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Qualitative Properties of Resolvent Families of Operators

by

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Introduction

The main purpose of this thesis is the study of qualitative properties of resolvent families of operators on Banach spaces and their applications to evolution equations. The concept of *resolvent families* was introduced by Da Prato and Ianelli in [**27**, Definition 1], as an extension of the notion of C_0-semigroups, to study the existence of mild solutions to the following integro-differential equations

$$\begin{cases} u'(t) = \displaystyle\int_0^t k(t-s)Au(s)\,ds, \quad t \geq 0, \\ u(0) = u_0, \end{cases} \tag{0.1}$$

where A is a closed linear operator defined on a Banach space X, $u_0 \in X$, $k \in L^1_{\text{loc}}(\mathbb{R}_+)$. By a resolvent family for (0.1) we mean a family of bounded linear operators $\{U(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ (here $\mathcal{B}(X)$ denotes the space of all bounded and linear operators on X) which satisfies the following properties:

a) $u(t) := U(t)x \in C([0,\infty), X)$, for all $x \in X$,
b) $u(t) := U(t)x \in C^1([0,\infty), X) \cap C([0,\infty), D(A))$ for all $x \in D(A)$,
c) there exist $M > 0$, and $\omega \in \mathbb{R}$ such that $\|U(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, and
d) the function u verifies problem (0.1).

The existence of a resolvent family to problem (0.1) allows us to solve the inhomogeneous problem

$$\begin{cases} u'(t) = \displaystyle\int_0^t k(t-s)Au(s)\,ds + f(t), \quad t \in [0,T], \\ u(0) = x, \end{cases} \tag{0.2}$$

for any $x \in X$ and $f \in C([0,T], X)$. In fact, if $\{U(t)\}_{t\geq 0}$ is a resolvent family for (0.1), then the solution to (0.2) is given in terms of its resolvent family by

$$u(t) = U(t)x + \int_0^t U(t-s)f(s)\,ds.$$

After that, this theory was developed rapidly. For instance, if A is a closed linear operator defined on X, $a \in L^1_{loc}(\mathbb{R}_+)$, and $f : \mathbb{R}_+ \to X$ is a continuous function, then the Volterra equation

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds, \quad t \in [0,T], T > 0, \tag{0.3}$$

is well-posed (which means that there exists a unique solution to (0.3)) if and only if the equation (0.3) admits a *resolvent family,* (see [**79**, Chapter I]) that is, there exists a strongly continuous family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ such that $S(t)$ commutes with A and satisfies the following *resolvent equation:*

$$S(t)x = x + \int_0^t a(t-s)AS(s)x\,ds, \quad t \geq 0, x \in X.$$

If $f \in W^{1,1}([0,T],X)$, then the solution to (0.3) is given in terms of its resolvent family by ([**79**, Proposition 1.2])

$$u(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)\,ds, t \in [0,T].$$

The examples above show that the solutions to certain abstract equations can be written in terms of its resolvent families. Some more general concepts such as integrated semigroups [**7**], [**9**], cosine (and sine) families [**8**], α-times resolvent [**50**], α-order resolvent [**53**], convoluted semigroups [**47**], integral resolvents [**37**], $(a,k)-$regularized families [**57**], among others, can be considered also as *resolvent families* because they play a crucial role in the representation of the solutions to certain integral, differential, integro-differential, among other equations. Therefore, the knowledge of properties of this resolvent families allows us to obtain important qualitative properties about these abstract equations.

This thesis is primarily focused to the study of the norm continuity and compactness of certain general resolvent families on Banach spaces. In addition, spectral mapping theorems for convoluted semigroups are given, and we introduce the concept of cosine and sine family on time scales.

In the following paragraphs, we provide a brief description of each chapter of this thesis:

Chapter 1 summarizes preliminaries and some notation used. Chapter 2 treats about the norm continuity of (a,k)-regularized families. The property of uniform continuity (or norm-continuity) for one-parameter families of bounded operators is a topic of increasing interest in recent researches, mainly because of their important role in the exploration of useful criteria for the existence of solutions to nonlinear partial differential equations when they are modeled as an abstract evolution equation on vector-valued spaces of functions, see e.g. the monographs [**8**] and [**79**]. The applications of uniform continuity are usually found in the use of fixed point arguments, which try to avoid hypothesis of compactness on the data of the problem, but where this hypothesis needs to be replaced by some better behavior on the family of bounded operators dealing with the well-posedness of the associated abstract linear problem. See e.g. [**5**, Remark 3.4], [**14**, Theorem 3.4], [**31**, Theorem 4.1], [**83**, Theorems 4.1 and 5.3] and [**87**] to cite a few references. Note that uniform continuity also plays a crucial role in investigating the stability of solutions to abstract Volterra equations [**20**, Theorem 2.9] and abstract Cauchy problems.

Recently, Z. Fan [**31**] and other authors (see e.g. [**83**, p.208, item (ii)]), obtained characterizations of compactness for families of bounded operators associated to a class of semilinear fractional Cauchy problem. In the searching of these characterizations, one of the difficult points is that they require the uniform continuity of the studied family [**31**, Theorem 3.6 and Theorem 3.7]. As remarked by Fan, the main problem is the non-existence of practical criteria that can assure uniform continuity of the given family of bounded operators.

The main goal of this chapter is to give a complete answer to this problem. Our framework will be the theory of (a,k)-regularized families [**57**]. We reformulate the above question as an inverse problem finding a class of scalar kernels (a,k) such that property of the uniform continuity holds.

An exhaustive study of uniform continuity is given not only for families of bounded operators associated to fractional Cauchy problems, but also for a very wide class of families of bounded operators $\{R(t)\}_{t\geq 0}$, namely, the class of (a,k)-regularized resolvent families [**57**]. This notion generalizes the theories of C_0-semigroups [**8**, Section 3.1], α-times integrated semigroups [**8**, Section 3.2], convoluted semigroups [**25**], cosine functions [**8**, Section 3.14], n-times integrated cosine families [**9**], resolvent families [**79**] α-resolvent families [**12**], among others. For example, if $a(t)=k(t)=1$ for all $t\geq 0$ and if $a(t)=t, k(t)=1$, the result is the well-known cases of strongly continuous semigroups and cosine operator functions, respectively. If $a(t)=1$ and $k(t)=t^n/n!$ then $R(t)$ is an n-integrated semigroup. Taking $a\in L^1_{loc}(\mathbb{R}_+)$ and $k(t)=1$ for all $t\geq 0$ we have that $R(t)$ is a resolvent family, which are the central object of study in the theory of abstract Volterra equations [**79**]. Finally, if $a(t)=t^{\alpha-1}/\Gamma(\alpha)$ $(\alpha>0)$ and $k(t)=1$ for all $t\geq 0$, then $R(t)$ corresponds to a α-resolvent family. An updated overview is given in [**63**, Section 2] and references therein.

Previous studies on uniform continuity for families of bounded operators have been done mainly in the case of C_0-semigroups [**51**]. K. Latrach, Paoli and Simonnet [**48**], [**49**] have studied the problem from different perspectives. See also [**55**] and [**56**] for a study in the case of resolvent families associated to Volterra equations. In the case of cosine and sine families of bounded operators, first studies are due to Travis and Webb [**80**, Proposition 4.1], [**81**, Proposition 2.4]. See also [**52**]. More recently, in [**39**] the authors have proven that when the semigroup generated by the linear part of some linear neutral partial functional differential equations in L^p-spaces is norm-continuous, then the semigroup solution associated to the neutral system is eventually norm-continuous.

It is well known that if a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of type (M,ω) defined on a Banach space X generated by an operator A is continuous in the operator norm for all $t>0$, then $\|(s+i\tau-A)^{-1}\| \to 0$ as $|\tau|\to\infty$ for all $s>\omega$. See [**15**], [**29**] and [**68**]. The converse is also true in Hilbert spaces H (see [**29**], [**33**], [**86**]) but it might fail in general Banach spaces (see for instance [**69**]). The recent paper [**21**] gives an example where the semigroup is nowhere continuous in operator norm but the resolvent tends to 0 along $2+i\tau$ almost logarithmically. However, the question of finding a similar characterization for (a,k)-regularized resolvent families under reasonable conditions on the kernels a and k remains as an open problem.

One of the main issues of this chapter in this thesis is the ability to solve this problem assuming that a and k are 2-regular (see Chapter 2 for definitions) and certain behavior of $\widehat{k}(\lambda)$ along the imaginary axis. More precisely, it is proved that the following assertions are equivalent:

(i) $\{R(t)\}_{t\geq 0}$ is continuous in $\mathcal{B}(H)$ for $t>0$,
(ii) $\lim\limits_{|\tau|\to\infty} \|\widehat{k}(s+i\tau)\big(I-\widehat{a}(s+i\tau)A\big)^{-1}\| = 0$ for some $s>\omega$.

This chapter is organized in the following way: Firstly, the definition of (a,k)-regularized resolvent families and their main properties, and the notion of Grothendieck space and the Dunford-Pettis property are recalled. A notion of regularity on the kernels will be also useful as well as an important result due to Lotz [**66**] that is the key to establishing one of our main results in the forthcoming sections. Next, a slightly surprising result is given: in Theorem 2.9 is shown that a strongly continuous (a,k)-regularized resolvent family on a class of Banach spaces containing all L^∞ spaces is necessarily uniformly continuous $(t\geq 0)$. This result generalizes a known theorem in the case of C_0-semigroups due to Lotz [**66**]. Then, we remark an interesting corollary: The result is also true in the case of certain families of bounded operators (called α-resolvent families) which play a central and decisive role in the development of qualitative properties for solutions to fractional partial differential equations. Next, to characterize those (a,k)-regularized resolvent families which have the property of being near $k(t)$ times the identity (i.e., $R(t)-k(t)I$ is compact for some positive value of t), early results on such property are due to Cuthbert [**26**], Henríquez [**40**] and Lutz [**67**] among other authors. It turns out that this property is equivalent to the compactness of the generator. This equivalence is proved in Theorem 2.14. Finally, an important characterization of uniform continuity (for $t > 0$) in case of Hilbert spaces (see Theorem 2.21) is given. This characterization constitutes a remarkable and nontrivial extension of previous results (see [**56**] and [**86**]) and will be useful in establishing practical criteria, e.g. on the compactness of (a, k)-regularized families of operators in general, and their specialization in different cases of interest. An example of this statement is given in Corollary 2.22 and further applications are indicated in Remark 2.23. The results of this chapter are part of the paper [**61**].

Chapter 3 is devoted to the compactness of fractional resolvent families. A fractional resolvent operator function endows the solution operator, defined by the inhomogeneous equation

$$D_t^\alpha u(t) = Au(t) + f(t, u(t)), \quad 0 < \alpha \leq 2, \tag{0.4}$$

by means of the variation of constants formula, with the compactness property, comparable with the finite-dimensional counterpart.

For $\alpha = 1$, the well known criterion for compactness of C_0-semigroups (see e.g. [**75**, Theorem 3.3, Chapter 2]), asserts that a C_0-semigroup $\{T(t)\}_{t\geq 0}$ generated by A is compact (for $t > 0$), if and only if, $T(t)$ is continuous in the uniform operator topology for $t > 0$, and the resolvent operator $(\lambda - A)^{-1}$ is compact for all $\lambda \in \rho(A)$, the resolvent set of A. This criterion has great importance in the study of existence of mild solutions for (0.4), because arguments to solve (0.4) using fixed points theorems of Schauder's type can be applied.

In case $\alpha = 2$, we find a similar situation assuming that A is the generator of a strongly continuous sine family $\{S(t)\}_{t\geq 0}$. In this case, the compactness criterion of sine family (see [**80**]), asserts that a sine family $S(t)$ is compact for all $t > 0$, if and only if, the resolvent operator $(\lambda^2 - A)^{-1}$ is compact for every $\lambda \in \rho(A)$. Observe that, in infinite dimensional Banach spaces, a cosine family $\{C(t)\}_{t\geq 0}$ cannot be compact.

In the last decade, the fractional differential equation (0.4), where the fractional derivative is understood in the Caputo sense, has been extensively studied. Equations with memory of type (0.4) are in connection with several applications in physics and viscoelasticity theory (see [**76**], [**79**] and references therein). The solution to equation (0.4) in the case $0 < \alpha < 1$ is essentially given by

$$u(t) = S_\alpha(t)u(0) + \int_0^t S_\alpha(t-s) f(s, u(s))\, ds,$$

where $\{S_\alpha(t)\}_{t\geq 0}$ is the $(\alpha, 1)$-resolvent family generated by A. Several properties of $\{S_\alpha(t)\}_{t\geq 0}$ have been studied in [**19**], [**50**], [**53**] among others. The compactness of $\{S_\alpha(t)\}_{t>0}$ was first studied by subordination methods, i.e., A is supposed to be a generator of a compact semi-group, and then compactness of the family $\{S_\alpha(t)\}_{t>0}$ is obtained, see Prüss [**78**, Corollary

2]. After that, Wang, Chen and Xiao [**83**], assuming that A is an almost sectorial operator and $(\lambda^\alpha - A)^{-1}$ is compact, proved that the family $\{S_\alpha(t)\}_{t>0}$ is continuous in the uniform operator topology for $t > 0$ [**83**, Theorem 3.2], and compact [**83**, Theorem 3.5]. The method relies on the use of functional calculus. Very recently, and under the hypothesis continuity in the uniform operator topology for $t > 0$, Fan [**31**] found out that the compactness of the resolvent operator $(\lambda^\alpha - A)^{-1}$ is necessary and sufficient for compactness of $\{S_\alpha(t)\}_{t>0}$. The proof follows a direct method having in mind the case $\alpha = 1$. However, the necessary condition has a flaw in their proof (see Remark 3.10 below), and therefore the problem of characterization of compactness remains open. The objective of this chapter is to provide a completely new approach to Fan's result, and to provide a complete characterization in the complementary case $1 < \alpha \leq 2$ for the associated family $R_\alpha(t) = (g_{\alpha-1} * S_\alpha)(t)$ that corresponds to the fractional counterpart of the sine functions for $\alpha = 2$ and that has not been studied previously in the literature. An application of these results to the existence of mild solutions to a semilinear fractional abstract equations with nonlocal initial conditions is also given. The results of this chapter are part of the paper [**60**].

In Chapter 4 we are interested in to obtain a version of the Spectral Mapping theorem for k-convoluted semigroups. It is well known that if A is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ defined on a Banach space X, then the spectral inclusion

$$\sigma(T(t)) \setminus \{0\} \quad \supseteq \quad e^{t\sigma(A)},$$

holds for all $t \geq 0$. Moreover, the equality holds for the point and residual spectrum, that is

$$\sigma_p(T(t)) \setminus \{0\} \quad = \quad e^{t\sigma_p(A)}, \tag{0.5}$$

$$\sigma_r(T(t)) \setminus \{0\} \quad = \quad e^{t\sigma_r(A)}, \tag{0.6}$$

for all $t \geq 0$. However, the converse inclusion for the entire spectrum does not hold in general [**30**, Chapter IV] because of the behavior of its approximate point spectrum. On the other hand, the works of C. Day [**28**], and G. Greiner and M. Müller [**36**], show that the equalities (0.5)-(0.6) and the corresponding equality for the approximate point spectrum hold for integrated semigroups by using an extrapolation theorem exposed by W. Arendt, F. Neubrander and U. Schlotterbeck (see [**10**]).

The k-convoluted semigroups, $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, are a natural extension of the concepts of n-integrated and α-integrated semigroups, $n \in \mathbb{N} \cup \{0\}$ and $\alpha > 0$, respectively.

In this chapter, a spectral mapping theorem for the point spectrum, approximate point spectrum and residual spectrum for an k-convoluted semigroup $\{R(t)\}_{t\geq 0}$ is proved. To be more precise, it is shown that if A generates a k-convoluted semigroup $\{R(t)\}_{t\geq 0}$, then

$$\sigma_p(R(t))\cup\{0\} = \left\{\int_0^t k(t-s)e^{\lambda s}\,ds : \lambda\in\sigma_p(A)\right\}\cup\{0\}, \tag{0.7}$$

$$\sigma_a(R(t))\cup\{0\} = \left\{\int_0^t k(t-s)e^{\lambda s}\,ds : \lambda\in\sigma_a(A)\right\}\cup\{0\}, \tag{0.8}$$

$$\sigma_r(R(t))\cup\{0\} = \left\{\int_0^t k(t-s)e^{\lambda s}\,ds : \lambda\in\sigma_r(A)\right\}\cup\{0\}, \tag{0.9}$$

for all $t\geq 0$. In this chapter the equalities (0.7), (0.8) and (0.9) are proved in case when A is the generator of a C_0-semigroup and a k-convoluted semigroup. The results of this chapter are part of the paper [**62**].

Finally, Chapter 5 is devoted to define and establish properties of cosine and sine functions on time scales. Recently, it was introduced in the literature the concept of C_0-semigroup on time scales, which encompass this concept in all the classical cases and also, for several interesting time scales such as the quantum scales, hybrid scales, among others (see [**41**]). We point out that this definition is very general and does not require the additivity property on the time scale. They employ the Laplace transform to address the definition of C_0-semigroup, which turns it much more general. On the other hand, it is a known fact that the theory of semigroups plays a crucial role to study the first order abstract Cauchy problem on time scales

$$\begin{cases} u^{\Delta}(t) &= Au(t), \quad t\in\mathbb{T}_0^+, \\ u(0) &= x, \end{cases} \tag{0.10}$$

and also, to study nonlinear first order abstract problem on time scales given by

$$\begin{cases} u^{\Delta}(t) &= Au(t)+f(t,u(t)), \quad t\in\mathbb{T}_0^+, \\ u(0) &= x, \end{cases} \tag{0.11}$$

where A is a closed linear operator in a Banach space X, $x\in X$ and $\mathbb{T}_0$ is a time scale such that $0\in\mathbb{T}_0$ and $\sup\mathbb{T}_0=+\infty$, and $\mathbb{T}_0^+=\mathbb{T}_0\cap\mathbb{R}^+$. The equations (0.10) and (0.11) are very important for applications and to study several interesting models.

On the other hand, the theory of abstract cosine and sine families plays an important role in the study of the existence of solutions to second order equations and to investigate different problems in several fields of knowledge. Due to this fact, this theory has been attracting the attention of several authors, see [**3**], [**22**], [**35**], [**43**], [**58**] and the references therein. However the theory of discrete abstract cosine functions has not been completely developed. For instance, the correspondence with the set of all discrete cosine functions defined by means of D'Alembert functional equations is still an open problem as well the description of the generator of abstract cosine has not been given yet. Also, to the best of our knowledge, the definition of abstract cosine functions on quantum or hybrid scales was not introduced in the literature until now. Therefore, one of the goals of this paper is to try to fulfill this lack of literature and to present a unified theory for abstract cosine functions which encompass the continuous, discrete and hybrid cases.

Therefore, motivated by these results, it is introduced in this chapter the concept of cosine and sine functions defined on time scales, in order to study the homogeneous abstract second order Cauchy problem on time scales

$$\begin{cases} u^{\Delta\Delta}(t) &= Au(t), \quad t \in \mathbb{T}_0^+, \\ u(0) &= x, \\ u^{\Delta}(0) &= y, \end{cases} \tag{0.12}$$

the inhomogeneous abstract second order Cauchy problem on time scales

$$\begin{cases} u^{\Delta\Delta}(t) &= Au(t) + f(t), \quad t \in \mathbb{T}_0^+, \\ u(0) &= x, \\ u^{\Delta}(0) &= y, \end{cases} \tag{0.13}$$

and the nonlinear abstract second order Cauchy problem on time scales

$$\begin{cases} u^{\Delta\Delta}(t) &= Au(t) + f(t, u(t)), \quad t \in \mathbb{T}_0^+, \\ u(0) &= x, \\ u^{\Delta}(0) &= y, \end{cases} \tag{0.14}$$

where A is a closed linear operator in a Banach space X, $x, y \in X$.

It is a known fact that when we are dealing with second order dynamic equations on time scales, we can formulate the problem using several different ways. Therefore, it is a big deal to find out an appropriate formulation to study each problem. In our case, since we are interested to study abstract cosine and sine functions as well as their properties, our equation given by (0.12) is the most appropriate to investigate this problem. See Remark 5.12 for details.

We recall the reader that the classical way to define cosine function is through the following property:

$$\begin{cases} 2C(t)C(s) &= C(t+s) + C(t-s), \quad t, s \in \mathbb{R}, \\ C(0) &= I. \end{cases} \tag{0.15}$$

However, if we define the abstract cosine function on time scales by the usual property (0.15), we need to require a very restrictive property to the time scale, in order to ensure that the abstract cosine on time scales is well-defined, which means, the time scale should satisfy the group property, which is defined as follows. If T has the group property, then the following two conditions are satisfied:

(1) $0 \in \mathbb{T}$,

(2) If $t, s \in \mathbb{T}$, then $t - s \in \mathbb{T}$.

Therefore, in order to avoid such restriction on the time scales, we present the definitions of the cosine and sine functions using Laplace transform on time scales. This approach is much more general and encompasses all time scales $\mathbb{T}_0$ satisfying $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = +\infty$. Further, using such approach, we are able to deal with several different types of time scale such as quantum scale, hybrid scales, among others. Although of all these advantages, to prove the results using only this general condition represents a big deal, because to prove mostly of the results concerning abstract cosine and sine functions, we need to present completely different and new arguments to the ones found in the literature, since these last ones usually employ the functional equation to get them.

This chapter is organized as follows. The first section is devoted to remember the concept of Laplace transform on time scales, its properties will be necessary to establish the main results. The second section is devoted to define and develop the concept of abstract cosine function on time scales, which generalizes the properties of the classical theory. Among others, we prove the following properties of the abstract cosine function:

(a) $\int_0^t (t - \sigma_{\mathbb{T}}(s))C(s)x\,\Delta s \in D(A)$ and $A\int_0^t (t - \sigma_{\mathbb{T}}(s))C(s)x\,\Delta s = C(t)x - x$ for all $x \in X, t \in \mathbb{T}_0^+$.

(b) If $x \in D(A)$, then $C(t)x \in D(A)$ and $AC(t)x = C(t)Ax$ for all $t \in \mathbb{T}_0^+$.

