A}$ when n tends to infinity. The topological ring $\mathcal{A}\{T\}$ is isomorphic to the completed tensor product $\mathcal{A}\widehat{\otimes}_k k[T]$ with respect to the given topology on $\mathcal{A}$ and the discrete topology on $k[T]$. The resulting continuous homomorphism

$$\exp(T\partial)\colon \mathcal{A} \to \mathcal{A}\{T\} \cong \mathcal{A}\widehat{\otimes}_k k[T]$$

determines through Kambayashi's definition a morphism of affine ind-k-schemes

$$\mathbb{G}_{a,k} \times_k \mathrm{Spf}(\mathcal{A}) \cong \mathrm{Spf}(\mathcal{A}\{T\}) \to \mathrm{Spf}(\mathcal{A})$$

which satisfies the axioms of an action of the additive group $\mathbb{G}_{a,k}$ on the affine ind-k-scheme $\mathrm{Spf}(\mathcal{A})$. We show conversely that every continuous homomorphism $\mathrm{e}\colon \mathcal{A} \to \mathcal{A}\{T\}$ which satisfies the axioms of a comorphism of a $\mathbb{G}_{a,k}$-action on an affine ind-k-scheme $\mathrm{Spf}(\mathcal{A})$ is the restricted exponential homomorphism $\exp(T\partial)$ associated to a topologically integrable k-derivation ∂ of $\mathcal{A}$ (see Theorem 2.26.)

One of the cornerstones of the algebraic theory of locally nilpotent derivations is the existence, for every nonzero such derivation ∂ of a k-algebra A, of a so-called local slice, that is, an element $s \in A$ such that $\partial(s) \in \ker(\partial) \setminus \{0\}$. Not every nonzero topologically integrable derivation k-derivation ∂ of a linearly topologized complete k-algebra $\mathcal{A}$ admits a local slice (see Example 2.13 for a counterexample). On the other hand, we establish that the theory of topologically integrable derivations with local slices closely resembles the usual finite-dimensional case: after an appropriate localization, these derivations admit a Dixmier-Reynolds operator (see Definition 2.15) which provides a retraction of $\mathcal{A}$ onto their kernels. In particular, we have the following result (see Proposition 2.16 and Corollary 2.18 for the general case).

Theorem. *Let $\mathcal{A}$ be linearly topologized complete k-algebra and let $\partial\colon \mathcal{A} \to \mathcal{A}$ be a topologically integrable derivation admitting a slice s such that $\partial(s) = 1$. Then $\mathcal{A} \cong (\ker\partial)\{s\}$ and $\exp(T\partial)$ coincides with the homomorphism of topological $(\ker\partial)$-algebras*

$$(\ker\partial)\{s\} \to (\ker\partial)\{s\}\{T\} \cong (\ker\partial)\{s,T\}, \quad s \mapsto s+T.$$

The paper is organized as follows. In Section 1 we collect and review essential definitions and results on the classes of topological groups, rings and modules which are relevant in the context of Kambayashi's definition of affine ind-schemes. In Section 2, we develop the basic algebraic theory of restricted exponential homomorphisms and their correspondence with topologically integrable iterated higher derivations. Section 3 is devoted to the geometric side of the picture: there, for the convenience of the reader, we first review the main steps of the construction of the affine ind-scheme associated to a linearly topologized complete ring, and then, we illustrate the resulting anti-equivalence between restricted exponential homomorphisms and additive group actions on affine ind-schemes.

1. PRELIMINARIES ON TOPOLOGICAL GROUPS AND RINGS

In this section we recall and gather general results on topological groups, rings and modules. Standard references for theses topics are for instance Bourbaki [3, Chapter III], [4, Chapter III] and Northcott [19].

Recall that a topological abelian group is an abelian group G endowed with a topology for which the map $G \times G \to G$, $(x, y) \mapsto x - y$ is continuous. The topology on G is called linear if G has a fundamental system of open neighborhoods of its neutral element 0 consisting of open subgroups. In what follows, we only consider topological abelian groups G endowed with a linear topology which satisfy the following additional condition:

($\star$) *There exists a fundamental system of open neighborhoods of the neutral element* 0 *consisting of a countable family of open subgroups.*

A topological group satisfying this property is in particular a first-countable topological space. For simplicity, we refer to them simply as topological groups and we refer those fundamental systems of open neighborhoods of the neutral element simply to as fundamental systems of open subgroups of G. Given such a fundamental system $(H_n)_{n\in\mathbb{N}}$ parametrized by the set $\mathbb{N}$ of non-negative integers, we always assume in addition that $H_0 = G$ and that $H_m \subseteq H_n$ whenever $m \geq n$.

A continuous homomorphism $f: G \to G'$ between topological groups is refered to as a *homomorphism of topological groups*. Note that such a homomorphism is automatically uniformly continuous in the sense of [3, II.2.1].

1.1. Separated completions of topological groups.

A topological group G is separated as a topological space if and only if the intersection of all open subgroups of G consists of the neutral element 0 only; hence, since every open subgroup of a topological group is also closed [3, III.2.1 Corollary to Proposition 4], if and only if $\{0\}$ is a closed subset of G.

Given a topological group G, the collection of topological groups G/H, where H ranges through the set Γ of open subgroups of G, together with the canonical surjective homomorphisms $p_{H',H}: G/H' \to G/H$ whenever $H' \subseteq H$, form an inverse system of topological groups when each G/H is endowed with the quotient topology, which is the discrete one as H is open. Note that with respect to these topologies, the canonical homomorphisms $p_H: G \to G/H$, $H \in \Gamma$, are homomorphisms of topological groups. The limit $\widehat{G} = \varprojlim_{H\in\Gamma} G/H$ of this system endowed with the inverse limit topology is a linearly topologized abelian group. We denote by $\widehat{p}_H: \widehat{G} \to G/H$, $H \in \Gamma$, the associated canonical continuous homomorphisms and by $c: G \to \widehat{G}$ the continuous homomorphism induced by the homomorphisms $p_H: G \to G/H$, $H \in \Gamma$.

Proposition 1.1. *Let G be a topological group and let $(H_n)_{n\in\mathbb{N}}$ be a fundamental system open subgroups of G. Then the following hold:*

1) The group $\widehat{G}$ is a separated topological group canonically isomorphic to the group $\varprojlim_{n\in\mathbb{N}} G/H_n$ endowed with the inverse limit topology,

2) The canonical projections $\widehat{p}_H: \widehat{G} \to G/H$ are surjective homomorphisms of topological groups,

3) The canonical map $c: G \to \widehat{G}$ is a homomorphism of topological groups whose image is a dense subgroup of $\widehat{G}$ and whose kernel is equal to the closure of $\{0\}$ in G. Furthermore, the induced morphism of topological groups $c: G \to c(G)$ is open.

Proof. Since all the G/H are endowed with the discrete topology, $\{0\}$ is closed in G/H and so, $\{0\}$ is closed in $\widehat{G}$ by definition of the inverse limit topology. This yields that $\widehat{G}$ is separated. Since $(H_n)_{n\in\mathbb{N}}$ is a cofinal subset of Γ, the canonical homomorphism $\widehat{G} \to \varprojlim_{n\in\mathbb{N}} G/H_n$ is an isomorphism of topological groups. A countable fundamental system of open subgroups of $\widehat{G}$ is given by the kernels of the projections $\widehat{p}_{H_n}$, $n \in \mathbb{N}$. This shows that $\widehat{G}$ is a topological group in our sense. Since each $p_{H',H}\colon G/H' \to G/H$, $H, H' \in \Gamma$, is surjective and $(H_n)_{n\in\mathbb{N}}$ is a countable cofinal subset of Γ, by Mittag-Leffler's theorem [3, II.3.5 Corollary I], the canonical homomorphisms $\widehat{p}_H\colon \widehat{G} \to G/H$ are all surjective. Assertion 3) is [3, III.7.3 Proposition 2]. □

Definition 1.2. The topological group $\widehat{G}$ is called the *separated completion* of the topological group G. We say that a topological group is *complete* if the canonical homomorphism $c\colon G \to \widehat{G}$ is an isomorphism of topological groups.

The separated separated completion $c\colon G \to \widehat{G}$ is characterized by the following universal property [3, III.3.4 Proposotion 8]: *For every homomorphism of topological groups $f\colon G \to G''$ where G'' is complete, there exists a unique homomorphism of topological groups $\widehat{f}\colon \widehat{G} \to G''$ such that $f = \widehat{f} \circ c$.*

Remark 1.3. A separated topological group G is metrizable. Indeed, given a countable fundamental system of open subgroups $(H_n)_{n\in\mathbb{N}}$, a metric d inducing the topology on G is for instance defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{2^n} & \text{if } x - y \in H_n \setminus H_{n+1}. \end{cases}$$

For such a group, the notion of completeness of Definition 1.2 is equivalent to the fact that the metric space (G, d) is a complete in the usual sense, see also Proposition 1.7 below.

Proposition 1.4. *Let G and G' be topological groups with respective separated completions $c\colon G \to \widehat{G}$ and $c'\colon G' \to \widehat{G'}$. Then for every homomorphism of topological groups $h\colon G \to G'$ there exists a unique homomorphism of topological groups $\widehat{h}\colon \widehat{G} \to \widehat{G'}$ such that $c' \circ h = \widehat{h} \circ c$.*

Conversely, every homomorphism of topological groups $\widehat{h}\colon \widehat{G} \to \widehat{G'}$ is uniquely determined by its restriction $\widehat{h} \circ c\colon G \to \widehat{G'}$ to G.

Proof. The first assertion is an immediate consequence of the universal property of separated completion. The second assertion follows from the fact that the image of the separated completion homomorphism $c\colon G \to \widehat{G}$ is dense. □

Lemma 1.5. *Let $(G_n)_{n\in\mathbb{N}}$ be an inverse system of complete topological groups with surjective transition homomorphisms $p_{m,n}\colon G_m \to G_n$ for every $m \geq n \geq 0$. Then the limit $\mathcal{G} = \varprojlim_{n\in\mathbb{N}} G_n$ endowed with the inverse limit topology is a complete topological group and each canonical projection $\widehat{p}_n\colon \mathcal{G} \to G_n$ is a surjective homomorphism of topological groups.*

Proof. The fact that $\mathcal{G}$ endowed with the inverse limit topology is a linearly topologized abelian group and the fact that the canonical projections $\widehat{p}_n\colon \mathcal{G} \to G_n$ are continuous homomorphisms are clear. The surjectivity of $\widehat{p}_n$ follows again from Mittag-Leffler's theorem [3, II.3.5 Corollary I]. A countable fundamental system of open subgroups of $\mathcal{G}$ is given for instance by the collection of inverse images of such fundamental systems of each G_n by the homomorphisms $\widehat{p}_n$. Finally, since each G_n is complete, it follows from [3, II.3.5 Corollary to Proposition 10] that $\mathcal{G}$ is complete. □

1.2. **Convergence and summability in topological groups.**

Definition 1.6. Let G be a topological group and let $(x_i)_{i\in I}$ be a family of elements of G parametrized by a countable infinite index set I. For every finite subset $J \subset I$, set $s_J = s_J((x_i)_{i\in I}) = \sum_{j\in J} x_j \in G$.

a) The family $(x_i)_{i\in I}$ is said to be *Cauchy* if for every open subgroup H of G there exists a finite subset $J(H)$ of I such that $x_i - x_j \in H$ for all $i, j \in I \setminus J(H)$.

b) The family $(x_i)_{i\in I}$ is said to *converge* to an element $x \in G$ if for every open subgroup H of G there exists a finite subset $J(H)$ such that $x_i - x \in H$ for all $i \in I \setminus J(n)$.

c) The family $(x_i)_{i\in I}$ is said to be *summable* of sum $s \in G$ if for every open subgroup H of G there exists a finite subset $J(H) \subset I$ such that $s_J - s \in H$ for every every finite subset $J \supset J(H)$ of I.

If G is separated then an element $x \in G$ to which a family $(x_i)_{i\in I}$ converges is unique if it exists, we call it the limit of the family $(x_i)_{i\in I}$. We say that a family $(x_i)_{i\in I}$ is convergent if it converges to an element $x \in G$. Similarly, an element $s \in G$ such that $(x_i)_{i\in I}$ is summable of sum $s \in G$ is unique if it exists. We call it the sum of the family $(x_i)_{i\in I}$ and we write $s = \sum_{i\in I} x_i$.

Proposition 1.7. [4, III.2.6 Proposition 5] *For a separated topological group G, the following conditions are equivalent:*

a) G is a complete topological group,

b) Every Cauchy family $(x_i)_{i\in I}$ of elements of G is convergent in G,

c) Every family $(x_i)_{i\in I}$ of element of G which converges to 0 is summable in G.

Definition 1.8. Let G and G' be topological groups, let $f_n\colon G \to G'$, $n \in \mathbb{N}$, be a sequence of homomomorphisms of groups and let $f\colon G \to G'$ be a homomorphism of groups.

a) The sequence $(f_n)_{n\in\mathbb{N}}$ is said to converge *pointwise* to f if for every $g \in G$ and every open subgroup H' of G', there exists an index n_0 such that $f_n(g) - f(g) \in H'$ for every integer $n \geq n_0$,

b) The sequence $(f_n)_{n\in\mathbb{N}}$ is said to converge *continuously* to f if every f_n, $n \in \mathbb{N}$, is continuous and for every $g \in G$ and every open subgroup H' of G', there exists an open subgroup H of G and an index n_0 such that $f_n(g + x) - f(g + x) \in H'$ for every element $x \in H$ and every integer $n \geq n_0$.

Clearly, a sequence $(f_n)_{n\in\mathbb{N}}$ which converges continuously to a homomorphism f converges pointwise to this homomorphism.

Lemma 1.9. *Let G be a topological group, let G' be separated topological group and let $f_n\colon G \to G'$, $n \in \mathbb{N}$, be a sequence of homomorphisms of topological groups. Then the following properties are equivalent:*

a) The sequence $(f_n)_{n\in\mathbb{N}}$ converges continuously to a homomorphism $f\colon G \to G'$,

b) There exists a homomorphism of topological groups $f\colon G \to G'$ such that the sequence $f_n - f$ converges continuously to the zero homomorphism,

c) The sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent to a homomorphism of topological groups $f\colon G \to G'$, and for every open subgroup H' of G', there exists an open subgroup H of G and an integer $n_0 \geq 0$ such that $(f_n - f)(H) \subset H'$ for every $n \geq n_0$.

In particular, if a sequence $(f_n)_{n\in\mathbb{N}}$ of homomorphisms of topological groups converges continuously to a homomorphism $f\colon G \to G'$, then f is continuous.

Proof. The implication b) $\Rightarrow$ a) is clear. Conversely, assume that the sequence $(f_n)_{n\in\mathbb{N}}$ converges continuously to a homomorphism $f\colon G\to G'$ and denote by $(h_n)_{n\in\mathbb{N}}$ the sequence of group homomorphisms defined by $h_n=f_n-f$ for every $n\in\mathbb{N}$. Applying the definition of continuous convergence to the point 0 of G, it follows that for every open subgroup H' of G', there exists an open subgroup H_1 of G such that $h_n(H_1)\subset H'$ for all sufficiently large n. On the other hand, since f_n is continuous for every n, there exists an open subgroup $H_2(n)$ of G such that $f_n(H_2(n))\subset H'$. Choosing n sufficiently large, we have $-f(x)=h_n(x)-f_n(x)\in H'$ for every $x\in H=H_1\cap H_2(n)$. Thus $f(H)\subset H'$ which shows that f is continuous at 0, hence continuous since it is a homomorphism of groups. Then $(h_n)_{n\in\mathbb{N}}$ is a sequence of homomorphisms of topological groups which converges continuously to the zero homomorphism. Thus, a) implies b).

Now assume that for some homomorphism of topological groups f the sequence $h_n=f_n-f$, $n\in\mathbb{N}$, converges continuously to the zero homomorphism. Applying the definition of continuous convergence to the element $0\in G$, we conclude that, for every open subgroup H' of G, there exists an open subgroup H of G and an integer n_0 such that $h_n(H)\subset H'$ for every $n\geq n_0$. So b) implies c). Conversely, assume that c) holds, let H' be an open subgroup of G' and let g be an element of G. Since by hypothesis the sequence $(f_n(g))_{n\in\mathbb{N}}$ converge to an element $f(g)$ of G, there exists an integer $n_1\geq 0$ such that $h_n(g)\in H'$ for every $n\geq n_1$. Since on the other hand there exists by hypothesis an open subgroup H of G and an integer $n_0\geq 0$ such that $h_n(H)\subset H'$ for every $n\geq n_0$, we conclude that $h_n(g+H)\subset H'$ for every $n\geq \max(n_0,n_1)$. So the sequence $(h_n)_{n\in\mathbb{N}}$ converges continuously to 0, which shows that c) implies b). To prove that f is continuous, since H' is an open subgroup of G', it also closed. It follows that for every $g\in H$, the limit $f(g)$ of the sequence $(f_n(g))_{n\in\mathbb{N}}$ belong to H', so that $f(H)\subset H'$. This shows that f is continuous at 0, hence continuous since it is a homomorphism. ☐

Lemma 1.10. *Let G be a topological group and let G' be a separated topological group with respective separated completions $c\colon G\to\widehat{G}$ and $c'\colon G'\to\widehat{G'}$. Let $f_n\colon G\to G'$, $n\in\mathbb{N}$, be a sequence of homomorphisms of topological groups, let $\widetilde{f_n}=c'\circ f_n\colon G\to\widehat{G'}$, $n\in\mathbb{N}$, and let $\widehat{f_n}\colon\widehat{G}\to\widehat{G'}$, $n\in\mathbb{N}$, be the sequence of homomorphisms of topological groups deduced from the sequence $(\widetilde{f}_n)_{n\in\mathbb{N}}$ by the universal property of the separated completion.*

If the sequence $(f_n)_{n\in\mathbb{N}}$ converges continuously to a homomorphism $f\colon G\to G'$ then the sequences $(\widetilde{f}_n)_{n\in\mathbb{N}}$ and $(\widehat{f}_n)_{n\in\mathbb{N}}$ converge continuously respectively to the homomorphism of topological groups $\widetilde{f}=c'\circ f$ and to the homomorphism of topological groups $\widehat{f}\colon\widehat{G}\to\widehat{G'}$ deduced from $\widetilde{f}$ by the universal property of the separated completion.

Proof. Note that $\widetilde{f_n}$ and $\widehat{f_n}$ are homomorphisms of topological groups for every $n\in\mathbb{N}$. By Lemma 1.9, f is a homomorphism of topological groups so that $\widetilde{f}$ and $\widehat{f}$ are homomorphisms of topological groups as well. Let $g\in G$ and let $\widehat{H'}$ be an open subgroup of $\widehat{G'}$. Then $H'={c'}^{-1}(\widehat{H'})$ is an open subgroup of G'. Since $(f_n)_{n\in\mathbb{N}}$ converges continuously to f, there exists an open subgroup H of G and an index n_0 such that $f_n(g+x)-f(g+x)\in H'$ for every element $x\in H$ and every integer $n>n_0$. Since $c'(H')\subset\widehat{H'}$, this implies that $\widetilde{f_n}(g+x)-\widetilde{f}(g+x)\in\widehat{H'}$, which shows that $(\widetilde{f_n})_{n\in\mathbb{N}}$ converges continuously to $\widetilde{f}$. Replacing G' by $\widehat{G'}$, the sequence f_n by the sequence $\widetilde{f_n}$ and the homomorphism f by $\widetilde{f}$ we can now assume that G' is complete. By Lemma 1.9, it remains to show that the sequence of group homomorphisms $(\widehat{h}_n)_{n\in\mathbb{N}}$ defined by $\widehat{h}_n=\widehat{f}_n-\widehat{f}$ converges continuously to the zero homomorphism on $\widehat{G}$. By definition of $(\widehat{h}_n)_{n\in\mathbb{N}}$, continuous convergence holds in restriction to the subgroup $G_0=c(G)$ of $\widehat{G}$. Since $\widehat{h}_n$ is uniformly continuous and G_0 is dense in

$\widehat{G}$, it follows that $(\widehat{h}_n)_{n\in\mathbb{N}}$ converges pointwise to the zero homomorphism on $\widehat{G}$. Let H' be an open subgroup of G'. Then there exist an integer n_0 and an open subgroup H of $\widehat{G}$ such that $\widehat{h}_n(z) \in H'$ for every $z \in H_0 = G_0 \cap H$ and every $n \geq n_0$. Since H_0 is dense in H and H is a first-countable topological space, for every $x \in H$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of H_0 which converges to x. Setting $y_n = x - x_n$, we deduce that the sequence $(\widehat{h}_i(y_j))_{(i,j)\in\mathbb{N}^2}$ converges to 0 in $\widehat{G'}$. This implies in particular that there exists a strictly increasing map $\varphi\colon \mathbb{N} \to \mathbb{N}$ and an integer $n_1 \geq 0$ such that $\widehat{h}_n(y_{\varphi(n)}) \in H'$ for every $n \geq n_1$. It follows that for every $n \geq \max(n_0, n_1)$, $\widehat{h}_n(x) = \widehat{h}_n(x_{\varphi(n)}) + \widehat{h}_n(y_{\varphi(n)})$ belongs to H', which shows, by Lemma 1.9 c), that the sequence $(\widehat{h}_n)_{n\in\mathbb{N}}$ converges continuously to the zero homomorphism on G. □

Corollary 1.11. *Let G be a topological group and let G' be separated topological group with separated completion $c'\colon G' \to \widehat{G'}$. Let $h_n\colon G \to G'$, $n \in \mathbb{N}$, be a sequence of homomorphisms of topological groups which converges continuously to the zero homomorphism. Then the sequence of homomorphisms $s_N = \sum_{n=0}^{N} c' \circ h_n$, $N \in \mathbb{N}$, converges continuously to the homomorphism*

$$s = \sum_{n\in\mathbb{N}} c' \circ h_n \colon G \to \widehat{G'}, \quad g \mapsto \sum_{n\in\mathbb{N}} c'(h_n(g)).$$

In particular, s is continuous.