(c) Let $x, y \in X$. Then $x \in D(A)$ and $Ax = y$ if, and only if, for all $t \in \mathbb{T}_0^+$ we have

$$\int_0^t (t - \sigma_{\mathbb{T}}(s))C(s)y\,\Delta s = C(t)x - x.$$

(d) If 0 is right-dense, then $D(A) = \left\{x \in X : \lim_{h \to 0^+} \dfrac{2(C(h)x - x)}{h^2} \text{ exists}\right\}$, and

$$Ax = \lim_{h \to 0^+} \frac{2(C(h)x - x)}{h^2}.$$

(e) If 0 and $\sigma_{\mathbb{T}}(0)$ are right-scattered, then

$$D(A) = \left\{x \in X : \frac{(C(\sigma_{\mathbb{T}}(\sigma_{\mathbb{T}}(0))) - C(\sigma_{\mathbb{T}}(0)))x}{\mu_{\mathbb{T}}(\sigma_{\mathbb{T}}(0))\mu(0)} + \frac{(C(0) - C(\sigma_{\mathbb{T}}(0)))x}{\mu_{\mathbb{T}}(0)^2} \quad \text{is well} - \text{defined}\right\},$$

and

$$Ax = \frac{(C(\sigma_{\mathbb{T}}(\sigma_{\mathbb{T}}(0))) - C(\sigma_{\mathbb{T}}(0)))x}{\mu(\sigma_{\mathbb{T}}(0))\mu_{\mathbb{T}}(0)} + \frac{(C(0) - C(\sigma_{\mathbb{T}}(0)))x}{\mu_{\mathbb{T}}(0)^2}.$$

(f) If 0 is right-scattered and $\sigma(0)$ is right-dense, then

$$D(A)=\Big\{x\in X:\lim_{h\to 0^+}\frac{(C(\sigma_{\mathbb{T}}(0)+h)-C(\sigma_{\mathbb{T}}(0)))x}{\mu_{\mathbb{T}}(0)h}+\frac{(C(0)-C(\sigma_{\mathbb{T}}(0))x}{\mu_{\mathbb{T}}(0)}\ \text{ exists}\Big\},$$

and

$$Ax=\lim_{h\to 0^+}\frac{(C(\sigma_{\mathbb{T}}(0)+h)-C(\sigma_{\mathbb{T}}(0)))x}{\mu_{\mathbb{T}}(0)h}+\frac{(C(0)-C(\sigma_{\mathbb{T}}(0))x}{\mu_{\mathbb{T}}(0)}.$$

In the previous properties, $\sigma_{\mathbb{T}}(t)$ and $\mu_{\mathbb{T}}(t)$ denote the forward jump operator and the graininess function, respectively. The properties (e) and (f) bring the definition of the generator A when 0 and $\sigma_{\mathbb{T}}(0)$ are right-scattered, and 0 is right-scattered and $\sigma_{\mathbb{T}}(0)$ is right-dense, respectively. These descriptions to the generator are very surprising and to the best of our knowledge, it has not been presented in the literature until now. Also, we point out that the cases (d), (e) and (f) are the only ones to consider, since the case when 0 is right-dense and $\sigma_{\mathbb{T}}(0)$ is right-scattered is not possible. See Remark 5.21 for details.

In the third section, our goal is to show how restrictive is the class of time scales which satisfies the group property. In order to do it, we prove several results describing such restriction. For instance, we prove that if the class of time scales satisfies the group property, then the hybrid time scales are not included in this class (see Theorem 5.23). Also, we show that if 0 is right-dense, then the only possibility for $\mathbb{T}$ is $\mathbb{R}$ (see Remark 5.25). The fourth section is devoted to introduce the concept of abstract sine function on time scales and prove its properties. Among others, we prove the following properties of the abstract sine function:

(a) If $x\in D(A)$, then $S(t)x\in D(A)$ and $AS(t)x=S(t)Ax$ for all $t\in\mathbb{T}_0^+$.

(b) $\int_0^t(t-\sigma_{\mathbb{T}}(s))S(s)x\,\Delta s\in D(A)$ and $A\int_0^t(t-\sigma_{\mathbb{T}}(s))S(s)x\,\Delta s=S(t)x-tx$ for all $x\in X, t\in\mathbb{T}_0^+$.

(c) Let $x,y\in X$. Then $x\in D(A)$ and $Ax=y$ if, and only if, for all $t\in\mathbb{T}_0^+$ we have:

$$\int_0^t(t-\sigma_{\mathbb{T}}(s))S(s)y\,\Delta s=S(t)x-tx.$$

Also, we prove a result which ensures that if $\mu(0)>0$ and the homogeneous Cauchy problem (0.12) has a solution, then A is a bounded linear map (see Theorem 5.34).

In the fifth section, we apply our results to study the inhomogeneous abstract second order Cauchy problem (0.13), obtaining a version of variation constant formula for the solution of the problem. Finally, in the last section we investigate the nonlinear abstract second order Cauchy problem on time scales (0.14). The results of this chapter are part of the paper [**70**].

CHAPTER 1

Preliminaries

This chapter contains some preliminaries used throughout the whole thesis. After presenting some notations and definitions, the operators of Riemann-Liouville fractional integration and Caputo fractional derivative are defined. Next, we introduce two special functions intimately related to fractional differential equations, in order to study the properties of the operator of Riemann-Liouville fractional differentiation. At the end of this chapter, we study the properties of the basic calculus on time scales.

Most notations used in this thesis are standard. Thus, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ denote the sets of natural, real, nonnegative and complex numbers, respectively.

We denote by $L^1_{loc}(\mathbb{R}_+)$ the set of locally integrable functions defined over $[0,\infty)$, and by $L^1_{loc}(\mathbb{R}_+, X)$ the Banach space of all locally (Bochner) integrable vector-valued functions. Let X, Y be Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, and we omit subscripts when there is no confusion. We denote by $\mathcal{B}(X,Y)$ the space of all bounded linear operators from X into Y, and by $\mathcal{B}(X)$ the space of all bounded linear operators from X into itself. If A is a linear operator on X, then $D(A)$, $\ker(A)$ and $\operatorname{ran}(A)$ denote respectively, domain of A, null space of A and range of A. Also, $\rho(A)$ and $\sigma(A)$ will denote the resolvent set and spectrum of A respectively, and $R(\lambda, A) = (\lambda I - A)^{-1}$ will represent the resolvent operator of A.

Definition 1.1. *For $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in L^1_{loc}(\mathbb{R}_+)$ we define the* ***finite convolution*** *between a and k as*

$$(a * k)(t) := \int_0^t k(t-s)a(s)\,ds.$$

Also we denote by a^{*n} the convolution of a with itself n-times, that is,

$$a^{*n}(t) = (a * a * \cdots * a)(t).$$

Definition 1.2. *For $\alpha > 0$, we define the function g_α as*

$$g_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0 \\ 0, & t \le 0 \end{cases}$$

where $\Gamma(\cdot)$ denotes the Gamma function. We also define $g_0(\cdot) = \delta_0$, the Dirac delta.

Definition 1.3. *We say that a resolvent family $\{R(t)\}_{t\geq 0}$ is* ***exponentially bounded*** *(or* ***of type*** *(M,ω)) if there exist constants $M\geq 0$ and $\omega\in\mathbb{R}$, such that*

$$\|R(t)\|\leq Me^{\omega t}\quad \textit{for all } t\geq 0.$$

Now, we give some definitons on fractional calculus.

Definition 1.4. *Given a continuous function f, the* ***Caputo fractional derivative*** *of order $\alpha>0$ is defined by*

$$D_t^\alpha f(t) := (g_{n-\alpha} * f^{(n)})(t) = \int_0^t g_{n-\alpha}(t-s) f^{(n)}(s)\, ds,$$

where $n=\lceil\alpha\rceil$ is the smallest integer greater that or equal to α.

For details in fractional calculus, we refer the reader to [**42**], [**46**], [**72**] and [**88**]. We notice that if $\alpha=m\in\mathbb{N}$, then $D_t^m = D^m = \frac{d^m}{dt^m}$.

Definition 1.5. [**88**] *Let $\alpha>0$. The α-order* ***Riemann-Liouville fractional integral*** *of u is defined by*

$$J^\alpha u(t) := \int_0^t g_\alpha(t-s) u(s)\, ds, \quad t\geq 0.$$

Also, we define $J^0 u(t) = u(t)$. Because of the convolution properties, the integral operators $\{J^\alpha\}_{\alpha\geq 0}$ satisfy the following semigroup law: $J^\alpha J^\beta = J^{\alpha+\beta}$ for all $\alpha,\beta\geq 0$.

The Caputo derivative operator D_t^α satisfies

$$\begin{aligned} D_t^\alpha J^\alpha u(t) &= u(t),\\ J^\alpha D_t^\alpha u(t) &= u(t) - \sum_{k=0}^{n-1} u^{(k)}(0) g_{k+1}(t), \end{aligned}$$

where $n=\lceil\alpha\rceil$.

For more detailed results on fractional calculus and fractional differential equations, we refer to [**1**], [**2**], [**4**], [**46**], [**59**], [**77**], [**84**], [**88**] and references therein.

Definition 1.6. *The* ***Laplace transform*** *of $f\in L^1_{loc}(\mathbb{R}_+,X)$ is defined by*

$$\widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\, dt, \quad \mathrm{Re}(\lambda)>\omega,$$

whenever the integral is absolutely convergent for $\mathrm{Re}(\lambda)>\omega$.

The reader can find in [**8**, Chapter 2] properties of the Laplace Transform for Bochner integrals. Also, we have the following properties for the fractional derivatives:

$$\widehat{D_t^{\alpha}u}(\lambda) \;=\; \lambda^{\alpha}\widehat{u}(\lambda) - \sum_{k=0}^{n-1} u^{(k)}(0)\lambda^{\alpha-1-k},$$

where $n = \lceil \alpha \rceil$ and $\lambda \in \mathbb{C}$.

Now, we present some basic concepts and properties about time scales which will be essential to prove the main results. The reader can find more details in [**17**], [**18**].

A ***time scale*** $\mathbb{T}$ is a closed and nonempty subset of $\mathbb{R}$. For every $t \in \mathbb{T}$, we define the ***forward*** and ***backward jump operators*** $\sigma_{\mathbb{T}}, \rho_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$, respectively, by:

$$\sigma_{\mathbb{T}}(t) = \inf\{s \in \mathbb{T} : s > t\} \;\text{ and }\; \rho_{\mathbb{T}}(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma_{\mathbb{T}}(t) > t$, we say that $t \in \mathbb{T}$ is ***right-scattered*** . If $t < \sup \mathbb{T}$ and $\sigma_{\mathbb{T}}(t) = t$, then t is called ***right-dense*** . Analogously, if $\rho_{\mathbb{T}}(t) < t$, we say that $t \in \mathbb{T}$ is ***left-scattered***, whereas if $t > \inf \mathbb{T}$ and $\rho_{\mathbb{T}}(t) = t$, then t is ***left-dense***. Also, we define the ***graininess function*** $\mu_{\mathbb{T}}(t) : \mathbb{T} \to [0, \infty)$ by $\mu_{\mathbb{T}}(t) := \sigma_{\mathbb{T}}(t) - t$.

We will denote a closed interval in $\mathbb{T}$ by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$, where $a, b \in \mathbb{T}$. Similarly, we can define the open intervals and half-open intervals, among others.

Definition 1.7. *A function $f : \mathbb{T} \to X$ is called* ***regulated*** *if its right-sided limit exists at right-dense points in $\mathbb{T}$, and its left-sided limit exists at left-dense points in $\mathbb{T}$.*

Definition 1.8. *A function $f : \mathbb{T} \to X$ is called* ***rd-continuous*** *if it is continuous at right-dense points in $\mathbb{T}$, and its left-sided limit exists at left-dense points in $\mathbb{T}$.*

We denote the class of all rd-continuous functions $f : \mathbb{T} \to X$ by $\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbb{T}, X)$.

To be able to remember the definition of Δ-derivative, we need to recall the concept of the set $\mathbb{T}^{\kappa}$, which is defined by:

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \setminus \{m\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximum } m, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

Definition 1.9. *For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$, we define the* ***delta derivative*** *of y to be the number (if it exists) $y^{\Delta}(t)$ with the following property: given $\epsilon > 0$, there exists a neighborhood U of t such that*

$$|y(\sigma_{\mathbb{T}}(t)) - y(s) - y^{\Delta}(t)(\sigma_{\mathbb{T}}(t) - s)| \leq \epsilon |\sigma_{\mathbb{T}}(t) - s|,$$

for all $s \in U$.

The delta-integral on time scales satisfies the same basic properties as the classical Riemann integral.

Theorem 1.10. [**17**, Theorem 1.75] *If $f \in \mathcal{C}_{rd}$ and $t \in \mathbb{T}^\kappa$, then*

$$\int_t^{\sigma_{\mathbb{T}}(t)} f(s)\,\Delta s = \mu_{\mathbb{T}}(t) f(t).$$

Theorem 1.11. [**17**, Theorem 1.77] *If $f, g \in \mathcal{C}_{rd}$, then*

(1) $\displaystyle\int_a^b f(\sigma_{\mathbb{T}}(t))g^\Delta(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\,\Delta t.$

(2) $\displaystyle\int_a^b f(t)g^\Delta(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma_{\mathbb{T}}(t))\,\Delta t.$

Next, we recall the notions and the results concerning Hilger complex plane and generalized exponential function. All the results can be found in [**17**].

Definition 1.12. *For $h > 0$, we define the* ***Hilger complex numbers****, the* ***Hilger real axis****, the* ***Hilger alternating axis*** *and the* ***Hilger imaginary circle*** *by:*

$$\begin{aligned}
\mathbb{C}_h &= \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}, \\
\mathbb{R}_h &= \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z > -\frac{1}{h} \right\}, \\
\mathbb{A}_h &= \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z < -\frac{1}{h} \right\}, \\
\mathbb{I}_h &= \left\{ z \in \mathbb{C}_h : \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\},
\end{aligned}$$

respectively. For $h = 0$, let $\mathbb{C}_0 = \mathbb{C}$, $\mathbb{R}_0 = \mathbb{R}$, $\mathbb{I}_0 = i\mathbb{R}$ and $\mathbb{A}_0 = \emptyset$.

Definition 1.13. *For $h > 0$ and $z \in \mathbb{C}_h$, we define the* ***Hilger real part*** *of z by:*

$$\mathrm{Re}_h(z) := \frac{|zh + 1| - 1}{h}.$$

We present the following notation used in Chapter 5:

$$\mathrm{Re}_\mu(\lambda)(t) := \mathrm{Re}_{\mu_{\mathbb{T}}(t)}(\lambda),$$

and $\mathrm{Re}_0(z) := \mathrm{Re}(z)$ in the usual sense.

Theorem 1.14. *If we define the circle plus addition $\oplus$ on $\mathbb{C}_h$ by:*

$$z \oplus w = z + w + zwh,$$

then $(\mathbb{C}_h, \oplus)$ is an Abelian group.

Definition 1.15. *If $z \in \mathbb{C}_h$, the **additive inverse** of z under the operation $\oplus$ is:*

$$\ominus z := \frac{-z}{1+zh},$$

*and we define the **circle minus substraction** $\ominus$ on $\mathbb{C}_h$ by:*

$$z \ominus w = z \oplus (\ominus w).$$

Definition 1.16. *We say that a function $p : \mathbb{T} \to \mathbb{R}$ is **regressive** provided that $1 + \mu_{\mathbb{T}}(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$.*

Definition 1.17. *For $p \in \mathcal{R}$, we define the **generalized exponential function** by:*

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu_{\mathbb{T}}(r)} p(r)\,\Delta r\right) \text{ for } s,t \in \mathbb{T}.$$

Here, the cylinder transformation ξ_h is defined by:

$$\xi_h(z) = \frac{1}{h}\log(1+zh),$$

where log *denotes the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.*

Finally, we present properties of the generalized exponential function on time scales.

Theorem 1.18. [**17**, Theorem 2.36] *If $p, q \in \mathcal{R}$, then:*

(1) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(2) $e_p^{\sigma}(t,s) = e_p(\sigma_{\mathbb{T}}(t), s) = (1+\mu(t)p(t))e_p(t,s)$;
(3) $\dfrac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;
(4) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(5) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(6) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
(7) $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$
(8) $\left(\dfrac{1}{e_p(t,s)}\right)^{\Delta} = -\dfrac{p(t)}{e_p(\sigma_{\mathbb{T}}(t), s)}$.

Lemma 1.19. [**54**, Lemma 5.1] *Let $\alpha > 0$, then for any fixed $s \in \mathbb{T}$, the following property is fulfilled:*

$$e_{\ominus\alpha}(t,s) \to 0 \quad as \quad t \to \infty.$$

CHAPTER 2

Norm Continuity

One-parameter strongly continuous families $\{R(t)\}_{t\geq 0}$ of bounded operators, defined on a Banach space, are useful instruments in the study of wide classes of abstract evolution equations, because of their important role in the determination of useful criteria for the existence of solutions to nonlinear partial differential equations, modeled as an abstract evolution equation on some vector-valued space of functions. In this Chapter, we give a complete answer to the question raised by some authors, studying a wide class of families of bounded operators named (a,k)-regularized resolvent families. We show conditions which ensure the uniform continuity of (a,k)-regularized resolvent families $R(t)$ for $t \geq 0$. Namely, it is shown that on certain Banach spaces (e.g., $L^\infty(S,\Sigma,\mu)$), each exponentially bounded (a,k)-regularized resolvent family is in fact uniformly continuous for $t \geq 0$. Also, we characterize families $R(t)$ such that $R(t) - k(t)I$ is a compact operator for all $t > 0$. Finally, we prove that in Hilbert spaces the uniform continuity of $R(t)$ for $t > 0$ (also called immediate norm continuity) is equivalent to the decay to zero of $\widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}$ along some imaginary axis. Our results widely generalize known properties for strongly continuous semigroups and cosine families of bounded operators.

1. (a,k)-regularized resolvent families

Let X be a Banach space, and A be a closed linear operator defined on X.

Definition 2.1. [**57**] *Let $k \in C(\mathbb{R}_+), k \neq 0$ and $a \in L^1_{loc}(\mathbb{R}_+), a \neq 0$ be given. A strongly continuous family $\{R(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called (a,k)-**regularized resolvent family** on X having A as its generator, if the following properties hold:*

(1) $R(0) = k(0)I$;

(2) $R(t)x \in D(A)$ *and* $R(t)Ax = AR(t)x$, *for all* $x \in D(A)$ *and* $t \geq 0$;

(3) $R(t)x = k(t)x + \int_0^t a(t-s)AR(s)x\,ds$, *for* $x \in D(A)$ *and* $t \geq 0$.

This notion generalizes the notions of C_0-semigroups, n-times integrated semigroups, k-convoluted semigroups, n-times integrated cosine families , n-times resolvent families, α-resolvent families, among others. For example, if $a(t) = k(t) \equiv 1$ for all $t \geq 0$ and if

$a(t) = t, k(t) \equiv 1$ for all $t \geq 0$, then we obtain strongly continuous semigroups and cosine operator functions, respectively [**8**]. If $a(t) \equiv 1$ and $k(t) = \frac{t^n}{n!}$ for all $t \geq 0$, then $R(t)$ is an n-times integrated semigroup [**7**]. Taking $a \in L^1_{loc}(\mathbb{R}_+)$ and $k(t) \equiv 1$ for all $t \geq 0$ we have that $R(t)$ is a resolvent family, which are the central object to study in the theory of abstract Volterra equations [**79**]. Finally, if $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}(\alpha > 0)$ and $k(t) \equiv 1$ for all $t \geq 0$, then $R(t)$ corresponds to an α-resolvent family [**50**].

It is well-known that if an (a, k)-regularized resolvent family exists, then it is unique [**57**]. Let $\{R(t)\}_{t\geq 0}$ be an (a, k)-regularized resolvent family with generator A such that

$$\|R(t)\| \leq Mk(t), \quad t \geq 0, \tag{2.1}$$

for some constant $M > 0$. Then, under certain hypothesis on the kernels a and k (see [**63**, Section 2] and references therein), we have

$$D(A) = \left\{ x \in X \,:\, \lim_{t\to 0^+} \frac{R(t)x - k(t)x}{(a * k)(t)} \text{ exists} \right\},$$

and

$$Ax = \lim_{t\to 0^+} \frac{R(t)x - k(t)x}{(a * k)(t)}. \tag{2.2}$$

We note that there is a one-to-one correspondence between (a, k)-regularized resolvent families and their generators.

For exponentially bounded (a, k)-regularized resolvent families, it is well known the following characterization.

Theorem 2.2. [**57**] *Let X be a Banach space and A be a closed and densely defined operator. The following assertions are equivalent:*

(1) *A is the generator of an (a, k)-regularized resolvent family of type (M, ω);*
(2) *For all $\lambda > \omega$, the resolvent set $\rho(A)$ contains the set $\left\{\frac{1}{\widehat{a}(\lambda)} : \lambda > \omega\right\}$ and*

$$\widehat{k}(\lambda)\,(I - \widehat{a}(\lambda)A)^{-1}\,x = \int_0^\infty e^{-\lambda t} R(t)x\,dt, \quad x \in X, \lambda > \omega.$$

Here, without loss of generality, we are assuming that a and k are Laplace transformable for $\lambda > \omega$.

We recall that a Banach space X is called a ***Grothendieck space*** if every weak* convergent sequence in X' converges weakly, where X' denotes the dual space of X.

Definition 2.3. *A Banach space X is said to have the* ***Dunford-Pettis property*** *if for all sequence $\{x_n\}_{n\geq 0}$ in X such that $x_n \to 0$ weakly in X and $x_n' \to 0$ weakly in X', we have $\langle x_n, x_n'\rangle \to 0$.*

For instance, the spaces $L^\infty(X,\Omega,\mu)$ where (X,Ω,μ) is a positive measure space, and $C(X)$ (where X is a compact $\sigma-$Stonian space) are Grothendieck spaces with the Dunford-Pettis property. Recall that X is ***Stonian*** if the closure of every open set is open, and it is ***σ-Stonian*** if the closure of every open F_σ-set is open. On the other hand, a Banach space E is ***injective*** if for every Banach space X and every subspace Y of X, each operator $T: Y \to E$ admits an extension $\widetilde{T}: X \to E$. Every injective Banach space is a Grothendieck space with the Dunford-Pettis property. Finally, a reflexive space does not have the Dunford-Pettis property, unless the space is finite dimensional.

Definition 2.4. [**79**] *Let $a \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable and $n \in \mathbb{N}$. The kernel $a(t)$ is called **n-regular** if there exists a constant $c > 0$ such that*

$$|\lambda^m \widehat{a}^{(m)}(\lambda)| \leq c|\widehat{a}(\lambda)|,$$

for all $Re(\lambda) > 0$ and $0 \leq m \leq n$.

For example, for $n \in \mathbb{N}$ fixed and $\alpha > 0$, the kernel $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ is n-regular. Also, for $a, b > 0$ and $n \in \mathbb{N}$ fixed, the kernel $a(t) = e^{bt}t^a$ is n-regular.

We finally recall the following result due to Lotz [**66**].