Proof. Let $\widetilde{h}_n = c' \circ h_n\colon G \to \widehat{G'}$. First note that since for every $g \in G$ the sequence $(h_n(g))_{n\in\mathbb{N}}$ converges to 0 in G', it follows from Proposition 1.7 that the family $(\widetilde{h}_n(g))_{n\in\mathbb{N}}$ of elements of $\widehat{G'}$ is summable, so that the map s is indeed well defined. Since every $\widetilde{h}_n$ is a homomorphism of groups, for every $g_1, g_2 \in G$ and every integer $N \in \mathbb{N}$, we have $s_N(g_1 + g_2) = s_N(g_1) + s_N(g_2)$. Since $\widehat{G'}$ is separated, this implies that $s(g_1 + g_2) = s(g_1) + s(g_2)$, showing that $s\colon G \to \widehat{G'}$ is a homomorphism. Let $\widehat{H'}$ be an open subgroup of $\widehat{G'}$. Since the sequence $(\widetilde{h}_n)_{n\in\mathbb{N}}$ converges continuously to the zero homomorphism, Lemma 1.9 implies that there exists an integer $n_0 \geq 0$ and an open subgroup H_1 of G such that $\widetilde{h}_n(H_1) \subset \widehat{H'}$ for every $n \geq n_0$. Since for every $n \in \mathbb{N}$, $\widetilde{h}_n$ is continuous, hence in particular continuous at 0, there exists an open subgroup H_2 of G such that $\widetilde{h}_n(H_2) \subset \widehat{H'}$ for every $n = 0, \ldots, n_0$. Putting $H = H_1 \cap H_2$, we have $\widetilde{h}_n(H) \subset \widehat{H'}$ for every $n \in \mathbb{N}$, which implies in turn that $s_N(H) \subset \widehat{H'}$ for every $N \in \mathbb{N}$. Since $\widehat{H'}$ is an open subgroup of $\widehat{G}$, it also closed. It follows that for every $g \in H$, the limit $s(g)$ of the sequence $(s_n(g))_{n\in\mathbb{N}}$ belong to $\widehat{H'}$, so that $s(H) \subset \widehat{H'}$ and $(s_N - s)(H) \subset \widehat{H'}$ for all $N \in \mathbb{N}$ so we have the hypothesis of c) Lemma 1.9 there fore s_N converge continuosly to the homomorphism s. The continuty of s also is a consequence of the last part of Lemma 1.9. □

1.3. **Recollection on topological rings and modules.** Recall that a commutative topological ring $\mathcal{A}$ is a topological abelian group endowed with a ring structure for which the multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous. A module M over a topological ring $\mathcal{A}$ is a called a topological $\mathcal{A}$-module if it is a topological abelian group and the scalar multiplication $\mathcal{A} \times M \to M$ is continuous, where $\mathcal{A} \times M$ is endowed with product topology. In the sequel, unless otherwise specified, the term *topological ring* (resp. *topological module*) will refer to a commutative topological ring $\mathcal{A}$ (resp. topological module M over some topological ring $\mathcal{A}$) endowed with a linear topology for which there exists a fundamental system of neighbourhoods of 0 consisting of a countable family $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ of ideals of $\mathcal{A}$ (resp. endowed with a linear topology with a fundamental system of neighbourhoods of 0 consisting of a countable family of open submodules $(M_n)_{n\in\mathbb{N}}$). We also always assume that $\mathfrak{a}_0 = \mathcal{A}$ and that $\mathfrak{a}_m \subseteq \mathfrak{a}_n$ whenever $m \geq m$ and similarly that $M_0 = M$ and $M_m \subseteq M_n$ whenever $m \geq n$. A continuous homomorphism $f\colon \mathcal{A} \to \mathcal{A}'$ between topological rings is refered to as a *homomorphism of topological rings*. We denote by $\mathcal{C}\mathrm{Hom}(\mathcal{A},\mathcal{B})$ the subgroup of the abelian group $\mathrm{Hom}(\mathcal{A},\mathcal{B})$ consisting of continuous homomorphisms. Similarly, a continuous homomorphism of topological modules $f\colon M \to N$ over a topological ring $\mathcal{A}$ is refered to as a *homomorphism of topological $\mathcal{A}$-modules* and we denote by $\mathcal{C}\mathrm{Hom}_{\mathcal{A}}(M,N)$ the $\mathcal{A}$-module of such homomorphisms.

Given a topological ring $\mathcal{A}$ (resp. a topological module M over a topological ring $\mathcal{A}$) the separated completion $\widehat{\mathcal{A}}$ of $\mathcal{A}$ (resp. $\widehat{M}$ of M) as a topological group carries the structure of a topological ring (resp.of a topological $\mathcal{A}$-module) and the canonical homomorphism of topological groups $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$ (resp. $c\colon M \to \widehat{M}$) is a homomorphism of topological rings (resp. of topological $\mathcal{A}$-modules). We say that $\mathcal{A}$ (resp. M) is a complete topological ring (resp. a complete topological $\mathcal{A}$-module) if $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$ (resp. $c\colon M \to \widehat{M}$) is an isomorphism.

For every complete topological ring $\mathcal{B}$ the composition with $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$ induces an isomorphism

$$c^*\colon \mathcal{C}\mathrm{Hom}(\widehat{\mathcal{A}},\mathcal{B}) \to \mathcal{C}\mathrm{Hom}(\mathcal{A},\mathcal{B}), \quad \widehat{f} \mapsto \widehat{f} \circ c.$$

Let $\mathcal{A}$ be a topological ring and let $\mathcal{B}$ be a complete topological ring, with fundamental systems $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ and $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ of open neighborhoods of 0, respectively. Set $A_n = \mathcal{A}/\mathfrak{a}_n$ and $B_m = \mathcal{B}/\mathfrak{b}_m$ so that we have $\widehat{\mathcal{A}} \cong \varprojlim_{n\in\mathbb{N}} A_n$ and $\mathcal{B} \cong \varprojlim_{m\in\mathbb{N}} B_m$. Every homomorphism of topological rings $f\colon \widehat{\mathcal{A}} \to \mathcal{B}$ is equivalently described by an inverse system of continuous homomorphisms $f_m\colon \widehat{\mathcal{A}} \to B_m$. The kernel of each such f_m being an open ideal of $\widehat{\mathcal{A}}$, it contains some open ideal $\mathfrak{a}_n$ and so, f_m factors through a homomorphism $f_{n,m}\colon A_n \to B_m$. Summing up, we have:

$$\begin{aligned}\mathcal{C}\mathrm{Hom}(\widehat{\mathcal{A}},\mathcal{B}) = \mathcal{C}\mathrm{Hom}(\varprojlim_{n\in\mathbb{N}} A_n, \varprojlim_{m\in\mathbb{N}} B_m) &\cong \varprojlim_{m\in\mathbb{N}}(\mathcal{C}\mathrm{Hom}(\varprojlim_{n\in\mathbb{N}} A_n, B_m)) \\ &\cong \varprojlim_{m\in\mathbb{N}}(\varinjlim_{n\in\mathbb{N}} \mathrm{Hom}(A_n, B_m)).\end{aligned}$$

1.3.1. *Completed tensor product.* We recall basic properties of completed tensor products of topological modules, see [4, III] and [10, 0.7.7].

Definition 1.12. ([4, III Exercise 28]) Let M and N be topological modules over a topological ring $\mathcal{A}$. The *completed tensor product* $M\widehat{\otimes}_{\mathcal{A}}N$ of M and N over $\mathcal{A}$ is the separated completion $\widehat{M \otimes_{\mathcal{A}} N}$ of the tensor product $M \otimes_{\mathcal{A}} N$ with respect to the linear topology generated by open neighborhoods of 0 of the form $U \otimes N + M \otimes V$, where U and V run respectively through the set of open $\mathcal{A}$-submodules of M and N.

We denote by $\tau\colon M \times N \to M\widehat{\otimes}_{\mathcal{A}}N$ the composition of the canonical homomorphism of topological $\mathcal{A}$-modules $M \times N \to M \otimes_{\mathcal{A}} N$, where $M \times N$ is endowed with the product topology, with the separated completion homomorphism $c\colon M \otimes_{\mathcal{A}} N \to M\widehat{\otimes}_{\mathcal{A}}N$

It follows from the universal properties of the tensor product and of the separated completion that the canonical homomorphism of topological $\mathcal{A}$-modules $\tau\colon M\times N \to M\widehat{\otimes}_{\mathcal{A}}N$ satisfies the following universal property: *For every continuous $\mathcal{A}$-bilinear homomorphism $\Phi\colon M \times N \to E$ into a complete topological $\mathcal{A}$-module E, there exists a unique homomorphism of topological $\mathcal{A}$-modules $\widehat{\varphi}\colon M\widehat{\otimes}_{\mathcal{A}}N \to E$ such that $\Phi = \widehat{\varphi} \circ \tau$.* As for the usual tensor product, this universal property implies the following associativity result whose proof is a direct adaptation of that of [5, II.3.8, Proposition 8]:

Lemma 1.13. *Let $\mathcal{A}$ be a topological ring, let M and $\mathcal{B}$ be respectively a topological $\mathcal{A}$-module and a topological $\mathcal{A}$-algebra and let N and P be topological $\mathcal{B}$-modules. Then there is a canonical isomorphism of complete topological $\mathcal{B}$-modules*

$$(M\widehat{\otimes}_{\mathcal{A}}N)\widehat{\otimes}_{\mathcal{B}}P \cong M\widehat{\otimes}_{\mathcal{A}}(N\widehat{\otimes}_{\mathcal{B}}P)$$

where $M\widehat{\otimes}_{\mathcal{A}}N$ is viewed as topological $\mathcal{B}$-module via the $\mathcal{B}$-module structure of N.

In the case where $M = \mathcal{B}_1$ and $N = \mathcal{B}_2$ are topological $\mathcal{A}$-algebras, the completed tensor product $\mathcal{B}_1\widehat{\otimes}_{\mathcal{A}}\mathcal{B}_2$ is a complete topological $\mathcal{A}$-algebra and the composition $\sigma_1 \colon \mathcal{B}_1 \to \mathcal{B}_1\widehat{\otimes}_{\mathcal{A}}\mathcal{B}_2$ (resp. $\sigma_2 \colon \mathcal{B}_2 \to \mathcal{B}_1\widehat{\otimes}_{\mathcal{A}}\mathcal{B}_2$) of $\mathrm{id}_{\mathcal{B}_1} \otimes 1 \colon \mathcal{B}_1 \to \mathcal{B}_1 \otimes_{\mathcal{A}} \mathcal{B}_2$ (resp. $1 \otimes \mathrm{id}_{\mathcal{B}_2} \colon \mathcal{B}_2 \to \mathcal{B}_1 \otimes_{\mathcal{A}} \mathcal{B}_2$) with the separated completion homomorphism $\mathcal{B}_1 \otimes_{\mathcal{A}} \mathcal{B}_2 \to \mathcal{B}_1\widehat{\otimes}_{\mathcal{A}}\mathcal{B}_2$ is a homomorphism of topological $\mathcal{A}$-algebras. The $\mathcal{A}$-algebra $\mathcal{B}_1\widehat{\otimes}_{\mathcal{A}}\mathcal{B}_2$ satisfies the following universal property: *For every complete topological $\mathcal{A}$-algebra $\mathcal{C}$ and every pair of homomorphisms of topological $\mathcal{A}$-algebras $f_i \colon \mathcal{B}_i \to \mathcal{C}$ there exists a unique homomorphism of topological $\mathcal{A}$-algebras $f \colon \mathcal{B}_1\widehat{\otimes}_{\mathcal{A}}\mathcal{B}_2 \to \mathcal{C}$ such that $f_i = f \circ \sigma_i$, $i = 1, 2$.*

In general, given a finitely generated algebra R over the field k, the covariant functor $R \otimes_k -$ which associates to a k-algebra A the k-algebra $R \otimes_k A$ is not representable in the category of k-algebras. The following example shows in contrast that the natural extension of $R\widehat{\otimes}_k-$ of $R \otimes_k -$ to the category of complete topological k-algebras is representable.

Example 1.14. Let R be a finitely generated algebra over a field k, both endowed with the discrete topology. Then the covariant functor

$$R\widehat{\otimes}_k- \colon (\mathrm{CTop}_{/k}) \to (\mathrm{Sets})$$

associating to a complete topological k-algebra $\mathcal{A}$ the completed tensor product $R\widehat{\otimes}_k\mathcal{A}$ is representable.

Proof. Since R is finitely generated, it is a countable k-vector space. We can thus write $R = \varinjlim_{n\in\mathbb{N}} V_n = \bigcup_{n\in\mathbb{N}} V_n$ where the V_n are an increasing sequence of finite dimensional k-vector spaces $V_0 \subset \cdots \subset V_n \subset V_{n+1} \subset \cdots$ which form an exhaustion of R. For every $n \leq n'$, the inclusion $V_n \subset V_{n'}$ induces a dual surjection $V_{n'}^{\vee} \to V_n^{\vee}$ between the duals of $V_{n'}$ and V_n respectively, hence a surjective k-algebra homomorphism $\mathrm{Sym}^{\cdot}(V_{n'}^{\vee}) \to \mathrm{Sym}^{\cdot}(V_n^{\vee})$ between the symmetric k-algebras of $V_{n'}^{\vee}$ and $V_n^{\vee}$, respectively. Viewing each $R_n = \mathrm{Sym}^{\cdot}(V_n^{\vee})$ as endowed with the discrete topology, the ring $\mathcal{R} = \varprojlim_{n\in\mathbb{N}} R_n$ endowed with the initial topology is a complete topological k-algebra whose isomorphism type is independent on the choice of a particular exhaustion $\{V_n\}_{n\in\mathbb{N}}$ of R by finite dimensional k-vector subspaces.

Now let $\mathcal{A} = \varprojlim_{m\in\mathbb{N}} A_m$ be a complete topological k-algebra. Since tensor product commutes with colimits and the k-vector spaces V_n are finite dimensional, we have natural isomorphisms of sets

$$\begin{aligned} R\widehat{\otimes}_k\mathcal{A} \cong \varprojlim_{m\in\mathbb{N}}(R \otimes_k A_m) &= \varprojlim_{m\in\mathbb{N}}((\varinjlim_{n\in\mathbb{N}} V_n) \otimes_k A_m) \\ &\cong \varprojlim_{m\in\mathbb{N}}(\varinjlim_{n\in\mathbb{N}}(V_n \otimes_k A_m)) \\ &\cong \varprojlim_{m\in\mathbb{N}}(\varinjlim_{n\in\mathbb{N}}(\mathrm{Hom}_{k-\mathrm{mod}}(V_n^{\vee}, A_m)). \end{aligned}$$

The universal property of symmetric algebras provides in turn natural isomorphisms

$$\mathrm{Hom}_{k-\mathrm{mod}}(V_n^{\vee}, A_m) \cong \mathrm{Hom}_{k-\mathrm{alg}}(\mathrm{Sym}^{\cdot}(V_n^{\vee}), A_m) = \mathrm{Hom}_{k-\mathrm{alg}}(R_n, A_m).$$

Summing up, we obtain for every $\mathcal{A}$ a natural isomorphism

$$\Phi_{\mathcal{A}} : \mathcal{C}\mathrm{Hom}_k(\mathcal{R},\mathcal{A}) = \varprojlim_{m\in\mathbb{N}} (\varinjlim_{n\in\mathbb{N}} (\mathrm{Hom}_{k-\mathrm{alg}}(R_n, A_m)) \xrightarrow{\cong} R\widehat{\otimes}_k\mathcal{A}.$$

These isomorphisms are easily seen from the construction to be functorial in $\mathcal{A}$, defining an isomorphism of covariant functors $\Phi\colon \mathcal{C}\mathrm{Hom}(\mathcal{R},-) \to R\widehat{\otimes}_k-$ which shows that $\mathcal{R}$ represents the functor $R\widehat{\otimes}_k-$.

The universal element $u = \Phi_{\mathcal{R}}(\mathrm{id}_{\mathcal{R}}) \in R\widehat{\otimes}_k\mathcal{R}$ can be described as follows. For every $n \in \mathbb{N}$, let $u_n \in R\otimes_k R_n = R\otimes_k \mathrm{Sym}^{\cdot}(V_n^{\vee})$ be the image by the natural homomorphism

$$V_n \otimes_k V_n^{\vee} \to R\otimes_k \mathrm{Sym}^{\cdot}(V_n^{\vee})$$

of the element corresponding to id_{V_n} via the isomorphism $\mathrm{Hom}_k(V_n, V_n) \cong V_n\otimes_k V_n^{\vee}$. The collection of elements $u_n \in R\otimes_k R_n$ is an inverse system with the respect to the projection homomorphisms $R\otimes_k R_{n'} \to R\otimes_k R_n$, $n \leq n'$ and we have $u = \varprojlim_{n\in\mathbb{N}} u_n \in \varprojlim_{n\in\mathbb{N}} R\otimes_k R_n = R\widehat{\otimes}_k\mathcal{R}$. ☐

1.3.2. *Separated completed localization.* In what follows by a *multiplicatively closed subset* of a ring A, we mean a subset S of A containing 1 and stable under multiplication. We now recall basic results on separated completed localizations of topological rings and modules, see [10, 0.7.6].

Definition 1.15. Let $\mathcal{A}$ be a be a topological ring and let $S \subset \mathcal{A}$ be a multiplicatively closed subset of $\mathcal{A}$. The *separated completed localization* $\widehat{S^{-1}\mathcal{A}}$ of $\mathcal{A}$ with respect to S is the separated completion of the usual localization $S^{-1}\mathcal{A}$ endowed with the topology co-induced by the localization homomorphism $j\colon \mathcal{A} \to S^{-1}\mathcal{A}$. The composition

$$\widetilde{j} = c\circ j\colon \mathcal{A} \xrightarrow{j} S^{-1}\mathcal{A} \xrightarrow{c} \widehat{S^{-1}\mathcal{A}}$$

of the usual localization homomorphism with the separated completion homomorphism is a homomorphism of topological ring, which we call *separated completed localization homomorphism* of $\mathcal{A}$ with respect to S.

Notation 1.16. Given a topological ring $\mathcal{A}$ and element $f \in \mathcal{A}$ (resp. a prime ideal $\mathfrak{p}$ of $\mathcal{A}$), we denote by $\widehat{\mathcal{A}_f}$ (resp. $\widehat{\mathcal{A}_{\mathfrak{p}}}$) the separated completed localization of $\mathcal{A}$ with respect to the multiplicatively closed subset $S = \{f^n\}_{n\geq 0}$ (resp. $S = \mathcal{A}\setminus\mathfrak{p}$).

The separated completed localization enjoys the following universal property:

Proposition 1.17. *With the notation above, let $\mathcal{B}$ be a complete topological ring and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism of topological rings such that $\varphi(S) \subset \mathcal{B}^*$. Then there exists a unique homomorphism of topological rings $\widehat{S^{-1}\varphi}\colon \widehat{S^{-1}\mathcal{A}} \to \mathcal{B}$ such that $\varphi = \widehat{S^{-1}\varphi}\circ\widetilde{j}$.*

Proof. By the universal property of the usual localization $j\colon \mathcal{A} \to S^{-1}\mathcal{A}$, the exists a unique homomorphism $S^{-1}\varphi\colon S^{-1}\mathcal{A} \to \mathcal{B}$ such that $\varphi = S^{-1}\varphi\circ j$. The homomorphism $S^{-1}\varphi$ is continuous for the topology on $S^{-1}\mathcal{A}$ co-induced by that on $\mathcal{A}$, and since $\mathcal{B}$ is complete, it follows that there exists a unique homomorphism of topological rings $\widehat{S^{-1}\varphi}\colon \widehat{S^{-1}\mathcal{A}} \to \mathcal{B}$ such that $S^{-1}\varphi = \widehat{S^{-1}\varphi}\circ c$. We then have $\varphi = S^{-1}\varphi\circ j = \widehat{S^{-1}\varphi}\circ c\circ j = \widehat{S^{-1}\varphi}\circ\widetilde{j}$. ☐

Lemma 1.18. *Let $\mathcal{A}$ be a topological ring with separated completion $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$, let $S \subset \mathcal{A}$ be a multiplicatively closed subset and let $\widehat{S} \subset \widehat{\mathcal{A}}$ be the closure of $c(S)$ in $\widehat{\mathcal{A}}$. Then there exists a canonical isomorphism $\widehat{S^{-1}\mathcal{A}} \cong \widehat{\widehat{S}^{-1}\widehat{\mathcal{A}}}$ of complete topological rings.*

Proof. Let $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ be a fundamental system of open ideals in $\mathcal{A}$, let $\mathrm{p}_n \colon \mathcal{A} \to A_n = \mathcal{A}/\mathfrak{a}_n$, $n \in \mathbb{N}$, be the quotient homomorphisms and let $S_n = \mathrm{p}_n(S) \subset A_n$. Then $\widehat{S} = \varprojlim_{n\in\mathbb{N}} S_n \subset \varprojlim_{n\in\mathbb{N}} A_n = \widehat{\mathcal{A}}$ so that, by definition,

$$\widehat{S^{-1}\mathcal{A}} \cong \varprojlim_{n\in\mathbb{N}} S_n^{-1}A_n \cong \widehat{\widehat{S}^{-1}\widehat{\mathcal{A}}}.$$

□

Corollary 1.19. *Let $\mathcal{A}$ be a topological ring, let $c\colon \mathcal{A} \to \widehat{\mathcal{A}}$ be its separated completion and let $S \subset \mathcal{A}$ be a multiplicatively closed subset. Then $\widehat{S^{-1}\mathcal{A}}$ is the zero ring if an only if 0 belongs to the closure $\widehat{S}$ of $c(S)$ in $\widehat{\mathcal{A}}$.*

Proof. In view of Lemma 1.18, we are reduced to the case where $\mathcal{A}$ is complete and S is closed in $\mathcal{A}$. Now if $0 \in S$ then $S^{-1}\mathcal{A}$ is the zero ring, and so $\widehat{S^{-1}\mathcal{A}}$ is the zero ring as well. Conversely, using the notation of the proof of Lemma 1.18, if $\widehat{S^{-1}\mathcal{A}} = \varprojlim_{n\in\mathbb{N}} S_n^{-1}A_n$ is the zero ring, then $S_n^{-1}A_n$ is the zero ring for every $n \in \mathbb{N}$, which implies that $0 \in S_n$ for every $n \in \mathbb{N}$. It follows that 0 belongs $\varprojlim_{n\in\mathbb{N}} S_n = S$ as S is closed. □

Example 1.20. Let $\mathcal{A} = \mathbb{C}[u]$ endowed with the u-adic topology, with fundamental system of neighbourhoods of 0 given by the ideals $\mathfrak{a}_n = u^n\mathbb{C}[u]$, $n \geq 0$, and let $S = \{u^m\}_{m\geq 0}$. We have then $\widehat{\mathcal{A}} \cong \mathbb{C}[[u]]$ endowed with the u-adic topology and $\widehat{S} = c(S) = \{u^m\}_{m\geq 0}$. Since $A_n = \mathbb{C}[u]/(u^n) = \mathbb{C}[[u]]/(u^n)$ the images $S_n = \pi_n(S) = \pi_n(\widehat{S})$, $n \geq 1$, all contain the element 0. Thus $S_n^{-1}A_n = \{0\}$ for every $n \geq 0$ from which it follows that $\widehat{S^{-1}\mathcal{A}}$ is the zero ring. On the other hand, we have $S^{-1}\mathcal{A} = \mathbb{C}[u^{\pm 1}]$ and $\widehat{S}^{-1}\widehat{\mathcal{A}} = \mathbb{C}[[u^{\pm 1}]] = \mathbb{C}((u))$ and the images of the ideal $\mathfrak{a}_n$ and $c(\mathfrak{a}_n)$ by the respective localization homomorphisms are all equal to the unit ideals in $\mathbb{C}[u^{\pm 1}]$ and $\mathbb{C}((u))$ respectively. The induced topologies on $S^{-1}\mathcal{A}$ and $\widehat{S}^{-1}\widehat{\mathcal{A}}$ are thus the trivial ones, which implies that the separated completions of these rings are both isomorphic to the zero ring.

Lemma 1.21. *Let $i\colon \mathcal{A} \to \mathcal{B}$ be an injective closed homomorphism of complete topological rings and let S be a multiplicatively closed subset of $\mathcal{A}$. Then $\widehat{S^{-1}i}\colon \widehat{S^{-1}A} \to \widehat{i(S)^{-1}\mathcal{B}}$ is an injective homomorphism of topological rings.*

Proof. Since $\mathcal{A}$ (resp. $\mathcal{B}$) is complete, the kernel of the separated completed localization homomorphism $\mathcal{A} \to \widehat{S^{-1}\mathcal{A}}$ (resp. $\mathcal{B} \to \widehat{i(S)^{-1}\mathcal{B}}$) consists of elements of $\mathcal{A}$ (resp. $\mathcal{B}$) which are anihilated by the multiplication by an element of the closure of S in $\mathcal{A}$ (resp. of the closure of $i(S)$ in $\mathcal{B}$). On the other hand, since i is a closed homomorphism of topological rings, $i(\mathcal{A})$ is complete subspace of $\mathcal{B}$, hence a closed subspace, so that the closures of $i(S)$ in $i(\mathcal{A})$ and $\mathcal{B}$ coincide. □

Remark 1.22. Note that the conclusion of Lemma 1.21 does not hold if $i\colon \mathcal{A} \to \mathcal{B}$ is not a closed homomorphism. For instance, the inclusion $\mathbb{C}[u] \to \mathbb{C}[[u]]$ where $\mathbb{C}[u]$ and $\mathbb{C}[[u]]$ are endowed respectively with the discrete topology and the u-adic topology is continuous but not closed. The separated completed localizations of these topological rings with respect to the multiplicatively closed subset $S = \{u^m\}_{m\geq 0}$ are respectively isomorphic to $\mathbb{C}[u, u^{-1}]$ endowed with the discrete topology and to the zero ring, so that $\widehat{S^{-1}i}$ is not injective in this case.