Theorem 2.5. [**66**, Theorem 10] *Let E be a Grothendieck space with the Dunford-Pettis property and let $(T_n) \subset \mathcal{B}(E)$ with $\lim_{n\to\infty} \|T_m(T_n - I)\| = 0$ for all $m \in \mathbb{N}$. If (T_n) tends to the identity in the strong operator topology, then $\lim_{n\to\infty} \|T_n - I\| = 0$. If, in addition, $\lim_{n\to\infty} \|(T_n - I)T_m\| = 0$ for every $m \in \mathbb{N}$, in particular, if all operators T_n commute, then it suffices to assume that (T_n) converges to the identity in the weak operator topology.*

Definition 2.6. *Let X be a complex Banach space. A strongly continuous family of bounded and linear operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be **uniformly continuous** if for all $s \geq 0$,*

$$\lim_{t\to s} \|S(t) - S(s)\|_{\mathcal{B}(X)} = 0. \tag{2.3}$$

This concept is also called ***norm-continuity*** for some authors ([**21**], [**29**], [**33**], [**69**]), but also it sometimes refers to the case that (2.3) holds for all $s > 0$. To distinguish between both cases, some authors say that $\{S(t)\}_{t\geq 0}$ is ***immediate norm continuous*** when refers to the continuity of $\{S(t)\}_{t\geq 0}$ in the uniform operator topology for $s > 0$.

2. Uniform continuity in L^∞ type spaces

In this section, we will assume that a and k are exponentially bounded functions, and hence Laplace transformable. Moreover, we suppose that $\widehat{k}(\lambda) \neq 0$ and $\widehat{a}(\lambda) \neq 0$ for all λ

sufficiently large. Our main result in this section shows that in certain classes of Banach spaces, the uniform continuity for (a,k)-regularized resolvent families is automatic. This result generalizes results of Lizama [**55**] and Lotz [**66**], and also provides new results. We will need two preliminary lemmata.

Lemma 2.7. *Let $\{R(t)\}_{t\geq 0}$ be a uniformly continuous (a,k)-regularized resolvent family with generator A. Assume that a and k are exponentially bounded functions, a is positive, and $|k(0)|\geq 1$. Then A must be a bounded operator and*

$$R(t) = k(t) + \sum_{n=1}^{\infty} A^n (a^{*n} * k)(t), \quad t \geq 0. \tag{2.4}$$

Proof. Fix $t>0$ and define

$$f(t) := \frac{1}{(1*a)(t)} \int_0^t a(t-s)R(s)\,ds.$$

Since $\{R(t)\}_{t\geq 0}$ is a uniformly continuous family, there exists $\delta>0$ such that for $0<s<\delta$, we have $\|R(s)-k(0)I\|<1$. Let $\tau\in(0,\delta)$ be fixed. Then,

$$\begin{aligned}
\left\| \frac{f(\tau)}{k(0)} - I \right\| &\leq \|f(\tau)-k(0)I\| \\
&= \left\| \frac{1}{(1*a)(\tau)} \int_0^\tau a(\tau-s)\big(R(s)-k(0)\big)\,ds \right\| \\
&< \frac{1}{(1*a)(\tau)} \int_0^\tau a(\tau-s)\,ds = 1.
\end{aligned}$$

Therefore, $\frac{1}{k(0)}f(\tau)$ is invertible on X. Let $x\in X$ be fixed. There exists $y\in X$ such that $x=f(\tau)y$. But, according to [**57**, Lemma 2.2] for $y\in X$ we have

$$f(\tau)y = \frac{1}{(1*a)(\tau)} \int_0^\tau a(\tau-s)R(s)y\,ds \in D(A).$$

Then, $D(A)=X$. Since A is closed, it implies that A is a bounded operator, proving the first assertion of the theorem.

From the hypothesis, we may assume that $a(t)\leq Me^{\lambda t}$ and $|k(t)|\leq Me^{\lambda t}$ for the same constants $M>0$ and $\lambda>0$. Denote $e_\lambda(t)=e^{\lambda t}, t\geq 0$ and observe that $e_\lambda^{*n}(t)=\frac{t^{n-1}}{(n-1)!}e_\lambda(t)$ for $n=2,3,...$ Hence, $|(a^{*n}*k)(t)|\leq M^{n+1}\frac{t^n}{n!}e_\lambda(t)$ for all $t\geq 0$ and $n\in\mathbb{N}$, and therefore

$$\sum_{n=1}^{\infty} \|A^n(a^{*n}*k)(t)\| \;\leq\; M\sum_{n=1}^{\infty} \|A\|^n M^n \frac{t^n}{n!} e_\lambda(t) \;=\; Me^{\|A\|Mt}e_\lambda(t).$$

This proves that the series in the right hand side of (2.4) converges. Define

$$S(t) := k(t) + \sum_{n=1}^{\infty} A^n (a^{*n} * k)(t), \quad t \geq 0.$$

It is easy to show that $S(t) = k(t) + A(a * S)(t)$. Now, by uniqueness, we conclude that $S(t) = R(t)$. It proves the formula (2.4). □

The following Lemma provides a converse of the above property. It is also new in the context of (a,k)-regularized resolvent families with $k \neq 1$.

Lemma 2.8. *Let $\{R(t)\}_{t\geq 0}$ be an strongly continuous (a,k)-regularized resolvent family with generator A. Assume that a and k are exponentially bounded, a is positive and $k \in C^1(\mathbb{R})$. If A is bounded, then $\{R(t)\}_{t\geq 0}$ is uniformly continuous.*

Proof. In order to see that the resolvent family $R(t)$ is uniformly continuous, we take $0 < t < s$ and observe that

$$\|R(t) - R(s)\| \;\; \leq \;\; \|k(t) - k(s)\| + \sum_{n=1}^{\infty} \|A\|^n |(a^{*n} * k)(t) - (a^{*n} * k)(s)|.$$

Since $k \in C^1(\mathbb{R})$, there exists $p \in C(\mathbb{R})$ such that $k(t) = \displaystyle\int_0^t p(r)\,dr + k(0)$. Hence,

$$\begin{aligned}
\|R(t) - R(s)\| \;\; &\leq \;\; \|k(t) - k(s)\| + \sum_{n=1}^{\infty} \|A\|^n \left| \big(a^{*n} * (p * 1)\big)(s) - \big(a^{*n} * (p*1)\big)(t) \right| \\
&\quad + |k(0)| \sum_{n=1}^{\infty} \|A\|^n \left| \big(a^{*n} * 1\big)(s) - \big(a^{*n} * 1\big)(t) \right| \\
&\leq \;\; \|k(t) - k(s)\| + \sum_{n=1}^{\infty} \|A\|^n \left| \int_t^s (a^{*n} * p)(v)\,dv \right| \\
&\quad + |k(0)| \sum_{n=1}^{\infty} \|A\|^n \left| \int_t^s a^{*n}(v)\,dv \right| \\
&\leq \;\; \|k(t) - k(s)\| + |t - s| \sum_{n=1}^{\infty} \|A\|^n \bigg(\sup_{t \leq v \leq s} |(a^{*n} * p)(v)| \\
&\quad + |k(0)| \sup_{t \leq v \leq s} |a^{*n}(v)| \bigg) \\
&\leq \;\; \|k(t) - k(s)\| + |t - s| \sum_{n=1}^{\infty} \|A\|^n \sup_{t \leq v \leq s} \left(\left| \int_0^v a^{*n}(\tau) p(v - \tau)\,d\tau \right| \right.
\end{aligned}$$

$$+|k(0)| \cdot |a^{*n}(v)|\Big).$$

Note that $|a^{*n}(\tau)| \leq \dfrac{M^n\tau^{n-1}}{(n-1)!}$ for $0 \leq \tau \leq s$, where $M := \sup\limits_{0\leq\tau\leq s} |a(\tau)|$. Hence, we obtain

$$\begin{aligned} \|R(t) - R(s)\| &\leq \|k(t) - k(s)\| + |t-s| \sum_{n=0}^{\infty} \frac{(\|A\|\, M)^{n+1} s^n}{n!} \sup_{t\leq v\leq s} \big(|k(v)| + |k(0)|\big) \\ &\leq \|k(t) - k(s)\| + |t-s| \cdot \sup_{t\leq v\leq s} \big(|k(v)| + |k(0)|\big) \cdot \|A\| M \cdot e^{\|A\|Ms}. \end{aligned}$$

This proves the lemma. □

Our main result in this section is the following theorem. It shows an interesting generalization of a result of Lotz in case that the kernel a is dominated by the kernel k in the sense of their Laplace transforms.

Theorem 2.9. *Let X be a Grothendieck space with the Dunford-Pettis property. Suppose that A generates an exponentially bounded (a,k)-regularized resolvent family $\{R(t)\}_{t\geq 0}$ on X. Assume that a and k are exponentially bounded, a is positive, $k \in C^1(\mathbb{R})$, and suppose that there exists a constant $M > 0$ such that $|\widehat{a}(\lambda)| \leq M|\widehat{k}(\lambda)|$ for all λ large enough. Then, $\{R(t)\}_{t\geq 0}$ is uniformly continuous on X.*

Proof. By hypothesis, there exists $\omega > 0$ such that the functional equation

$$\widehat{R}(\lambda)\widehat{R}(\mu) = \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)} \frac{1}{\frac{1}{\widehat{a}(\lambda)} - \frac{1}{\widehat{a}(\mu)}} \widehat{R}(\mu) - \frac{\widehat{k}(\mu)}{\widehat{a}(\mu)} \frac{1}{\frac{1}{\widehat{a}(\lambda)} - \frac{1}{\widehat{a}(\mu)}} \widehat{R}(\lambda), \quad \lambda, \mu > \omega$$

holds, see [**63**, Equation 3.8]. Here $\widehat{R}(\lambda) = \widehat{k}(\lambda)\,(I - \widehat{a}(\lambda)A)^{-1}$ by Theorem 2.2. Then, we get the formula

$$\big(\widehat{a}(\mu) - \widehat{a}(\lambda)\big)\big(\widehat{R}(\lambda) - \widehat{k}(\lambda)\big)\widehat{R}(\mu) = \widehat{a}(\lambda)\widehat{k}(\lambda)\widehat{R}(\mu) - \widehat{a}(\lambda)\widehat{k}(\mu)\widehat{R}(\lambda), \tag{2.5}$$

for all $\lambda, \mu > \omega$. Let us define $T(\lambda) := \frac{1}{\widehat{k}(\lambda)}\widehat{R}(\lambda) = (I - \widehat{a}(\lambda)A)^{-1}$, $\lambda > \omega$. Replacing in (2.5), we obtain the identity

$$\widehat{a}(\mu)(T(\lambda) - I)T(\mu) \;=\; \widehat{a}(\lambda)T(\mu) - \frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}\widehat{R}(\lambda) + \frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}\widehat{R}(\lambda)T(\mu) - \widehat{a}(\lambda)T(\mu).$$

Therefore,

$$\|\widehat{a}(\mu)(T(\lambda) - I)T(\mu)\| \leq \|\widehat{a}(\lambda)T(\mu)\| + \left|\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}\right| \|\widehat{R}(\lambda)\| + \left|\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}\right| \|\widehat{R}(\lambda)T(\mu)\| + \|\widehat{a}(\lambda)T(\mu)\|$$

and, by hypothesis and the fact that $|\widehat{a}(\lambda)| \to 0$ and $\|\widehat{R}(\lambda)\| \to 0$ as $\lambda \to +\infty$, we obtain for each $\mu > \omega$ fixed, that

$$\lim_{\lambda \to +\infty} \|\widehat{a}(\mu)(T(\lambda) - I)T(\mu)\| = 0.$$

In particular, we have that

$$\lim_{\lambda \to +\infty} \|(T(\lambda) - I)T(\mu)\| = 0,$$

for μ fixed. From Theorem 2.5, there exists $\lambda_1 > \omega$ such that $T(\lambda_1)$ is invertible on X, that is, $T(\lambda_1)^{-1} \in B(X)$. Therefore, A is a bounded operator and, by Lemma 2.8, we conclude that the family $\{R(t)\}_{t\geq 0}$ is uniformly continuous. $\square$

In case $a(t) = k(t) = 1$, we recover the following result due to Lotz [**66**].

Corollary 2.10. *Let X be a Grothendieck space with the Dunford-Pettis property. Then, every strongly continuous one-parameter semigroup of operators on X is uniformly continuous.*

In case $k(t) = 1$ and $a \in L^1_{loc}(\mathbb{R}_+)$ is a Laplace transformable kernel, we recover [**55**, Theorem 3.2] as follows.

Corollary 2.11. *Let X be a Grothendieck space with the Dunford-Pettis property. Then, every strongly continuous resolvent family of operators on X is uniformly continuous.*

We consider for $\alpha > 0$ the function $g_\alpha(t)$ (see Definition 1.2), whose Laplace transform is $\widehat{g_\alpha}(\lambda) = \lambda^{-\alpha}$. Next, we consider the fractional abstract Cauchy problem

$$D_t^\alpha u(t) = Au(t), \quad t > 0, \tag{2.6}$$

where A is a closed linear operator defined on a Banach space X, and D_t^α denotes the Caputo fractional derivative (see Definition 1.4). Recall that if A generates a $(g_\alpha,1)$-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$, then the solution to (2.6) is given by $u(t) = S_\alpha(t)u_0$ whenever $u_0 \in D(A)$. See [**12**].

Corollary 2.12. *Let X be a Grothendieck space with the Dunford-Pettis property. Let $\alpha > 1$ and suppose that A generates an exponentially bounded $(g_\alpha, 1)$-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ on X. Then, $\{S_\alpha(t)\}_{t\geq 0}$ is uniformly continuous on X.*

Remark 2.13. It is known that the above result holds for $\alpha > 2$ without restriction on the Banach space X. See [**12**, Corollary 3.4].

It is interesting to observe that for integral resolvents, i.e. in case $a = k$, we also obtain automatically uniform continuity for the class of Grothendieck spaces with the Dunford-Pettis property. An special case is $a(t) = k(t) = t$ corresponding to a sine family [**8**, Section 3.15].

3. (a,k)-regularized families with compact generator

In Section 2, it was proved that if an operator A is bounded, then the (a,k)-resolvent family generated by A is given by

$$R(t) = k(t)I + \sum_{n=1}^{\infty} A^n (a^{*n} * k)(t), \ t > 0.$$

In this section, we develop some aspects of (a,k)-regularized resolvent families of bounded linear operators on a Banach space which have the property of being near $k(t)$ times the identity (i.e., $R(t) - k(t)I$ is compact for some positive value of t). First results on such property are due to Cuthbert [**26**], Henríquez [**40**], Lizama [**55**] and Lutz [**67**]. The following result generalizes all the above mentioned papers.

Theorem 2.14. *Let $\{R(t)\}_{t\geq 0}$ be an strongly continuous (a,k)-resolvent family of type (M,ω) with generator A. Suppose that the kernels a,k are exponentially bounded functions, a is positive, and $|k(0)| \geq 1$. Then, the following assertions are equivalent:*

(1) *$R(t) - k(t)I$ is compact for all $t > 0$.*
(2) *A is a compact operator.*

Proof. Suppose that A is compact. Since the set of compact operators is a closed subspace of $\mathcal{B}(X)$, we have by (2.4) that

$$R(t) - k(t)I = \sum_{n=1}^{\infty} k(t)^{*n} A^n = \lim_{N\to\infty} \sum_{n=1}^{N} k(t)^{*n} A^n,$$

and hence, $R(t) - k(t)I$ is a compact operator.

Conversely, suppose that $R(t) - k(t)I$ is compact for all $t > 0$. According to the hypothesis, we have that for all $\mathrm{Re}(\lambda) > \omega$, the operator $(I - \widehat{a}(\lambda)A)$ is invertible and

$$\int_0^{\infty} e^{-\lambda t} R(t)\,dt = \widehat{R}(\lambda) = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}$$

for $\mathrm{Re}(\lambda) > \omega$. For $x \in X$, define $H(\lambda)x := \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}x$. We have

$$\begin{aligned} \lambda H(\lambda)x - \lambda\widehat{k}(\lambda)x &= \int_0^{\infty} \lambda e^{-\lambda t} R(t)x\,dt - \lambda\widehat{k}(\lambda)x \\ &= \int_0^{\infty} \lambda e^{-\lambda t} R(t)x\,dt - \int_0^{\infty} \lambda e^{-\lambda t} k(t)x\,dt \\ &= \int_0^{\infty} \lambda e^{-\lambda t} \big(R(t) - k(t)\big)x\,dt. \end{aligned}$$

Hence by [**85**, Corollary 2.3], we obtain that $\lambda H(\lambda)x - \lambda\widehat{k}(\lambda)x$ is a compact operator. From the identity

$$\lambda H(\lambda)x - \lambda\widehat{k}(\lambda) = -\lambda\widehat{k}(\lambda)\Big(I - \frac{H(\lambda)}{\widehat{k}(\lambda)}\Big),$$

we obtain that $\Big(I - \frac{H(\lambda)}{\widehat{k}(\lambda)}\Big)$ is a compact operator, and this implies that $\operatorname{ran}\Big(\frac{H(\lambda)}{\widehat{k}(\lambda)}\Big)$ is closed. On the other hand, $\operatorname{ran}\Big(\frac{H(\lambda)}{\widehat{k}(\lambda)}\Big) = D(A)$ is dense on X. Therefore $D(A) = \overline{D(A)} = X$, concluding that A is a bounded operator. Next, we observe that the following identity holds:

$$A = \big(\lambda H(\lambda)x - \lambda\widehat{k}(\lambda)I\big)\frac{(I - \widehat{a}(\lambda)A)}{\lambda\widehat{k}(\lambda)\widehat{a}(\lambda)},$$

and this implies that A is a compact operator. The proof is complete. □

4. Immediate norm continuity in Hilbert spaces

Since reflexive spaces do not have the Dunford-Pettis property, we cannot apply Theorem 2.9 to characterize the uniform continuity for $t > 0$ of (a, k)-regularized resolvent families in general Banach spaces. Of course, in this case the generator A is not necessarily bounded. This is one of the reasons why a characterization only in terms of the generator is desirable but unfortunately it is difficult to obtain in general Banach spaces. However, we can obtain a positive result extending an important result due to O. El Mennaoui and K.-J. Engel [**29**] valid for the case $a(t) \equiv k(t) \equiv 1$ to the case of (a, k)-regularized resolvent families in Hilbert spaces (see also [**56**] for the case of resolvent families).

Let A be a closed operator with domain $D(A)$ densely defined, $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$. Moreover, suppose a, k are 2-regular kernels.

Definition 2.15. *Let $k \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable. We say that k is an* ***admissible kernel*** *if there exists $\lim_{\lambda\to i\tau} \widehat{k}(\lambda) = \widehat{k}(i\tau)$ for all $|\tau| \geq 1$, and satisfies the following condition:*

(H) *There exists a constant $M > 0$ such that, for all $|\tau| \geq 1$,*

$$\frac{1}{|\tau\,\widehat{k}(i\tau)|} \leq M.$$

Example 2.16. For instance, the function $k(t) = g_\alpha(t)$ is an admissible kernel for $0 < \alpha \leq 1$, but fails to be admissible for $\alpha > 1$. Moreover, is easy to check that $k(t)$ is 2-regular (see Definition 2.4).

To prove our main result in this section, we need the following lemmata. The first Lemma, corresponds to a general result for strongly continuous families of bounded operators.

Lemma 2.17. *Let $\{R(t)\}_{t\geq 0}$ be a strongly continuous family of type (M,σ) defined in a Hilbert space H. Then for any $x\in H$ and $\omega>\sigma$, $\|\widehat{R}(\omega+iu)x\|$ and $\|\widehat{R}(\omega+iu)^*x\|$ are in $L^2(\mathbb{R},H)$, viewed as functions of $u\in\mathbb{R}$.*

Proof. Without loss of generality, we can suppose that $\sigma\geq 0$. Let $\omega>\sigma$ be given and define $R_1(t):=e^{-\omega t}R(t)$. Then $\|R_1(t)\|\leq Me^{-(\omega-\sigma)t}$ for $t\geq 0$. Let $x\in H$ be fixed, and note that $\chi_{[0,\infty)}(\cdot)R_1(\cdot)x$ is in $L^2(\mathbb{R},H)$, where $\chi_{[0,\infty)}(\cdot)$ denotes the characteristic function. In fact, we have

$$\|\chi_{[0,\infty)}(\cdot)R_1(\cdot)x\|_2^2 \;\leq\; \int_0^\infty \|Me^{-(\omega-\sigma)t}x\|^2\,dt \leq \frac{M^2\|x\|^2}{2(\omega-\sigma)}. \tag{2.7}$$

On the other hand, because $\{R(t)\}_{t\geq 0}$ is a family of type (M,σ), its Laplace transform $\widehat{R}(\lambda)$ is well-defined for all $\operatorname{Re}(\lambda)>\sigma$ and is holomorphic there. Hence, we have for all $x\in H$ and $s\in\mathbb{R}$,

$$\begin{aligned}\widehat{R}(\omega+is)x &= \int_0^\infty e^{-(\omega+is)t}R(t)x\,dt \;=\; \int_0^\infty e^{-ist}R_1(t)x\,dt\\ &= \int_{-\infty}^\infty e^{-ist}\chi_{[0,\infty)}(t)R_1(t)x\,dt \;=\; \mathcal{F}\big(\chi_{[0,\infty)}(\cdot)R_1(\cdot)\big)(s).\end{aligned}$$

It follows from (2.7) and the Plancherel Theorem that $\widehat{R}(\omega+i(\cdot))x\in L^2(\mathbb{R},H)$. Analogously, we can prove that $\widehat{R}(\omega+i(\cdot))^*x\in L^2(\mathbb{R},H)$. This proves the lemma. □

Lemma 2.18. *Let $a,k\in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable and A be a closed linear operator defined on a Banach space X. Assume that $H(\lambda):=\widehat{k}(\lambda)(I-\widehat{a}(\lambda)A)^{-1}$ exists for all $\operatorname{Re}(\lambda)>\omega$. Then, there exist functions $f_i(\lambda)$, $i=1,2$ and $h_j(\lambda)$, $j=1,2,3$, such that:*

1. $H'(\lambda)=f_1(\lambda)H(\lambda)+f_2(\lambda)H(\lambda)^2$,
2. $H''(\lambda)=h_1(\lambda)H(\lambda)+h_2(\lambda)H(\lambda)^2+h_3(\lambda)H(\lambda)^3$,

for all $\operatorname{Re}(\lambda)>\omega$.

Proof. A computation shows that for all $\operatorname{Re}(\lambda)>\omega$, we have

$$f_1(\lambda)=\frac{\widehat{k}'(\lambda)}{\widehat{k}(\lambda)}-\frac{\widehat{a}'(\lambda)}{\widehat{a}(\lambda)},\qquad f_2(\lambda)=\frac{\widehat{a}'(\lambda)}{\widehat{k}(\lambda)\widehat{a}(\lambda)}, \tag{2.8}$$

and

$$h_1(\lambda) = \frac{\widehat{k}''(\lambda)}{\widehat{k}(\lambda)}-\frac{2\widehat{k}'(\lambda)\widehat{a}'(\lambda)}{\widehat{k}(\lambda)\widehat{a}(\lambda)}+\frac{2\widehat{a}'(\lambda)^2}{\widehat{a}(\lambda)^2}-\frac{\widehat{a}''(\lambda)}{\widehat{a}(\lambda)}, \tag{2.9}$$

$$h_2(\lambda) = \frac{\widehat{a}'(\lambda)\widehat{k}'(\lambda)}{\widehat{k}(\lambda)^2\widehat{a}(\lambda)}-\frac{4\widehat{a}'(\lambda)^2}{\widehat{k}(\lambda)\widehat{a}(\lambda)^2}+\frac{\widehat{a}''(\lambda)}{\widehat{k}(\lambda)\widehat{a}(\lambda)}, \tag{2.10}$$

$$h_3(\lambda) \quad = \quad \frac{2\widehat{a}'(\lambda)^2}{\widehat{k}(\lambda)^2\widehat{a}(\lambda)^2}. \tag{2.11}$$

This proves the lemma. □

Lemma 2.19. *Let $a, k \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable and 2-regular, and suppose that k is an admissible kernel. Then, there exists a constant $M > 0$ such that:*

(1) $|\lambda f_1(\lambda)| < M$ *and* $|f_2(\lambda)| < M$ *for all* $\mathrm{Re}(\lambda) > \omega$*;*
(2) $\sup_{|\tau|\geq N} |h_3(s+i\tau)| < M$ *for all* $s > \omega$ *and* $N \geq 1$*;*
(3) $\int_{|\tau|\geq N} |h_j(s+i\tau)|^j \, d\tau < M$ *for all* $s > \omega$ *and* $N \geq 1$, $j = 1, 2$.

Proof. It is a direct consequence of formulas (2.8) - (2.11). □

Lemma 2.20. [**29**] *Let X be a Banach space and let $R : [0, \infty) \to X$ be a function which is continuous for $t > 0$. If there exist $M > 0, \omega \in \mathbb{R}$ such that $\|R(t)\| \leq Me^{\omega t}$, then for every $\mu > \omega$,*

$$\lim_{|\tau|\to\infty} \|\widehat{R}(\mu + i\tau)\| = 0.$$

Our main result in this section is the following characterization. It extends the main result in [**56**, Theorem 2.2], proved in the case $k(t) \equiv 1$. See also [**86**] for the same characterization in case of C_0-semigroups, i.e., $k(t) \equiv 1$ and $a(t) \equiv 1$.

Theorem 2.21. *Let A be a closed linear operator defined in a Hilbert space H with dense domain $D(A)$. Assume that A generates a strongly continuous (a, k)-regularized resolvent $R(t)$ of type (M, ω), with $M > 0, \omega \in \mathbb{R}$. Also, suppose that $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$ are Laplace transformable, 2-regular kernels, and that k is admissible. Then, the following conditions are equivalent:*

(1) $\{R(t)\}_{t\geq 0}$ *is continuous in* $\mathcal{B}(H)$ *for* $t > 0$,
(2) $\lim_{|\tau|\to\infty} ||\widehat{k}(s+i\tau)\,(I - \widehat{a}(s+i\tau)A)^{-1}\,|| = 0$ *for some* $s > \omega$.

Proof. (1) $\implies$ (2). It follows from Lemma 2.20.

(2) $\implies$ (1). Let $x \in H$ be fixed and $\mu > \omega$. Since $\|R(t)e^{-\mu t}x\| \leq Me^{-(\mu-\omega)t}\|x\|$, the function $t \mapsto \chi_{[0,\infty)}(t)R(t)e^{-\mu t}$ is in $L^2(\mathbb{R}, H)$ for all $\mu > \omega$ (compare with the inequality (2.7)). Since H is a Hilbert space, by Plancherel theorem is well known that the Fourier transform is an unitary operator on $L^2(\mathbb{R}, H)$, thus we obtain

$$\mathcal{F}\big(\chi_{[0,\infty)}(\cdot)R(\cdot)e^{-\mu\cdot}x\big) = \widehat{R}(\mu + i\tau)x,$$

and hence, for $t > 0$ and each $x \in H$,

$$R(t)e^{-\mu t}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t}\widehat{R}(\mu + i\tau)x\,d\tau. \tag{2.12}$$

Clearly, the resolvent $R(t)$ is continuous in $\mathcal{B}(H)$ for $t > 0$ if and only if $R(t)e^{-\mu t}$ is continuous in $\mathcal{B}(H)$ for $t > 0$. Next, note that for each $x \in H$ we have

$$\widehat{R}(\mu + i\tau)x = \widehat{k}(\mu + i\tau)\left(I - \widehat{a}(\mu + i\tau)A\right)^{-1}x,$$

and observe that if $|\tau| \to \infty$, then we get $\widehat{a}(\mu + i\tau) \to 0$ and $\widehat{k}(\mu + i\tau) \to 0$, whence $\lim_{|\tau|\to\infty} \widehat{R}(\mu + i\tau)x = 0$. Applying this to (2.12) and integrating by parts, we have

$$R(t)e^{-\mu t}x = \frac{-1}{2\pi t}\int_{-\infty}^{\infty} e^{i\tau t}\widehat{R}'(\mu + i\tau)x\,d\tau.$$

for $t > 0$ and $x \in H$. Now, by Lemma 2.18 (1) and Lemma 2.19 (1) with $H(\lambda) := \widehat{R}(\lambda)$, we have $\lim_{|\tau|\to\infty} \widehat{R}'(\mu + i\tau) = 0$, whence, again integrating by parts, we get

$$R(t)e^{-\mu t}x = \frac{1}{2\pi t^2}\int_{-\infty}^{\infty} e^{i\tau t}\widehat{R}''(\mu + i\tau)x\,d\tau, \quad x \in H,\ t > 0. \tag{2.13}$$

Next, we show that the operator family $\{t^2 R(t)e^{-\mu t}\}_{t>0}$ is continuous in $\mathcal{B}(H)$ for $t > 0$: Indeed, formula (2.13) shows

$$\begin{aligned}
\|t^2R(t)e^{-\mu t}x - s^2R(s)e^{-\mu s}x\| &= \frac{1}{\pi}\Big\|\int_{-\infty}^{\infty}(e^{i\tau t} - e^{i\tau s})\widehat{R}''(\mu + i\tau)x\,d\tau\Big\| \\
&\le \frac{1}{\pi}\Big\|\int_{|\tau|\ge N}(e^{i\tau t} - e^{i\tau s})\widehat{R}''(\mu + i\tau)x\,d\tau\Big\| \\
&\quad + \frac{1}{\pi}\int_{|\tau|\le N}|e^{i\tau t} - e^{i\tau s}|\|\widehat{R}''(\mu + i\tau)x\|\,d\tau \\
&=: I_1(N) + I_2(N).
\end{aligned}$$

First, we estimate $I_1(N)$: Let $\epsilon > 0$ and take $x^* \in H$. Using Lemma 2.18 (2), and the Cauchy-Schwarz and Hölder inequalities, we have

$$\begin{aligned}
|\langle I_1(N), x^*\rangle| \le\ & \sup_{|\tau|\ge N}\|\widehat{R}(\mu + i\tau)\| \cdot \bigg(\int_{|\tau|\ge N}|h_1(\mu + i\tau)|\,d\tau \cdot \|x\|\|x^*\| \\
& + \Big(\int_{|\tau|\ge N}\|\widehat{R}(\mu + i\tau)x\|^2\,d\tau\Big)^{1/2}\Big(\int_{|\tau|\ge N}\|h_2(\mu + i\tau)x^*\|^2\,d\tau\Big)^{1/2} \\
& + \sup_{|\tau|\ge N}\|h_3(\mu + i\tau)\| \cdot \\
& \Big(\int_{|\tau|\ge N}\|\widehat{R}(\mu + i\tau)x\|^2\,d\tau\Big)^{1/2}\Big(\int_{|\tau|\ge N}\|\widehat{R}(\mu + i\tau)^*x^*\|^2\,d\tau\Big)^{1/2}\bigg).
\end{aligned}$$

Next, by the Plancherel Theorem for the Hilbert space valued Fourier transform, and Lemma 2.19 (2)-(3), we get

$$\begin{aligned} |\langle I_1(N), x^*\rangle| &\leq \sup_{|\tau|\geq N} \|\widehat{R}(\mu+i\tau)\| \left(M\|x\|\|x^*\| + \left(2\pi\int_0^\infty \|e^{-\mu t}R(t)x\|^2\,dt\right)^{1/2} M\|x^*\| \right. \\ &\quad \left. + M\cdot\left(2\pi\int_0^\infty \|e^{-\mu t}R(t)x\|^2\,dt\right)^{1/2}\left(2\pi\int_0^\infty \|e^{-\mu t}R(t)^*x^*\|^2\,dt\right)^{1/2}\right). \end{aligned}$$

Since $\{R(t)\}_{t\geq 0}$ is of type (M,ω), we have that $\{e^{-\mu t}R(t)\}_{t>0}$ and $\{e^{-\mu t}R(t)^*\}_{t>0}$ are exponentially bounded families of type $(M,\omega-\mu)$, and that there exists a positive constant $C>0$ such that

$$\int_0^\infty \|e^{-\mu t}R(t)x\|^2\,dt \leq C^2\|x\|, \qquad \int_0^\infty \|e^{-\mu t}R(t)^*x^*\|^2\,dt \leq C^2\|x\|,$$

for all $x\in H$. Combining the above with the Hahn-Banach Theorem, we obtain the existence of a constant $K>0$ such that

$$\begin{aligned} I_1(N) &= \sup_{\|x^*\|\leq 1} |\langle \int_{|\tau|\geq N} (e^{i\tau t}-e^{i\tau s})\widehat{R}''(\mu+i\tau)x\,d\tau, x^*\rangle| \\ &\leq K\cdot\sup_{|\tau|\geq N}\|\widehat{R}(\mu+i\tau)\|\|x\|. \end{aligned}$$

Since $\lim\limits_{|\tau|\to\infty}\|\widehat{R}(\mu+i\tau)\|=0$, there exists $N>0$ such that

$$K\cdot\sup_{|\tau|\geq N}\|\widehat{R}(\mu+i\tau)\|<\epsilon$$

which yields the estimate $I_1(N)<\epsilon\|x\|$ for each $x\in H$.

To estimate $I_2(N)$, we observe that $|e^{i\alpha}-1|^2 = 4\sin^2(\alpha/2), \alpha\in\mathbb{R}$. Therefore, for the above fixed N we have

$$\begin{aligned} I_2(N) &= \int_{|\tau|\leq N} |e^{i\tau t}-e^{i\tau s}|\|\widehat{R}''(\mu+i\tau)x\|\,d\tau \\ &\leq \left(\int_{|\tau|\leq N}|e^{i\tau(t-s)}-1|^2\,d\tau\right)^{1/2}\left(\int_{|\tau|\leq N}\|\widehat{R}''(\mu+i\tau)x\|^2\,d\tau\right)^{1/2} \\ &= \left(4\int_{|\tau|\leq N}\left|\sin^2\left(\frac{(s-t)\tau}{2}\right)\right|\right)^{1/2}\left(\int_{|\tau|\leq N}\|\widehat{R}''(\mu+i\tau)x\|^2\,d\tau\right)^{1/2} \\ &\leq \left(\int_{|\tau|\leq N}|\tau|^2|s-t|^2\,d\tau\right)^{1/2}\left(\int_{|\tau|\leq N}\|\widehat{R}''(\mu+i\tau)x\|^2\,d\tau\right)^{1/2} \\ &\leq |s-t|\left(\frac{2N^3}{3}\right)^{1/2}\left(\int_{|\tau|\leq N}\|\widehat{R}''(\mu+i\tau)x\|^2\,d\tau\right)^{1/2}. \end{aligned}$$

Since $\widehat{R}''(\mu+i\tau)$ is a continuous function and the integral is defined over a compact subset of $\mathbb{R}$, there exists a constant $C'>0$ such that $\|\widehat{R}''(\mu+i\tau)x\| \le C'\|x\|$; this implies

$$I_2(N) \;\le\; |s-t|\Big(\frac{2N^3}{3}\Big)^{1/2} \cdot (2N)^{1/2} C'\|x\| \;\le\; |s-t| K' N^2 \|x\|.$$

By using these estimates for $I_1(N), I_2(N)$, we get

$$||t^2 R(t) e^{-\mu t} - s^2 R(s) e^{-\mu s}|| < 2\epsilon,$$

for all $|s-t|<\delta$. This completes the proof. □

We finish this chapter with a direct application to results of Fan [**31**].

Corollary 2.22. *Let A be a closed linear operator defined in a Hilbert space H with dense domain $D(A)$. Assume that A generates an α-regularized resolvent $S_\alpha(t)$ of type (M,ω) for some $0<\alpha<1$ and suppose*

$$\lim_{|\tau|\to\infty} \|(s+i\tau)^{\alpha-1}\big((s+i\tau)^\alpha - A\big)^{-1}\| = 0$$

for some $s>\omega$. Then, $S_\alpha(t)$ is compact for $t>0$, if and only if $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho(A)$.

Remark 2.23. In a Hilbert space, all the results on Section 4 of [**31**] remain true when the hypothesis on the given operator A, as generator of an analytic compact α-regularized resolvent $(0<\alpha<1)$, is replaced by

$$\lim_{|\tau|\to\infty} \|(s+i\tau)^{\alpha-1}\big((s+i\tau)^\alpha - A\big)^{-1}\| = 0$$

for some $s>\omega$.

Remark 2.24. In the case $a(t)=k(t)\equiv 1$, it is known that the characterization obtained in Theorem 2.21 cannot be extended to Banach spaces (see [**21**] and [**69**] for instance). However, we can naturally ask: Does it exist a class of kernels $(a,k)\neq(1,1)$ where this characterization remains true in general Banach spaces?

CHAPTER 3

Compactness

Fractional resolvent families are useful instruments in the study of abstract models for partial differential equations describing anomalous diffusion. In this chapter, we study and characterize the compactness of resolvent families of operators associated to fractional differential equations, for the case $0 < \alpha \leq 2$. The compactness of $\{S_\alpha(t)\}_{t\geq 0}$ was studied by using different methods, see Prüss [**78**, Corollary 2], Wang, Chen and Xiao [**83**, Theorem 3.5], and Fan [**31**], under the hypothesis of continuity in the uniform operator topology. The objectives of this chapter are: to provide a completely new approach to Fan's result in the case $0 < \alpha \leq 1$, and to provide a complete characterization in the complementary case $1 < \alpha \leq 2$ for the associated family $R_\alpha(t) = (g_{\alpha-1} * S_\alpha)(t)$, fractional counterpart of the sine functions and not studied previously. Finally, we show a new application in the study of existence of mild solutions for a class of semilinear fractional differential equations with non-local initial conditions. The results of this Chapter were recently published in [**60**].

1. Fractional resolvent families

The following definition was first introduced in [**13**] although implicitly in [**57**] and [**79**].

Definition 3.1. *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We call A the **generator of an** $(\alpha,1)$**-resolvent family** if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : \mathrm{Re}(\lambda) > \omega\} \subseteq \rho(A)$, and*

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\,dt, \quad \mathrm{Re}(\lambda) > \omega, x \in X. \tag{3.1}$$

In this case, the family $\{S_\alpha(t)\}_{t\geq 0}$ is called $(\alpha,1)$-***resolvent family generated by*** A.

The next definition was introduced in [**6**] after previous work in [**13**].

Definition 3.2. *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We call A the **generator of an** (α,α)**-resolvent family** if there exist $\omega \geq 0$ and a strongly continuous function $R_\alpha : \mathbb{R}_+ \to \mathcal{B}(X)$ such that, $\{\lambda^\alpha :$*

$\mathrm{Re}(\lambda) > \omega\} \subseteq \rho(A)$ *and*

$$(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} R_\alpha(t)x\,dt, \quad \mathrm{Re}(\lambda) > \omega, x \in X. \tag{3.2}$$

In this case the family $\{R_\alpha(t)\}_{t\geq 0}$ is called (α,α)-***resolvent family generated by*** A.

Because of the uniqueness of the Laplace transform, an $(1,1)$-resolvent family is a C_0-semigroup, a $(2,1)$-resolvent family corresponds to a cosine family and a $(2,2)$-resolvent family is a sine family, see [**8**].

Remark 3.3. If A is the generator of an $(\alpha,1)$-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$, then by [**57**, Proposition 3.1 and Lemma 2.2], we have that the family $\{S_\alpha(t)\}_{t\geq 0}$ verifies the following properties:

a) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(0) = I$;
b) $S_\alpha(t)A \subset AS_\alpha(t)$ for $t \geq 0$;
c) for $x \in D(A)$, the resolvent equation

$$S_\alpha(t)x = x + \int_0^t g_\alpha(t-s)S_\alpha(s)Ax\,ds$$

holds for all $t \geq 0$.

Similarly, an (α,α)-resolvent family $\{R_\alpha(t)\}_{t\geq 0}$ verifies:

a) $R_\alpha(t)$ is strongly continuous for $t \geq 0$ and $R_\alpha(0) = g_\alpha(0)I$;
b) $R_\alpha(t)A \subset AR_\alpha(t)$ for $t \geq 0$;
c) for $x \in D(A)$, the resolvent equation

$$R_\alpha(t)x = g_\alpha(t)x + \int_0^t g_\alpha(t-s)R_\alpha(s)Ax\,ds$$

holds for all $t \geq 0$.

Finally, we recall that a strongly continuous family $\{T(t)\}_{t\geq 0}$ is ***exponentially bounded*** if $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$, with $M > 0$ and $\omega \in \mathbb{R}$ (Definition 1.3).

2. Characterization of compactness

Definition 3.4. *Let X, Y be Banach spaces. We say that an operator $T \in \mathcal{B}(X,Y)$ is* ***compact*** *if $\overline{T(B_X)}$ is a compact subset of Y, where $B_X = \{x \in X : \|x\| \leq 1\}$.*

Definition 3.5. *We say that the resolvent family $\{S_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is* ***compact*** *if for every $t > 0$, the operator $S_\alpha(t)$ is a compact operator. In the same way, $\{R_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is compact if $R_\alpha(t)$ is compact for every $t > 0$.*

The next Theorem was proved recently in [**31**, Theorem 3.6]. Unfortunately the proof in [**31**] does not allow to obtain that the compactness of the resolvent $(\lambda^\alpha - A)^{-1}$ implies the compactness of the $(\alpha, 1)$-resolvent family $\{S_\alpha(t)\}_{t>0}$, because there is a logical flaw in the proof (see Remark 3.10 below). Here we prove, by a completely different method, the desired characterization.

Our method of proof relies on two main ingredients. The first of them is a theorem due to Weis [**85**] that asserts, roughly speaking, that the integral of a family of compact operators is a compact operator:

Lemma 3.6. [**85**, Corollary 2.3] *Let (Ω, μ) be a measure space and $\Omega \ni t \mapsto T_t \in \mathcal{B}(X, Y)$ be a strongly integrable function, i.e.,*

$$Tx = \int_\Omega T_t x \, d\mu(t) \tag{3.3}$$

exists for all $x \in X$ as a Bochner integral and $\int_\Omega \|T_t\| \, d\mu(t) < \infty$. If μ-almost all T_t in (3.3) *are compact, then T is compact.*

The second one is a theorem due to Haase [**38**]. This result gives direct inversion of the Laplace transform for one-parameter families of operators, when the family is regularized by finite convolution with a locally integrable kernel:

Lemma 3.7. [**38**, Proposition 2.1] *Let X, Y be Banach spaces, let $S : [0, \infty) \to \mathcal{B}(X, Y)$ be strongly continuous, and let $a \in L^1_{loc}[0, \infty)$ be a scalar function, both a and S of finite exponential type. Then for every $\omega > \omega_0(S), \omega_0(a)$ one has*

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\omega - iN}^{\omega + iN} e^{\lambda t} \widehat{(a * S)}(\lambda) \, d\lambda = (a * S)(t),$$

in $\mathcal{B}(X, Y)$, uniformly in t from compact subsets of $[0, \infty)$.

Theorem 3.8. *Let $0 < \alpha \leq 1$ and $\{S_\alpha(t)\}_{t\geq 0}$ be an $(\alpha, 1)$-resolvent family of type (M, ω) generated by A. Suppose that $S_\alpha(t)$ is continuous in the uniform operator topology for all $t > 0$. Then, the following assertions are equivalent:*

(1) *$S_\alpha(t)$ is a compact operator for all $t > 0$.*
(2) *$(\lambda - A)^{-1}$ is a compact operator for all $\lambda > \omega^{1/\alpha}$.*

Proof. (1) $\implies$ (2) Suppose that $\{S_\alpha(t)\}_{t>0}$ is compact and let $\lambda > \omega$ be fixed. Then, by (3.1) we have

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} = \int_0^\infty e^{-\lambda t} S_\alpha(t) \, dt,$$

where the integral in the right-hand side exists in the Bochner sense, because $\{S_\alpha(t)\}_{t>0}$ is continuous in the uniform operator topology, by hypothesis. Then, by Lemma 3.6, we conclude that $(\lambda^\alpha - A)^{-1}$ is a compact operator.

(2) $\implies$ (1) The case $\alpha = 1$ follows from [**75**].

For the case $0 < \alpha < 1$, let $t > 0$ be fixed. By (3.1) and (3.2), and the uniqueness of the Laplace transform, we have the relation

$$S_\alpha(t) = (g_{1-\alpha} * R_\alpha)(t). \tag{3.4}$$

It follows that $g_{1-\alpha} \in L^1_{\mathrm{loc}}[0,\infty)$, and therefore, by Lemma 3.7 and (3.4) we obtain

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\omega-iN}^{\omega+iN} e^{\lambda t} (\widehat{g_{1-\alpha} * R_\alpha})(\lambda)\, d\lambda = (g_{1-\alpha} * R_\alpha)(t) = S_\alpha(t),$$

in $\mathcal{B}(X)$. Therefore,

$$\frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1}\, d\lambda = S_\alpha(t),$$

where Γ is the path consisting of the vertical line $\{\omega + it : t \in \mathbb{R}\}$. By hypothesis and Lemma 3.6, we conclude that $S_\alpha(t)$ is compact. $\square$

Remark 3.9. Theorem 3.8 extends the compactness criterion for semigroup operator functions, see e.g. [**75**], [**30**, Chapter II, Theorem 4.29] and [**34**].