Definition 1.23. Let $\mathcal{A}$ be a be a topological ring, let M be a topological $\mathcal{A}$-module and let $S \subset \mathcal{A}$ multiplicatively closed subset of $\mathcal{A}$. The *separated completed localization* $\widehat{S^{-1}M}$ of

M with respect to S is the separated completion of the $S^{-1}\mathcal{A}$-module $S^{-1}M = S^{-1}\mathcal{A} \otimes_{\mathcal{A}} M$ with respect to the topology co-induced by the localization homomorphism $j_M \colon M \to S^{-1}M$

The natural structure of topological $S^{-1}\mathcal{A}$-module on $S^{-1}M$ induces a structure of complete topological $\widehat{S^{-1}\mathcal{A}}$-module on $\widehat{S^{-1}M}$. The composition

$$\widetilde{j}_M = c \circ j_M \colon M \stackrel{j_M}{\to} S^{-1}M \stackrel{c}{\to} \widehat{S^{-1}M}$$

of the usual localization homomorphism with the separated completion homomorphism is a homomorphism of topological modules which we call the *separated completed localization homomorphism* of M with respect to S.

For every homomorphism of topological $\mathcal{A}$-modules $f \colon M \to N$, we denote by $\widehat{S^{-1}f} \colon \widehat{S^{-1}M} \to \widehat{S^{-1}N}$ the homomorphism of topological $\widehat{S^{-1}\mathcal{A}}$-modules induced by the universal properties of usual localization and separated completion.

1.4. **Restricted power series.** We recall properties of restricted power series rings with coefficients in a topological ring following [4, III.4.2], see also [10, 0.7.5].

Definition 1.24. Let $\mathcal{A}$ be a topological ring with separated completion $c \colon \mathcal{A} \to \widehat{\mathcal{A}}$ and let $T_1, \ldots, T_r$ be a collection of indeterminates.

The ring of *restricted power series* with coefficients in $\widehat{\mathcal{A}}$ is the separated completion $\widehat{\mathcal{A}}\{T_1, \ldots, T_r\}$ of the polynomial ring $\mathcal{A}[T_1, \ldots, T_r]$ endowed with the topology generated by the ideals $\mathfrak{a}[T_1, \ldots, T_r]$, where $\mathfrak{a}$ runs through the set of open ideals of $\mathcal{A}$.

We denote by $i_0 \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}\{T_1, \ldots, T_r\}$ the homomorphism of topological rings deduced by the universal property of $c \colon \mathcal{A} \to \widehat{\mathcal{A}}$ from the composition of the inclusion $\mathcal{A} \hookrightarrow \mathcal{A}[T_1, \ldots, T_r]$ as the subring of constant polynomials with the separated completion homomorphism $\mathcal{A}[T_1, \ldots, T_r] \to \widehat{\mathcal{A}}\{T_1, \ldots, T_r\}$. The elements in the image of i_0 are said to be *constant restricted power series*.

Letting $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ be a fundamental system of open ideals of $\mathcal{A}$, it follows from the definition that

$$\widehat{\mathcal{A}}\{T_1, \ldots, T_r\} \cong \varprojlim_{n\in\mathbb{N}} (\mathcal{A}/\mathfrak{a}_n)[T_1, \ldots, T_r].$$

Identifying a polynomial in $\widehat{\mathcal{A}}[T_1, \ldots, T_r]$ with the family $(a_I)_{I\in\mathbb{N}^r}$ of its coefficients, we see that the elements of $\widehat{\mathcal{A}}\{T_1, \ldots, T_r\}$ are represented by families $(a_I)_{I\in\mathbb{N}^r}$ of elements of $\widehat{\mathcal{A}}$ which converge to 0 in the sense of Definition 1.6, namely, a family of elements $(a_I)_{I\in\mathbb{N}^r}$ of $\widehat{\mathcal{A}}$ represents a restricted power series if and only if for every open ideal $\mathfrak{a}$ of $\widehat{\mathcal{A}}$, all but finitely many of the a_I belong to $\mathfrak{a}$.

Following [4, III.4.2], we henceforth identify $\widehat{\mathcal{A}}\{T_1, \ldots, T_r\}$ with the $\widehat{\mathcal{A}}$-subalgebra of the algebra of formal power series with coefficients in $\widehat{\mathcal{A}}$ consisting of formal power series

$$\sum_{I=(i_1,\ldots,i_r)\in\mathbb{N}^r} a_I T_1^{i_1} \cdots T_r^{i_r}$$

such that the family $(a_I)_{I\in\mathbb{N}^r}$ converges to 0. Note that such a family is summable in $\widehat{\mathcal{A}}$ by Proposition 1.7, so that with our identification, the family $(a_I T_1^{i_1} \cdots T_r^{i_r})_{I\in\mathbb{N}^r}$ of elements of $\widehat{\mathcal{A}}\{T_1, \ldots, T_r\}$ is summable, with sum $\sum_{I\in\mathbb{N}^r} a_I T_1^{i_1} \cdots T_r^{i_r}$. Recall [3, 5.3 Proposition 2 and Theorem 2] that for every partition $(J_\lambda)_{\lambda\in L}$ of $\mathbb{N}^r$, the subfamily $(a_I)_{I\in J_\lambda}$ is summable, say of sum $s_\lambda \in \widehat{\mathcal{A}}$, and the family $(s_\lambda)_{\lambda\in L}$ is summable, with the same sum as the family

$(a_I)_{I\in\mathbb{N}^r}$, so that, with our identification, we have

$$\sum_{I\in\mathbb{N}^r} a_I T_1^{i_1}\cdots T_r^{i_r} = \sum_{\lambda\in L}\sum_{I\in J_\lambda} a_I T_1^{i_1}\cdots T_r^{i_r}.$$

Proposition 1.25. *The ring $\widehat{\mathcal{A}}\{T_1,\ldots,T_r\}$ satisfies the following universal property: for every continuous ring homomorphism $f\colon \mathcal{A}\to\mathcal{B}$ to a complete topological ring $\mathcal{B}$ and every choice of r elements $b_1,\ldots,b_r$ of $\mathcal{B}$, there exists a unique continuous ring homomorphism $\overline{f}\colon \widehat{\mathcal{A}}\{T_1,\ldots,T_r\}\to\mathcal{B}$ such that $\overline{f}|_{c(\mathcal{A})}=\widehat{f}$ and such that $\overline{f}(T_i)=b_i$ for every $i=1,\ldots,r$.*

Proof. This follows from [4, III.4.2 Proposition 4] and the universal property of the separated completion homomorphism $c\colon \mathcal{A}\to\widehat{\mathcal{A}}$. □

Corollary 1.26. *Let $\mathcal{A}$ be a complete topological ring and let $\mathcal{B}$ be a complete topological $\mathcal{A}$-algebra. Then for every $r\geq 1$, there are canonical isomorphisms*

$$\mathcal{C}\mathrm{Hom}_{\mathcal{A}-\mathrm{alg}}(\mathcal{A}\{T_1,\ldots,T_r\},\mathcal{B})\cong\mathcal{C}\mathrm{Hom}_{\mathcal{A}-\mathrm{mod}}(\mathcal{A}^{\oplus r},\mathcal{B})\cong\mathcal{B}^{\oplus r}.$$

Notation 1.27. Given a complete topological ring $\mathcal{A}$ and a subset $J\subset\{1,\ldots,r\}$, we denote by

$$\pi_{(1,J)}\colon \mathcal{A}\{T_i\}_{i\in\{1,\ldots,r\}}\to\mathcal{A}\{T_i\}_{i\notin J}$$

the unique homomorphism of topological $\mathcal{A}$-algebras defined by $\pi_{(1,J)}(T_i)=1$ if $i\in J$ and $\pi_{(1,J)}(T_i)=T_i$ otherwise. For $J=\{1,\ldots,r\}$, we denote the corresponding homomorphism $\mathcal{A}\{T_1,\ldots,T_r\}\to\mathcal{A}$ simply by $\pi_{(1,\ldots,1)}$.

For any collection $a_1,\ldots,a_r$ of elements of $\mathcal{A}$, we denote by $\lambda(a_1,\ldots,a_r)$ the unique endomorphisms of topological $\mathcal{A}$-algebras of $\mathcal{A}\{T_1,\ldots,T_r\}$ defined by $T_i\mapsto a_iT_i$, $i=1,\ldots,r$.

Finally, we denote by $\Delta\colon \mathcal{A}\{T_1,\ldots,T_r\}\to\mathcal{A}\{T\}$ the unique homomorphism of topological $\mathcal{A}$-algebras that maps T_i to T for every $i=1,\ldots,r$.

It follows from the definition of the completed tensor product that we have canonical isomorphisms

$$\widehat{\mathcal{A}}\{T_1,\ldots,T_r\}\cong\mathcal{A}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T_1,\ldots,T_r]\cong\widehat{\mathcal{A}}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T_1,\ldots,T_r]$$

where $\mathbb{Z}[T_1,\ldots,T_r]$ is endowed with the discrete topology. The following lemma is then a straightforward consequence of Lemma 1.13.

Lemma 1.28. *For every complete topological ring $\mathcal{A}$ and every set of variables $T_1,\ldots,T_s,T_{s+1},\ldots,T_r$, there exist canonical isomorphisms of complete topological $\mathcal{A}$-algebras*

$$\mathcal{A}\{T_1,\ldots,T_s,T_{s+1},\ldots T_r\}\cong\mathcal{A}\{T_1,\ldots,T_r\}\widehat{\otimes}_{\mathcal{A}}\mathcal{A}\{T_{s+1},\ldots,T_r\}\cong\mathcal{A}\{T_1,\ldots,T_s\}\{T_{s+1},\ldots T_r\}.$$

Lemma 1.29. *Let $\mathcal{B}$ be the limit of a countable inverse system $(\mathcal{B}_n)_{n\in\mathbb{N}}$ of complete topological rings with surjective continuous transition homomorphisms $p_{m,n}\colon\mathcal{B}_m\to\mathcal{B}_n$ for every $m\geq n\geq 0$ and let $T_1,\ldots,T_r$ be indeterminates. Then the canonical homomorphism of complete topological rings*

$$\mathcal{B}\{T_1,\ldots,T_r\}\cong(\varprojlim_{n\in\mathbb{N}}\mathcal{B}_n)\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T_1,\ldots,T_r]\to\varprojlim_{n\in\mathbb{N}}(\mathcal{B}_n\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T_1,\ldots T_r])\cong\varprojlim_{n\in\mathbb{N}}(\mathcal{B}_n\{T_1,\ldots,T_r\})$$

is an isomorphism.

Proof. Let $p_n\colon \mathcal{B}\to\mathcal{B}_n$, $n\in\mathbb{N}$ be the canonical projection homomorphisms. By definition, elements of $\mathcal{B}\{T_1,\ldots,T_r\}$ are represented by families $(b_I)_{I\in\mathbb{N}^r}$ of elements of $\mathcal{B}$ which converge to 0 in $\mathcal{B}$. Since the projection homomorphisms p_n are surjective, it follows from the definition of the topology on $\mathcal{B}$ that these families are in one-to-one correspondence with collections of families $(b_{n,I})_{I\in\mathbb{N}^r}$ of elements of $\mathcal{B}_n$, $n\in\mathbb{N}$, such that $(b_{n,I})_{I\in\mathbb{N}^r}$ converges to 0 in $\mathcal{B}_n$ for every $n\in\mathbb{N}$ and such that $b_{n,I}=p_{m,n}(b_{m,I})$ for every $m\geq n\geq 0$ and every $I\in\mathbb{N}^r$. □

Lemma 1.30. *Let $\mathcal{A}$ be a topological ring, let $\mathcal{B}$ be a separated topological ring with separated completion $c\colon \mathcal{B}\to\widehat{\mathcal{B}}$ and let $h_n\colon \mathcal{A}\to\mathcal{B}$ be a sequence of homomorphisms of groups which converges pointwise to the zero homomorphism. Then the map*

$$s\colon \mathcal{A}\to\widehat{\mathcal{B}}\{T\},\quad a\mapsto\sum_{n\in\mathbb{N}}c(h_n(a))T^n$$

is a well-defined homomorphism of groups and the following assertions are equivalent:

a) The homomorphism $s\colon \mathcal{A}\to\widehat{\mathcal{B}}\{T\}$ is continuous,

b) Every h_n, $n\in\mathbb{N}$, is continuous and the sequence $(h_n)_{n\in\mathbb{N}}$ converges continuously to the zero homomorphism.

Proof. Let $u_n\colon \mathcal{A}\to\widehat{\mathcal{B}}\{T\}$, $n\in\mathbb{N}$, be the sequence of homomorphisms of groups defined by $u_n(a)=c(h_n(a))T^n$. Since the sequence $(h_n)_{n\in\mathbb{N}}$ converges pointwise to the zero homomorphism, it follows from the definition of the topology on $\widehat{\mathcal{B}}\{T\}$ that the sequence $(u_n)_{n\in\mathbb{N}}$ converges pointwise to the zero homomorphism. Arguing as in the proof of Corollary 1.11, we conclude that the map s is a well-defined homomorphism of groups, and that the sequence of homomorphisms $(s_N)_{N\in\mathbb{N}}$ defined by $s_N=\sum_{n=0}^N u_n$ converges pointwise to s.

If each h_n, $n\in\mathbb{N}$, is continuous and the sequence $(h_n)_{n\in\mathbb{N}}$ converges continuously to the zero homomorphism, then s is a homomorphism of topological groups by Corollary 1.11. Conversely, assume that s is continuous. By definition of the topology on

$$\widehat{\mathcal{B}}\{T\}\subset\widehat{\mathcal{B}}[[T]]\cong\prod_{n\in\mathbb{N}}\widehat{\mathcal{B}},$$

where the isomorphism $\widehat{\mathcal{B}}[[T]]\cong\prod_{n\in\mathbb{N}}\widehat{\mathcal{B}}$ is induced by the product topology on $\prod_{n\in\mathbb{N}}\widehat{\mathcal{B}}$, every projection $p_n\colon \widehat{\mathcal{B}}\{T\}\to\widehat{\mathcal{B}}$, $(b_n)_{n\in\mathbb{N}}\mapsto b_n$, $n\in\mathbb{N}$, is a homomorphism of topological groups. It follows that $p_n\circ s=c\circ h_n$ is continuous for every $n\in\mathbb{N}$, which implies in turn that h_n is continuous since $c\colon \mathcal{B}\to\widehat{\mathcal{B}}$ is open onto its image. Furthermore, for every open ideal $\widehat{\mathfrak{b}}$ of $\widehat{\mathcal{B}}$, there exists an index $n\geq 0$ such that $c(h_n(a))\in\widehat{\mathfrak{b}}$ for every $a\in\mathcal{A}$ and every $n\geq n_0$. This implies that the sequence $(c\circ h_n)_{n\in\mathbb{N}}$ converges continuously to the zero homomorphism, hence that the sequence $(h_n)_{n\in\mathbb{N}}$ converges to the zero homomorphism. □

Lemma 1.31. *Let $\mathcal{A}$ be topological ring with separated completion $c\colon \mathcal{A}\to\widehat{\mathcal{A}}$ and let $S\subset\mathcal{A}$ be a multiplicatively closed subset. Let $T_1,\ldots,T_r$ be a set of indeterminates and let $\widehat{S}_0\subset\widehat{\mathcal{A}}\{T_1,\ldots,T_r\}$ be the image of S by the composition of c with the inclusion $i_0\colon \widehat{\mathcal{A}}\hookrightarrow\widehat{\mathcal{A}}\{T_1,\ldots,T_r\}$. Then there exists a canonical isomorphism of complete topological $\widehat{S^{-1}\mathcal{A}}$-algebras*

$$\widehat{S^{-1}\mathcal{A}}\{T_1,\ldots,T_r\}\cong(\widehat{S}_0^{-1}(\widehat{\mathcal{A}\{T_1,\ldots,T_r\}})),$$

where $(\widehat{S}_0^{-1}(\widehat{\mathcal{A}\{T_1,\ldots,T_r\}}))$ *is viewed as an* $\widehat{S^{-1}\mathcal{A}}$*-algebra via the unique homomorphism of topological rings deduced from the homomorphism of topological rings*

$$\mathcal{A} \stackrel{i_0\circ c}{\to} \widehat{\mathcal{A}}\{T_1,\ldots,T_r\} \to (\widehat{S}_0^{-1}(\widehat{\mathcal{A}\{T_1,\ldots,T_r\}}))$$

by the universal property of separated completed localization.

Proof. Indeed, it follows from the definition of the separated completed localization and the definition of the restricted power series rings that these topological rings are both isomorphic, as topological $\widehat{S^{-1}\mathcal{A}}$-algebras, to the separated completion of the ring $S^{-1}\mathcal{A}[T_1,\ldots,T_r] = S^{-1}\mathcal{A}\otimes_{\mathbb{Z}}\mathbb{Z}[T_1,\ldots,T_r]$ with respect to the topology generated by the open ideals $S^{-1}\mathfrak{a}[T_1,\ldots,T_r]$, where $\mathfrak{a}$ ranges through the set of open ideals of $\mathcal{A}$. □

Notation 1.32. Given a multiplicatively closed subset S of a topological ring $\mathcal{A}$, we denote by

$$\widetilde{j}_{T_1,\ldots,T_r} : \widehat{\mathcal{A}}\{T_1,\ldots,T_r\} \to \widehat{S^{-1}\mathcal{A}}\{T_1,\ldots,T_r\}$$

the homomorphism of topological $\widehat{\mathcal{A}}$-algebras such that $\widetilde{j}_{T_1,\ldots,T_r}(T_i) = T_i$ for every $i = 1,\ldots,r$.

2. Restricted exponential homomorphisms and topologically integrable derivations

In this section, we develop the basic algebraic theory of restricted exponential homomorphisms, which are the counterpart for topological rings of the co-action homomorphisms $e\colon B \to B\otimes_{\mathbb{Z}}\mathbb{Z}[T]$ of the Hopf algebra $\mathbb{Z}[T]$ of the additive group scheme $\mathbb{G}_{a,\mathbb{Z}}$ in the category of rings. We establish a one-to-one correspondence between restricted exponential homomorphisms and suitable systems of continuous iterated higher derivations which extends the classical correspondence between algebraic exponential homomorphisms $e\colon B\to B\otimes_{\mathbb{Z}}\mathbb{Z}[T]$ and locally finite iterative higher derivations of the ring B [18, 2, 8].

2.1. Restricted exponential homomorphisms.

Recall that the ring $\mathbb{Z}[T]$, where T is an indeterminate, carries the structure of a cocommutative Hopf algebra whose comuliplication, coinverse and counit are given respectively by the following $\mathbb{Z}$-algebra homomorphisms:

$$\begin{array}{ll} m\colon \mathbb{Z}[T]\to\mathbb{Z}[T]\otimes_{\mathbb{Z}}\mathbb{Z}[T]\cong\mathbb{Z}[T,T'], & T\mapsto T+T' \\ \iota\colon \mathbb{Z}[T]\to\mathbb{Z}[T], & T\mapsto -T \\ \epsilon\colon \mathbb{Z}[T]\to\mathbb{Z}, & T\mapsto 0. \end{array}$$

Given any complete topological ring $\mathcal{A}$, the complete topological ring $\mathcal{A}\{T\} = \mathcal{A}\widehat{\otimes}\mathbb{Z}[T]$ inherits the structure of a cocommutative topological Hopf $\mathcal{A}$-algebra with comultiplication $\mathrm{id}_{\mathcal{A}}\widehat{\otimes}m$, coinverse $\mathrm{id}_{\mathcal{A}}\widehat{\otimes}\iota$ and counit $\mathrm{id}_{\mathcal{A}}\widehat{\otimes}\epsilon$.

Definition 2.1. Let $\mathcal{A}$ be a complete topological ring and let $\mathcal{B}$ be a complete topological $\mathcal{A}$-algebra. A *restricted exponential* $\mathcal{A}$*-homomorphism* is a homomorphism of topological $\mathcal{A}$-algebras

$$\mathrm{e}\colon \mathcal{B}\to\mathcal{B}\{T\} = \mathcal{B}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T]$$

which defines a coaction of the Hopf $\mathcal{A}$-algebra $\mathcal{A}\{T\}$ on $\mathcal{B}$. This means equivalently that the following diagrams of $\mathcal{A}$-algebra homomorphisms are commutative:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\mathrm{e}} & \mathcal{B}\{T\} \\ {\scriptstyle \mathrm{e}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{B}}\widehat{\otimes} m} \\ \mathcal{B}\{T\} & \xrightarrow{\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}} & \mathcal{B}\{T'\}\{T\} = \mathcal{B}\{T',T\} \end{array} \qquad \begin{array}{ccc} \mathcal{B} & \xrightarrow{\mathrm{e}} & \mathcal{B}\{T\} \\ & {\scriptstyle \mathrm{id}_{\mathcal{B}}}\searrow & \downarrow{\scriptstyle q=\mathrm{id}_{\mathcal{B}}\widehat{\otimes}\epsilon} \\ & & \mathcal{B} = \mathcal{B}\{T\}/T\mathcal{B}\{T\}. \end{array}$$

Let $p_i\colon \mathcal{B}[[T]] = \prod_{i\in\mathbb{N}} \mathcal{B} \to \mathcal{B}$ denote the i-th projection. Then by definition of the topology on $\mathcal{B}\{T\}$, the composition $\mathrm{e}_i = p_i \circ \mathrm{e}\colon \mathcal{B} \to \mathcal{B}$ is a homomorphism of topological $\mathcal{A}$-modules for every $i \in \mathbb{N}$ for which we can write

$$\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i.$$

The commutativity of the right hand side diagram of Definition 2.1 means that $\mathrm{e}_0 = \mathrm{id}_{\mathcal{B}}$. On the other hand, with the identifications made, the homomorphisms $\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}$ and $\mathrm{id}_{\mathcal{B}}\widehat{\otimes} m$ are given by

$$\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}\colon \mathcal{B}\{T\} \to \mathcal{B}\{T',T\},\ \sum_{i\in\mathbb{N}} b_i T^i \mapsto \sum_{i\in\mathbb{N}} \mathrm{e}(b_i) T^i = \sum_{(i,j)\in\mathbb{N}^2} \mathrm{e}_j(b_i) T'^j T^i$$

$$\mathrm{id}_{\mathcal{B}}\widehat{\otimes} m\colon \mathcal{B}\{T\} \to \mathcal{B}\{T',T\},\ \sum_{i\in\mathbb{N}} b_i T^i \mapsto \sum_{i\in\mathbb{N}} b_i (T'+T)^i.$$

The commutativity of the left hand side diagram in Definition 2.1 says that in $\mathcal{B}\{T,T'\}$, we have

$$\sum_{(i,j)\in\mathbb{N}^2} (\mathrm{e}_j \circ \mathrm{e}_i) T'^j T^i = \sum_{\ell\in\mathbb{N}} \mathrm{e}_\ell (T'+T)^\ell. \tag{2.1}$$

In the next subsections, we first establish general properties of restricted exponential homomorphisms e. Then we discuss in more detail the properties of their associated collections $(\mathrm{e}_i)_{i\in\mathbb{N}}$ of homomorphisms of topological modules. In what follows, unless otherwise specified all topological modules and rings and are assumed to be modules and algebras over a fixed topological ring $\mathcal{A}$, and homomorphisms between these are assumed to be homomorphisms of $\mathcal{A}$-modules and $\mathcal{A}$-algebras respectively.