Remark 3.10. The proof of [**31**, Theorem 3.6] in (2) $\implies$ (1) uses [**31**, Lemma 3.4]. However, one of the hypothesis of such Lemma is precisely (1).

Remark 3.11. Useful criteria for continuity of $S_\alpha(t)$ in the uniform operator topology can be found in the work of Fan [**31**]. For example, this property is true for the class of analytic resolvents, see [**31**, Lemma 3.8].

Our second main result completely characterizes the compactness of (α,α)-resolvent families in the range $1 < \alpha \leq 2$. In contrast with the case $0 < \alpha \leq 1$, it is remarkable that we obtain here a characterization solely in terms of properties of the generator A.

Theorem 3.12. *Let $1 < \alpha \leq 2$ and A be the generator of an $(\alpha, 1)$-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ of type (M, ω). Then, A generates an (α,α)-resolvent family $\{R_\alpha(t)\}_{t\geq 0}$ of type $\left(\frac{M}{\omega^{\alpha-1}}, \omega\right)$, and the following assertions are equivalent:*

(1) *$R_\alpha(t)$ is a compact operator for all $t > 0$.*
(2) *$(\lambda - A)^{-1}$ is a compact operator for all $\lambda > \omega^{1/\alpha}$.*

Proof. We first prove that A generates an (α,α)-resolvent family $\{R_\alpha(t)\}_{t\geq 0}$ of type $\left(\frac{M}{\omega^{\alpha-1}},\omega\right)$. By hypothesis, we have $\|S_\alpha(t)\|\leq Me^{\omega t}$ for all $t\geq 0$, and define

$$R_\alpha(t):=(g_{\alpha-1}*S_\alpha)(t), \tag{3.5}$$

for all $t\geq 0$. We obtain

$$\begin{aligned}\|R_\alpha(t)\| &\leq M\int_0^t g_{\alpha-1}(s)e^{\omega(t-s)}\,ds\\ &\leq \frac{Me^{\omega t}}{\Gamma(\alpha-1)}\int_0^\infty s^{\alpha-2}e^{-\omega s}\,ds\leq \frac{Me^{\omega t}}{\omega^{\alpha-1}}.\end{aligned}$$

In particular, we conclude that $R_\alpha(t)$ is Laplace transformable and, for $\lambda>\omega$, we have

$$\widehat{R}_\alpha(\lambda)=\frac{1}{\lambda^{\alpha-1}}\widehat{S}_\alpha(\lambda)=(\lambda^\alpha-A)^{-1}.$$

Therefore, by definition, A is generator of $R_\alpha(t)$ and it is an (α,α)-resolvent family. This proves the first claim.

(1) $\implies$ (2) Suppose that $\{R_\alpha(t)\}_{t>0}$ is compact. First, we prove that $R_\alpha(t)$ is continuous in the uniform operator topology for $t>0$. We can assume that $\omega>0$, and observe that for $t>s$, by (3.5) we have

$$\begin{aligned}R_\alpha(t)-R_\alpha(s) &= \int_s^t g_{\alpha-1}(t-r)S_\alpha(r)\,dr+\int_0^s \big(g_{\alpha-1}(t-r)-g_{\alpha-1}(s-r)\big)S_\alpha(r)\,dr\\ &=: I_1+I_2.\end{aligned} \tag{3.6}$$

Observe that

$$\|I_1\| \leq \int_s^t g_{\alpha-1}(t-r)\|S_\alpha(r)\|\,dr \leq Me^{\omega t}\int_s^t g_{\alpha-1}(t-r)\,dr.$$

Because $\alpha>1$, we have $g_\alpha(0)=0$, and we obtain

$$\|I_1\| \leq Me^{\omega t}g_\alpha(t-s). \tag{3.7}$$

On the other hand,

$$\begin{aligned}\|I_2\| &\leq \int_0^s |g_{\alpha-1}(t-r)-g_{\alpha-1}(s-r)|\|S_\alpha(r)\|\,dr\\ &\leq Me^{\omega s}\int_0^s |g_{\alpha-1}(t-r)-g_{\alpha-1}(s-r)|\,dr\\ &= Me^{\omega s}\int_0^s |g_{\alpha-1}(t-s+r)-g_{\alpha-1}(r)|\,dr.\end{aligned}$$

Note that $g_{\alpha-1}$ is decreasing for $\alpha<2$, we have $g_{\alpha-1}(r)-g_{\alpha-1}(t-s+r)>0$, obtaining

$$\|I_2\| \leq Me^{\omega s}\int_0^s \big(g_{\alpha-1}(r)-g_{\alpha-1}(t-s+r)\big)\,dr$$

$$= \quad Me^{\omega s}\big(g_\alpha(s) - g_\alpha(t) + g_\alpha(t-s)\big). \tag{3.8}$$

Observe that in the case $\alpha = 2$ we have $I_2 \equiv 0$ because $g_1(t) \equiv 1$. Combining (3.7) with (3.8), and replacing in (3.6), we obtain the assertion. Then, by Definition 3.2 for $\lambda > \omega$, we have

$$(\lambda^\alpha - A)^{-1} = \int_0^\infty e^{-\lambda t} R_\alpha(t)\, dt,$$

and the integral in the right-hand side exists in the Bochner sense, because $\{R_\alpha(t)\}_{t>0}$ is continuous in the uniform operator topology. Therefore, by Lemma 3.6 and the compactness of $\{R_\alpha(t)\}_{t>0}$, we conclude that $(\lambda^\alpha - A)^{-1}$ is a compact operator.

(2) $\implies$ (1) Let $t > 0$ be fixed. Since $\alpha > 1$, it follows that $g_{\alpha-1} \in L^1_{\text{loc}}[0,\infty)$ and hence, by Lemma 3.7 and (3.5), we obtain

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\omega-iN}^{\omega+iN} e^{\lambda t} (\widehat{g_{\alpha-1} * S_\alpha})(\lambda)\, d\lambda = (g_{\alpha-1} * S_\alpha)(t) = R_\alpha(t),$$

in $\mathcal{B}(X)$. Therefore,

$$\frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda^\alpha - A)^{-1}\, d\lambda = R_\alpha(t)$$

where Γ is the path consisting of the vertical line $\{\omega + it : t \in \mathbb{R}\}$. By hypothesis and Lemma 3.6, we conclude that $R_\alpha(t)$ is a compact operator. $\square$

Remark 3.13. In the case $\alpha = 2$, the preceding Theorem extends the compactness criterion for sine operator functions in [**80**]. See also [**82**, Theorem 10.1.1].

3. Example: A problem with non-local initial condition

In this section, we present one example which does not aim at generality, but indicate how our theorems can be applied to more concrete problems. For other examples, see [**31**, Theorem 4.1] and [**83**, Theorem 5.3].

Example 3.14. Let $T > 0$ be given. We study the semilinear problem

$$D_t^\alpha u(t) = Au(t) + J_t^{1-\alpha} f(t, u(t)), \quad 0 < \alpha < 1, \quad t \in T := [0, T], \tag{3.9}$$

with nonlocal initial condition

$$u(0) + g(u) = u_0, \tag{3.10}$$

where $f : [0,T] \times X \to X$ and $g : C(I,X) \to C(I,X)$ are continuous. Here, D_t^α denotes the Caputo fractional derivative (see Definition 1.4).

The concept of nonlocal initial condition has been introduced to extend the study of classical initial value problems. This notion is more precise for describing nature phenomena than the classical notion because additional information is taken into account. For the importance of nonlocal conditions in different fields, the reader is referred to [**64**] and the references therein.

Let A be the generator of an $(\alpha, 1)$ - resolvent family $S_\alpha(t)$. Then it is known that the mild solution of (3.9) is defined by means of the variation of constants formula:

$$u(t) = S_\alpha(t)\big(u_0 - g(u)\big) + \int_0^t S_\alpha(t-s) f(s, u(s))\, ds, \quad t \in I.$$

See, for instance, [**31**, Section 4]. We will make the following assumptions:

(H1) f satisfies the Carathéodory condition, that is $f(\cdot, u)$ is strongly measurable for each $u \in X$ and $f(t, \cdot)$ is continuous for each $t \in I$.

(H2) There exists a continuous function $\mu : I \to \mathbb{R}_+$ such that

$$\|f(t,u)\| \le \mu(t)\|u\|, \quad \forall\, t \in I,\ u \in C(I, X).$$

(H3) $g : C(I, X) \to C(I, X)$ is continuous and there exists $L_g > 0$ such that

$$\|g(u) - g(v)\| < L_g \|u - v\|,\ \forall u, v \in C(I, X).$$

We prove the following existence theorem. The method of proof combines ideas from [**31**] and [**59**].

Theorem 3.15. *Let A be the generator of an $(\alpha, 1)$-resolvent family $S_\alpha(t)$ of type (M, ω). Suppose that $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda > \omega$, and that $S_\alpha(t)$ is continuous in the uniform operator topology for all $t > 0$. Then, under assumptions* (H1) – (H3)*, the system* (3.9) – (3.10) *has at least one mild solution.*

Proof. Define the operator $G : C(I, X) \to C(I, X)$ by

$$(Gu)(t) := S_\alpha(t)\big(u_0 - g(u)\big) + \int_0^t S_\alpha(t-s) f(s, u(s))\, ds.$$

Let $B_r := \{u \in C(I, X) : \|u\| \le r\}$. The proof will be conducted into several steps.

Step 1. First, we show that Γ sends bounded sets of $C(I, X)$ into bounded sets of $C(I, X)$. In other words, for any given $r > 0$, there exists $\xi > 0$ such that $GB_r \subset B_\xi$. Let $u \in B_r$ and $N := \sup\limits_{u \in B_r} \|g(u)\|$, then

$$\|Gu(t)\| \ \le\ M\|S_\alpha(t)\|(\|u_0\| + \|g(u)\|) + M\int_0^t \|S_\alpha(t-s)\|\|f(s, u(s))\|\, ds$$

$$\begin{aligned}
&\leq \; Me^{\omega t}(\|u_0\| + N) + M\int_0^t e^{\omega(t-s)}\mu(s)\|u(s)\|\,ds \\
&\leq \; Me^{\omega T}(\|u_0\| + N) + Mr\|\mu\|\frac{e^{\omega T}}{\omega} =: \xi.
\end{aligned}$$

Thus, $GB_r \subset B_\xi$.

Step 2. Next, we show that G is a continuous operator: Let $u_n, u \in B_r$ such that $u_n \to u$ in $C(I, X)$. Then, we have

$$\begin{aligned}
\|Gu_n(t) - Gu(t)\| \;&\leq\; \|S_\alpha(t)\|(\|g(u_n) - g(u)\|) + \\
&\quad \int_0^t \|S_\alpha(t-s)\|\|f(s, u_n(s)) - f(s, u(s))\|\,ds \\
&\leq\; Me^{\omega t}L_g\|u_n - u\| + M\int_0^t e^{\omega(t-s)}\|f(s, u_n(s)) - f(s, u(s))\|\,ds \\
&\leq\; Me^{\omega T}L_g\|u_n - u\| + M\int_0^t e^{\omega(t-s)}\mu(s)(\|u_n(s)\| + \|u(s)\|)\,ds \\
&\leq\; Me^{\omega T}L_g\|u_n - u\| + 2rM\int_0^t e^{\omega(t-s)}\mu(s)\,ds.
\end{aligned}$$

Choose n large enough such that $\|u_n - u\| < \epsilon$. Also note that $e^{\omega(t-s)}\mu(s)$ is integrable on I. So, by the dominated convergence theorem,

$$\int_0^t e^{\omega(t-s)}\|f(s, u_n(s)) - f(s, u(s))\|\,ds \to 0 \quad \text{as } n \to \infty;$$

which shows that G is continuous.

Step 3. Now, we show that G sends bounded sets of $C(I, X)$ into equicontinuous sets of $C(I, X)$: Let $u \in B_r$ with $r > 0$, and take $t_2 < t_1 \in I$. Then, we have

$$\begin{aligned}
\|Gu(t_1) - Gu(t_2)\| \;&\leq\; \|(S_\alpha(t_1) - S_\alpha(t_2))(u_0 - g(u))\| + \int_{t_2}^{t_1} \|S_\alpha(t_1 - s)f(s, u(s))\|\,ds \\
&\quad + \int_0^{t_2} \|(S_\alpha(t_1 - s) - S_\alpha(t_2 - s))f(s, u(s))\|\,ds \\
&=: \; I_1 + I_2 + I_3.
\end{aligned}$$

We have

$$I_1 \leq \|S_\alpha(t_1) - S_\alpha(t_2)\|\|(u_0 - g(u))\|,$$

and because of the uniform continuity of $S_\alpha(t)$ for $t > 0$, we obtain $\lim\limits_{t_1 \to t_2} I_1 = 0$.

Next, we have

$$\begin{aligned} I_2 &\leq \int_{t_2}^{t_1} e^{\omega(t_1-s)}\mu(s)\|u(s)\|\,ds \\ &\leq r\|\mu\|e^{\omega T}(t_1-t_2), \end{aligned}$$

therefore, $\lim_{t_1\to t_2} I_2 = 0$.
Finally, we have

$$\begin{aligned} I_3 &\leq \int_0^{t_2} \|S_\alpha(t_1-s)-S_\alpha(t_2-s)\|\|f(s,u(s))\|\,ds \\ &\leq \int_0^{t_2} \|S_\alpha(t_1-s)-S_\alpha(t_2-s)\|\mu(s)\|u(s)\|\,ds \\ &\leq r\int_0^{t_2} \|S_\alpha(t_1-s)-S_\alpha(t_2-s)\|\mu(s)\,ds. \end{aligned}$$

Now observe that

$$\|S_\alpha(t_1-\cdot)-S_\alpha(t_2-\cdot)\|\mu(s) \leq 2Me^{\omega T}\mu(\cdot) \in L^1(I,\mathbb{R}),$$

and $S_\alpha(t_1-s)-S_\alpha(t_2-s) \to 0$ in $\mathcal{B}(X)$, as $t_1 \to t_2$. Thus $\lim_{t_1\to t_2} I_3 = 0$ by the dominated convergence theorem.

Step 4. G maps B_r into relatively compact sets in X: In view of the hypothesis and Theorem 3.8, we have that $S_\alpha(t)$ is compact for all $t > 0$, and hence, we deduce that the set

$$\mathcal{K} = \{S_\alpha(t-s)f(s,u(s)) : u \in C(I,X), 0 \leq s \leq t\}$$

is relatively compact for each $t \in I$ (see the proof of [**31**, Theorem 4.1] for details). Then, the set $\overline{\mathrm{conv}(K)}$ is compact. Moreover, for $u \in B_r$, using the Mean-Value Theorem for Bochner integrals, we obtain

$$G(u(t)) \in t\,\overline{\mathrm{conv}(K)},\ \text{for all } t \in [0,T].$$

Therefore, the set $\{Gu(t); u \in B_r\}$ is relatively compact in X for every $t \in [0,T]$. From Steps 1-4, we deduce that G is continuous and compact by the Arzela-Ascoli's Theorem.

Step 5. Consider the set

$$\Omega := \{u \in B_r : u = \lambda G u,\ 0 < \lambda < 1\}.$$

Clearly, $\Omega \neq \emptyset$ since $0 \in \Omega$, so let $u \in \Omega$. Then, we have

$$\|u(t)\| \leq \lambda\left(Me^{\alpha t}(\|u_0\|+\|g(u)\|) + M\int_0^t e^{\omega(t-s)}\|f(s,u(s)\|\,ds\right)$$

$$\begin{aligned}
&\leq \lambda\left(Me^{\alpha t}(\|u_0\| + N) + Mr\int_0^t e^{\omega(t-s)}\mu(s)\,ds\right)\\
&\leq Me^{\alpha t}(\|u_0\| + N) + Mr\|\mu\|\frac{e^{\omega T}}{\omega}.
\end{aligned}$$

Thus, Ω is bounded.

So, by the Leray-Schauder theorem, G has a fixed point. The proof is complete. $\square$

CHAPTER 4

Spectral Mapping Theorem

In this chapter, we prove the spectral mapping theorem for the point spectrum of a k-convoluted semigroup $\{R(t)\}_{t\geq 0}$, and a version of the spectral mapping theorem for approximate point spectrum and residual spectrum of $R(t)$ under the hypothesis that its generator also generates a C_0-semigroup.

1. k-convoluted semigroups

This section is devoted to preliminaries and some properties of k-convoluted semigroups.

Definition 4.1. *Let A be a closed linear operator and $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. A strongly continuous family $\{R(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called a k-**convoluted semigroup** generated by A if the following properties hold:*

(1) *$R(t)x \in D(A)$ and $R(t)Ax = AR(t)x$ for all $x \in D(A)$ and $t \geq 0$;*
(2) *$\int_0^t R(s)x\,ds \in D(A)$ for all $t \geq 0$ and $x \in X$, and*

$$R(t)x = \int_0^t k(s)x\,ds + A\int_0^t R(s)x\,ds, \quad x \in X,\ t \geq 0. \tag{4.1}$$

We notice that if $k(t) = \frac{t^{n-1}}{n!}, n \in \mathbb{N}$, the k-convoluted semigroup corresponds to the n-times integrated semigroup introduced by Arendt [**7**], and a 0-times integrated semigroup is the same as a C_0-semigroup. If A generates a C_0-semigroup, then A generates an n-times integrated semigroup for all $n \in \mathbb{N}$. If $k(t) = g_\alpha(t), \alpha \geq 0$, the k-convoluted semigroup is an α-times integrated semigroup (see [**71**]). The concept of k-convoluted semigroups was introduced by I. Cioranescu in [**24**]. On the other hand, we notice that a k-convoluted semigroup also corresponds to an $(1, 1 * k)$-regularized family (see [**57**] and Definition 2.1).

Proposition 4.2. [**65**, Theorem 5.3] *Let $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. If $\{R(t)\}_{t\geq 0}$ is a k-convoluted semigroup with generator A on a Banach space X, then*

$$\int_0^t e^{\lambda(t-s)}R(s)x\,ds \in D(A)$$

and

$$(\lambda - A)\int_0^t e^{\lambda(t-s)}R(s)x\,ds = \int_0^t k(t-s)e^{\lambda s}x\,ds - R(t)x, \tag{4.2}$$

for all $x \in X$.

For further use, we quote below the following lemma which is a direct consequence of Definition 4.1 (2).

Lemma 4.3. *Let $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. If $\{R(t)\}_{t\geq 0}$ is a k-convoluted semigroup with generator A on a Banach space X, then for all $t \geq 0$, $\overline{R(t)X} \subseteq \overline{D(A)}$.*

A k-convoluted semigroup verifies the functional equation given in the following result.

Theorem 4.4. [**44**, Remark 4.3 and Proposition 5.3] *If $\{R(t)\}_{t\geq 0}$ is a k-convoluted semigroup with generator A on a Banach space X, then for $s, t \geq 0$ and $x \in X$ we have*

$$R(t)R(s)x = \int_t^{t+s} k(t+s-r)R(r)x\,dr - \int_0^s k(t+s-r)R(r)x\,dr. \tag{4.3}$$

2. Spectral Mapping Theorem

Definition 4.5. *Let X be a Banach space, and $A \in \mathcal{B}(X)$ a closed operator. We define the* ***point spectrum****,* ***approximate point spectrum****, and* ***residual spectrum*** *of A respectively, as follows:*

$$\begin{aligned}
\sigma_p(A) &= \big\{\lambda \in \mathbb{C} : \ker(\lambda I - A) \neq \{0\}\big\},\\
\sigma_a(A) &= \big\{\lambda \in \mathbb{C} : (\lambda I - A)\,\text{is not injective, or}\ \mathrm{ran}(\lambda I - A)\,\text{is not closed in}\ X\big\},\\
\sigma_r(A) &= \big\{\lambda \in \mathbb{C} : \ker(\lambda I - A) = \{0\}\,\text{and}\ \overline{\mathrm{ran}(\lambda I - A)} \neq X\big\}.
\end{aligned}$$

In this section we shall prove that for a k-convoluted semigroup $\{R(t)\}_{t\geq 0}$ whose generator A generates also a C_0-semigroup the spectral mapping theorem holds for the point, approximate point and residual spectrum, that is, we have the following equalities:

$$\begin{aligned}
\sigma_p(R(t)) \cup \{0\} &= \left\{\int_0^t k(t-s)e^{\lambda s}\,ds : \lambda \in \sigma_p(A)\right\} \cup \{0\},\\
\sigma_a(R(t)) \cup \{0\} &= \left\{\int_0^t k(t-s)e^{\lambda s}\,ds : \lambda \in \sigma_a(A)\right\} \cup \{0\},\\
\sigma_r(R(t)) \cup \{0\} &= \left\{\int_0^t k(t-s)e^{\lambda s}\,ds : \lambda \in \sigma_r(A)\right\} \cup \{0\}.
\end{aligned}$$

In the sequel, we consider the following characterization of $\sigma_a(A)$.

Lemma 4.6. [**30**, Lemma 4.1.9] *For a closed operator $A : D(A) \subset X \to X$, and a number $\lambda \in \mathbb{C}$, one has $\lambda \in \sigma_a(A)$, if and only if, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset D(A)$, such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0$.*

From [**65**, Theorems 5.3 and 5.5-5.7], we have the following result.