2.2. **Basic properties of restricted exponential automorphisms.**

2.2.1. *Rings of invariants and associated restricted exponential homomorphisms.*

Definition 2.2. Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. We say that en element $b \in \mathcal{B}$ is e-*invariant* if $\mathrm{e}(b) = i_0(b)$. We denote by

$$\mathcal{B}^{\mathrm{e}} = \mathrm{Ker}(\mathrm{e} - i_0) \subseteq \mathcal{B}$$

the subset of all e-invariant elements of $\mathcal{B}$, endowed with the induced topology.

Proposition 2.3. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. Then the following hold:*

a) The set $\mathcal{B}^{\mathrm{e}}$ is a complete topological subring of $\mathcal{B}$.

b) For every $i \geq 1$, the homomorphism $\mathrm{e}_i\colon \mathcal{B} \to \mathcal{B}$ is a homomorphism of topological $\mathcal{B}^{\mathrm{e}}$-modules.

c) If $\mathcal{B}$ admits a fundamental system $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ of open prime ideals of $\mathcal{B}$ then $\mathcal{B}^{\mathrm{e}}$ is factorially closed in $\mathcal{B}$. In particular, every invertible element of $\mathcal{B}$ is contained in $\mathcal{B}^{\mathrm{e}}$.

Proof. The fact that $\mathcal{B}^{\mathrm{e}}$ is a subgring of $\mathcal{B}$ is clear. Since $\mathcal{B}\{T\}$ is complete, hence separated, $\{0\}$ is a closed subset of $\mathcal{B}\{T\}$. Since $\mathrm{e} - i_0 \colon \mathcal{B} \to \mathcal{B}\{T\}$ is a homomorphism of topological groups, $\mathcal{B}^{\mathrm{e}}$ is a closed subgroup of $\mathcal{B}$, hence a complete topological group since $\mathcal{B}$ is complete. Assertion b) is clear from the definition of $\mathcal{B}^{\mathrm{e}}$ and the homomorphisms e_i.

Now let $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ be a fundamental system of open prime ideals of $\mathcal{B}$ and let

$$\pi_n \colon \mathcal{B}\{T\} = \varprojlim_{n\in\mathbb{N}} (\mathcal{B}/\mathfrak{b}_n)[T] \to (\mathcal{B}/\mathfrak{b}_n)[T], \quad n \in \mathbb{N},$$

be the canonical projections. Write $\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i$, where $\mathrm{e}_0 = \mathrm{id}_{\mathcal{B}}$, and let $b, b' \in \mathcal{B}$. Assume that $\mathrm{e}(bb') = \mathrm{e}(b)\mathrm{e}(b') = bb'$. Then for every $n \geq 0$, we have

$$\pi_n(\mathrm{e}(b)\mathrm{e}(b')) = \left(\sum_{i\in\mathbb{N}} \pi_n(\mathrm{e}_i(b))T^i\right)\left(\sum_{i\in\mathbb{N}} \pi_n(\mathrm{e}_i(b'))T^i\right) = \pi_n(bb') = \pi_n(b)\pi_n(b')$$

in the integral domain $(\mathcal{B}/\mathfrak{b}_n)[T]$. It follows that $\pi_n(\mathrm{e}_i(b)) = \pi_n(\mathrm{e}_i(b')) = 0$ for every $i \geq 1$. This implies that for every $i \geq 1$, $\mathrm{e}_i(b)$ and $\mathrm{e}_i(b')$ belong to $\bigcap_{n\geq 1} \mathfrak{b}_n = \{0\}$ as $\mathcal{B}$ is separated. Thus $\mathrm{e}(b) = b$ and $\mathrm{e}(b') = b'$. Finally, if $b \in \mathcal{B}$ is invertible, then $bb^{-1} = 1 \in \mathcal{B}^{\mathrm{e}}$ and so, b and b^{-1} belong to $\mathcal{B}^{\mathrm{e}}$. ☐

Example 2.4. If the topology on $\mathcal{B}$ is the discrete one, the existence of a fundamental system of open prime ideals of $\mathcal{B}$ is equivalent to the property that $\mathcal{B}$ is an integral domain. This is not true in general, and the conclusion of assertion b) in Proposition 2.3 does not hold under the weaker assumption that $\mathcal{B}$ is integral. Indeed, let $\mathcal{B} = \varprojlim_{n\in\mathbb{N}} \mathbb{C}[u]/(u^n) \cong \mathbb{C}[[u]]$ be the completion of $\mathbb{C}[u]$ for the u-adic topology. The homomorphism of $\mathbb{C}$-algebras $\mathbb{C}[u] \to \mathbb{C}[[u]]\{T\}$ defined by

$$u \mapsto u\sum_{i\in\mathbb{N}} (uT)^i = \sum_{i\in\mathbb{N}} u^{i+1}T^i$$

induces a uniquely determined continuous homomorphism $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ which satisfies the axioms of a restricted exponential $\mathbb{C}$-homomorphism. The ring of invariants $\mathcal{B}^{e}$ is equal to the subring of constant formal power series. In particular, the invertible element $1-u$ of $\mathcal{B}$ does not belong to $\mathcal{B}^{\mathrm{e}}$, so that $\mathcal{B}^{\mathrm{e}}$ is not factorially closed in $\mathcal{B}$.

Proposition 2.5. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. Then for every element $a \in \mathcal{B}^{\mathrm{e}}$, the homomorphism*

$$\mathrm{e}_{\lambda(a)} := \lambda(a) \circ \mathrm{e}\colon \mathcal{B} \xrightarrow{\mathrm{e}} \mathcal{B}\{T\} \xrightarrow{T\mapsto aT} \mathcal{B}\{T\}$$

is a restricted exponential homomorphism.

Proof. Since the homomorphisms e and $\lambda(a)$ are continuous, so is $\mathrm{e}_{\lambda(a)}$. The commutativity of the right hand side diagram in Definition 2.1 for $\mathrm{e}_{\lambda(a)}$ is clear. Writing $\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i$, we have

$$\mathrm{e}_{\lambda(a)} = \sum_{i\in\mathbb{N}} \mathrm{e}_i(aT)^i = \sum_{i\in\mathbb{N}} (a^i\mathrm{e}_i)T^i,$$

where $a^i\mathrm{e}_i$ is the homomorphism defined by $b \mapsto a^i\mathrm{e}_i(b)$ for every $b \in \mathcal{B}$. Since $a \in \mathcal{B}^{\mathrm{e}}$, $a^i \in \mathcal{B}^{\mathrm{e}}$ for every $i \geq 0$. Since by Proposition 2.3 b) each e_i, $i \geq 1$, is a homomorphism of topological $\mathcal{B}^{\mathrm{e}}$-modules, applying (2.1), we obtain

$$\begin{aligned}
(\mathrm{e}_{\lambda(a)}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}) \circ \mathrm{e}_{\lambda(a)} &= \textstyle\sum_{i\in\mathbb{N}} (\mathrm{e}_{\lambda(a)} \circ (a^i\mathrm{e}_i))T^i \\
&= \textstyle\sum_{(i,j)\in\mathbb{N}^2} (\mathrm{e}_j \circ (a^i e_i))(aT')^j T^i \\
&= \textstyle\sum_{(i,j)\in\mathbb{N}^2} (\mathrm{e}_j \circ \mathrm{e}_i)(aT')^j (aT)^i \\
&= \textstyle\sum_{\ell\in\mathbb{N}} \mathrm{e}_\ell(aT' + aT)^\ell = (\mathrm{id}_{\mathcal{B}}\widehat{\otimes} m) \circ \mathrm{e}_{\lambda(a)}.
\end{aligned}$$

This shows that the commutativity of left hand side diagram in Definition 2.1 is satisfied for $\mathrm{e}_{\lambda(a)}$. □

Proposition 2.6. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon \mathcal{B}\to\mathcal{B}\{T\}$ be a restricted exponential homomorphism. Let $S\subset\mathcal{B}^{\mathrm{e}}$ be a multiplicatively closed subset, let $\widetilde{j}\colon\mathcal{B}\to\widehat{S^{-1}\mathcal{B}}$ be the separated completed localization homomorphism and let $\widetilde{j}_T\colon\mathcal{B}\{T\}\to\widehat{S^{-1}\mathcal{B}}\{T\}$ be the induced homomorphism.*

Then there exists a unique restricted exponential homomorphism $\widehat{S^{-1}\mathrm{e}}\colon\widehat{S^{-1}\mathcal{B}}\to\widehat{S^{-1}\mathcal{B}}\{T\}$ such that $\widetilde{j}_T\circ\mathrm{e}=\widehat{S^{-1}\mathrm{e}}\circ\widetilde{j}$

Proof. We identify $S\subset\mathcal{B}$ with $i_0(S)\subset\mathcal{B}\{T\}$ and $S^{-1}\widehat{(\mathcal{B}\{T\})}$ with $\widehat{S^{-1}\mathcal{B}}\{T\}$ by the canonical isomorphism of Lemma 1.31. Since $S\subset\mathcal{B}^{\mathrm{e}}$,we have $\mathrm{e}(S)=S\subset\mathcal{B}\{T\}$ so that by the universal property of separated completed localization, there exists a unique homomorphism of topological rings

$$\widehat{S^{-1}e}\colon\widehat{S^{-1}\mathcal{B}}\to\widehat{S^{-1}\mathcal{B}}\{T\}$$

such that $\widetilde{j}_T\circ\mathrm{e}=\widehat{S^{-1}\mathrm{e}}\circ\widetilde{j}$. Write $\mathrm{e}=\sum_{i\in\mathbb{N}}\mathrm{e}_iT^i$. Since $S\subset\mathcal{B}^{\mathrm{e}}$ and since e_i is a homomorphism of topological $\mathcal{B}^{\mathrm{e}}$-modules for every $i\geq 0$ by Proposition 2.3 b), it follows that each e_i, $i\in\mathbb{N}$, induces a uniquely determined homomorphism of topological $S^{-1}\mathcal{B}^{\mathrm{e}}$-modules $S^{-1}\mathrm{e}_i\colon S^{-1}\mathcal{B}\to S^{-1}\mathcal{B}$, hence by the universal property of separated completed localization, a homomorphism $\widehat{S^{-1}\mathrm{e}_i}\colon\widehat{S^{-1}\mathcal{B}}\to\widehat{S^{-1}\mathcal{B}}$ of topological $\widehat{S^{-1}\mathcal{B}^{\mathrm{e}}}$-modules. By construction, we then have

$$\widehat{S^{-1}\mathrm{e}}=\sum_{i\in\mathbb{N}}\widehat{S^{-1}\mathrm{e}_i}T^i.$$

To show that $\widehat{S^{-1}\mathrm{e}}$ is a restricted exponential homomorphism, it is enough to check the commutativity of the two diagrams of Definition 2.1 in restriction to the dense image of $S^{-1}\mathcal{B}$ in $\widehat{S^{-1}\mathcal{B}}$ by the separated completion morphism $c\colon S^{-1}\mathcal{B}\to\widehat{S^{-1}\mathcal{B}}$. Let $x=s^{-1}b\in S^{-1}\mathcal{B}$, where $b\in\mathcal{B}$. Then by definition of $\widehat{S^{-1}\mathrm{e}}$, we have

$$\widehat{S^{-1}\mathrm{e}}(c(x))=\sum_{i\in\mathbb{N}}\widehat{S^{-1}\mathrm{e}_i}(c(x))T^i=c_T\big(\sum_{i\in\mathbb{N}}(S^{-1}\mathrm{e}_i)(x)T^i\big)=c_T\big(\sum_{i\in\mathbb{N}}s^{-1}\mathrm{e}_i(b)T^i\big),$$

where $c_T\colon S^{-1}\mathcal{B}\{T\}\to\widehat{S^{-1}\mathcal{B}}\{T\}$ denotes the separated completion homomorphism. This immediately implies the commutativity of the right hand side diagram of Definition 2.1. On the other hand, letting $c_{T,T'}\colon S^{-1}\mathcal{B}\{T,T'\}\to\widehat{S^{-1}\mathcal{B}}\{T,T'\}$ be the separated completion homomorphism, we have

$$\begin{aligned}
(\widehat{S^{-1}e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})(\widehat{S^{-1}e}(c(x)) &= \textstyle\sum_{i\in\mathbb{N}}\widehat{S^{-1}e}(c(s^{-1}e_i(b)))T^i\\
&= \textstyle\sum_{(i,j)\in\mathbb{N}}\widehat{S^{-1}e_j}(c(s^{-1}e_i(b)))T'^jT^i\\
&= c_{T,T'}(\textstyle\sum_{(i,j)\in\mathbb{N}^2}S^{-1}e_j((s^{-1}e_i(b)))T'^jT^i)\\
&= c_{T,T'}(\textstyle\sum_{(i,j)\in\mathbb{N}^2}s^{-1}e_j(e_i(b))T'^jT^i)\\
&= c_{T,T'}(\textstyle\sum_{\ell\in\mathbb{N}}s^{-1}e_\ell(b)(T'+T)^\ell)\\
&= c_{T,T'}(\textstyle\sum_{\ell\in\mathbb{N}}S^{-1}e_\ell(x)(T'+T)^\ell)\\
&= (\textstyle\sum_{\ell\in\mathbb{N}}\widehat{S^{-1}e_\ell}(c(x))(T'+T)^\ell)=(\mathrm{id}_{\mathcal{B}}\widehat{\otimes}m)(\widehat{S^{-1}e}(c(x)),
\end{aligned}$$

which shows the commutativity of the left hand side diagram. □

Proposition 2.7. *Let $\pi\colon \mathcal{B} \to \mathcal{C}$ be a surjective open homomorphism of complete topological rings. Let $I = \mathrm{Ker}(\pi)$ and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism. Assume that $\mathrm{e}(I) \subset i_0(I)\mathcal{B}\{T\}$. Then there exists a unique restricted exponential homomorphism $\bar{\mathrm{e}}\colon \mathcal{C} \to \mathcal{C}\{T\}$ such that $\pi_T \circ \mathrm{e} = \bar{\mathrm{e}} \circ \pi$, where $\pi_T\colon \mathcal{B}\{T\} \to \mathcal{C}\{T\}$ is the unique homomorphism of topological $\mathcal{B}$-algebras which maps T to T.*

Proof. The assumptions imply that e induces a unique homomorphism of rings

$$\widetilde{\mathrm{e}}\colon \mathcal{C} = \mathcal{B}/I \to \mathcal{B}\{T\}/i_0(I)\mathcal{B}\{T\} \cong \mathcal{B}/I \otimes \mathcal{B}\{T\} \cong \mathcal{C} \otimes \mathcal{B}\{T\}$$

such that $(\pi \otimes \mathrm{id}_{\mathcal{B}\{T\}}) \circ \mathrm{e} = \widetilde{\mathrm{e}} \circ \pi$. Since π is open, $\widetilde{\mathrm{e}}$ is continuous when we endow the ring $\mathcal{C} \otimes \mathcal{B}\{T\}$ with the linear topology generated by open ideals of the form $U_{\mathcal{C}} \otimes \mathcal{B}\{T\} + \mathcal{C} \otimes V_{\mathcal{B}\{T\}}$ where $U_{\mathcal{C}}$ and $V_{\mathcal{B}\{T\}}$ run through the sets of open ideals of $\mathcal{C}$ and $\mathcal{B}\{T\}$ respectively. Note also that $\pi \otimes \mathrm{id}_{\mathcal{B}\{T\}}$ is an open homomorphism of topological rings. The composition

$$\bar{\mathrm{e}}\colon \mathcal{C} \to \mathcal{C}\widehat{\otimes}\mathcal{B}\{T\} \cong \mathcal{C}\{T\}$$

of $\widetilde{\mathrm{e}}$ with the separated completion homomorphism $c\colon \mathcal{C} \otimes \mathcal{B}\{T\} \to \mathcal{C}\widehat{\otimes}\mathcal{B}\{T\}$ is then a homomorphism of topological rings. The commutativity of the two diagrams in Definition 2.1 is straightforward to check. □

2.2.2. *Operations on restricted exponential homomorphism.*
Recall (cf. Notation 1.27) that for a complete topological ring $\mathcal{B}$, $\Delta\colon \mathcal{B}\{T, T'\} \to \mathcal{B}\{T''\}$ denotes the unique continuous $\mathcal{B}$-algebra homomorphism which maps T and T' to T''.

Proposition 2.8. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ and $\mathrm{e}'\colon \mathcal{B} \to \mathcal{B}\{T'\}$ be restricted exponential homomorphisms such that the following diagram commutes*

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\ \mathrm{e}'\ } & \mathcal{B}\{T\} \\
\Big\downarrow{\scriptstyle \mathrm{e}} & & \Big\downarrow{\scriptstyle \mathrm{e}'\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}} \\
\mathcal{B}\{T'\} & \xrightarrow[\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T']}]{} & \mathcal{B}\{T, T'\}
\end{array}$$

Then the map $\mathrm{e}''\colon \mathcal{B} \to \mathcal{B}\{T''\}$ defined by

$$\mathrm{e}'' = \Delta \circ (\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T']}) \circ \mathrm{e}' = \Delta \circ (\mathrm{e}'\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}) \circ \mathrm{e}$$

is a restricted exponential homomorphism.

Proof. Being the composition of homomorphisms of topological rings, e'' is a homomorphism of topological rings. Denoting T'' by T_0, we have, by definition of e'',

$$\mathrm{e}'' = \sum_{n \in \mathbb{N}} \mathrm{e}''_n T_0^n = \sum_{(i,j) \in \mathbb{N}^2} (\mathrm{e}'_j \circ \mathrm{e}_i) T_0^{i+j} = \sum_{(i,j) \in \mathbb{N}^2} (\mathrm{e}_j \circ \mathrm{e}'_i) T_0^{i+j}. \tag{2.2}$$

This implies in particular that $\mathrm{e}''_0 = \mathrm{e}'_0 \circ \mathrm{e}_0 = \mathrm{id}_{\mathcal{B}}$, hence that the commutativity of the right hand side diagram of Definition 2.1 holds for e''. Combining Equation (2.2) above with

Equation (2.1) for e and e', we obtain on the other hand that

$$\begin{aligned}(\mathrm{id}_{\mathcal{B}}\widehat{\otimes}m)\circ\mathrm{e}'' &= \textstyle\sum_{(i,j)\in\mathbb{N}^2}(\mathrm{e}'_j\circ\mathrm{e}_i)(T_0+T'_0)^{i+j}\\ &= \textstyle\sum_{(i,k,\ell)\in\mathbb{N}^3}(\mathrm{e}'_\ell\circ\mathrm{e}'_k\circ\mathrm{e}_i)T_0'^{\ell}T_0^k(T_0+T'_0)^i\\ &= \textstyle\sum_{(i,k,\ell)\in\mathbb{N}^3}(\mathrm{e}_i\circ\mathrm{e}'_\ell\circ\mathrm{e}'_k)(T_0+T'_0)^iT_0^{\ell}T_0^k\\ &= \textstyle\sum_{(m,n,k,\ell)\in\mathbb{N}^4}(\mathrm{e}_n\circ\mathrm{e}_m\circ\mathrm{e}'_\ell\circ\mathrm{e}'_k)T_0'^{n}T_0^{m}T_0'^{\ell}T_0^k\\ &= \textstyle\sum_{(m,n,k,\ell)\in\mathbb{N}^4}(\mathrm{e}_n\circ\mathrm{e}'_\ell\circ\mathrm{e}_m\circ\mathrm{e}'_k)T_0'^{n+\ell}T_0^{k+m}\\ &= \textstyle\sum_{(i,n,\ell)\in\mathbb{N}^3}(\mathrm{e}_n\circ\mathrm{e}'_\ell\circ\mathrm{e}''_i)T_0'^{n+\ell}T_0^i\\ &= \textstyle\sum_{(i,j)\in\mathbb{N}^2}(\mathrm{e}''_j\circ\mathrm{e}''_i)T_0'^{j}T_0^i\\ &= (\mathrm{e}''\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T_0]})\circ\mathrm{e}'',\end{aligned}$$

which shows that e'' is a restricted exponential homomorphism. □

Proposition 2.9. *Let $(\mathcal{B}_n)_{n\in\mathbb{N}}$ be a countable inverse system of complete topological rings with continuous and surjective transition homomorphisms $p_{m,n}\colon\mathcal{B}_m\to\mathcal{B}_n$ for every $m\geq n\geq 0$. Let $\mathcal{B}$ be its limit and let $p_n\colon\mathcal{B}\to\mathcal{B}_n$, $n\in\mathbb{N}$, be the canonical continuous projections. Let $\mathrm{e}_n\colon\mathcal{B}_n\to\mathcal{B}_n\{T\}$, $n\in\mathbb{N}$, be a collection of restricted exponential homomorphisms such that*

$$\mathrm{e}_n\circ p_{m,n}=(p_{n,m}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})\circ\mathrm{e}_n\qquad\forall m\geq n\geq 0.$$

Then there exists a unique restricted exponential homomorphism $\underline{\mathrm{e}}=\varprojlim_{n\in\mathbb{N}}\mathrm{e}_n\colon\mathcal{B}\to\mathcal{B}\{T\}$ such that $\mathrm{e}_n\circ p_n=(p_n\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})\circ\underline{\mathrm{e}}$ for every $n\in\mathbb{N}$.

Proof. The hypothesis combined with Lemma 1.29 implies the existence of a unique homomorphism of topological rings

$$\underline{\mathrm{e}}=\varprojlim_{n\in\mathbb{N}}\mathrm{e}_n\colon\mathcal{B}=\varprojlim_{n\in\mathbb{N}}\mathcal{B}_n\to\varprojlim_{n\in\mathbb{N}}\mathcal{B}_n\{T\}\cong\mathcal{B}\{T\}$$

such that $\mathrm{e}_n\circ p_n=(p_n\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})\circ\underline{\mathrm{e}}$ for every $n\in\mathbb{N}$. The equality $\mathrm{e}_{n,0}=\mathrm{id}_{\mathcal{B}}$ for every $n\in\mathbb{N}$ implies that $\underline{\mathrm{e}}_0=\mathrm{id}_{\mathcal{B}}$. Similarly, since $(\mathrm{id}_{\mathcal{B}}\widehat{\otimes}m)\circ\mathrm{e}_n=(\mathrm{e}_n\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})\circ\mathrm{e}_n$ for every $n\in\mathbb{N}$, it follows that $(\mathrm{id}_{\mathcal{B}}\widehat{\otimes}m)\circ\underline{\mathrm{e}}=(\underline{\mathrm{e}}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})\circ\underline{\mathrm{e}}$, showing that $\underline{\mathrm{e}}$ is a restricted exponential homomorphism. □

2.2.3. *Restricted exponential homomorphisms and automorphisms.*

Proposition 2.10. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon\mathcal{B}\to\mathcal{B}\{T\}$ be a restricted exponential homomorphism. Then for every continuous ring automorphism α of $\mathcal{B}$, the composition*

$${}^{\alpha}\mathrm{e}:=(\alpha\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})\circ\mathrm{e}\circ\alpha^{-1}\colon\mathcal{B}\to\mathcal{B}\{T\}$$

is a restricted exponential homomorphism.

Proof. It is clear that ${}^{\alpha}\mathrm{e}$ is a homomorphism of topological rings. By definition, we have

$${}^{\alpha}\mathrm{e}=\sum_{i\in\mathbb{N}}{}^{\alpha}\mathrm{e}_iT^i=\sum_{i\in\mathbb{N}}(\alpha\circ\mathrm{e}_i\circ\alpha^{-1})T^i.$$

Since $\mathrm{e}_0=\mathrm{id}_{\mathcal{B}}$, we have ${}^{\alpha}\mathrm{e}_0=\mathrm{id}_{\mathcal{B}}$ showing that commutativity of the right hand side diagram of Definition 2.1 holds for ${}^{\alpha}\mathrm{e}(b)$. On the other hand, applying Equation (2.1), we have

$$\begin{aligned}(\mathrm{id}_{\mathcal{B}}\widehat{\otimes}m)\circ{}^{\alpha}\mathrm{e} &= \textstyle\sum_{\ell\in\mathbb{N}}(\alpha\circ\mathrm{e}_{\ell}\circ\alpha^{-1})(T+T')^{\ell}\\ &= \textstyle\sum_{(i,j)\in\mathbb{N}^2}(\alpha\circ\mathrm{e}_i\circ\mathrm{e}_j\circ\alpha^{-1})T'^{j}T^{i}\\ &= \textstyle\sum_{(i,j)\in\mathbb{N}^2}\big((\alpha\circ\mathrm{e}_i\circ\alpha^{-1})\circ(\alpha\circ\mathrm{e}_j\circ\alpha^{-1})\big)T'^{j}T^{i}\\ &= ({}^{\alpha}\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})\circ{}^{\alpha}\mathrm{e},\end{aligned}$$

which shows that ${}^{\alpha}\mathrm{e}$ is a restricted exponential homomorphism. □

Proposition 2.11. *Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon\mathcal{B}\to\mathcal{B}\{T\}$ be a restricted exponential homomorphism. Then the compositions*

$$\varphi=\pi_{(1)}\circ e\colon \mathcal{B}\xrightarrow{\mathrm{e}}\mathcal{B}\{T\}\xrightarrow{T\mapsto 1}\mathcal{B}\quad\textit{and}\quad \psi=\pi_{(-1)}\circ e\colon \mathcal{B}\xrightarrow{\mathrm{e}}\mathcal{B}\{T\}\xrightarrow{T\mapsto -1}\mathcal{B}$$

are continuous ring automorphisms of $\mathcal{B}$ inverse to each other.

Proof. The homomorphisms φ and ψ are clearly continuous. Note that by definition, $\psi=\pi_{(1)}\circ\mathrm{e}_{\lambda(-1)}$. Furthermore, letting $\pi_{(1,1)}=\pi_{(1)}\circ\Delta\colon\mathcal{B}\{T,T'\}\to\mathcal{B}$ be the unique continuous $\mathcal{B}$-algebra homomorphism that maps T and T' to 1, we have

$$\psi\circ\varphi=\pi_{(1,1)}\circ(\mathrm{e}_{\lambda(-1)}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})\circ e\quad\text{and}\quad\varphi\circ\psi=\pi_{(1,1)}\circ(\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T']})\circ\mathrm{e}_{\lambda(-1)}.$$

Let $f\colon\mathcal{B}\to\mathcal{B}\{T,T'\}$ be the composition of e with the unique continuous $\mathcal{B}$-algebra homomorphism $\mathcal{B}\{T\}\to\mathcal{B}\{T,T'\}$ that maps T to $T-T'$. Since e is a restricted exponential homomorphism, Equation (2.1) implies that

$$(\mathrm{e}_{\lambda(-1)}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})\circ e=f=(\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T']})\circ\mathrm{e}_{\lambda(-1)}.$$

Since $\pi_{(1,1)}\circ f=\mathrm{id}_{\mathcal{B}}$, the assertion follows. □

2.3. **Sliced restricted exponential homomorphisms.**

Definition 2.12. Let $\mathcal{B}$ be a complete topological ring and let $\mathrm{e}\colon\mathcal{B}\to\mathcal{B}\{T\}$ be a restricted exponential homomorphism. A *local slice* for e is an element $s\in\mathcal{B}$ such that $\mathrm{e}(s)\in\mathcal{B}\{T\}$ is a polynomial of degree 1.

Example 2.13. Let $\mathcal{B}=\mathbb{C}[u]$ endowed with the discrete topology, so that $\mathcal{B}\{T\}=\mathbb{C}[u][T]$. Then $\mathrm{e}\colon\mathbb{C}[u]\to\mathbb{C}[u][T]$ defined by $P(u)\mapsto P(u+T)$ is a restricted exponential homomorphism which has the element $s=u$ as a slice. On the other hand, letting $\mathcal{B}=\mathbb{C}[[u]]$ endowed with the u-adic topology, the restricted exponential homomorphism

$$\mathrm{e}\colon\mathbb{C}[[u]]\to\mathbb{C}[[u]]\{T\},\quad u\mapsto u\sum_{i\in\mathbb{N}}(uT)^i=\sum_{i\in\mathbb{N}}u^{i+1}T^i$$

of Example 2.4 does not admit a local slice.

Let $\mathrm{e}=\sum_{i\in\mathbb{N}}\mathrm{e}_iT^i\colon\mathcal{B}\to\mathcal{B}\{T\}$ be a restricted exponential homomorphism and let s be a local slice. Then, by definition, we have $\mathrm{e}(s)=\mathrm{e}_0(s)+\mathrm{e}_1(s)T=s+\mathrm{e}_1(s)T$. Applying $\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]}$ to this last equation we obtain

$$s+\mathrm{e}_1(s)(T+T')=((\mathrm{id}_{\mathcal{B}}\widehat{\otimes}m)\circ\mathrm{e})(s)=((\mathrm{e}\widehat{\otimes}\mathrm{id}_{\mathbb{Z}[T]})\circ\mathrm{e})(s)=\mathrm{e}(s)+\mathrm{e}(\mathrm{e}_1(s))T=s+\mathrm{e}_1(s)T'+\mathrm{e}(\mathrm{e}_1(s))T$$

in $\mathcal{B}\{T,T'\}$. It follows that $s_1:=\mathrm{e}_1(s)$ belongs to $\mathcal{B}^{\mathrm{e}}$.

Let

$$\tilde{j}_{s_1}\colon\mathcal{B}\to\widehat{\mathcal{B}_{s_1}}=\widehat{S^{-1}\mathcal{B}}\quad\text{and}\quad\bar{j}_{s_1}\colon\mathcal{B}^{\mathrm{e}}\to\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}=\widehat{S^{-1}\mathcal{B}^{\mathrm{e}}}$$

be the separated completed localization homomorphisms of $\mathcal{B}$ and $\mathcal{B}^{\mathrm{e}}$ with respect to the multiplicative subset $S=\{s_1^n\}_{n\geq 0}$ of $\mathcal{B}^{\mathrm{e}}\subseteq\mathcal{B}$. Since $\mathcal{B}^{\mathrm{e}}$ is closed in $\mathcal{B}$ by Proposition 2.3, the homomorphism of topological rings $\widehat{i}_{s_1}\colon\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}\to\widehat{\mathcal{B}_{s_1}}$ induced by the inclusion

$i\colon \mathcal{B}^{\mathrm{e}} \hookrightarrow \mathcal{B}$ is injective by Lemma 1.21. We henceforth consider $\widehat{\mathcal{B}_{s_1}}$ as a $\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}$-algebra via this homomorphism. By Proposition 2.6, there exists a unique restricted exponential homomorphism $\widehat{\mathrm{e}_{s_1}} := \widehat{S^{-1}e}\colon \widehat{\mathcal{B}_{s_1}} \to \widehat{\mathcal{B}_{s_1}}\{T\}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\mathrm{e}} & \mathcal{B}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T] \\ {\scriptstyle \widetilde{j}_{s_1}}\downarrow & & \downarrow{\scriptstyle \widetilde{j}_{s_1}\widehat{\otimes}\mathrm{id}} \\ \widehat{\mathcal{B}_{s_1}} & \xrightarrow{\widehat{\mathrm{e}_{s_1}}} & \widehat{\mathcal{B}_{s_1}}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T]. \end{array}$$

Let σ be the image of the element $s_1^{-1}s \in \mathcal{B}_{s_1}$ in $\widehat{\mathcal{B}_{s_1}}$ by the separated completion homomorphism.

Lemma 2.14. *The image $\sigma \in \widehat{\mathcal{B}_{s_1}}$ of the element $s_1^{-1}s \in \mathcal{B}_{s_1}$ by the separated completion homomorphism is a regular element of $\widehat{\mathcal{B}_{s_1}}$. Furthermore, if $\widehat{\mathcal{B}_{s_1}}$ is not the zero ring, then $\widehat{\mathrm{e}_{s_1}}(\sigma) = \sigma + T$.*

Proof. If $\widehat{\mathcal{B}_{s_1}}$ is the zero ring, there is nothing to prove. We can thus assume that $\widehat{\mathcal{B}_{s_1}} \neq \{0\}$. By Corollary 1.19, 0 does not belong to the closure of S in $\mathcal{B}$, so that the image of s_1 in $\widehat{\mathcal{B}_{s_1}}$ is a nonzero invertible element. By definition of $\widehat{\mathrm{e}_{s_1}}$, we have $\widehat{\mathrm{e}_{s_1}}(\sigma) = \sigma + T$, so that σ is a local slice for $\widehat{\mathrm{e}_{s_1}}$. Now let $b \in \widehat{\mathcal{B}_{s_1}}$ be an element such that $\sigma b = 0$. Then we have

$$0 = \widehat{\mathrm{e}_{s_1}}(\sigma b) = \widehat{\mathrm{e}_{s_1}}(\sigma)\widehat{\mathrm{e}_{s_1}}(b) = (\sigma + T)\sum_{i\in\mathbb{N}} \widehat{\mathrm{e}_{s_1}}_i(b)T^i = \sigma b + \sum_{i\geq 1}(\sigma\widehat{\mathrm{e}_{s_1}}_i(b) + \widehat{\mathrm{e}_{s_1}}_{i-1}(b))T^i,$$

from which we infer by induction that

$$b = \widehat{\mathrm{e}_{s_1}}_0(b) = (-1)^i\widehat{\mathrm{e}_{s_1}}_i(b)\sigma^i \text{ for every } i \in \mathbb{N}.$$

Since $\widehat{\mathrm{e}_{s_1}}(b) \in \widehat{\mathcal{B}_{s_1}}\{T\}$, the sequence $(\widehat{\mathrm{e}_{s_1}}_i(b))_{i\in\mathbb{N}}$ converges to 0 in $\widehat{\mathcal{B}_{s_1}}$ by definition of the topology on $\widehat{\mathcal{B}_{s_1}}\{T\}$. This implies in turn that the sequence$((-1)^i\widehat{\mathrm{e}_{s_1}}_i(b)\sigma^i)_{i\in\mathbb{N}}$ also converges to 0, hence that $b = 0$. So σ is a regular element of $\widehat{\mathcal{B}_{s_1}}$. □

We let

$$\upsilon_{-\sigma}\colon \widehat{\mathcal{B}_{s_1}}\{T\} \to \widehat{\mathcal{B}_{s_1}} \quad \text{and} \quad \upsilon_{\sigma}\colon \widehat{\mathcal{B}_{s_1}}\{T\} \to \widehat{\mathcal{B}_{s_1}}$$

be the unique homomorphisms of topological $\widehat{\mathcal{B}_{s_1}}$-algebras mapping T to $-\sigma$ and σ respectively.

Definition 2.15. With the notation above, we call the topological ring homomorphisms

$$R_s = \upsilon_{-\sigma} \circ \widehat{\mathrm{e}_{s_1}}\colon \widehat{\mathcal{B}_{s_1}} \to \widehat{\mathcal{B}_{s_1}} \quad \text{and} \quad \theta_s = \upsilon_{\sigma} \circ (\widehat{i_{s_1}}\widehat{\otimes}\mathrm{id})\colon \widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}\{T\} \to \widehat{\mathcal{B}_{s_1}}$$

the *Dixmier-Reynolds homomomorphism* and the *cylinder homomorphism* associated to the local slice s.

Proposition 2.16. *Let $\mathcal{B}$ be a complete topological ring, let $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ be a restricted exponential homomorphism and let s be a local slice for e. Then the following hold:*

a) The homomorphism $R_s\colon \widehat{\mathcal{B}_{s_1}} \to \widehat{\mathcal{B}_{s_1}}$ is a homomorphism of topological $\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}$-algebras with image equal to $\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}$ and such that $R_s(R_s(b)b') = R_s(b)R_s(b')$ for every $b, b' \in \widehat{\mathcal{B}_{s_1}}$.

b) The homomorphism $\theta_s\colon \widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}\{T\} \to \widehat{\mathcal{B}_{s_1}}$ is an isomorphism of topological $\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}$-algebras.

Proof. Since by Proposition 2.3 $\mathcal{B}^{\mathrm{e}}$ is closed in $\mathcal{B}$, it follows that 0 belongs to closure of $S = \{s_1^n\}_{n\geq 0}$ in $\mathcal{B}^{\mathrm{e}}$ if and only if it belongs to the closure $\overline{S}$ of S in $\mathcal{B}$. This implies that $\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}$ and $\widehat{\mathcal{B}_{s_1}}$ are equal to the zero ring if and only if $0 \in \overline{S}$. If $0 \in \overline{S}$ then assertions a) and b) hold trivially. We thus assume from now on that $0 \notin \overline{S}$ so that the images of s_1 in $\widehat{\mathcal{B}^{\mathrm{e}}_{s_1}}$ and $\widehat{\mathcal{B}_{s_1}}$ are non-zero invertible elements. By Lemma 2.14, σ is a local slice for $\widehat{\mathrm{e}_{s_1}}$. Replacing $\mathcal{B}$ by $\widehat{\mathcal{B}_{s_1}}$, e by $\widehat{\mathrm{e}_{s_1}}$ and s by σ, we may thus assume without loss of generality from the very beginning that $\mathrm{e}(s) = s + T$.

To prove assertion a), we first observe that since $\mathcal{B}^{\mathrm{e}} = \mathrm{Ker}(\mathrm{e} - i_0)$, where $i_0 \colon \mathcal{B} \to \mathcal{B}\{T\}$ is the inclusion of $\mathcal{B}$ as the subring of constant restricted power series, we have $R_s|_{\mathcal{B}^{\mathrm{e}}} = \mathrm{id}_{\mathcal{B}^{\mathrm{e}}}$ so that R_s is indeed a $\mathcal{B}^{\mathrm{e}}$-algebra homomorphism. Now given an element $b \in \mathcal{B}$, we have $R_s(b) = \sum_{i\in\mathbb{N}} \mathrm{e}_i(b)(-s)^i$ and then

$$\begin{array}{rcl}(\mathrm{e} \circ R_s)(b) & = & \mathrm{e}(\sum_{i\in\mathbb{N}} \mathrm{e}_i(b)(-s)^i) \\ & = & \sum_{i\in\mathbb{N}} \mathrm{e}(\mathrm{e}_i(b))(-\mathrm{e}(s))^i \\ & = & \sum_{i\in\mathbb{N}} \mathrm{e}(\mathrm{e}_i(b))(-s-T)^i \\ & = & \sum_{(i,j)\in\mathbb{N}} (\mathrm{e}_i \circ \mathrm{e}_j)(b)T^j(s-T)^i \\ & = & \sum_{\ell\in\mathbb{N}} \mathrm{e}_\ell(b)(T-s-T)^\ell = R_s(b).\end{array}$$

This shows that $\mathrm{e} \circ R_s = R_s$ hence that the image of R_s is contained in $\mathcal{B}^{\mathrm{e}} \subset \mathcal{B}$. Finally, since $R_s|_{\mathcal{B}^{\mathrm{e}}} = \mathrm{id}_{\mathcal{B}^{\mathrm{e}}}$, we have for every $b, b' \in \mathcal{B}$, $R_s(R_s(b)b') = R_s(R_s(b))R_s(b') = R_s(b)R_s(b')$. This shows a).

To prove assertion b), we observe that the composition

$$\big(\upsilon_s \circ (\mathrm{id}_{\mathbb{Z}[T]} \widehat{\otimes} \upsilon_{-s})\big) \circ (\mathrm{id} \widehat{\otimes} m) \circ \mathrm{e} \colon \mathcal{B} \to \mathcal{B}\{T\} \to \mathcal{B}\{T, T'\} \to \mathcal{B}$$

is equal to the identity. Writing $\upsilon_s \widehat{\otimes} \upsilon_{-s}$ for $\upsilon_s \circ (\mathrm{id}_{\mathbb{Z}[T]} \widehat{\otimes} \upsilon_{-s})$ for simplicity, it follows that for every element $b \in \mathcal{B}$, we have

$$\begin{array}{rcl}b = ((\upsilon_s \widehat{\otimes} \upsilon_{-s}) \circ (\mathrm{id} \widehat{\otimes} m) \circ \mathrm{e})(b) & = & ((\upsilon_s \widehat{\otimes} \upsilon_{-s}) \circ (\mathrm{e} \widehat{\otimes} \mathrm{id}) \circ \mathrm{e})(b) \\ & = & (\upsilon_s \widehat{\otimes} \upsilon_{-s})(\sum_{(i,j)\in\mathbb{N}^2} (\mathrm{e}_j \circ \mathrm{e}_i)(b)T'^j T^i) \\ & = & \sum_{i\in\mathbb{N}} (\upsilon_{-s} \circ \mathrm{e})(\mathrm{e}_i(b))s^i \\ & = & \sum_{i\in\mathbb{N}} R_s(e_i(b))s^i\end{array}$$

Since $R_s(e_i(b)) \in \mathcal{B}^{\mathrm{e}}$ for every $i \in \mathbb{N}$, this implies that θ is surjective. Suppose that θ is not injective and let $a = \sum_{i\in\mathbb{N}} a_i T^i \in \mathcal{B}^{\mathrm{e}}\{T\}$ be a nonzero element of minimal order $\mathrm{ord}_0(a) = \min\{i,\ a_i \neq 0\}$ such that $\theta(a) = \sum_{i\in\mathbb{N}} a_i s^i = 0$. Since by Lemma 2.14 s is a regular element of $\mathcal{B}$, the minimality of $\mathrm{ord}_0(a)$ implies that that $a_0 \neq 0$. On the other hand, since $R_s(s) = 0$, we have

$$a_0 = \sum_{i\in\mathbb{N}} a_i R_s(s)^i = R_s(\theta(a)) = 0,$$

a contradiction. This shows that θ is injective, hence an isomorphism. □

Remark 2.17. For a restricted exponential homomorphism $\mathrm{e} \colon \mathcal{B} \to \mathcal{B}\{T\}$ with a local slice s such that $\mathrm{e}(s) = s + s_1 T$, assertion a) in Proposition 2.16 says that the homomorphism of topological rings

$$R_s = \upsilon_{-s_1^{-1}s} \circ \widehat{\mathrm{e}_{s_1}} \colon \widehat{\mathcal{B}_{s_1}} \to \widehat{\mathcal{B}_{s_1}}$$

of Definition 2.15 is an idempotent endomorphism that satisfies the property of a Reynolds operator with values in the subalgebra of $\widehat{\mathrm{e}_{s_1}}$-invariant elements of $\widehat{\mathcal{B}_{s_1}}$. The composition $\pi_s \colon \mathcal{B} \to \widehat{\mathcal{B}_{s_1}}$ of the separated completed localization homomorphism $\mathcal{B} \to \widehat{\mathcal{B}_{s_1}}$ with R_s is the analog in the context of restricted exponential homomorphisms of the Dixmier map introduced in [8, 1.1.9].

Corollary 2.18. *Let $\mathcal{B}$ be a complete topological ring and let* $\mathrm{e}\colon \mathcal{B} \to \mathcal{B}\{T\}$ *be a restricted exponential homomorphism. Assume that* e *has a local slice* $s \in \mathcal{B}$ *such that* $\mathrm{e}(s) = s + T$*. Then* $\mathcal{B} \cong \mathcal{B}^{\mathrm{e}}\{s\}$ *and* e *coincides with the homomorphism of topological* $\mathcal{B}^{\mathrm{e}}$*-algebras*

$$\mathcal{B}^{\mathrm{e}}\{s\} \to \mathcal{B}^{\mathrm{e}}\{s\}\{T\} \cong \mathcal{B}^{\mathrm{e}}\{s, T\}, \quad s \mapsto s + T.$$

2.4. **Topologically integrable iterated higher derivations.** In this subsection, we extend to arbitrary complete topological rings the correspondence between locally finite iterative higher derivations and exponential homomorphisms which classically holds for discretely topologized rings.

2.4.1. *Continuous iterated higher derivations.*

Definition 2.19. Let $\mathcal{A}$ be a topological ring, let $\mathcal{B}$ be a topological $\mathcal{A}$-algebra and M be a topological $\mathcal{B}$-module. A continuous $\mathcal{A}$-derivation of $\mathcal{B}$ into M is a homomorphism of topological $\mathcal{A}$-modules $\partial\colon \mathcal{B} \to M$ which satisfies the Leibniz rule $\partial(bb') = b \cdot \partial(b') + \partial(b) \cdot b'$ for all $b, b' \in B$.

Lemma 2.20. *With the notation of Definition 2.19, let* $c_{\mathcal{B}}\colon \mathcal{B} \to \widehat{\mathcal{B}}$ *and* $c_M\colon M \to \widehat{M}$ *be the separated completions of* $\mathcal{B}$ *and* M *respectively. Then for every continuous* $\mathcal{A}$*-derivation* $\partial\colon \mathcal{B} \to M$ *there exists a unique continuous* $\mathcal{A}$*-derivation* $\widehat{\partial}\colon \widehat{\mathcal{B}} \to \widehat{M}$ *such that* $\partial \circ c_M = \widehat{\partial} \circ c_B$*.*

Proof. The existence of a unique homomorphism of topological $\mathcal{A}$-modules $\widehat{\partial}\colon \widehat{\mathcal{B}} \to \widehat{M}$ such that $c_M \circ \partial = \widehat{\partial} \circ c_{\mathcal{B}}$ follows from Proposition 1.4. By construction, $\widehat{\partial}$ is the homomorphism obtained from $\widetilde{\partial} = c_M \circ \partial$ by the universal property of the separated completion homomorphism $c_{\mathcal{B}}$. Since $\widetilde{\partial}$ satisfies the Leibniz rule, it follows that $\widehat{\partial}$ satisfies the Leibniz rule in restriction to the image of $c_{\mathcal{B}}$, hence on $\widehat{\mathcal{B}}$ because $\widehat{\partial}$ is uniformly continuous and $c_{\mathcal{B}}(\mathcal{B})$ is dense in $\mathcal{B}$. □

Definition 2.21. Let $\mathcal{A}$ be a topological ring and let $\mathcal{B}$ be a topological $\mathcal{A}$-algebra. A continuous *iterated higher* $\mathcal{A}$*-derivation* of $\mathcal{B}$ is a collection $D = \left\{D^{(i)}\right\}_{i \geq 0}$ of homomorphisms of topological $\mathcal{A}$-modules $D^{(i)}\colon \mathcal{B} \to \mathcal{B}$ which satisfy the following properties:

(1) The homomorphism $D^{(0)}$ is the identity homomorphism of $\mathcal{B}$,
(2) For every $i \geq 0$, the *Leibniz rule* $D^{(i)}(bb') = \sum_{j=0}^{i} D^{(j)}(b)D^{(i-j)}(b')$ holds for every pair of elements $b, b' \in \mathcal{B}$,
(3) For every $i, j \geq 0$, $D^{(i)} \circ D^{(j)} = \binom{i+j}{i} D^{(i+j)}$.