Lemma 4.7. *Let $\{R(t)\}_{t\geq 0}$ be a k-convoluted semigroup with generator A on a Banach space X. Then, we have*

$$\sigma(R(t)) \cup \{0\} \supseteq \left\{ \int_0^t k(t-s)e^{\lambda s}\, ds : \lambda \in \sigma(A) \right\} \cup \{0\}, \tag{4.4}$$

and the following inclusions hold:

$$\sigma_p(R(t)) \cup \{0\} \ \supseteq \ \left\{ \int_0^t k(t-s)e^{\lambda s}\, ds : \lambda \in \sigma_p(A) \right\} \cup \{0\} \tag{4.5}$$

$$\sigma_a(R(t)) \cup \{0\} \ \supseteq \ \left\{ \int_0^t k(t-s)e^{\lambda s}\, ds : \lambda \in \sigma_a(A) \right\} \cup \{0\}. \tag{4.6}$$

Moreover, if A is densely defined, then

$$\sigma_r(R(t)) \cup \{0\} \ \supseteq \ \left\{ \int_0^t k(t-s)e^{\lambda s}\, ds : \lambda \in \sigma_r(A) \right\} \cup \{0\}. \tag{4.7}$$

Lemma 4.8. *Let A be a closed linear operator with $\rho(A) \neq \emptyset$, $k \in C^1(\mathbb{R}_+)$, and let $\{R(t)\}_{t\geq 0}$ be a k-convoluted semigroup generated by A. If $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a bounded sequence such that $R(\lambda, A)x_n \to 0$ for $\lambda \in \rho(A)$, then $R^2(t)x_n \to 0$ for all $t \geq 0$.*

Proof. By (4.1), we have

$$(\lambda - A)\int_0^t R(r)x\, dr = \lambda \int_0^t R(r)x\, dr - R(t)x + (1 * k)(t)x, \quad x \in X, t \geq 0.$$

Observe that the operator $T_1 : X \to X$ defined by

$$T_1(t)x := (\lambda - A)\int_0^t k(t-r)R(r)x\, dr,$$

is bounded for each $x \in X$. In fact, by integration by parts, we obtain

$$\begin{aligned}
T_1(t)x &= \lambda \int_0^t k(t-r)R(r)x\, dr - A\int_0^t k(t-r)R(r)x\, dr \\
&= \lambda \int_0^t k(t-r)R(r)x\, dr - A\left((1 * R)(r)k(t-r)\Big|_{r=0}^{r=t} + \int_0^t k'(t-r)(1 * R)(r)x\, dr \right) \\
&= \lambda \int_0^t k(t-r)R(r)x\, dr - A(1 * R)(t)k(0)x - \int_0^t k'(t-r)A(1 * R)(r)x\, dr
\end{aligned}$$

$$\begin{aligned}
&= \lambda \int_0^t k(t-r)R(r)x\,dr - k(0)R(t)x + k(t)(1*k)(t)x \\
&\quad - \int_0^t k'(t-r)R(r)x\,dr + \int_0^t k'(t-r)(1*k)(r)x\,dr,
\end{aligned}$$

where we have used (4.1) in the last identity. Hence, it shows that T_1 is a bounded operator. Similarly, the operator $T_2(\cdot), T_3(\cdot) : X \to X$ defined by

$$\begin{aligned}
T_2(t)x &:= (\lambda - A)\int_0^t k(t+s-r)R(r)x\,dr, \quad \text{for } s \geq 0 \text{ fixed}, \\
T_3(s)x &:= (\lambda - A)\int_0^s k(t+r)R(s-r)x\,dr \\
&= (\lambda - A)\int_0^s k(t+s-r)R(r)x\,dr, \quad \text{for } t \geq 0 \text{ fixed},
\end{aligned}$$

for $t, s \geq 0$, are bounded operators, since in this case we obtain by using (4.1) that

$$\begin{aligned}
T_2(t)x &= \lambda \int_0^t k(t+s-r)R(r)x\,dr - k(s)R(t)x + k(s)(1*k)(t)x \\
&\quad - \int_0^t k'(t+s-r)R(r)x\,dr + \int_0^t k'(t+s-r)(1*k)(r)x\,dr, \\
T_3(s)x &= \lambda \int_0^s k(t+r)R(s-r)x\,dr - k(t)R(s)x + k(t)(1*k)(s)x \\
&\quad - \int_0^s k'(t+r)R(s-r)x\,dr + \int_0^s k'(t+r)(1*k)(s-r)x\,dr.
\end{aligned}$$

On the other hand, since k' is continuous on $\mathbb{R}_+$, we get that

$$x \mapsto (\lambda - A)\left(\int_t^{t+s} k(t+s-r)R(r)x\,dr - \int_0^s k(t+s-r)R(r)x\,dr\right), \quad x \in X, \tag{4.8}$$

is a bounded operator on X because (4.8) can be written as

$$T_1(t+s)x - T_2(t)x - T_3(s)x, \quad x \in X.$$

We notice that (4.8) is precisely $(\lambda - A)R(s)R(t)x$ by Theorem 4.4, and therefore, since $R(t)$ commutes with A, it follows that if $R(\lambda, A)x_n \to 0$, then $R(s)R(t)x_n \to 0$. □

Lemma 4.9. *Let $\{R(t)\}_{t\geq 0}$ be a k-convoluted semigroup with generator A. Then, for $\mu \neq 0$ and any $t_0 \geq 0$, A is bounded on* $\ker(R(t_0) - \mu)$.

Proof. The closed subspace $K := \ker(R(t_0) - \mu)$ is both $R(t)$ and A invariant, so we may restrict this family to K. By the Corollary 4.3, we have $\overline{R(\lambda, A)K} = \overline{D(A|_K)} = \overline{R(t_0)K} = K$. We need to show that there exists $c > 0$ such that for all $x \in K$ we have $\|R(\lambda, A)x\| \geq c\|x\|$. Suppose that this is not true, and there is a norm one sequence $\{x_n\} \subset K$ such that $R(\lambda, A)x_n \to 0$. By Lemma 4.8 we have $R^2(t)x_n \to 0$ for all $t \geq 0$, which contradicts $(R(t_0) - \mu)x_n \to 0$. Therefore, there is $c > 0$ such that

$$\|R(\lambda, A)x\| \geq c\|x\|, \quad x \in K. \tag{4.9}$$

Now, we claim that $R(\lambda, A)\ker(R(t_0) - \mu)$ is closed. For this, we consider $x \in \overline{R(\lambda, A)K}$, then there exists a sequence $\{z_n\} \subset K$ such that $R(\lambda, A)z_n \to x$. From (4.9) we have

$$\|z_i - z_j\| \leq \frac{1}{c}\|R(\lambda, A)z_i - R(\lambda, A)z_j\|$$

which goes to 0 as $i, j \to \infty$, thus $\{z_n\}$ is a Cauchy sequence, and so there exists $x_0 \in K$ such that $z_n \to z_0$. Then, we have

$$\begin{aligned}
\|R(\lambda, A)z_0 - x\| &= \|R(\lambda, A)z_0 - R(\lambda, A)z_n + R(\lambda, A)z_n - x\| \\
&\leq \|R(\lambda, A)z_0 - R(\lambda, A)z_n\| + \|R(\lambda, A)z_n - x\| \\
&\leq \|R(\lambda, A)\|\|z_0 - z_n\| + \|R(\lambda, A)z_n - x\|,
\end{aligned}$$

and taking $n \to \infty$, we obtain $R(\lambda, A)z_0 = x$, therefore $R(\lambda, A)\ker(R(t_0) - \mu)$ is closed, and so $R(\lambda, A)$ is surjective. Hence, for x in K, $y = R(\lambda, A)x$ and from (4.9) we get

$$\|y\| \geq c\|(\lambda I - A)y\| = c\|\lambda y - Ay\|,$$

which implies

$$\|Ay\| = \|Ay - \lambda y + \lambda y\| \leq \|Ay - \lambda y\| + \|\lambda y\|.$$

Therefore, $\|Ax\| \leq (|\lambda| + \frac{1}{c})\|x\|$, concluding the proof. □

We notice that in the following result, we have the spectral mapping theorem for the point spectrum without the assumption that A generates also a C_0-semigroup.

Theorem 4.10. *For $\{R(t)\}_{t\geq 0}$ a k-convoluted semigroup with generator A, we have*

$$\sigma_p(R(t)) \cup \{0\} = \left\{ \int_0^t k(t-s)e^{\lambda s}\, ds : \lambda \in \sigma_p(A) \right\} \cup \{0\}. \tag{4.10}$$

Proof. The $\supseteq$ inclusion holds by Lemma 4.7.

To obtain the $\subseteq$ inclusion, let $\mu \in \sigma_p(R(t_0)) \setminus \{0\} \subset \sigma(R(t_0)) \setminus \{0\}$, and set $K = \ker(R(t_0) - \mu)$. By the proof of Lemma 4.9, we can restrict $R(t)$ to $K = \ker(R(t_0) - \mu)$; also, we have that $A|_K$ is bounded on K. Hence, $A|_K$ has a nonempty spectrum. Take $\nu \in \sigma(A|_K)$. We also have that $\sigma(R(t_0)|_K) = \{\mu\}$, so by (4.4) we have $\int_0^{t_0} k(t_0 - s)e^{\nu s}\, ds \in \sigma(R(t_0)|_K) = \{\mu\}$, and therefore

$$\mu = \int_0^{t_0} k(t_0 - s)e^{\nu s}\, ds.$$

Let $0 \neq x_0 \in K$. From the identity (4.2) and evaluating at $\lambda = \nu$ we have

$$\begin{aligned}(\nu - A)\int_0^{t_0} e^{\nu(t_0-s)}R(t_0)x_0\,ds &= \int_0^{t_0} k(t_0-s)e^{\nu s}x_0\,ds - R(t_0)x_0\\ &= \int_0^{t_0} k(t_0-s)e^{\nu s}x_0\,ds - \mu x_0\\ &= 0,\end{aligned}$$

therefore $\nu \in \sigma_p(A)$ and thus $\mu \in \left\{\int_0^{t_0} k(t_0-s)e^{\lambda s}\,ds : \lambda \in \sigma_p(A)\right\}$. $\square$

Remark 4.11. We can obtain an alternative proof of (4.10) (see [**28**, Theorem 2.6]), without considering equation (4.2), as follows: we define an analytic function $f : \sigma(A|_K) \to \mathbb{C}$ as

$$\lambda \mapsto \int_0^{t_0} k(t_0-s)e^{\lambda s}\,ds.$$

Since A is bounded, we have $\sigma(A|_K) \subset B(0, \|A|_K\|)$, where

$$B(0, \|A|_K\|) = \{x \in X : \|x\| \leq \|A|_K\|\},$$

and thus f is defined over a compact subset of $\mathbb{C}$. Hence, by properties of analytic functions, $\sigma(A|_K)$ is a finite set of isolated points. Moreover, for $x \in K$ we have

$$R(t_0)\big|_K x = \mu x = \int_0^{t_0} k(t_0-s)e^{\nu s}x\,ds.$$

By the spectral theorem, we only need to prove that $g(\lambda) = f(\lambda) - \mu$ has a zero of finite multiplicity at $\lambda = \nu$. For this, by differentiation we get

$$g''(\lambda) = g(\lambda) + \lambda g'(\lambda) + \mu,$$

and, as $\mu \neq 0$, we conclude that at least one of $g'(\nu)$, $g''(\nu)$ is not zero. Hence, the multiplicity of the zero of g at $\lambda = \nu$ is at most two, and, we have:

$$g(\lambda) = \begin{cases} \lambda^2 h(\lambda), h(0) \neq 0, & \nu = 0\\ (\lambda - \nu)h(\lambda), h(\nu) \neq 0, & \nu \neq 0.\end{cases}$$

We conclude that $\nu \in \sigma_p(A)$.

Theorem 4.12. *For $\{R(t)\}_{t\geq 0}$ a k-convoluted semigroup on a Banach space X whose generator A generates a C_0-semigroup, we have*

$$\sigma_a(R(t)) \cup \{0\} = \left\{\int_0^t k(t-s)e^{\lambda s}\,ds : \lambda \in \sigma_a(A)\right\} \cup \{0\}. \tag{4.11}$$

PROOF. Since A generates a C_0-semigroup, we can follow the construction in [**73**, A-I, Section 3.6]. For $t_0 \geq 0$ fixed and $\mu \in \sigma_a(R(t_0))$, let

$$\begin{aligned} m(X) &:= \big\{\{x_n\}_{n\in\mathbb{N}} : \sup_n \|x_n\| < \infty\big\}, \\ p(X) &:= \big\{\{x_n\} \in m(X) : (R(t_0)-\mu)x_n \to 0\big\}, \\ D(A_1) &:= \big\{\{R(\lambda, A)x_n\} : \{x_n\} \in p(X)\big\}, \end{aligned}$$

where in $m(X)$ we have $\|x_n\|_{m(X)} = \sup_n \|x_n\|_X$. Because of the resolvent equation, the last set is independent of $\lambda \in \rho(A)$. Also, set

$$\begin{aligned} R_1(t)\{x_n\} &:= \{R(t)x_n\} \quad \text{for } \{x_n\} \in p(X), \\ A_1\{x_n\} &:= \{Ax_n\} \quad \text{for } \{x_n\} \in D(A_1); \end{aligned}$$

we have that $p(X)$ is both $R_1(t)$-invariant and A_1-invariant; also, we note that $R_1(t)$ is uniformly continuous because of the uniform continuity of $R(t)$. Thus, we have

$$\|R_1(t)\{x_n\}\|_{p(X)} = \sup_n \|R(t)x_n\|_X, \qquad \|R(t)x_n\|_X \leq \|R(t)\|_{\mathcal{B}(X)} \sup_n \|x_n\|_X,$$

therefore, $\|R_1(t)\|_{\mathcal{B}(p(X))} \leq \|R(t)\|_{\mathcal{B}(X)}$ and $R_1(t)$ is exponentially bounded. Moreover, as a consequence of the definition, the resolvent equation holds for $R_1(t)$. Consequently, $R_1(t)$ is a k-convoluted semigroup with generator A_1. Moreover, for $\{x_n\}$ such that $(R(t_0)-\mu)x_n \to 0$ on X, we take $\{x_n\}_r = \{x_{n+r}\}$ and we obtain $(R_1(t_0)-\mu)(x_n)_r \to 0$ as $r \to \infty$ on $p(X)$, and $\mu \in \sigma_a(R_1(t_0))$.

Next, we take

$$F = \big\{\{x_n\} \in p(X) : x_n \to 0 \text{ on } X\big\}$$

and let $c(X) = p(X)/F$; so, F is $R_1(t)$-invariant. We have to show that F is A_1-invariant and closed in $p(X)$. Let $\{x_n\} \in D(A_1)$ with $x_n \to 0$ on X. We have that $R(\lambda, A)A$ is a bounded operator on X and, as $A_1\{x_n\} \in p(X)$ and the fact that A and $R(t)$ commute, we obtain $R(\lambda, A)Ax_n \to 0$ and $(R(t_0)-\mu)Ax_n \to 0$ on X, and therefore $Ax_n \to 0$ on X, obtaining the first part of the claim. Now, let $\{x_n\}_r \in F$ with $\{x_n\}_r \to \{x_n\}$ on $p(E)$; we want to show that $x_n \to 0$ on X. In fact, for arbitrary $\epsilon > 0$, let r_ϵ and n_ϵ such that

$$\begin{aligned} \sup_n \|(x_n)_r - x_n\|_X &< \epsilon/2 \qquad \text{for } r > r_\epsilon, \\ \|(x_n)_{r_\epsilon}\|_X &< \epsilon/2 \qquad \text{for } n > n_\epsilon, \end{aligned}$$

and thus $\|x_n\|_X < \epsilon$, obtaining the second part of the claim.

Therefore, for $\{\widetilde{x_n}\} = \{x_n\}/F$, we have $\widetilde{R_1}(t)\{\widetilde{x_n}\} = R_1(t)\{\widetilde{x_n}\}$ and $\widetilde{A_1}\{\widetilde{x_n}\} = A_1\{\widetilde{x_n}\}$ which means that $\widetilde{R_1}(t)$ is a k-convoluted semigroup with generator $\widetilde{A_1}$. Moreover, if $(R_1(t)-$

$\mu)\{x_n\}_r \to 0$ as $r \to \infty$ on $p(X)$, then $(\widetilde{R_1}(t) - \mu)\{\widetilde{x_n}\} \to 0$ on $c(X)$, and by Theorem 4.10, there exist $\lambda \in \sigma_p(\widetilde{A})$ and $\{\widetilde{y_n}\} \neq 0$ such that $(\lambda - \widetilde{A_1})\{\widetilde{y_n}\} \to 0$; hence, $(\lambda - A)y_n \to 0$ on X. Also, $\|\{\widetilde{y_n}\}\|_{c(X)} \neq 0$, which implies that there is a subsequence $\{y_{n_s}\}$ on X and $c > 0$ such that $\|y_{n_s}\|_X > c$ for all s and thus $\lambda \in \sigma_a(A)$, concluding the proof. $\square$

For a Banach space X, we define the sun dual of X by $X^\odot := \overline{D(A')}^{X'}$. If A is a closed and linear operator defined on X we define $A^\odot$ as the part of A' in $X^\odot$, where A' denotes the adjoint operator of A. Let $\{R(t)\}_{t\geq 0}$ be a k-convoluted semigroup generated by A, and we denote its dual by $\{R(t)'\}_{t\geq 0}$. Since that in general, $\{R(t)'\}_{t\geq 0}$ may not be strongly continuous on X', we restrict it to $X^\odot$ to obtain a strongly continuous family $\{R^\odot(t)\}_{t\geq 0}$ called the *sun dual resolvent family*, which is defined by $R^\odot(t) = R(t)'|_{X^\odot}, t \geq 0$.

By the Hahn-Banach theorem, we recall that $\sigma_r(A) = \sigma_p(A')$, provided the adjoint A' of A is well defined, i.e. A is densely defined.

Theorem 4.13. *For $\{R(t)\}_{t\geq 0}$ a k-convoluted semigroup on a Banach space X such that $\|R(t)\| \leq M(1 * k)(t)$ and whose generator A generates a C_0-semigroup, we have*

$$\sigma_r(R(t)) \cup \{0\} = \left\{ \int_0^t k(t-s)e^{\lambda s}\, ds : \lambda \in \sigma_r(A) \right\} \cup \{0\}. \tag{4.12}$$

Proof. By hypothesis over A, this operator is densely defined, and so we may apply [**65**, Theorem 4.1] to obtain that the sun dual operator $R(t)^\odot$ is strongly continuous with generator $A^\odot$. Moreover, by definition of the sun dual, $R(t)^\odot = R(t)'\big|_{X^\odot}$, where $R(t)'$ denotes the dual of $R(t)$, and by [**65**, Proposition 5.1], we have $\sigma_p(R(t)^\odot) = \sigma_p(R(t)')$ and $\sigma_p(A^\odot) = \sigma_p(A')$, where A' denotes the adjoint operator of A. Now, $\mu \in \sigma_r(R(t_0)) \setminus \{0\}$ implies $\mu \in \sigma_p(R(t_0)') \setminus \{0\}$, and by Theorem 4.10, there exists $\lambda \in \sigma_p(A') = \sigma_r(A)$ such that

$$\mu = \int_0^{t_0} k(t_0 - s)e^{\lambda s}\, ds,$$

concluding the proof. $\square$

Remark 4.14. We notice that if A generates a k-convoluted semigroup $\{R(t)\}_{t\geq 0}$ on a Banach space X and A has dense domain in X, then the Theorem 4.13 is also true without the assumption that A is the generator of a C_0-semigroup.

CHAPTER 5

Abstract cosine and sine functions on time scales

Abstract cosine and sine functions defined on a Banach space, are useful tools in the study of wide classes of abstract evolution equations. In this Chapter, we introduce a definition of cosine and sine functions on time scales, which unify the continuous, discrete and cases which are between these ones, by means of using Laplace transforms on time scales. This approach is much more general and encompasses all time scales $\mathbb{T}_0$ satisfying $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = +\infty$. Further, using such approach, we are able to deal with several different types of time scales. We study the relationship between the cosine function on time scales and its infinitesimal generator, proving several properties concerning it. Next, we prove several properties of the time scales which have the semigroup property and show how it is restrictive. More precisely, we prove that a definition by means of the functional equation is not possible for a wide class of time scales (see Lemma 5.23), and for a wide class of graininess functions (see Theorem 5.29). For instance, the obtained results to this type of problem would be very restrictive, and it would not include hybrid time scales. Moreover, it would not include even the quantum scale, which has several applications in quantum physics. Also, we study the sine functions on time scales, presenting their main properties. Finally, we apply our theory to study the homogeneous and inhomogeneous abstract Cauchy problem on time scales in Banach spaces.

1. Laplace transform on time scales

Let $\mathbb{T}_0$ be a time scale such that $0 \in \mathbb{T}$ and $\sup \mathbb{T}_0 = +\infty$, and let us denote by $\mathbb{T}_0^+ = \mathbb{T}_0 \cap \mathbb{R}_+$. Also, note that if $\lambda \in \mathcal{R}$ is constant, then $\ominus\lambda \in \mathcal{R}$ and $e_{\ominus\lambda}(t,0)$ is well defined on $\mathbb{T}_0$ (see [**17**]). Now, we recall the notion of Laplace transform on time scales and we present important properties which will be essential to our purposes.

Definition 5.1. Assume $x : \mathbb{T}_0 \to \mathbb{R}$ is a regulated function. Then, the ***Laplace transform on time scales*** of x is defined by

$$\widehat{x}(\lambda) = \mathcal{L}\{x\}(\lambda) := \int_0^\infty x(t) e^{\sigma}_{\ominus\lambda}(t,0)\, \Delta t,$$

for $\lambda \in D\{x\}$, where

$$D\{x\} = \left\{\lambda \in \mathbb{C} : \int_0^\infty x(t)e^\sigma_{\ominus\lambda}(t,0)\,\Delta t \text{ exists}\right\}.$$

Theorem 5.2. [**17**, Theorem 3.84] *Assume f and g are regulated functions on $\mathbb{T}_0$, and α and β are constants. Then*

$$\mathcal{L}\{\alpha x + \beta y\}(\lambda) = \alpha\mathcal{L}\{x\}(\lambda) + \beta\mathcal{L}\{y\}(\lambda)$$

for $\lambda \in D\{x\} \cap D\{y\}$.

Theorem 5.3. [**17**, Lemma 3.85] *If $\lambda \in \mathbb{C}$ is regressive, then*

$$e^\sigma_{\ominus\lambda}(t,0) = \frac{e_{\ominus\lambda}(t,0)}{1+\mu(t)\lambda} = -\frac{(\ominus\lambda)(t)}{\lambda}e_{\ominus\lambda}(t,0).$$

Theorem 5.4. [**41**, Theorem 3.12] *Let $f : \mathbb{T}_0^+ \to X$ be an rd-continuous and delta-differentiable function. If $\lambda \in \mathbb{C}$ is such that*

$$\lim_{t\to\infty} f(t)e_{\ominus\lambda}(t,0) = 0$$

when $\mathrm{Re}_\mu(\lambda)(t) > 0$ for all $t \in \mathbb{T}_0^+$ and $\widehat{f^\Delta}(\lambda)$ exists, then $\widehat{f}(\lambda)$ exists and $\widehat{f^\Delta}(\lambda) = \lambda\widehat{f}(\lambda) - f(0)$.

Corollary 5.5. [**41**, Corollary 3.13] *Let $f \in \mathcal{C}_{rd}(\mathbb{T}_0^+, X)$ and $F(t) = \int_0^t f(s)\,\Delta s$. Let $\lambda \in \mathbb{C} \setminus \{0\}$ for all $t \in \mathbb{T}_0^+$ such that $\widehat{f}(\lambda)$ exists and $\lim_{t\to\infty} f(t)e_{\ominus\lambda}(t,0) = 0$, then $\widehat{F}(\lambda) = \widehat{f}(\lambda)/\lambda$.*

In the sequel, we recall the notion of strongly rd-continuous functions, and C_0-semigroups on time scales and its infinitesimal generator A. We denote by $\mathcal{B}(X)$ the space consisting of all bounded operators from X into itself, endowed with the strong topology. See [**41**] for more details.