Note that the first two properties imply in particular that $\partial = D^{(1)}\colon \mathcal{B} \to \mathcal{B}$ is a continuous $\mathcal{A}$-derivation of $\mathcal{B}$ into itself. If $\mathcal{A}$ contains the field $\mathbb{Q}$ then the third property implies that $D^{(i)} = \frac{1}{i!}\partial^i$ for every $i \geq 0$, where ∂^i denotes the i-th iterate of ∂. In this case, a continuous iterated higher $\mathcal{A}$-derivation is then uniquely determined by a continuous $\mathcal{A}$-derivation ∂ of $\mathcal{B}$ into itself. The notion of higher derivation was first introduced by Hasse and Schmidt in [11].

Definition 2.22. Let $\mathcal{A}$ be a topological ring and let $\mathcal{B}$ be a topological $\mathcal{A}$-algebra. A *topologically integrable iterated higher* $\mathcal{A}$*-derivation* of $\mathcal{B}$ (an $\mathcal{A}$-TIIHD for short) is a continuous iterated higher $\mathcal{A}$-derivation $D = \left\{D^{(i)}\right\}_{i \geq 0}$ such that the sequence of homomorphims of topological $\mathcal{A}$-modules $(D^{(i)})_{n \in \mathbb{N}}$ converges continuously to the zero endomorphism of $\mathcal{B}$.

When $\mathcal{A}$ contains the field $\mathbb{Q}$, we say that a continuous $\mathcal{A}$-derivation ∂ is *topologically integrable* if its associated continuous iterated higher $\mathcal{A}$-derivation $D = \left\{\frac{1}{i!}\partial^i\right\}_{i\geq 0}$ is topologically integrable, or equivalently, if the sequence of homomorphims of topological $\mathcal{A}$-modules $(\partial^i)_{i\in\mathbb{N}}$ converges continuously to the zero endomorphism of $\mathcal{B}$.

Remark 2.23. When $\mathcal{A}$ and $\mathcal{B}$ are topological rings endowed with the discrete topology, the condition that the sequence $(D^{(i)})_{n\in\mathbb{N}}$ is continuously convergent to the zero map means equivalently that for every element $b \in B$ there exists an integer i_0 such that $D^{(i)}(b) = 0$ for every $i \geq i_0$. We thus recover in this case the classical notions of locally finite iterated higher $\mathcal{A}$-derivation of $\mathcal{B}$ ([18], [2]) and locally nilpotent $\mathcal{A}$-derivation of $\mathcal{B}$ ([8]).

Lemma 2.24. *With the notation of Definition 2.21, let $c\colon \mathcal{B} \to \widehat{\mathcal{B}}$ be the separated completion of $\mathcal{B}$. Then for every iterated higher $\mathcal{A}$-derivation $D = \left\{D^{(i)}\right\}_{i\geq 0}$ of $\mathcal{B}$, there exists a unique iterated higher $\mathcal{A}$-derivation $\widehat{D} = \left\{\widehat{D}^{(i)}\right\}_{i\geq 0}$ of $\widehat{\mathcal{B}}$ such that $\widehat{D}^{(i)} \circ c = c \circ D^{(i)}$ for every $i \geq 0$.*

Furthermore, if D is topologically integrable, then so is $\widehat{D}$.

Proof. The existence of a unique collection of homomorphism of topological $\mathcal{A}$-modules $\widehat{D}^{(i)}$ such that $\widehat{D}^{(i)} \circ c = c \circ D^{(i)}$ for every $i \geq 0$ follows from the universal property of the separated completion. The fact that $\widehat{D} = \left\{\widehat{D}^{(i)}\right\}_{i\geq 0}$ satisfies the properties of an iterated higher $\mathcal{A}$-derivation follows from the same argument as in the proof of Lemma 2.20. The second assertion follows from Lemma 1.10. ☐

Example 2.25. Let $\mathcal{A}$ be a complete topological ring and let $\mathcal{A}[T]$ be endowed with the topology generated by the ideals $\mathfrak{a}[T]$, where $\mathfrak{a}$ runs through the set of open ideals of $\mathcal{A}$. Let $\Delta\colon \mathcal{A}[T] \to \mathcal{A}[T,S]$ be the homomorphism of topological $\mathcal{A}$-algebras $T \mapsto T+S$ and let for every $i \geq 0$,

$$D^{(i)} = \left(\frac{1}{i!}\frac{\partial^i}{\partial S^i}|_{S=0}\right) \circ \Delta\colon \mathcal{A}[T] \to \mathcal{A}[T]$$

be the homomorphism of $\mathcal{A}$-modules which associates to a polynomial $P(T) \in \mathcal{A}[T]$ the i-th coefficient of the Taylor expansion at 0 of $P(T+S)$ with respect to the variable S. In particular, $D^{(1)}$ is simply the $\mathcal{A}$-derivation $\frac{\partial}{\partial T}$ of $\mathcal{A}[T]$.

It is straightforward to check that $D = \{D^{(i)}\}_{i\geq 0}$ is a continuous locally finite iterated higher $\mathcal{A}$-derivation of $\mathcal{A}[T]$. In particular, the sequence $(D^{(i)})_{i\in\mathbb{N}}$ is pointwise convergent to the zero homomorphism. Since on the other hand $D^{(i)}(\mathfrak{a}\mathcal{A}[T]) \subseteq \mathfrak{a}\mathcal{A}[T]$ for every $i \geq 0$ and every open ideal $\mathfrak{a}$ of $\mathcal{A}$, it follows from Lemma 1.9 that the sequence $(D^{(i)})_{i\in\mathbb{N}}$ converges continuously to the zero map (see Exemple 1.9). The collection $D = \{D^{(i)}\}_{i\geq 0}$ is thus a topologically integrable iterated higher $\mathcal{A}$-derivation of $\mathcal{A}[T]$. By Lemma 2.24, its canonical extension $\widehat{D} = \{\widehat{D}^{(i)}\}_{i\geq 0}$ to the separated completion $\mathcal{A}\{T\}$ of $\mathcal{A}[T]$ is a topologically integrable iterated higher $\mathcal{A}$-derivation of $\mathcal{A}\{T\}$.

2.4.2. *The correspondence between restricted exponential homomorphisms and topologically integrable iterated higher derivations.*

Theorem 2.26. *Let $\mathcal{A}$ be a complete topological ring let $\mathcal{B}$ be a complete topological $\mathcal{A}$-algebra. Then there exists a one-to-one correspondence between restricted exponential $\mathcal{A}$-homomorphisms $\mathcal{B} \to \mathcal{B}\{T\}$ and topologically integrable iterated higher $\mathcal{A}$-derivations of $\mathcal{B}$.*

The correspondence is defined as follows:

1) Given a restricted exponential homomorphism $\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i \colon \mathcal{B} \to \mathcal{B}\{T\}$, it follows from the identities (2.1) expressing the commutativity of the two diagrams in Definition 2.1 that the collection of homomorphisms of topological $\mathcal{A}$-modules $D^{(i)} = \mathrm{e}_i$, $i \in \mathbb{N}$, is a continuous iterated higher $\mathcal{A}$-derivation of $\mathcal{B}$. Since e is continuous, it follows in turn from Lemma 1.30 that the so-defined sequence $(D^{(i)})_{i\in\mathbb{N}}$ converges continuously to the zero homomorphism. This shows that $D = \{D^{(i)}\}_{i\in\mathbb{N}}$ is a topologically integrable iterated higher $\mathcal{A}$-derivation of $\mathcal{B}$.

2) Conversely, given a topologically integrable iterated higher $\mathcal{A}$-derivation $D = \{D^{(i)}\}_{i\in\mathbb{N}}$ of $\mathcal{B}$, it follows from Lemma 1.30 again that the $\mathcal{A}$-module homomorphism

$$e = \exp(TD) := \sum_{i\in\mathbb{N}} D^{(i)} T^i \colon \mathcal{B} \to \mathcal{B}\{T\}$$

is well-defined and continuous. The properties of an iterated higher $\mathcal{A}$-derivation listed in Definition 2.21 guarantee precisely that e satisfies the axioms of a restricted exponential homomorphism.

In the case where the base topological ring $\mathcal{A}$ contains the field $\mathbb{Q}$, the fact that a continuous iterated higher $\mathcal{A}$-derivation $D = \{D^{(i)}\}_{i\in\mathbb{N}}$, is uniquely determined by the continuous $\mathcal{A}$-derivation $\partial = D^{(1)}$ of $\mathcal{B}$ implies in turn that a restricted exponential $\mathcal{A}$-homomorphism $\mathrm{e} = \sum_{i\in\mathbb{N}} \mathrm{e}_i T^i \colon \mathcal{B} \to \mathcal{B}\{T\}$ is uniquely determined by the topologically integrable $\mathcal{A}$-derivation

$$\partial = \mathrm{e}_1 = \tfrac{\partial}{\partial T}|_{T=0} \circ \mathrm{e}.$$

The following example illustrates the importance of continuous convergence in the correspondence between topologically integrable $\mathcal{A}$-derivation and restricted exponential homomorphisms.

Example 2.27. Let $\mathcal{A}$ be a complete topological $\mathbf{k}$-algebra of finite type containing $\mathbb{Q}$, with fundamental system of open ideals $(\mathfrak{a}_n)_{n\in\mathbb{N}}$, and let $\mathcal{A}[(X_i)_{i\in\mathbb{N}}] = \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}[(X_i)_{i\in\mathbb{N}}]$ be the polynomial ring in countably many variables X_i, $i \in \mathbb{N}$. Let $\mathcal{B}$ be the separated completion of $\mathcal{A}[(X_i)_{i\in\mathbb{N}}]$ with respect to the topology induced by the fundamental system of open ideals

$$\bar{\mathfrak{a}}_n = \mathfrak{a}_n \otimes_{\mathbb{Z}} \mathbb{Z}[(X_i)_{i\in\mathbb{N}}] + \mathcal{A} \otimes_{\mathbb{Z}} (X_i)_{i\geq n}, \quad n \in \mathbb{N}.$$

Note that since $\mathcal{A}$ is separated and $\bigcap_{i\in\mathbb{N}} (X_i)_{i\geq n} = \{0\}$, $\mathcal{A}[(X_i)_{i\in\mathbb{N}}]$ is a separated topological ring so that the separated completion homomorphism $c\colon \mathcal{A}[(X_i)_{i\in\mathbb{N}}] \to \mathcal{B}$ is injective.

Let ∂ be the $\mathcal{A}$-derivation of $\mathcal{A}[(X_i)_{i\in\mathbb{N}}]$ defined by

$$\partial(X_0) = X_1, \quad \partial(X_{2i-1}) = X_{2i+1}, \text{ and } \quad \partial(X_{2i}) = X_{2i-2} \quad \forall i \geq 1.$$

It is easily seen that ∂ is continuous and that the sequence of homomorphisms $(\partial^i)_{i\in\mathbb{N}}$ converges pointwise to the zero homomorphism. However, it does not converge continuously to the zero homomorphism. Indeed, since $\partial^\ell(X_{2\ell}) = X_0$, it follows that for any given $i \in \mathbb{N}$ there cannot exist any integer n_0 such that $\partial^n(\bar{\mathfrak{a}}_i) \subseteq \bar{\mathfrak{a}}_1$ for every $n \geq n_0$. Since $c\colon \mathcal{A}[(X_i)_{i\in\mathbb{N}}] \to \mathcal{B}$ is injective, this implies in turn that the associated $\mathcal{A}$-derivation $\widehat{\partial}$ of $\mathcal{B}$ is not topologically integrable. The associated $\mathcal{A}$-algebra homomorphism $\exp(T\widehat{\partial})\colon \mathcal{B} \to \mathcal{B}\{T\}$ is well-defined but not continuous, hence is not a restricted exponential homomorphism.

We now briefly translate some of the main basic properties of restricted exponential homomorphisms established in subsection 2.2 to the language of topologically integrable derivations. We first observe that the ring of invariants $\mathcal{B}^{\mathrm{e}}$ of the restricted exponential homomorphism associated to a topologically integrable $\mathcal{A}$-derivation ∂ of $\mathcal{B}$ is equal to the kernel of ∂.

Proposition 2.28. *Let $\mathcal{A}$ be a topological ring containing $\mathbb{Q}$, let $\mathcal{B}$ be a complete topological $\mathcal{A}$-algebra and let ∂ be a topologically integrable $\mathcal{A}$-derivation of $\mathcal{B}$. Then the following hold :*

a) For every element $f \in \mathrm{Ker}(\partial)$, the $\mathcal{A}$-derivation $f\partial$ of $\mathcal{B}$ is topologically integrable.

b) For every multiplicatively closed subset S of $\mathrm{Ker}(\partial)$, the $\mathcal{A}$-derivation $\widehat{S^{-1}\partial}$ of $\widehat{S^{-1}\mathcal{B}}$ is topologically integrable.

c) For every surjective open homomorphism of complete topological rings $\pi \colon \mathcal{B} \to \mathcal{C}$ such that $\partial(\mathrm{Ker}\,\pi) \subset \mathrm{Ker}\,\pi$, the induced $\mathcal{A}$-derivation $\overline{\partial}$ of $\mathcal{C} \cong \mathcal{B}/\,\mathrm{Ker}\,\pi$ is topologically integrable.

d) For every topologically integrable $\mathcal{A}$-derivation ∂' of $\mathcal{B}$ such $\partial \circ \partial' = \partial' \circ \partial$, the $\mathcal{A}$-derivation $\partial'' = \partial + \partial'$ of $\mathcal{B}$ is topologically integrable.

Proof. The assertions follow respectively from Propositions 2.5, 2.6, 2.7 and 2.8 □

Similarly, the following proposition is a consequence of Proposition 2.9:

Proposition 2.29. *Let $\mathcal{A}$ be a topological ring containing $\mathbb{Q}$ and let $(\mathcal{B}_n)_{n\in\mathbb{N}}$ be a countable inverse system of complete topological $\mathcal{A}$-algebras with continuous and surjective transition homomorphisms $p_{m,n} \colon \mathcal{B}_m \to \mathcal{B}_n$ for every $m \geq n \geq 0$. Let $\mathcal{B}$ be its limit and let $p_n \colon \mathcal{B} \to \mathcal{B}_n$, $n \in \mathbb{N}$, be the canonical continuous projections. Let $\partial_n \colon \mathcal{B}_n \to \mathcal{B}_n$, $n \in \mathbb{N}$, be a sequence of topologically integrable $\mathcal{A}$-derivations such that $\partial_n \circ p_{m,n} = p_{m,n} \circ \partial_m$ for all $m \geq n \geq 0$. Then there exists a unique topologically integrable $\mathcal{A}$-derivation $\partial = \varprojlim_{n\in\mathbb{N}} \partial_n$ of $\mathcal{B}$ such that $\partial_n \circ p_n = p_n \circ \partial$ for every $n \in \mathbb{N}$.*

Proposition 2.16 b) can be translated as follows:

Proposition 2.30. *Let $\mathcal{A}$ be a topological ring containing $\mathbb{Q}$, let $\mathcal{B}$ be a complete topological $\mathcal{A}$-algebra and let ∂ be a topologically integrable $\mathcal{A}$-derivation of $\mathcal{B}$. If $\mathrm{Ker}\,\partial^2 \setminus \mathrm{Ker}\,\partial$ is not empty then for every $s \in \mathrm{Ker}\,\partial^2 \setminus \mathrm{Ker}\,\partial$ with $\partial(s) = s_1 \in \mathrm{Ker}\,\partial$, there exists an isomorphism of topological $\widehat{(\mathrm{Ker}\,\partial)}_{s_1}$-algebras*

$$\widehat{\mathcal{B}_{s_1}} \stackrel{\cong}{\to} \widehat{(\mathrm{Ker}\,\partial)}_{s_1}\{S\}$$

which maps the induced topologically integrable $\widehat{(\mathrm{Ker}\,\partial)}_{s_1}$-derivation $\widehat{\partial_{s_1}}$ of $\widehat{\mathcal{B}_{s_1}}$ onto the topologically integrable $\widehat{(\mathrm{Ker}\,\partial)}_{s_1}$-derivation $\frac{\partial}{\partial S}$ of $\widehat{(\mathrm{Ker}\,\partial)}_{s_1}\{S\}$.

In contrast with the case of usual locally nilpotent derivations of discrete topological rings containing $\mathbb{Q}$, there exist topologically integrable derivations ∂ of complete topological rings containing $\mathbb{Q}$ for which $\mathrm{Ker}\,\partial^n = \mathrm{Ker}\,\partial$ for every $n \geq 1$ so that in particular $\mathrm{Ker}\,\partial^2 \setminus \mathrm{Ker}\,\partial = \emptyset$. This is the case for instance for the topologically integrable derivation $u^2\frac{\partial}{\partial u}$ of $\mathbb{C}[[u]]$ corresponding to the restricted exponential homomorphism of Example 2.13. The following example provides another illustration of this phenomenon.

Example 2.31. As in Example 2.27, let $\mathcal{A}$ be a complete topological ring containing $\mathbb{Q}$ with fundamental system of open ideals $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ and let $\mathcal{B}$ be the separated completion of the polynomial ring $\mathcal{A}[(X_i)_{i\in\mathbb{N}}] = \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}[(X_i)_{i\in\mathbb{N}}]$ in countably many variables with respect to the topology induced by the fundamental system of open ideals

$$\overline{\mathfrak{a}}_n = \mathfrak{a}_n \otimes_{\mathbb{Z}} \mathbb{Z}[(X_i)_{i\in\mathbb{N}}] + \mathcal{A} \otimes_{\mathbb{Z}} (X_i)_{i\geq n}, \quad n \in \mathbb{N}.$$

Then the $\mathcal{A}$-derivation $\widehat{\partial}_+$ of $\mathcal{B}$ induced by the $\mathcal{A}$-derivation of $\mathcal{A}[(X_i)_{i\in\mathbb{N}}]$ defined by

$$\partial_+(X_i) = (i+1)X_{i+1}, \quad \forall i \geq 0,$$

is topologically integrable with $\operatorname{Ker}\widehat{\partial}_+^n = \mathcal{A}$ for all $n \geq 1$.

Proof. For every $n \geq 1$, let $\widehat{\mathcal{B}}_n = \mathcal{A}\{X_0, \ldots, X_{n-1}\}$ be the separated completion of

$$\mathcal{A}[X_0, \ldots, X_{n-1}] = \mathcal{A}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[X_0, \ldots, X_{n-1}]$$

with respect to the topology generated by the ideals $\mathfrak{a}_i\mathcal{A}[X_0, \ldots, X_{n-1}]$, $i \in \mathbb{N}$. The topological rings $\widehat{\mathcal{B}}_0 = \mathcal{A}$ and $\widehat{\mathcal{B}}_n$, $n \geq 1$, form an inverse system of complete topological $\mathcal{A}$-algebras for the collection of continuous and surjective transition homomorphisms $p_{m,n} \colon \widehat{\mathcal{B}}_m \to \widehat{\mathcal{B}}_n$ with kernels $(X_n, \ldots, X_{m-1})\widehat{\mathcal{B}}_m$, $m \geq n \geq 0$, whose limit is canonically isomorphic to $\mathcal{B}$. We denote by

$$p_n \colon \mathcal{B} = \varprojlim_{n\in\mathbb{N}} \widehat{\mathcal{B}}_n \to \widehat{\mathcal{B}}_n, \quad n \in \mathbb{N}$$

the canonical continuous surjective homomorphisms. For every $n \geq 1$, the $\mathcal{A}$-derivation $\widehat{\delta}_n \colon \widehat{\mathcal{B}}_n \to \widehat{\mathcal{B}}_{n+1}$ induced by the $\mathcal{A}$-derivation

$$\delta_n = \sum_{i=0}^{n-1}(i+1)X_{i+1}\frac{\partial}{\partial X_i} \colon \mathcal{A}[X_0, \ldots, X_{n-1}] \to \mathcal{A}[X_0, \ldots, X_n]$$

is continuous and the composition of $\widehat{\delta}_n$ with the projection $p_{n+1,n} \colon \widehat{\mathcal{B}}_{n+1} \to \widehat{\mathcal{B}}_n$ is a topologically integrable $\mathcal{A}$-derivation $\widehat{\partial}_{+,n}$ of $\widehat{\mathcal{B}}_n$. Since by construction $p_n \circ \widehat{\partial}_+ = \widehat{\partial}_{+,n} \circ p_n$ for every $n \in \mathbb{N}$, it follows from Proposition 2.29 that $\widehat{\partial}_+$ is topologically integrable.

Using the canonical isomorphisms $\mathcal{A}\{X_0, \ldots, X_{n-1}\} \cong \mathcal{A}\{X_0, \ldots, X_{n-2}\}\{X_{n-1}\}$ of Lemma 1.28, it is straightforward to check by induction on n that for every $n \geq 1$, the kernel of the $\mathcal{A}$-derivation $\widehat{\delta}_n$ is equal to $\mathcal{A}$. Since $p_{n+1} \circ \widehat{\partial}_+ = \widehat{\delta}_n \circ p_n$ for every $n \in \mathbb{N}$, this implies that $\operatorname{Ker}\widehat{\partial}_+ = \mathcal{A}$. Finally, if $\operatorname{Ker}\widehat{\partial}_+^n \setminus \operatorname{Ker}\widehat{\partial}_+ \neq \emptyset$ for some $n \geq 2$, then there would exist an element $s \in \operatorname{Ker}\widehat{\partial}_+^2 \setminus \operatorname{Ker}\widehat{\partial}_+$. Letting $s_1 = \widehat{\partial}_+(s)$, it would follow from Proposition 2.30 that $\widehat{\mathcal{B}_{s_1}} \cong \widehat{\mathcal{A}_{s_1}}\{S\}$, which is impossible. Thus $\operatorname{Ker}\widehat{\partial}_+^n = \mathcal{A}$ for every $n \geq 1$, which completes the proof. □

3. Geometric interpretation: additive group actions on affine ind-schemes

In this section, we recall a construction due to Kambayashi which associates to every complete topological ring $\mathcal{A}$ a locally topologically ringed space $(\operatorname{Spf}(\mathcal{A}), \mathcal{O}_{\operatorname{Spf}(\mathcal{A})})$ called the affine ind-scheme of $\mathcal{A}$. We then establish that restricted exponential homomorphisms correspond through this construction to actions of the additive group ind-scheme on affine ind-schemes.

As in the previous sections, we use the term topological ring to refer to a linearly topologized ring $\mathcal{A}$ which admits a fundamental system of open neighborhoods of 0 consisting of a countable family $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ of ideals of $\mathcal{A}$.

3.1. **Recollection on affine ind-schemes.** We review the basic steps of the construction of the affine ind-scheme associated to a complete topological ring $\mathcal{A}$ following Kambayashi [12, 13].

Definition 3.1. A *topologically ringed space* is a ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ such that for every open subset V of $\mathfrak{X}$, $\mathcal{O}_{\mathfrak{X}}(V)$ is a topological ring and such that for every pair of open subsets $V' \subseteq V$ of $\mathfrak{X}$, the restriction homomorphism $\mathcal{O}_{\mathfrak{X}}(V) \to \mathcal{O}_{\mathfrak{X}}(V')$ is a continuous homomorphism of topological rings.

A *morphism of topologically ringed spaces* from $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ to $(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ is a pair $(f, f^\sharp)$ consisting of a continuous map $f\colon \mathfrak{X} \to \mathfrak{Y}$ and a homomorphism $f^\sharp\colon \mathcal{O}_{\mathfrak{Y}} \to f_*\mathcal{O}_{\mathfrak{X}}$ of sheaves of rings on $\mathfrak{Y}$ such that for every open subset V of $\mathfrak{Y}$ the homomorphism $f^\sharp(V)\colon \mathcal{O}_{\mathfrak{Y}}(V) \to f_*\mathcal{O}_{\mathfrak{X}}(V) = \mathcal{O}_{\mathfrak{X}}(f^{-1}(V))$ is a continuous homomorphism of topological rings.

Let $\mathcal{A}$ be a complete topological ring with a fundamental system $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ of open neighborhoods of 0. For every $m \geq n \geq 0$, let $\pi_n\colon \mathcal{A} \to A_n = \mathcal{A}/\mathfrak{a}_n$ and $\pi_{m,n}\colon A_m \to A_n$ be the quotient morphisms and let $(j_n, j_n^\sharp)\colon X_n = (\mathrm{Spec}(A_n), \mathcal{O}_{\mathrm{Spec}(A_n)}) \to X = (\mathrm{Spec}(\mathcal{A}), \mathcal{O}_{\mathrm{Spec}(\mathcal{A})})$ and

$$(j_{m,n}, j_{m,n}^\sharp)\colon X_n = (\mathrm{Spec}(A_n), \mathcal{O}_{\mathrm{Spec}(A_n)}) \to X_m = (\mathrm{Spec}(A_m), \mathcal{O}_{\mathrm{Spec}(A_m)})$$

be the corresponding closed immersion of schemes.

We let $\mathrm{Spf}(\mathcal{A})$ be the subset of $\mathrm{Spec}(\mathcal{A})$ consisting of open prime ideals of $\mathcal{A}$, endowed with the subspace topology induced by the usual Zariski topology on $\mathrm{Spec}(\mathcal{A})$. For every $n \geq 0$, the structure sheaf of X_n induces a sheaf of rings $\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n} = (j_{n,*}\mathcal{O}_{\mathrm{Spec}(A_n)})|_{\mathrm{Spf}(\mathcal{A})}$ on $\mathrm{Spf}(\mathcal{A})$. Since $j_n = j_m \circ j_{m,n}$ for every $m \geq n$, the collection of homomorphisms

$$j_{m,n}^\sharp\colon \mathcal{O}_{\mathrm{Spec}(A_m)} \to (j_{m,n})_*\mathcal{O}_{\mathrm{Spec}(A_n)}$$

induces an inverse system of homomorphisms of sheaves of rings $\varphi_{m,n}\colon \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),m} \to \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n}$. Considering each of the sheaves $\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n}$ as sheaves of discrete topological rings on $\mathrm{Spf}(\mathcal{A})$, we let $\mathcal{O}_{\mathrm{Spf}(\mathcal{A})}$ be the limit of this inverse system in the category of sheaves of topological rings. So, for every open subset $V = U \cap \mathrm{Spf}(\mathcal{A})$ of $\mathrm{Spf}(\mathcal{A})$, where U is a Zariski open subset of $\mathrm{Spec}(\mathcal{A})$, we have

$$\mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(V) = (\varprojlim_{n\in\mathbb{N}} \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n})(V) = \varprojlim_{n\in\mathbb{N}} \mathcal{O}_{\mathrm{Spec}(A_n)}(j_n^{-1}(U))$$

endowed with the initial topology.

Definition 3.2. The *affine ind-scheme of a complete topological ring* $\mathcal{A}$ is the topologically ringed space $(\mathrm{Spf}(\mathcal{A}), \mathcal{O}_{\mathrm{Spf}(\mathcal{A})})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be a complete topological rings with respective fundamental systems $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ and $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ of open neighborhoods of 0, and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism of topological rings. The continuity of φ implies that the image of the restriction to $\mathrm{Spf}(\mathcal{B}) \subseteq \mathrm{Spec}(B)$ of the continuous map $\mathrm{Spec}(\mathcal{B}) \to \mathrm{Spec}(\mathcal{A})$ is contained in the subset $\mathrm{Spf}(\mathcal{A}) \subseteq \mathrm{Spec}(\mathcal{A})$. Furthermore, for every ideal $\mathfrak{b}_n \subseteq \mathcal{B}$, there exists an index $n' = n'(n)$ such that $\mathfrak{a}_{n'} \subseteq \varphi^{-1}(\mathfrak{b}_n)$. It follows that φ induces a morphism of schemes $\alpha_{n,n'}\colon \mathrm{Spec}(\mathcal{B}/\mathfrak{b}_n) \to \mathrm{Spec}(\mathcal{A}/\mathfrak{a}_{n'})$ and that, letting $\alpha\colon \mathrm{Spf}(\mathcal{B}) \to \mathrm{Spf}(\mathcal{A})$ be the continuous map induced by φ, the following diagram of continuous maps of topological spaces commutes

$$\begin{array}{ccc}
\mathrm{Spf}(\mathcal{B}) & \xrightarrow{\ \alpha\ } & \mathrm{Spf}(\mathcal{A}) \\
{\scriptstyle j_{\mathcal{B},n}}\uparrow & & \uparrow{\scriptstyle j_{\mathcal{A},n'}} \\
\mathrm{Spec}(\mathcal{B}/\mathfrak{b}_n) & \xrightarrow{\ \alpha_{n,n'}\ } & \mathrm{Spec}(\mathcal{A}/\mathfrak{a}'_n).
\end{array}$$

Pushing forward the homomorphism $\alpha^\sharp_{n,n'}\colon \mathcal{O}_{\mathrm{Spec}(\mathcal{A}/\mathfrak{a}_{n'})} \to (\alpha_{n,n'})_*\mathcal{O}_{\mathrm{Spec}(\mathcal{B}/\mathfrak{b}_n)}$ by $j_{\mathcal{A},n'}$ gives a homomorphism of sheaves of rings

$$\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n'} = (j_{\mathcal{A},n'})_*\mathcal{O}_{\mathrm{Spec}(\mathcal{A}/\mathfrak{a}_{n'})} \to \alpha_*(j_{\mathcal{B},n})_*\mathcal{O}_{\mathrm{Spec}(\mathcal{B}/\mathfrak{b}_n)} = \alpha_*\mathcal{O}_{\mathrm{Spf}(\mathcal{B}),n}$$

on $\mathrm{Spf}(\mathcal{A})$. The homomorphism of sheaves of topological rings

$$\mathcal{O}_{\mathrm{Spf}(\mathcal{A})} = \varprojlim_{n'\in\mathbb{N}} \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n'} \to \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n'} \to \alpha_*\mathcal{O}_{\mathrm{Spf}(\mathcal{B}),n}$$

is continuous and independent on the choice of an index n' such that $\mathfrak{a}_{n'} \subseteq \varphi^{-1}(\mathfrak{b}_n)$. This yields in turn a canonical continuous homomorphism of sheaves of topological rings

$$\alpha^\sharp\colon \mathcal{O}_{\mathrm{Spf}(\mathcal{A})} \to \varprojlim_{n\in\mathbb{N}} \alpha_*\mathcal{O}_{\mathrm{Spf}(\mathcal{B}),n} = \alpha_*\mathcal{O}_{\mathrm{Spf}(\mathcal{B})}.$$

The morphism of topologically ringed spaces

$$(\alpha,\alpha^\sharp)\colon (\mathrm{Spf}(\mathcal{B}),\mathcal{O}_{\mathrm{Spf}(\mathcal{B})}) \to (\mathrm{Spf}(\mathcal{A}),\mathcal{O}_{\mathrm{Spf}(\mathcal{A})})$$

is called the *morphism of affine ind-schemes associated to* φ. We henceforth denote it simply by $\mathrm{Spf}(\varphi)$.

Remark 3.3. Let $\mathcal{A}$ be a complete topological ring with a fundamental system $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ of open neighborhoods of 0. Since a prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathcal{A})$ is open if and only it is equal to $\pi_n^{-1}(\pi_n(\mathfrak{p}))$ for some $n \geq 0$, the set $\mathrm{Spf}(\mathcal{A}) \subset \mathrm{Spec}(\mathcal{A})$ is equal to the union of the images of the closed immersions $j_n\colon \mathrm{Spec}(A_n) \hookrightarrow \mathrm{Spec}(\mathcal{A})$, $n \geq 0$. Furthermore the induced canonical map $\varinjlim_{n\in\mathbb{N}} \mathrm{Spec}(A_n) \to \mathrm{Spf}(\mathcal{A})$ is bijective and continuous with respect to the final topology on $\varinjlim_{n\in\mathbb{N}} \mathrm{Spec}(A_n)$ and the Zariski topology on $\mathrm{Spf}(\mathcal{A})$. Note however that this canonical map is in general not a homeomorphism, i.e. the final topology on $\varinjlim_{n\in\mathbb{N}} \mathrm{Spec}(A_n)$ is strictly finer than the Zariski topology, see [22].

For every $f \in \mathcal{A}$, we let

$$\mathfrak{D}(f) = \mathrm{Spf}(\mathcal{A}) \cap D(f) = \{\mathfrak{p} \in \mathrm{Spf}(\mathcal{A}) \mid f \notin \mathfrak{p}\}$$

where $D(f)$ is the usual principal open subset $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid f \notin \mathfrak{p}\}$ of $\mathrm{Spec}(\mathcal{A})$. These open subsets $\mathfrak{D}(f)$ form a basis of the Zariski topology on $\mathrm{Spf}(\mathcal{A})$. Let $\widehat{\mathcal{A}_f} = \varprojlim_{n\in\mathbb{N}}(A_n)_{\pi_n(f)}$ be the separated completed localization of $\mathcal{A}$ with respect to the multiplicatively closed set $\{f^n\}_{n\in\mathbb{N}}$ (see Definition 1.15 and Notation 1.16). Then the collection of canonical projections

$$\widehat{\mathcal{A}_f} \to (A_n)_{\pi_n(f)} = \mathcal{O}_{\mathrm{Spec}(A_n)}(D(\pi_n(f))) = \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n}(\mathfrak{D}(f)), \quad n \geq 0$$

induces a canonical isomorphism of topological rings $\widehat{\mathcal{A}}_f \to \varprojlim_{n\in\mathbb{N}} \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),n}(\mathfrak{D}(f)) = \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(\mathfrak{D}(f))$. Since the canonical homomorphism $\mathcal{A} \to \widehat{\mathcal{A}_1} = \varprojlim_{n\in\mathbb{N}} A_n$ is an isomorphism because $\mathcal{A}$ is complete, we have in particular $\mathcal{A} \cong \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(\mathrm{Spf}(\mathcal{A}))$ and the restriction homomorphism

$$\mathcal{A} \cong \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(\mathrm{Spf}(\mathcal{A})) \to \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(\mathfrak{D}(f)) = \widehat{\mathcal{A}_f}$$

coincides with the separated completed localization homomorphism $\tilde{j}_f\colon \mathcal{A} \to \widehat{\mathcal{A}_f}$. The morphism of affine ind-schemes $\mathrm{Spf}(\tilde{j}_f) = (\tilde{j}_f, \tilde{j}_f^\sharp)\colon \mathrm{Spf}(\widehat{\mathcal{A}_f}) \to \mathrm{Spf}(\mathcal{A})$ is an open immersion with image equal to $\mathfrak{D}(f)$.

Given a point $\mathfrak{p} \in \mathrm{Spf}(\mathcal{A})$, the stalk $\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),\mathfrak{p}}$ of $\mathcal{O}_{\mathrm{Spf}(\mathcal{A})}$ at $\mathfrak{p}$ can be described as follows. Let S be the multiplicatively closed subset of $\mathcal{A}$ consisting of elements f such that $\mathfrak{p} \in \mathfrak{D}(f)$. Then S is a directed set under the relation $f \leq g$ if and only if $\mathfrak{D}(g) \subseteq \mathfrak{D}(f)$, equivalently, if and only if the separated completed localization homomorphism $\tilde{j}_g \colon \mathcal{A} \to \widehat{\mathcal{A}_g}$ factors through $\tilde{j}_f \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{A}_f}$. Since by definition of the Zariski topology on $\mathrm{Spf}(\mathcal{A})$, the open subsets $\mathfrak{D}(f)$, $f \in S$, form a cofinal subset of the set of all open neighborhoods of $\mathfrak{p}$ in $\mathrm{Spf}(\mathcal{A})$, we have a canonical isomorphism of rings

$$\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),\mathfrak{p}} = \varinjlim_{U \ni \mathfrak{p}} \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(U) \cong \varinjlim_{f \in S} \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}(\mathfrak{D}(f)) \cong \varinjlim_{f \in S} \widehat{\mathcal{A}_f}.$$

By [13, Theorem 2.2.3.], the ring $\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),\mathfrak{p}}$ is local and the canonical homomorphism $h \colon \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),\mathfrak{p}} \to \widehat{\mathcal{A}_{\mathfrak{p}}}$ induced by the collection of compatible canonical homomorphisms $\widehat{\mathcal{A}_f} \to \widehat{\mathcal{A}_{\mathfrak{p}}}$ has dense image in the local topological ring $\widehat{\mathcal{A}_{\mathfrak{p}}}$. Moreover, the image in $\mathcal{O}_{\mathrm{Spf}(\mathcal{A}),\mathfrak{p}}$ by the canonical homomorphism $\widehat{\mathcal{A}_f} \to \mathcal{O}_{\mathrm{Spf}(\mathcal{A}),\mathfrak{p}}$ of an element $a_f = (a_{f,n}) \in \widehat{\mathcal{A}_f} = \varprojlim_{n \in \mathbb{N}} (A_n)_{\pi_n(f)}$ belongs to the kernel of h if and only for every $n \geq 0$ then exists $g = g(n, f) \geq f$ such that $a_{f,n}$ belongs to the kernel of the restriction homomorphism

$$(A_n)_{\pi_n(f)} = \mathcal{O}_{\mathrm{Spec}(A_n)}(D(\pi_n(f))) \to \mathcal{O}_{\mathrm{Spec}(A_n)}(D(\pi_n(g))) = (A_n)_{\pi_n(g)}$$

corresponding to the inclusion of principal open subsets $D(\pi_n(g)) \subset D(\pi_n(f))$ of $\mathrm{Spec}(A_n)$.

It follows in particular that the affine ind-scheme $\mathrm{Spf}(\mathcal{A}) = (\mathrm{Spf}(\mathcal{A}), \mathcal{O}_{\mathrm{Spf}(\mathcal{A})})$ of a complete topological ring $\mathcal{A}$ is a locally topologically ringed space. One can also check that the morphism of topologically ringed spaces $\mathrm{Spf}(\varphi) \colon \mathrm{Spf}(\mathcal{B}) \to \mathrm{Spf}(\mathcal{A})$ associated to a continuous homomorphism of complete topological rings $\varphi \colon \mathcal{A} \to \mathcal{B}$ is a morphism of locally topologically ringed spaces.

Definition 3.4. An *affine ind-scheme* is a locally topologically ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ isomorphic to the affine ind-scheme $(\mathrm{Spf}(\mathcal{A}), \mathcal{O}_{\mathrm{Spf}(\mathcal{A})})$ of a complete topological ring $\mathcal{A}$.

A *morphism between affine ind-schemes* $(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ and $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a morphism of locally topologically ringed spaces

$$(f, f^\sharp) \colon (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \cong (\mathrm{Spf}(\mathcal{B}), \mathcal{O}_{\mathrm{Spf}(\mathcal{B})}) \to (\mathrm{Spf}(\mathcal{A}), \mathcal{O}_{\mathrm{Spf}(\mathcal{A})}) \cong (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$$

which is induced by a continuous homomorphism of complete topological rings $\varphi \colon \mathcal{A} \to \mathcal{B}$.

Given an affine ind-scheme $\mathfrak{S} = \mathrm{Spf}(\mathcal{A})$, an affine ind-$\mathfrak{S}$-scheme is an affine ind-scheme $\mathfrak{X} \cong \mathrm{Spf}(\mathcal{B})$ with a morphism of affine ind-schemes $f \colon \mathfrak{X} \to \mathfrak{S}$. A morphism between affine ind-$\mathfrak{S}$-schemes $g \colon \mathfrak{Y} \to \mathfrak{S}$ and $f \colon \mathfrak{X} \to \mathfrak{S}$ is a morphism of affine-ind-schemes $h \colon \mathfrak{Y} \to \mathfrak{X}$ such that $g = f \circ h$. The category $(\mathrm{AffInd}_{/\mathfrak{S}})$ of affine ind-$\mathfrak{S}$-schemes is by construction anti-equivalent to the category of complete topological $\mathcal{A}$-algebras $\varphi \colon \mathcal{A} \to \mathcal{B}$. Given two such affine ind-$\mathfrak{S}$-schemes $\mathfrak{X} \cong \mathrm{Spf}(\mathcal{B})$ and $\mathfrak{X}' \cong \mathrm{Spf}(\mathcal{B}')$ corresponding to complete topological $\mathcal{A}$-algebras $\varphi \colon \mathcal{A} \to \mathcal{B}$ and $\varphi' \colon \mathcal{A} \to \mathcal{B}'$ respectively, it follows from the universal property of the completed tensor product (see subsection 1.3.1) that the affine ind-$\mathfrak{S}$-scheme $\mathrm{Spf}(\mathcal{B} \widehat{\otimes}_{\mathcal{A}} \mathcal{B}')$ together with the projections morphisms $p_1 \colon \mathrm{Spf}(\mathcal{B} \widehat{\otimes}_{\mathcal{A}} \mathcal{B}') \to \mathrm{Spf}(\mathcal{B})$ and $p_2 \colon \mathrm{Spf}(\mathcal{B} \widehat{\otimes}_{\mathcal{A}} \mathcal{B}') \to \mathrm{Spf}(\mathcal{B}')$ induced respectively by the canonical homomorphisms $\sigma_1 \colon \mathcal{B} \to \mathcal{B} \widehat{\otimes}_{\mathcal{A}} \mathcal{B}'$ and $\sigma_2 \colon \mathcal{B} \to \mathcal{B} \widehat{\otimes}_{\mathcal{A}} \mathcal{B}'$ is the fibered product of $\mathfrak{X}$ and $\mathfrak{X}'$ in the category of affine ind-$\mathfrak{S}$-schemes. We denote it by $\mathfrak{X} \widehat{\times}_{\mathfrak{S}} \mathfrak{X}'$.

Example 3.5. Let R be a finitely generated algebra over a field k and let $X = \mathrm{Spec}(R)$ be the associated affine k-scheme of finite type. Then the functor

$$\widehat{F} = \mathrm{Mor}(X, \mathbb{A}^1_k) : (\mathrm{AffIndSch}/_k)^\circ \to (\mathrm{Sets})$$

which associates to every affine ind-k-scheme $\mathfrak{S}$ the set of morphisms of affine ind-$\mathfrak{S}$-schemes from $X\widehat{\times}_k\mathfrak{S}$ to $\mathbb{A}^1_k\widehat{\times}_k\mathfrak{S}$ is representable.

Proof. Indeed, given an affine ind-k-scheme $\mathfrak{S} = \mathrm{Spf}(\mathcal{A})$, we have

$$\widehat{F}(\mathfrak{S}) = \mathrm{Hom}_{\mathfrak{S}}(X\widehat{\times}_k\mathfrak{S}, \mathbb{A}_k\widehat{\times}_k\mathfrak{S}) \cong \mathcal{C}\mathrm{Hom}_{\mathcal{A}-\mathrm{alg}}(\mathcal{A}\{T\}, R\widehat{\otimes}_k\mathcal{A}) \cong R\widehat{\otimes}_k\mathcal{A},$$

where the last isomorphism follows from Corollary 1.26. So $\widehat{F}$ coincides via the anti-equivalence between the category of affine ind-k-schemes and the category of complete topological k-algebras to the covariant functor $R\widehat{\otimes}_k-$. By Example 1.14 and its proof, the latter is represented by the complete topological k-algebra $\mathcal{R} = \varprojlim_{n\in\mathbb{N}} \mathrm{Sym}^{\cdot}(V_n^{\vee})$, where $\{V_n\}_{n\in\mathbb{N}}$ is any exhaustion of R by finite dimensional k-vector subspaces. It follows that $\widehat{F}$ is represented by the affine ind-k-scheme $\mathfrak{X} = \mathrm{Spf}(\mathcal{R})$. The universal element $u \in R\widehat{\otimes}_k\mathcal{R}$ defined in the proof of Example 1.14 corresponds in turn to a morphism of affine ind-k-schemes $\upsilon\colon X\widehat{\times}_k\mathfrak{X} \to \mathbb{A}^1_k$.

By construction, the set $\mathfrak{X}(k) = \mathrm{Mor}_k(\mathrm{Spec}(k), \mathfrak{X})$ of k-rational points of $\mathfrak{X}$ is equal to the union of the sets of k-rational points of the schemes $\mathrm{Spec}(\mathrm{Sym}^{\cdot}(V_n^{\vee}))$, hence to $\bigcup_{n\in\mathbb{N}} V_n = R$. The map $\upsilon(k)\colon (X\widehat{\times}_k\mathfrak{X})(k) = X(k)\times\mathfrak{X}(k) \to \mathbb{A}^1_k(k) = k$ is the universal evaluation map which associates to pair (x, f) consisting of a k-rational point x of X and a k-rational point $f \in R$ of $\mathfrak{X}$ the element $\upsilon(x, f) = f(x)$ of k. □

3.2. **Additive group ind-scheme actions.** We now give the geometric interpretation of restricted exponential homomorphisms as comorphisms of actions of the additive group ind-scheme on affine ind-schemes.

Let $\mathcal{A}$ be a complete topological ring and let $\mathfrak{S} = \mathrm{Spf}(\mathcal{A})$ be its associated affine ind-scheme. Let $\mathcal{A}\{T\} \cong \mathcal{A}\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[T]$ be the ring of restricted power series in one variable over $\mathcal{A}$, let $i_0\colon \mathcal{A} \to \mathcal{A}\{T\}$ be the canonical inclusion homomorphism of $\mathcal{A}$ as the subring of constants restricted power series (see subsection 1.4). Recall (c.f Subsection 2.1) that $(\mathcal{A}\{T\}, m, \iota, \epsilon)$ is a cocommutative topological Hopf $\mathcal{A}$-algebra, where $m\colon \mathcal{A}\{T\} \to \mathcal{A}\{T, T'\}$ is the unique continuous homomorphism of topological $\mathcal{A}$-algebras that maps T to $T + T'$, $\iota\colon \mathcal{A}\{T\} \to \mathcal{A}\{T\}$ is the unique continuous homomorphism of topological $\mathcal{A}$-algebras that maps T to $-T$ and $\epsilon\colon \mathcal{A}\{T\} \to \mathcal{A}$ is the unique continuous homomorphism of topological $\mathcal{A}$-algebras that maps T to $0 \in \mathcal{A}$.