Definition 5.6. A function $T : \mathbb{T}_0^+ \to \mathcal{B}(X)$ is ***strongly rd-continuous*** if one of the following conditions are satisfied:

(1) If 0 is right-dense, then $\|T(t)x - x\| \to 0$ as $t \to 0^+$.
(2) If 0 is right-scattered, then $\|T(0)x - x\| = 0$.

Definition 5.7. We say that $T : \mathbb{T}_0^+ \to \mathcal{B}(X)$ is a ***C_0-semigroup with infinitesimal generator*** A if the following conditions are satisfied:

(1) $T(0) = I$, and for every $x \in X$, the function $t \mapsto T(t)x$ is strongly rd-continuous.

(2) There exists λ_0 such that $(\lambda_0, \infty)_{\mathbb{T}} \subset \rho(A)$, $\lambda \in D\{T\}$, and

$$\widehat{T}(\lambda)x = \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)T(t)x\,\Delta t = (\lambda - A)^{-1}x, \quad x \in X,$$

for all $\mathrm{Re}_\mu(\lambda)(t) > \lambda_0$ and $t \in \mathbb{T}_0^+$.

Definition 5.8. Given an abstract C_0-semigroup $T : \mathbb{T}_0^+ \to \mathcal{B}(X)$ on time scales, we define the ***infinitesimal generator*** A by means of:

$$D(A) = \begin{cases} x \in X : \lim\limits_{t\to 0} \dfrac{T(t)x - x}{t} \text{ exists}, & \text{if 0 is right-dense} \\ \\ x \in X : \dfrac{T(\sigma(0))x - x}{\mu(0)} \text{ exists}, & \text{if 0 is right-scattered} \end{cases}$$

and

$$Ax = \begin{cases} \lim\limits_{t\to 0} \dfrac{T(t)x - x}{t}, & \text{if 0 is right-dense} \\ \\ \dfrac{T(\sigma(0))x - x}{\mu(0)}, & \text{if 0 is right-scattered} \end{cases}$$

In what follows, we present some properties of abstract C_0-semigroups on time scales.

Theorem 5.9. [**41**, Theorem 3.15] *Let A be a closed linear operator in X, and $f, g \in L^1_{loc}(\mathbb{T}_0^+, X)$ such that $\omega \in D\{f\} \cap D\{g\}$. Then, the following assertions are equivalent:*

(1) *$f(t) \in D(A)$ and $Af(t) = g(t)$ a.e. on $\mathbb{T}_0^+$.*
(2) *$\widehat{f}(\lambda) \in D(A)$ and $A\widehat{f}(\lambda) = \widehat{g}(\lambda)$ whenever $\mathrm{Re}_\mu(\lambda)(t) > \mathrm{Re}_\mu(\omega)(t)$ for all $t \in \mathbb{T}_0^+$.*

Theorem 5.10. [**41**, Corollary 3.16] *Let A be a linear operator in X with nonempty resolvent set, and let $T(t) \in \mathcal{B}(X)$. The following assertions are equivalent:*

(1) *$(\lambda - A)^{-1}T(t) = T(t)(\lambda - A)^{-1}$ for all $\lambda \in \rho(A)$.*
(2) *$(\lambda - A)^{-1}T(t) = T(t)(\lambda - A)^{-1}$ for some $\lambda \in \rho(A)$.*
(3) *For all $x \in D(A)$, $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$.*

Theorem 5.11. [**41**, Theorem 4.9] *Let T be a C_0-semigroup with infinitesimal generator A. Let $\mathrm{Re}_\mu(c)(t) \in D\{T\}$ for every $t \in \mathbb{T}_0^+$. Then, the following conditions hold:*

(1) *Let $B \in \mathcal{B}(X)$ such that $B\widehat{T}(\lambda) = \widehat{T}(\lambda)B$ whenever $\mathrm{Re}_\mu(\lambda)(t) > \mathrm{Re}_\mu(c)(t)$ for every $t \in \mathbb{T}_0^+$. Then $BT(t) = T(t)B$ for all $t \in \mathbb{T}_0^+$.*
(2) *In particular, $T(t)T(s) = T(s)T(t)$ for all $t, s \in \mathbb{T}_0^+$.*

2. Abstract cosine function on time scales

Let $\mathbb{T}_0$ be a time scale such that $0 \in \mathbb{T}$ and $\sup \mathbb{T}_0 = +\infty$, and A be a closed linear operator in a Banach space X. In this section, our goal is to study the following abstract Cauchy second order problem on time scales:

$$\begin{cases} u^{\Delta\Delta}(t) &= Au(t), \quad t \in \mathbb{T}_0^+, \\ u(0) &= x, \\ u^{\Delta}(0) &= y, \end{cases} \tag{5.1}$$

where $x, y \in X$. We denote the class of functions $f : \mathbb{T} \to X$ that are twice delta-differentiable and the second delta-derivative is rd-continuous by $\mathcal{C}^2_{rd} = \mathcal{C}^2_{rd}(\mathbb{T}, X)$.

Remark 5.12. A big deal when one is studying a second-order dynamic equation on time scales is to choose the appropriate formulation of the problem, because it is a known fact that there are several ways to formulate a second order dynamic equation on time scales. For instance, a very usual way to define the problem (5.1) is through the following formulation:

$$\begin{cases} u^{\Delta\Delta}(t) &= Au^{\sigma}(t), \quad t \in \mathbb{T}_0^+, \\ u(0) &= x, \\ u^{\Delta}(0) &= y, \end{cases} \tag{5.2}$$

but we will show here that this formulation is not appropriate in the case of abstract cosine function on time scales (see Remark 5.17). Moreover, if we look into the definition of hyperbolic cosine and sine functions, and cosine and sine functions, they are considered by problem with a formulation similar to the equation (5.1), not using the one described by equation (5.2) (see [**17**]). Therefore, the most natural is to consider the problem using the formulation (5.1) instead of (5.2), since here we are interested in the study of abstract cosine and sine functions on time scales. Also, another natural question which appears when we are dealing with second order dynamic equations on time scales concerns about the possibility to consider the problem using nabla and delta derivatives, instead of only delta-derivatives or only nabla-derivatives. In this case, our problem would have the following formulation

$$\begin{cases} u^{\nabla\Delta}(t) &= Au^{\sigma}(t), \quad t \in \mathbb{T}_0^+, \\ u(0) &= x, \\ u^{\nabla}(0) &= y. \end{cases} \tag{5.3}$$

The formulation given by equation (5.3) is not appropriate in our case again, because the well-known property described in Corollary 5.5 does not remain true if we define $F(t) =$

$\int_0^t f(s)\,\nabla s$ instead of $F(t) = \int_0^t f(s)\,\Delta s$. This fact makes that the usual and classical properties of abstract cosine function on time scales do not coincide in the case $\mathbb{T} = \mathbb{R}$, which is not interesting for our purpose of generalization and extension of the continuous, discrete, hybrid results, among others. Finally, the other motivation for not to consider the equation (5.1) follows from the fact that neither the hyperbolic cosine and sine on time scales, nor cosine and sine functions on time scales are considered through equations of this type. See [**17**], for details.

Definition 5.13. A ***classical solution*** is a function $u \in \mathcal{C}^2_{rd}(\mathbb{T}_0^+, X)$ such that $u(t) \in D(A)$ for every $t \in \mathbb{T}_0^+$ and satisfies the problem (5.1).

In the sequel, we introduce a more general definition of the solution for the problem (5.1).

Definition 5.14. A ***mild solution*** is a function $u \in \mathcal{C}_{rd}(\mathbb{T}_0^+, X)$ such that

$$\int_0^t \int_0^s u(r)\,\Delta r\,\Delta s = \int_0^t (t - \sigma(s))u(s)\,\Delta s \in D(A)$$

and for all $t \in \mathbb{T}_0^+$,

$$u(t) = x + ty + A\int_0^t (t - \sigma(s))u(s)\,\Delta s. \tag{5.4}$$

In what follows, we present a strong connection between a mild solution and a classical solution of the problem (5.1).

Theorem 5.15. *A mild solution u of the problem* (5.1) *is a classical solution if, and only if, $u \in \mathcal{C}^2_{rd}(\mathbb{T}_0^+, X)$.*

Proof. If u is a classical solution of (5.1), then by the definition, it follows directly that $u \in \mathcal{C}^2_{rd}(\mathbb{T}_0^+, X)$, obtaining the desired result. Conversely, assume that $u \in \mathcal{C}^2_{rd}(\mathbb{T}_0^+, X)$ is a mild solution of (5.1) and let $t \in \mathbb{T}_0^+$. Integrating and applying the Fundamental Theorem of Calculus for Δ-integrals [**18**, Theorem 5.34] in the equation (5.1), we have

$$u^{\Delta}(t) - y = A\int_0^t u(s)\,\Delta s, \tag{5.5}$$

and then, integrating and applying [**18**, Theorem 5.34] again in (5.5), we obtain

$$\begin{aligned} u(t) - x - ty &= A\int_0^t \int_0^s u(r)\,\Delta r\,\Delta s \\ &= A\int_0^t \int_{\sigma(r)}^t u(r)\,\Delta s\,\Delta r \end{aligned}$$

$$= \quad A\int_0^t (t-\sigma(r))u(r)\,\Delta r,$$

where in the second equality, we use the change of order of integration (see [**16**] for details), concluding the proof. □

Now, we present a result which brings an important property of a mild solution of (5.1).

Theorem 5.16. *Let $u \in \mathcal{C}_{rd}(\mathbb{T}_0^+, X)$, $\omega \in D\{u\}$. Assume that $\lambda \in \mathbb{C}$ is such that*

$$\lim_{t\to\infty} u(t)e_{\ominus\lambda}(t,0) = 0, \quad \lim_{t\to\infty} e_{\ominus\lambda}(t,0)\int_0^t u(s)\,\Delta s = 0, \tag{5.6}$$

for every λ with $\mathrm{Re}_\mu(\lambda)(t) > \mathrm{Re}_\mu(\omega)(t)$ for all $t \in \mathbb{T}_0^+$. Then, u is a mild solution of (5.1) *if, and only if,*

$$\widehat{u}(\lambda) \in D(A) \ \text{ and } \ \lambda x + y = (\lambda^2 - A)\widehat{u}(\lambda).$$

for all λ which satisfies (5.6).

Proof. Define $v(t) := \displaystyle\int_0^t u(s)\,\Delta s$ and $w(t) = \displaystyle\int_0^t v(s)\,\Delta s$, for every $t \in \mathbb{T}_0^+$. Therefore, by hypothesis and by Corollary 5.5, we get

$$\widehat{v}(\lambda) = \frac{\widehat{u}(\lambda)}{\lambda} \quad \text{and} \quad \widehat{w}(\lambda) = \frac{\widehat{v}(\lambda)}{\lambda};$$

It implies that

$$\widehat{w}(\lambda) = \frac{\widehat{u}(\lambda)}{\lambda^2}.$$

On the other hand, we have

$$\frac{\widehat{u}(\lambda)}{\lambda^2} = \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t v(s)\,\Delta s\,\Delta t. \tag{5.7}$$

Then, by hypothesis and by Theorem 5.9, we get

$$\frac{\widehat{u}(\lambda)}{\lambda^2} = \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t v(s)\,\Delta s\,\Delta t \in D(A).$$

Also, if u is a mild solution of (5.1) and applying Theorem 5.9 again, we have

$$\begin{aligned} \widehat{u}(\lambda) &= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)u(t)\,\Delta t \\ &= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)x\,\Delta t + \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)ty\,\Delta t \\ &\quad + A\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t (t-\sigma(s))u(s)\,\Delta s\,\Delta t. \end{aligned} \tag{5.8}$$

Now, note that by properties of the exponential function (Theorem 1.18 and Lemma 1.19),

$$\int_0^\infty e_{\ominus\lambda}^\sigma(t,0)x\,\Delta t = \int_0^\infty -\frac{\ominus\lambda}{\lambda}e_{\ominus\lambda}(t,0)x\,\Delta t = -\frac{1}{\lambda}\big(e_{\ominus\lambda}(t,0)x\big)_{t=0}^{t\to+\infty} = \frac{x}{\lambda}. \tag{5.9}$$

Next, using integration by parts (Theorem 1.11), we have

$$\begin{aligned}\int_0^\infty e_{\ominus\lambda}^\sigma(t,0)ty\,\Delta t &= \int_0^\infty -\frac{\ominus\lambda}{\lambda}e_{\ominus\lambda}(t,0)ty\,\Delta t = \int_0^\infty -\frac{1}{\lambda}e_{\ominus\lambda}^\Delta(t,0)ty\,\Delta t\\ &= -\frac{1}{\lambda}\big(e_{\ominus\lambda}(t,0)ty\big)_{t=0}^{t\to+\infty} + \frac{1}{\lambda}\int_0^\infty e_{\ominus\lambda}^\sigma(t,0)y\,\Delta t\\ &= \frac{1}{\lambda}\int_0^\infty -\frac{\ominus\lambda}{\lambda}e_{\ominus\lambda}(t,0)y\,\Delta t\\ &= -\frac{1}{\lambda^2}\big(e_{\ominus\lambda}(t,0)y\big)_{t=0}^{t\to+\infty} = \frac{y}{\lambda^2}. \end{aligned}\tag{5.10}$$

Finally, we have

$$\begin{aligned}A\int_0^\infty e_{\ominus\lambda}^\sigma(t,0)\int_0^t(t-s)u(s)\,\Delta s\,\Delta t &= A\int_0^\infty e_{\ominus\lambda}^\sigma(t,0)\int_0^t\int_0^s u(r)\,\nabla r\,\Delta s\,\Delta t\\ &= A\int_0^\infty e_{\ominus\lambda}^\sigma(t,0)\int_0^t v(s)\,\Delta s\,\Delta t\\ &= A\,\frac{\widehat{u}(\lambda)}{\lambda^2}. \end{aligned}\tag{5.11}$$

Combining (5.8) with (5.9), (5.10) and (5.11), we get

$$\widehat{u}(\lambda) = \frac{x}{\lambda} + \frac{y}{\lambda^2} + A\frac{\widehat{u}(\lambda)}{\lambda^2}, \tag{5.12}$$

obtaining the result. To prove the converse, we follow the same idea. $\square$

Remark 5.17. If we consider equation (5.2) instead of (5.1) then we do not get the equation (5.12), but we obtain

$$\widehat{u}(\lambda) = \frac{x}{\lambda} + \frac{y}{\lambda^2} + A\frac{\widehat{u^\sigma}(\lambda)}{\lambda^2}. \tag{5.13}$$

In the case of $\mathbb{T}=\mathbb{Z}$ using this property $\widehat{u^\sigma}(\lambda) = \lambda(\widehat{u}(\lambda)-u(0))$, (by [**17**, p. 122]) we have

$$\begin{aligned}\widehat{u}(\lambda) &= \frac{x}{\lambda} + \frac{y}{\lambda^2} + A\frac{\lambda(\widehat{u}(\lambda)-u(0))}{\lambda^2},\\ &= (I-A)\frac{x}{\lambda} + \frac{y}{\lambda^2} + A\frac{(\widehat{u}(\lambda))}{\lambda}.\end{aligned}$$

Therefore, if $y=0$, then we have

$$(\lambda - A)\widehat{u}(\lambda) = (I-A)x,$$

hence $\widehat{u}(\lambda) = (\lambda - A)^{-1}(I - A)x$ instead of $\widehat{u}(\lambda) = (\lambda^2 - A)^{-1}x$, which is the expected solution to the second order Cauchy problem. Thus, it does not make sense to consider the formulation of the problem by equation (5.2).

As an immediate consequence, we have the following result.

Corollary 5.18. *Let $u \in \mathcal{C}_{rd}(\mathbb{T}_0^+, X)$, $\omega \in D\{u\}$ and suppose that the conditions* (5.6) *hold for every λ with $\mathrm{Re}_\mu(\lambda)(t) > \mathrm{Re}_\mu(\omega)(t)$ for all $t \in \mathbb{T}_0^+$. Then, u is a mild solution of the problem* (5.1), *if and only if,*

$$\widehat{u}(\lambda) \in D(A) \ \text{ and } \ \widehat{u}(\lambda) = \lambda(\lambda^2 - A)^{-1}x + (\lambda^2 - A)^{-1}y. \tag{5.14}$$

Now, let us present the definition of cosine function on time scales. From now on, we assume $u \in \mathcal{C}_{rd}(\mathbb{T}_0^+, X)$, $\omega \in D\{u\}$, and $\mathrm{Re}_\mu(\lambda)(t) > \mathrm{Re}_\mu(\omega)(t)$.

Definition 5.19. We say that a strongly rd-continuous function $C : \mathbb{T}_0^+ \to \mathcal{B}(X)$ is a ***cosine function with infinitesimal generator*** A if the following condition is satisfied: there exists ω such that $(\omega, \infty)_\mathbb{T} \subset \rho(A)$, $\lambda^2 \in D\{C\}$, and

$$\widehat{C}(\lambda)x = \int_0^\infty e^\sigma_{\ominus\lambda}(t, 0)C(t)x\,\Delta t = \lambda(\lambda^2 - A)^{-1}x, \quad x \in X,$$

for all $\mathrm{Re}_\mu(\lambda^2)(t) > \omega$ and $t \in \mathbb{T}_0$.

Next, we present some fundamental properties of a cosine function C on time scales, and their relations with the generator A.

Proposition 5.20. *Let $C : \mathbb{T}_0^+ \to \mathcal{B}(X)$ be a cosine function on X and let A be its generator. Then the following assertions hold:*

a) $\int_0^t (t - \sigma(s))C(s)x\,\Delta s \in D(A)$ *and* $A\int_0^t (t - \sigma(s))C(s)x\,\Delta s = C(t)x - x$ *for all* $x \in X, t \in \mathbb{T}_0^+$.
b) *Let $x \in D(A)$, $\mathrm{Re}_\mu(c)(t) \in D\{C\}$ and suppose $A\widehat{C}(\lambda) = \widehat{C}(\lambda)A$ when $\mathrm{Re}_\mu(\lambda^2)(t) > \mathrm{Re}_\mu(c)(t)$, then $C(t)x \in D(A)$ and $AC(t)x = C(t)Ax$ for all $t \in \mathbb{T}_0^+$.*
c) *Let $x, y \in X$, then $x \in D(A)$ and $Ax = y$ if, and only if, $\int_0^t (t - \sigma(s))C(s)y\,\Delta s = C(t)x - x$ for all $t \in \mathbb{T}_0^+$.*
d) *If 0 is right-dense, then* $D(A) = \left\{x \in X : \lim\limits_{h \to 0^+} \dfrac{2(C(h)x - x)}{h^2} \text{ exists}\right\}$, *and*

$$Ax = \lim_{h \to 0^+} \frac{2(C(h)x - x)}{h^2}.$$

e) *If* 0 *and* $\sigma(0)$ *are right-scattered, then*

$$D(A)=\Big\{x\in X:\frac{\big(C(\sigma(\sigma(0)))-C(\sigma(0))\big)x}{\mu(\sigma(0))\mu(0)}+\frac{\big(C(0)-C(\sigma(0))\big)x}{\mu(0)^2}\text{ is well}-\text{defined}\Big\},$$

and

$$Ax=\frac{\big(C(\sigma(\sigma(0)))-C(\sigma(0))\big)x}{\mu(\sigma(0))\mu(0)}+\frac{\big(C(0)-C(\sigma(0))\big)x}{\mu(0)^2}.$$

f) *If* 0 *is right-scattered and* $\sigma(0)$ *is right-dense, then*

$$D(A)=\Big\{x\in X:\lim_{h\to 0^+}\frac{\big(C(\sigma(0)+h)-C(\sigma(0))\big)x}{\mu(0)h}+\frac{\big(C(0)-C(\sigma(0)\big)x}{\mu(0)}\text{ exists}\Big\},$$

and

$$Ax=\lim_{h\to 0^+}\frac{\big(C(\sigma(0)+h)-C(\sigma(0))\big)x}{\mu(0)h}+\frac{(C(0)-C(\sigma(0))x}{\mu(0)}.$$

Proof. a) Let $\lambda\in\mathbb{C}$ such that the condition (5.6) is satisfied, $\mathrm{Re}_\mu(\lambda)(t)>\mathrm{Re}_\mu(c)(t)$ and $\mathrm{Re}_\mu(\lambda^2)(t)>\mathrm{Re}_\mu(c)(t)$ for all $t\in\mathbb{T}_0^+$. Considering $y=0$ in the equation (5.14), it follows by the definition of the abstract cosine function that C satisfies the equation (5.14). Then by Corollary 5.18, it follows that C is a mild solution of (5.1) for $y=0$. Therefore, the result follows by applying the definition of a mild solution.

b) For $x\in X$ and $\mathrm{Re}_\mu(\lambda)(t)>\mathrm{Re}_\mu(c)(t)$ for $t\in\mathbb{T}_0^+$, we have

$$\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)C(t)Ax\,\Delta t=\widehat{C}(\lambda)Ax=A\widehat{C}(\lambda)x=\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)AC(t)x\,\Delta t,$$

which implies by the uniqueness of Laplace transform [**41**, Theorem 3.14], that $AC(t)x=C(t)Ax$ for all $t\in\mathbb{T}_0^+$.

c) Define $C_1(t)=\int_0^t C(s)\,\nabla s$, and also $C_2(t)=\int_0^t C_1(s)\,\Delta s$. Let $\lambda\in\mathbb{C}$ such that the condition (5.6) is satisfied, $\mathrm{Re}_\mu(\lambda)(t)>\mathrm{Re}_\mu(c)(t)$ and $\mathrm{Re}_\mu(\lambda^2)(t)>\mathrm{Re}_\mu(c)(t)$ for all $t\in\mathbb{T}_0^+$. By hypothesis and by Corollary 5.5, we get:

$$\widehat{C_1}(\lambda)=\frac{\widehat{C}(\lambda)}{\lambda},\quad \widehat{C_2}(\lambda)=\frac{\widehat{C_1}(\lambda)}{\lambda},\quad\text{and}\quad \widehat{C_2}(\lambda)=\frac{\widehat{C}(\lambda)}{\lambda^2}. \tag{5.15}$$

Assume that $\int_0^t(t-\sigma(s))C(s)y\,\Delta s=C(t)x-x$. Using Laplace transform, we get

$$\begin{aligned}\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t(t-\sigma(s))C(s)y\,\Delta s\,\Delta t &= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t\int_0^s C(r)y\,\Delta r\,\Delta s\,\Delta t\\ &= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t C_1(s)y\,\Delta s\,\Delta t\\ &= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)C_2(t)y\,\Delta t\end{aligned}$$

$$= \frac{1}{\lambda}(\lambda^2 - A)^{-1}y, \tag{5.16}$$

by (5.15) and the definition of cosine function. On the other hand, applying again the definition of cosine function and using (5.9), we have

$$\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\big(C(t)x - x\big)\,\Delta t = \lambda(\lambda^2 - A)^{-1}x - \frac{x}{\lambda},$$

which implies, by (5.16) and by the uniqueness of the Laplace transform [**41**, Theorem 3.14],

$$\begin{aligned}
\frac{1}{\lambda}(\lambda^2 - A)^{-1}y &= \lambda(\lambda^2 - A)^{-1}x - \frac{x}{\lambda}\\
(\lambda^2 - A)^{-1}y &= \lambda^2(\lambda^2 - A)^{-1}x - x\\
(\lambda^2 - A)(\lambda^2 - A)^{-1}y &= (\lambda^2 - A)\lambda^2(\lambda^2 - A)^{-1}x - (\lambda^2 - A)x\\
y &= \lambda^2 x - (\lambda^2 - A)x \;=\; Ax
\end{aligned}$$

for $\mathrm{Re}_\mu(\lambda^2)(t) > \mathrm{Re}_\mu(\omega)(t)$, proving the first part of the claim.