The affine ind-$\mathfrak{S}$-scheme $\mathrm{Spf}(i_0)\colon \mathrm{Spf}(\mathcal{A}\{T\}) \to \mathfrak{S}$ is then an abelian group object in the category of affine ind-$\mathfrak{S}$-schemes, with respective group law and neutral section given by the morphisms

$$\mathrm{Spf}(m)\colon \mathrm{Spf}(\mathcal{A}\{T\})\widehat{\times}_{\mathfrak{S}}\mathrm{Spf}(\mathcal{A}\{T\}) \cong \mathrm{Spf}(\mathcal{A}\{T, T'\}) \to \mathrm{Spf}(\mathcal{A}\{T\})$$

and $\mathrm{Spf}(\epsilon)\colon \mathfrak{S} = \mathrm{Spf}(\mathcal{A}) \to \mathrm{Spf}(\mathcal{A}\{T\})$, which is isomorphic to the affine ind-$\mathfrak{S}$-group scheme $\mathbb{G}_{a,\mathbb{Z}}\widehat{\times}_{\mathbb{Z}}\mathfrak{S}$, where $\mathbb{G}_{a,\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[T])$ is the usual additive group scheme. We henceforth denote this affine ind-$\mathfrak{S}$-group scheme by $\mathbb{G}_{a,\mathfrak{S}}$ and call it the *additive group ind-scheme over* $\mathfrak{S}$.

Given any complete topological $\mathcal{A}$-algebra $\varphi\colon \mathcal{A} \to \mathcal{B}$, it follows from Proposition 1.25 that the map $\mathcal{B} \to \mathrm{Hom}_{\mathcal{B}\text{-alg}}(\mathcal{A}\{T\}, \mathcal{B})$, $b \mapsto \overline{\varphi}_b$, where $\overline{\varphi}_b$ is the unique continuous $\mathcal{A}$-algebra homomorphism $\bar{\varphi}_b\colon \mathcal{A}\{T\} \to \mathcal{B}$ such $\bar{\varphi}_b(T) = b$, is an isomorphism of topological abelian groups. This implies in turn that the affine ind-$\mathfrak{S}$-group scheme $\mathbb{G}_{a,\mathfrak{S}}$ represents the covariant functor

$$\begin{array}{rcl} \Gamma\colon (\mathrm{AffInd}_{/\mathfrak{S}})^{\mathrm{opp}} & \to & (\mathrm{TopAbGrps}) \\ (\mathrm{Spf}(\mathcal{B}), \mathcal{O}_{\mathrm{Spf}(\mathcal{B})}) & \mapsto & \Gamma(\mathrm{Spf}(\mathcal{B}), \mathcal{O}_{\mathrm{Spf}(\mathcal{B})}) = \mathcal{O}_{\mathrm{Spf}(\mathcal{B})}(\mathrm{Spf}(\mathcal{B})) = \mathcal{B} \end{array}$$

from the opposite category of affine ind-$\mathfrak{S}$-schemes to the category of topological abelian groups.

Now let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a complete topological $\mathcal{A}$-algebra and let $\mathfrak{X} = \mathrm{Spf}(\mathcal{B})$ be the affine ind-$\mathfrak{S}$-scheme of $\mathcal{B}$. Since $\mathcal{B}$ is complete we have a canonical isomorphism $\mathcal{B}\{T\} \cong \mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{A}\{T\}$. Let $e\colon \mathcal{B} \to \mathcal{B}\{T\} \cong \mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{A}\{T\}$ be a restricted exponential $\mathcal{A}$-homomorphism as in Definition 2.1. Then $\mathrm{Spf}(\mathcal{B}\{T\}) \cong \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X}$ and the morphism of ind-$\mathfrak{S}$-schemes $\mu := \mathrm{Spf}(e)\colon \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} \to \mathfrak{X}$ satisfies the axioms of an action of the ind-$\mathfrak{S}$-group scheme $\mathbb{G}_{a,\mathfrak{S}}$ on $\mathfrak{X}$, namely, the commutativity of the following two diagrams

$$\begin{array}{ccc} \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} & \xrightarrow{id\times\mu} & \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} \\ \downarrow{\scriptstyle m\times id} & & \downarrow{\scriptstyle \mu} \\ \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} & \xrightarrow{\mu} & \mathfrak{X} \end{array} \qquad \begin{array}{ccc} \mathfrak{S}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} & \xrightarrow{\mathrm{Spf}(\epsilon)\times id} & \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} \\ & \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle \mu} \\ & & \mathfrak{X} \end{array}$$

Conversely, given an action $\mu\colon \mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} \to \mathfrak{X}$ of $\mathbb{G}_{a,\mathfrak{S}}$ on $\mathfrak{X} = \mathrm{Spf}(\mathcal{B})$, the homomorphism of complete topological $\mathcal{A}$-algebras

$$e = \mu^\sharp(\mathrm{Spf}(\mathcal{B}))\colon \mathcal{B} = \mathcal{O}_{\mathrm{Spf}(\mathcal{B})}(\mathrm{Spf}(\mathcal{B})) \to \mu_*\mathcal{O}_{\mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X}}(\mathrm{Spf}(\mathcal{B})) \cong \mu_*\mathcal{O}_{\mathrm{Spf}(\mathcal{B}\{T\})}(\mathrm{Spf}(\mathcal{B})) \cong \mathcal{B}\{T\}$$

satisfies the axioms of a restricted exponential $\mathcal{A}$-homomorphism. In other words, via the anti-equivalence between the category of affine ind-$\mathfrak{S}$schemes $f\colon \mathfrak{X} \to \mathfrak{S}$ and the category of complete topological $\mathcal{A}$-algebras $\varphi\colon \mathcal{A} \to \mathcal{B}$, restricted exponential $\mathcal{A}$-homomorphisms $e\colon \mathcal{B} \to \mathcal{B}\{T\}$ correspond to $\mathbb{G}_{a,\mathfrak{S}}$-actions on the affine ind-$\mathfrak{S}$-scheme $\mathrm{Spf}(\mathcal{B})$.

Combined with Theorem 2.26, this yields the following extension of the classical correspondence between $\mathbb{G}_{a,S}$-actions on an affine scheme $X = \mathrm{Spec}(B)$ over an affine scheme $S = \mathrm{Spec}(A)$ and locally finite higher iterative A-derivations (A-LFHID) of B:

Theorem 3.6. *Let $\mathcal{A}$ be a complete topological ring and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a complete topological $\mathcal{A}$-algebra. Let $\mathfrak{S} = \mathrm{Spf}(\mathcal{A})$ and let $f\colon \mathfrak{X} = \mathrm{Spf}(\mathcal{B}) \to \mathfrak{S}$ be the corresponding affine ind-scheme and affine ind-$\mathfrak{S}$-scheme respectively.*

Then actions $\mathbb{G}_{a,\mathfrak{S}}\widehat{\times}_{\mathfrak{S}}\mathfrak{X} \to \mathfrak{X}$ of the additive group ind-scheme $\mathbb{G}_{a,\mathfrak{S}}$ on $\mathfrak{X}$ are in one-to-one correspondence with topologically integrable iterated higher $\mathcal{A}$-derivations $D = \{D^{(i)}\}_{i\geq 0}$ of $\mathcal{B}$.

We now consider examples of affine ind-schemes with actions of the additive group ind-scheme. A first natural example is given by the affine ind-scheme $\mathrm{Mor}(X, \mathbb{A}^1_k)$ associated to an affine k-scheme of finite type X endowed with a non-trivial $\mathbb{G}_{a,k}$-action.

Example 3.7. Let $X = \mathrm{Spec}(R)$ be an affine scheme of finite type over a field k of characteristic zero endowed with a non-trivial $\mathbb{G}_{a,k}$-action $\mu : \mathbb{G}_{a,k} \times_k X \to X$. Let $\mathfrak{X} = \mathrm{Mor}(X, \mathbb{A}^1_k)$ be the ind-scheme of Example 3.5 and let $\widehat{\mu} : \mathbb{G}_{a,k}\widehat{\times}_k\mathfrak{X} \to \mathfrak{X}$ be the morphism of functors given by the composition at the source by the k-automorphisms of X associated to the $\mathbb{G}_{a,k}$-action μ. Then $\widehat{\mu}$ is a morphism of affine ind-schemes which defines a $\mathbb{G}_{a,k}$-action on $\mathfrak{X}$.

Proof. The assertion is an immediate consequence of Yoneda embedding lemma. Nevertheless, let us give a constructive argument following the lines of that of Example 3.5. Let δ be the non-zero locally nilpotent k-derivation corresponding the $\mathbb{G}_{a,k}$-action μ. Then R admits an exhaustion by a countable family $\mathcal{W} = \{W_n\}_{n\in\mathbb{N}}$ of finite dimensional δ-stable k-vector subspaces. Indeed, given any exhaustion of R by a countable family $\mathcal{V} = \{V_n\}_{n\in\mathbb{N}}$

of finite dimensional k-vector subspaces, the fact that δ is locally nilpotent implies that for every $n \in \mathbb{N}$, the k-vector subspace W_n generated by the elements $\delta^m(f)$, $m \geq 0$, where the elements f run through a k-basis of V_n is finite dimensional and δ-stable. Furthermore, since $V_n \subseteq V_m$ for every $m \geq n$ and $V_n \subseteq W_n$, we have $V_n \subseteq W_n \subseteq W_m$ so that the W_m forms an increasing exhaustion of R by δ-stable finite dimensional k-vector subspaces.

Let $\mathcal{W} = \{W_m\}$ be a δ-stable exhaustion of R as above. Then, for every $m \in \mathbb{N}$, the restriction of δ to W_m is a nilpotent linear endomorphism δ_m of W_m. The dual endomorphism $\delta_m^\vee$ of $W_m^\vee$ defines a unique k-derivation ∂_m of the symmetric algebra $\mathrm{Sym}^{\cdot}(W_m^\vee)$ of $W_m^\vee$, which is locally nilpotent. The collection of so-defined locally nilpotent k-derivations ∂_m of the k-algebras $\mathrm{Sym}^{\cdot}(W_m^\vee)$ form an inverse system with respect to the surjective projection homomorphisms $p_{m,n}\colon \mathrm{Sym}^{\cdot}(W_m^\vee) \to \mathrm{Sym}^{\cdot}(W_n^\vee)$ associated to the inclusions $W_n \subseteq W_m$, $m \geq n$. By Proposition 2.29, there exists a unique topologically integrable k-derivation ∂ of the topological k-algebra $\mathcal{R} = \varprojlim_{n\in\mathbb{N}} \mathrm{Sym}^{\cdot}(W_n^\vee)$ such that for every $n \in \mathbb{N}$, we have $\partial_n \circ p_n = p_n \circ \partial$, where $p_n\colon \mathcal{R} \to \mathrm{Sym}^{\cdot}(W_n^\vee)$ is the canonical continuous projection.

By Example 3.5, the affine ind-scheme $\mathfrak{X} = \mathrm{Spf}(\mathcal{R})$ of $\mathcal{R}$ represents the functor $\mathrm{Mor}(X, \mathbb{A}^1_k)$. The $\mathbb{G}_{a,k}$-action $\widehat{\mu} : \mathbb{G}_{a,k} \widehat{\times}_k \mathfrak{X} \to \mathfrak{X}$ is then that associated to the topologically integrable k-derivations ∂ of $\mathcal{R}$. Note that the k-derivation $\Delta = -\delta \widehat{\otimes}_k \mathrm{id}_{\mathcal{R}} + \mathrm{id}_R \widehat{\otimes}_k \partial$ of $R \widehat{\otimes}_k \mathcal{R}$ is also topologically integrable. It defines a $\mathbb{G}_{a,k}$-action on the affine ind-k-scheme $X \widehat{\times}_k \mathfrak{X} = \mathrm{Spf}(R \widehat{\otimes}_k \mathcal{R})$ for which, by construction, the universal evaluation morphism $\upsilon\colon X \widehat{\times}_k \mathfrak{X} \to \mathbb{A}^1_k$ of Example 3.5 is $\mathbb{G}_{a,k}$-invariant. □

Remark 3.8. With the notation of Examples 3.5 and 3.7, the restriction to the set $\mathfrak{X}(k) = R$ of k-rational points of $\mathfrak{X}$ of the $\mathbb{G}_{a,k}$-action $\widehat{\mu} : \mathbb{G}_{a,k} \widehat{\times}_k \mathfrak{X} \to \mathfrak{X}$ coincides with the contragredient representation of $(k, +)$ on R defined for every f in R by

$$t \cdot f = \exp((-t)\delta)(f) = \sum_{n\geq 0} (-1)^n \frac{t^n}{n!} \delta^n(f).$$

In particular, for every k-rational point x of X, we have $(t \cdot f)(x) = f((-t) \cdot x)$ and hence

$$\upsilon(t \cdot (x, f)) = (t \cdot f)(t \cdot x) = f(x) = \upsilon(x, f).$$

Example 3.9. As a concrete illustration of Example 3.7, consider the locally nilpotent k-derivation $\delta = \partial/\partial x$ of $R = k[x]$ corresponding to the action of $\mathbb{G}_{a,k}$ on $\mathbb{A}^1_k$ by translations and the exhaustion of R by the δ-stable subspaces

$$W_n = k[x]_{\leq n} = k\langle x^0, \ldots, x^n\rangle, \quad n \in \mathbb{N},$$

consisting of polynomials of degree less than or equal to n. For every $n \in \mathbb{N}$, the algebra $\mathrm{Sym}^{\cdot}(W_n^\vee)$ is isomorphic to the polynomial ring $k[X_0, \ldots, X_n]$, where $(X_0, \ldots, X_n, \ldots)$ is the family of elements of the dual $R^\vee$ of R as a k-vector space defined by $X_i(x_j) = \delta_{i,j}$ for every $i, j \in \mathbb{N}$. The complete topological k-algebra $\mathcal{R} = \varprojlim_{n\in\mathbb{N}} \mathrm{Sym}^{\cdot}(W_n^\vee)$ is isomorphic to the separated completion of the polynomial ring $k[(X_i)_{i\in\mathbb{N}}]$ with respect to the topology induced by the fundamental system of open ideals $\mathfrak{a}_n = (X_i)_{i\geq n} k[(X_i)_{i\in\mathbb{N}}]$. The universal element $u \in R \widehat{\otimes}_k \mathcal{R}$ of the proof of Example 1.14 can be represented by the formal power series $\sum_{n\in\mathbb{N}} x^n X_n \in k[x][[(X_i)_{i\in\mathbb{N}}]]$.

On the other hand, the k-derivation ∂_n of $\mathrm{Sym}^{\cdot}(W_n^\vee)$ is given by $\partial_n(X_i) = (i+1)X_{i+1}$, if $i \leq n-1$ and $\partial_n(X_n) = 0$. The corresponding topologically integrable k-derivation $\partial = \varprojlim_{n\in\mathbb{N}} \partial_n$ of $\mathcal{R}$ induced by the inverse system of locally nilpotent k-derivations ∂_n, $n \in \mathbb{N}$, coincides with the topologically integrable k-derivation with trivial kernel k of Example 2.31. Note that in contrast, for the topologically integrable k-derivation $\Delta =$

$-\delta\widehat{\otimes}_k \mathrm{id}_{\mathcal{A}} + \mathrm{id}_R\widehat{\otimes}_k\partial$ of $R\widehat{\otimes}_k\mathcal{R}$, we have

$$\Delta(u) = \sum_{n\geq 0}\Delta(x^nX_n) = \sum_{n\geq 1}(-nx^{n-1}X_n + (n+1)x^nX_{n+1}) = 0.$$

For an affine scheme X of finite over a field k, there are many natural affine ind-k-schemes that can be constructed from the ind-k-scheme $\mathfrak{X} = \mathrm{Mor}(X, \mathbb{A}^1_k)$ (see e.g. [9]). These include for instance the ind-k-scheme $\mathrm{Mor}_k(X, Y)$ where Y is any affine k-scheme and the ind-k-scheme $\mathrm{Aut}_k(X)$ of k-automorphisms of X. Since every non-trivial $\mathbb{G}_{a,k}$-action on X gives rise to a non-trivial $\mathbb{G}_{a,k}$-action on these affine ind-k-schemes, this provides a large supply of natural affine ind-k-schemes with interesting natural $\mathbb{G}_{a,k}$-actions.

Another family of examples is given by the following ind-scheme counterpart of Danielewski hypersurfaces.

Example 3.10. Let $\mathcal{A}$ be an integral complete topological algebra over a field k of characteristic zero. Let $\mathcal{A}\{y,z\}$ be the restricted power series ring in two variables over $\mathcal{A}$. Let $x \in \mathcal{A}$ be a non-zero element and let $P(y) \in \mathcal{A}\{y\}$ be a non-zero restricted power series. Then there exists a unique continuous $\mathcal{A}$-derivation ∂ of $\mathcal{A}\{y,z\}$ such that, $\partial(y) = x$ and $\partial(z) = P'(y)$, where $P'(y) \in \mathcal{A}\{y\}$ denote the derivative of the restricted power series $P(y)$. The so defined $\mathcal{A}$-derivation ∂ is topologically integrable. Indeed, let $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ be a fundamental system of open ideals of $\mathcal{A}$ so that we have by definition $\mathcal{A}\{y,z\} = \varprojlim_{n\in\mathbb{N}} A_n[y,z]$, where $A_n = \mathcal{A}/\mathfrak{a}_n$. Let $x_n \in A_n$ and $P_n(y) \in A_n[y]$ denote the respective residue classes of $x \in \mathcal{A}$ and $P(y) \in \mathcal{A}\{y\}$, $n \geq 0$. Then we have $\partial = \varprojlim_{n\in\mathbb{N}} \partial_n$, where ∂_n is the triangular, hence locally nilpotent, A_n-derivation of $A_n[y,z]$ defined by $\partial_n(y) = x_n$ and $\partial_n(z) = P'_n(y)$. Thus ∂ is topologically integrable by Proposition 2.29. Note that the element $xz - P(y)$ belongs to the kernel $\mathrm{Ker}\partial$ of ∂, in particular, in contrast with the example considered in Example 2.31, $\mathcal{A}$ is a proper sub-algebra of $\mathrm{Ker}\partial$.

Now assume in addition that x and $P(y)$ are chosen so that the principal ideal $I = (xz - P(y))$ of $\mathcal{A}\{y,z\}$ is prime and closed. Then $\mathcal{B} = \mathcal{A}\{y,z\}/I$ is a topological $\mathcal{A}$-algebra when endowed with the quotient topology, and the quotient homomorphism $q\colon \mathcal{A}\{y,z\} \to \mathcal{B}$ is continuous. Since $\partial(I) \subset I$, ∂ induces a topologically integrable $\mathcal{A}$-derivation $\overline{\partial}$ of $\mathcal{B}$. Letting $\mathfrak{X} = \mathrm{Spf}(\mathcal{A})$, the homomorphism $\mathcal{A}\{y,z\} \to \mathcal{B}$ corresponds to a closed embedding of affine ind-$\mathfrak{X}$-schemes

$$\mathfrak{Y} = \mathrm{Spf}(\mathcal{B}) \hookrightarrow \mathbb{A}^2_{\mathfrak{X}} = \mathrm{Spf}(\mathcal{A}\{y,z\})$$

which is equivariant for the $\mathbb{G}_{a,\mathfrak{X}}$-actions on $\mathfrak{Y}$ and $\mathbb{A}^2_{\mathfrak{X}}$ associated to $\overline{\partial}$ and ∂ respectively.

For a concrete illustration, consider the completion $\mathcal{A}$ of the polynomial ring $k[(X_i)_{i\in\mathbb{N}}]$ in countably many variables with respect to the topology generated by the ideals $\mathfrak{a}_n = (X_i - 1)_{i>n}$. Let $x \in \mathcal{A}$ be the element represented by the Cauchy sequence $x_n = \prod_{i=0}^n X_i \in k[X_0, \ldots, X_n]$. Choosing for the restricted power series $P(y)$ a non-constant polynomial $P(y) \in k[y] \subset \mathcal{A}\{y\}$, we obtain an associated affine ind-$\mathfrak{X}$-scheme $\mathfrak{Y} \subset \mathbb{A}^2_{\mathfrak{X}}$ which is a colimit of so-called *Danielewski varieties* in $\mathbb{A}^{n+3}_k$ defined by equations of the form $\prod_{i=0}^n X_i z = P(y)$ (see e.g. [7]). One can also choose non-polynomial restricted power-series $P(y) \in \mathcal{A}\{y\}$, for instance, the one $P(y) = \sum_{i=0}^{\infty}(X_i - 1)^i y^{i+1}$.

REFERENCES

[1] M. Artin, A. Grothendieck and J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas, Tome 1: Théorie des topos*, Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4). Lecture Notes in Mathematics 269, Springer 1972.

[2] A. Crachiola, L. Makar-Limanov, *On the rigidity of small domains*, J. Algebra, 284 (2005), pp. 1-12.

[3] N. Bourbaki, *Elements of mathematics. General topology*, Chapters 1-4. Translated from the French. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966
[4] N. Bourbaki, *Elements of mathematics. Commutative algebra*, Chapters 1-7. Translated from the French. Translated from the French. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass., 1972.
[5] N. Bourbaki, *Elements of mathematics. Algebra I*, Chapters 1-3. Translated from the French. Hermann, Paris; Addison-Wesley Publishing Co., Reading Mass., 1974.
[6] R. Díaz, and A. Liendo, *On toric ind-varieties and pro-affine semigroups*, J. Algebra 569 (2021) 442–465.
[7] A. Dubouloz, *Additive group actions on Danielewski varieties and the Cancellation Problem*, Math. Z. 255 (2007), no. 1, 77-93.
[8] G. Freudenburg, *Algebraic theory of locally nilpotent derivations.* Encyclopaedia of Mathematical Sciences 136, Invariant Theory and Algebraic Transformation Groups 7, Springer, 2017.
[9] J.-P. Furter and H. Kraft *On the geometry of the automorphism groups of affine varieties*, arXiv:1809.04175 [math.AG] (2018).
[10] A. Grothendieck and J. Dieudonné, Jean *Éléments de géométrie algébrique: I. Le langage des schémas*, Grundlehren der Mathematischen Wissenschaften (in French). 166 (2nd ed.). Berlin; New York: Springer-Verlag. ISBN 978-3-540-05113-8.
[11] H. Hasse, F.K. Schmidt,*Noch eine Bergründung der Theorie der höheren Differential quotienten in einemalgebraischen Functionenkörper einer Unbestimmten,*J. Reine Angew. Math. 177 (1937) 215–237
[12] T. Kambayashi, *Pro-affine algebras, ind-affine groups and the Jacobian Problem*, Journal of Algebra, 185 (1996), 481-501.
[13] T. Kambayashi, *Some basic results on pro-afine algebras and ind-afine schemes*, Osaka J. Math. 40 (2003), no. 3, 621-638.
[14] T. Kambayashi, *Inverse limits of polynomial rings,* Osaka J. Math., 41 (2004), 617-624.
[15] T. Kambayashi and M. Miyanishi, *On two recent views of the Jacobian conjecture*, Affine algebraic geometry, 113-138, Contemp. Math., 369, Amer. Math. Soc., Providence, RI, 2005.
[16] S. Kumar, *Kac-Moody groups, their ag varieties and representation theory, Progress in Mathematics*, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.
[17] H. Kraft and A. Regeta, *Automorphisms of the Lie algebra of vector fields on affine n-space*, J. Eur. Math. Soc. (JEMS) 19 (2017), no. 5, 1577-1588.
[18] M. Miyanishi,*A remark on an iterative infinite higher derivation*, J. Math. Kyoto Univ. 8 (1968) 411-415.
[19] D. G Northcott, *Lessons on rings, modules and multiplicities*, Cambridge University Press Bentley House, 1968.
[20] I. R. Shafarevich, *On some infinite-dimensional groups*, Rend. Mat. Appl. (5) 25 (1-2) (1966) 208-212.
[21] I. R. Shafarevich, *On some infinite-dimensional groups. II*, Math. USSR Izvestija 18 (1982), 185-194.
[22] I. Stampfli, *On the topologies on ind-varieties and related irreducibility questions*, J. Algebra 372 (2012), 531-541.

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