Conversely, let $x, y \in X$ be such that $x \in D(A)$ and $Ax = y$. Notice that

$$\begin{aligned}
A\widehat{C}(\lambda)x &= A\lambda(\lambda^2 - A)^{-1}x \;=\; \lambda(\lambda^2 - A)^{-1}Ax \;=\; \lambda(\lambda^2 - A)^{-1}y\\
&= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)C(t)y\,\Delta t \;=\; \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)C(t)Ax\,\Delta t \;=\; \widehat{C}(\lambda)Ax,
\end{aligned}$$

and by part a), we have

$$\begin{aligned}
C(t)x - x &= A\int_0^t (t - \sigma(s))C(s)x\,\Delta s \;=\; \int_0^t (t - \sigma(s))AC(s)x\,\Delta s\\
&= \int_0^t (t - \sigma(s))C(s)Ax\,\Delta s \;=\; \int_0^t (t - \sigma(s))C(s)y\,\Delta s
\end{aligned}$$

where we used assertion b).

d) Let $x \in D(A)$ and $Ax = y$. From assertion c), it follows that

$$\begin{aligned}
\frac{2}{t^2}(C(t)x - x) - y &= \frac{2}{t^2}\int_0^t (t - \sigma(s))C(s)y\,\Delta s - y \qquad (5.17)\\
&= \frac{2}{t^2}\int_0^t (t - \sigma(s))\big(C(s)y - y\big)\,\Delta s.
\end{aligned}$$

Since 0 is right-dense, then it is possible to find a sequence $\{\delta_n\} \subset \mathbb{T}_0^+$ such that $\delta_n \to 0^+$ as $n \to \infty$. Since σ is rd-continuous and 0 is right-dense, it follows that $\lim_{n\to\infty}\sigma(\delta_n) = \sigma(0) = 0$. Hence, by (5.17) and by applying L'Hopital, we have

$$\lim_{n\to+\infty}\frac{2(C(\sigma(\delta_n))x - x)}{(\sigma(\delta_n))^2} - y \;=\; \lim_{n\to+\infty}\frac{2}{(\sigma(\delta_n))^2}\int_0^{\sigma(\delta_n)} (\sigma(\delta_n) - \sigma(s))(C(s)y - y)\,\Delta s$$

$$
\begin{aligned}
&= \lim_{n\to+\infty}\frac{2}{(\sigma(\delta_n))^2}\int_0^{\delta_n}(\sigma(\delta_n)-\sigma(s))(C(s)y-y)\,\Delta s\\
&\quad+\lim_{n\to+\infty}\frac{2}{(\sigma(\delta_n))^2}\int_{\delta_n}^{\sigma(\delta_n)}(\sigma(\delta_n)-\sigma(s))(C(s)y-y)\,\Delta s\\
&= \lim_{n\to+\infty}\frac{1}{\sigma(\delta_n)}(\sigma(\delta_n)-\sigma(\delta_n))(C(\sigma(\delta_n)y-y)\\
&\quad+\lim_{n\to+\infty}\frac{1}{(\sigma(\delta_n))^2}(\sigma(\delta_n)-\sigma(\delta_n))(C(\delta_n)y-y)\mu(\delta_n)\\
&= 0.
\end{aligned}
$$

Therefore,

$$
\lim_{\delta\to0^+}\frac{2}{t^2}(C(t)x-x)=y. \tag{5.18}
$$

Conversely, let $x, y \in X$ be such that (5.18) holds. From part a), we have

$$
A\frac{2}{t^2}\int_0^t(t-s)C(s)x\,\Delta s=\frac{2}{t^2}\big(C(t)x-x\big)\to y \quad \text{as } t\to0^+.
$$

Therefore, by (5.18) and since A is closed, it follows that $x\in D(A)$ and $Ax=y$.

e) Let $x \in D(A)$, and $Ax = y$. From assertion c), it follows that

$$
\begin{aligned}
&\frac{\big(C(\sigma(\sigma(0)))-C(\sigma(0))\big)x}{\mu(\sigma(0))\mu(0)}+\frac{(C(0)-C(\sigma(0)))x}{\mu(0)^2}\\
&= \frac{1}{\mu(\sigma(0))\mu(0)}\left[\int_0^{\sigma(\sigma(0))}(\sigma(\sigma(0))-\sigma(s))C(s)y\,\Delta s-\int_0^{\sigma(0)}(\sigma(0)-\sigma(s))C(s)y\,\Delta s\right]\\
&\quad+\frac{1}{\mu(0)^2}\int_0^{\sigma(0)}(\sigma(0)-\sigma(s))C(s)y\,\Delta s\\
&= \frac{1}{\mu(\sigma(0))\mu(0)}\left[\int_0^{\sigma(0)}(\sigma(\sigma(0))-\sigma(s))C(s)y\,\Delta s+\int_{\sigma(0)}^{\sigma(\sigma(0))}(\sigma(\sigma(0))-\sigma(s))C(s)y\,\Delta s\right.\\
&\quad\left.-\int_0^{\sigma(0)}(\sigma(0)-\sigma(s))C(s)y\,\Delta s\right]+\frac{1}{\mu(0)^2}(\sigma(0)-\sigma(0))C(0)\mu(0)y\\
&= \frac{1}{\mu(\sigma(0))\mu(0)}\Big[\big(\sigma(\sigma(0))-\sigma(0)\big)\mu(0)C(0)y+\big(\sigma(\sigma(0))-\sigma(\sigma(0))\big)\mu(\sigma(0))C(\sigma(0))y\\
&\quad-\frac{1}{\mu(\sigma(0))\mu(0)}(\sigma(0)-\sigma(0))\mu(0)C(0)y\Big]\\
&= \frac{1}{\mu(\sigma(0))\mu(0)}\mu(\sigma(0))\mu(0)C(0)y=y,
\end{aligned}
$$

proving the claim.

Conversely, let $x, y \in X$ be such that

$$\frac{\big(C(\sigma(\sigma(0))) - C(\sigma(0))\big)x}{\mu(\sigma(0))\mu(0)} + \frac{(C(0) - C(\sigma(0)))x}{\mu(0)^2} = y.$$

From part a), we have

$$\begin{aligned} & A\frac{1}{\mu(\sigma(0))\mu(0)}\left[\int_0^{\sigma(\sigma(0))}(\sigma(\sigma(0)) - \sigma(s))C(s)y\,\Delta s - \int_0^{\sigma(0)}(\sigma(0) - \sigma(s))C(s)y\,\Delta s\right] \\ & -A\frac{1}{\mu(0)^2}\int_0^{\sigma(0)}(\sigma(0) - \sigma(s))C(s)y\,\Delta s \\ = \; & \frac{(C(\sigma(\sigma(0))) - C(\sigma(0)))x}{\mu(\sigma(0))\mu(0)} + \frac{(C(0) - C(\sigma(0)))x}{\mu(0)^2} = y, \end{aligned}$$

getting the desired result. Therefore, $x \in D(A)$ and $Ax = y$.

f) Let $x \in D(A)$, and $Ax = y$. From part c) and using Theorem 1.11 (citar), it follows that

$$\begin{aligned} \frac{C(\sigma(0) + h)x - C(\sigma(0))x}{\mu(0)h} &= \frac{1}{\mu(0)h}\Big(C(\sigma(0) + h)x - x - C(\sigma(0))x + x\Big) \\ &= \frac{1}{\mu(0)h}\bigg(\int_0^{\sigma(0)+h}(\sigma(0) + h - \sigma(s))C(s)y\,\Delta s \\ &\quad - \int_0^{\sigma(0)}(\sigma(0) - \sigma(s))C(s)y\,\Delta s\bigg) \\ &= \frac{1}{\mu(0)h}\bigg(\int_0^{\sigma(0)} hC(s)y\,\Delta s + \int_{\sigma(0)}^{\sigma(0)+h}(\sigma(0) + h - \sigma(s))C(s)y\,\Delta s\bigg) \\ &= \frac{1}{\mu(0)h}\bigg(h\mu(0)C(0)y + \int_{\sigma(0)}^{\sigma(0)+h}(\sigma(0) + h - \sigma(s))C(s)y\,\Delta s\bigg) \\ &= y + \frac{1}{\mu(0)h}\bigg(\int_{\sigma(0)}^{\sigma(0)+h}(\sigma(0) + h - \sigma(s))C(s)y\,\Delta s\bigg), \end{aligned}$$

from which, applying limit when $h \to 0^+$ and using L'Hopital theorem, we get

$$\begin{aligned} \lim_{h\to 0^+}\frac{C(\sigma(0) + h)x - C(\sigma(0))x}{\mu(0)h} &= \lim_{h\to 0^+}\left[y + \int_{\sigma(0)}^{\sigma(0)+h}(\sigma(0) + h - \sigma(s))C(s)y\,\Delta s\right] \\ &= y + \lim_{h\to 0^+}\frac{1}{\mu(0)}(\sigma(0) + h - \sigma(\sigma(0)))C(\sigma(0))y \\ &= y + \lim_{h\to 0^+}\frac{1}{\mu(0)}(\sigma(0) + h - \sigma(0))C(\sigma(0))y \\ &= y + \lim_{h\to 0^+}\frac{1}{\mu(0)}hC(\sigma(0))y \;=\; y, \end{aligned} \tag{5.19}$$

since $\sigma(0)$ is right-dense. On the other hand, we get

$$\begin{aligned}
-\frac{(C(\sigma(0))-C(0))x}{\mu(0)^2} &= -\frac{1}{\mu(0)^2}\int_0^{\sigma(0)}(\sigma(0)-\sigma(s))C(s)y\,\Delta s \\
&= -\frac{1}{\mu(0)^2}(\sigma(0)-\sigma(0))\mu(0)C(0)y \\
&= 0.
\end{aligned}$$

Therefore, we have

$$\lim_{h\to 0^+}\frac{(C(\sigma(0)+h)-C(\sigma(0)))x}{\mu(0)h}+\frac{(C(0)-C(\sigma(0))x}{\mu(0)}=y, \tag{5.20}$$

obtaining the result. Conversely, suppose that x, y satisfy (5.20). From part a), we have

$$\begin{aligned}
&A\left[\frac{1}{\mu(0)h}\left(\int_0^{\sigma(0)}hC(s)y\,\Delta s+\int_{\sigma(0)}^{\sigma(0)+h}(\sigma(0)+h-\sigma(s))C(s)y\,\Delta s\right)\right] \\
&-A\left[\frac{1}{\mu(0)^2}\int_0^{\sigma(0)}(\sigma(0)-\sigma(s))C(s)y\,\Delta s\right] \\
&\qquad = \frac{(C(\sigma(0)+h)-C(\sigma(0)))x}{\mu(0)h}+\frac{(C(0)-C(\sigma(0))x}{\mu(0)}\to y \quad \text{as } t\to 0^+.
\end{aligned}$$

Therefore, by (5.19) and since A is closed, it follows that $x \in D(A)$ and $Ax = y$. □

Remark 5.21. Notice that the Theorem 5.16 describes all the possibilities for the definition of the generator A, since the case where 0 is right-dense and $\sigma(0)$ is right-scattered at the same time is not possible. Indeed, if 0 is right-dense, then $\sigma(0) = 0$ and, therefore, $\sigma(0)$ has to be right-dense.

3. Time scales with the semigroup property

In this section, let us show that to require that the time scales have the group property restricts a lot the class of time scales, which we can deal.

Among others, we will prove in this section that if the time scale has the group property, then it cannot be *hybrid* for instance, because the only possibilities for the time scale with this property are to have only right-dense points or to have only right-scattered points. Also, in this section, we show that if 0 is right-dense, then the unique possibility for $\mathbb{T}$ is $\mathbb{R}$.

Notice that the classical definition of cosine function is very strong and restrictive by means of the theory of time scales. More precisely, the abstract cosine function is defined by:

$$\begin{cases} 2C(t)C(s) &= C(t+s) + C(t-s), \quad t, s \in \mathbb{R}, \\ \qquad C(0) &= I, \end{cases} \tag{5.21}$$

Notice that if we define the cosine functions on time scales as (5.21), then in order to ensure that the abstract cosine function is well defined, it is necessary to require that the time scale has the group property, which means that the following conditions are satisfied:

(1) $0 \in \mathbb{T}$;
(2) If $a, b \in \mathbb{T}$, then $a - b \in \mathbb{T}$.

These properties on the time scales are too strong and restricts the class of time scales as we will see in the next results.

Theorem 5.22. *If $\mathbb{T}$ has the group property, then for every $a, b \in \mathbb{T}$, we have $a + b \in \mathbb{T}$.*

Proof. In fact, since $0 \in \mathbb{T}$ and $a, b \in \mathbb{T}$ implies that $a - b \in \mathbb{T}$. Then, clearly $-a, -b \in \mathbb{T}$ and therefore, $a - (-b) = a + b \in \mathbb{T}$. □

Theorem 5.23. *If $\mathbb{T}$ has the group property, then every point in $\mathbb{T}$ is right-dense or every point in $\mathbb{T}$ is right-scattered.*

Proof. Suppose that there exist $a, b \in \mathbb{T}$ such that a is right-dense and b is right-scattered. Since a is right-dense, there exists a sequence $\{t_n\} \subset \mathbb{T}$ such that $t_n \to a^+$ as $n \to \infty$. It implies that the sequence $s_n := t_n - a \in \mathbb{T}$ for each $n \in \mathbb{N}$ and converges to zero. Therefore, $s_n + b \in \mathbb{T}$ for each $n \in \mathbb{N}$ and converges to b^+ as $n \to \infty$. Therefore, b is right-dense, which is a contradiction, proving the Theorem. □

Notice that the last property is very restrictive. For example, it shows that time scales which fulfill the group property cannot be hybrid. It excludes the time scale $\mathbb{T} = \bigcup_{k=-\infty}^{+\infty} [a_k, b_k]$, for $a_k < b_k$ for each $k \in \mathbb{Z}$, which is very important to study population models, for instance. See [**17**], [**23**], for more details.

Lemma 5.24. *If $\mathbb{T}$ has the group property, and there exists $0 \neq a \in \mathbb{T}$, then*

$$\sup \mathbb{T} = +\infty \qquad \text{and} \qquad \inf \mathbb{T} = -\infty.$$

Proof. It follows directly from the definition. □

Remark 5.25. Notice that if $\mathbb{T}$ has the group property and $a \neq 0 \in \mathbb{T}$ is right-dense, then by Theorem 5.23 and Lemma 5.24, we have that $\mathbb{T} = \mathbb{R}$.

In the sequel, we prove a property of the forward jump operator when $\mathbb{T}$ has the group property.

Theorem 5.26. *Suppose that $\mathbb{T}$ has the group property. Then*

$$\sigma(a+b) = \sigma(a) + b \quad \text{and} \quad \sigma(a+b) = \sigma(b) + a \tag{5.22}$$

for every $a, b \in \mathbb{T}$.

Proof. If a is right-dense, then by Theorem 5.23, $a+b$ is also right-dense. Therefore, the equality (5.22) follows. If a is right-scattered, then $a+b$ is also right-scattered by Theorem 5.23. Also, notice that

$$a + b < \sigma(a) + b \tag{5.23}$$

and $\sigma(a) + b \in \mathbb{T}$. Then, we have

$$\sigma(a+b) \leq \sigma(a) + b, \tag{5.24}$$

by (5.23) and by the definition of the forward jump operator. On the other hand, notice that

$$a = a + b - b < \sigma(a+b) - b. \tag{5.25}$$

Since $\sigma(a+b) - b \in \mathbb{T}$, $\sigma(a) \leq \sigma(a+b) - b$ by (5.25). Therefore, we have

$$\sigma(a) + b \leq \sigma(a+b). \tag{5.26}$$

Combining (5.24) and (5.26), we get

$$\sigma(a+b) = \sigma(a) + b,$$

concluding the result. The other equality is proved analogously. □

Analogously, we prove the equality $\rho(a+b) = \rho(b) + a$, for every $a, b \in \mathbb{T}$, whenever $\mathbb{T}$ has the group property. By the results above, we conclude that the graininess function is constant.

Corollary 5.27. *Suppose that $\mathbb{T}$ has the group property, then for every $a, b \in \mathbb{T}$, we have $\mu(a) = \mu(b)$.*

Proof. By Lemma 5.26, we have:

$$\sigma(a+b) = \sigma(a) + b = \sigma(b) + a.$$

Therefore,

$$\mu(a) = \sigma(a) - a = \sigma(b) - b = \mu(b),$$

obtaining the result. $\square$

The next result follows the same way as Corollary 5.27. Therefore, we omit its proof.

Corollary 5.28. *Suppose $\mathbb{T}$ has the group property, then for every $a, b \in \mathbb{T}$, we have $\nu(a) = \nu(b)$.*

Finally, we prove a property of exponential function on time scales when $\mathbb{T}$ has the semigroup property.

Theorem 5.29. *Let $\mathbb{T}$ be a time scale with the group property. If $p \in \mathcal{R}$ is constant, then:*

$$e_p(t+a, a) = e_p(t, 0), \quad \text{for all } t, a \in \mathbb{T}.$$

Proof. By the definition,

$$e_p(t+a, a) = \exp\left(\int_a^{t+a} \xi_{\mu(r)}(p)\,\Delta r\right).$$

Let us consider two cases: if $\mathbb{T}$ has only right-dense points or $\mathbb{T}$ has only right-scattered points. Notice that by Lemma 5.23, these are the only cases to consider. If $\mathbb{T}$ has only right-dense points, then,

$$\begin{aligned} e_p(t+a, a) &= \exp\left(\int_a^{t+a} \xi_{\mu(r)}(p)\,\Delta r\right) \\ &= \exp\left(\int_a^{t+a} p\,\Delta r\right) \\ &= \exp\left(p \cdot (t+a-a)\right) \\ &= \exp(pt) \\ &= e_p(t, 0). \end{aligned}$$

If $\mathbb{T}$ has only right-scattered points, then,

$$\begin{aligned} e_p(t+a, a) &= \exp\left(\int_a^{t+a} \xi_{\mu(r)}(p)\,\Delta r\right) \\ &= \exp\left(\int_a^{t+a} \frac{1}{\mu(r)} \log(1 + p\mu(r))\,\Delta r\right) \end{aligned}$$

$$\begin{aligned} &= \exp\left(\int_0^t \frac{1}{\mu(r+a)}\log(1+p\mu(r+a))\,\Delta r\right) \\ &= \exp\left(\int_0^t \frac{1}{\mu(r)}\log(1+p\mu(r))\,\Delta r\right) \\ &= e_p(t,0), \end{aligned}$$

by Corollary 5.27, achieving the result. □

Corollary 5.30. *If $\mathbb{T}$ has the group property, and $p \in \mathcal{R}$ is constant, then*

$$e_{\ominus p}(t+a,a) = e_{\ominus p}(t,0),$$

for every $a, t \in \mathbb{T}$.

Corollary 5.31. *If $\mathbb{T}$ has the group property, and $p \in \mathcal{R}$ is constant, then*

$$e^{\sigma}_{\ominus p}(t+a,a) = e^{\sigma}_{\ominus p}(t,0),$$

for every $a, t \in \mathbb{T}$.

Proof. Notice that, by Corollaries 5.27 and 5.30,

$$\begin{aligned} e^{\sigma}_{\ominus p}(t+a,a) &= \big(1+\mu(t+a)\ominus p\big)e_{\ominus p}(t+a,a) \\ &= \big(1+\mu(t)\ominus p\big)e_{\ominus p}(t,0) \\ &= e^{\sigma}_{\ominus p}(t,0), \end{aligned}$$

concluding the proof. □

4. Abstract sine function on time scales

This section is devoted to study the abstract sine function on time scales.

Definition 5.32. We say that a strongly rd-continuous function $S : \mathbb{T}_0^+ \to \mathcal{B}(X)$ is a ***sine function with infinitesimal generator*** A if the following condition is satisfied: there exists ω such that $(\omega,\infty)_{\mathbb{T}} \subset \rho(A)$, $\lambda^2 \in D\{S\}$, and

$$\widehat{S}(\lambda)x = \int_0^{\infty} e^{\sigma}_{\ominus\lambda}(t,0)S(t)x\,\Delta t = (\lambda^2 - A)^{-1}x, \quad x \in X,$$

for all $\operatorname{Re}_{\mu}(\lambda^2)(t) > \omega$ and $t \in \mathbb{T}_0$.

If A generates a cosine function C, then:

$$(\lambda^2 - A)^{-1} = \frac{1}{\lambda}\int_0^{\infty} e^{\sigma}_{\ominus\lambda}(t,0)C(t)\,\Delta t = \int_0^{\infty} e^{\sigma}_{\ominus\lambda}(t,0)\int_0^t C(s)\,\Delta s\,\Delta t,$$

for $\mathrm{Re}_\mu(\lambda^2)(t) > \omega$. Thus, A generates the abstract sine function S given by $S(t)x := \int_0^t C(s)x\,\Delta s$. It implies that this definition is consistent with Definition 5.19.

Next, we establish some relations between a sine function and its generator.

Proposition 5.33. *Let $S : \mathbb{T}_0^+ \to \mathcal{B}(X)$ be a sine function on X and let A be its generator. Then, the following assertions hold:*

a) $\int_0^t (t-\sigma(s))S(s)x\,\Delta s \in D(A)$ *and* $A\int_0^t (t-\sigma(s))S(s)x\,\Delta s = S(t)x - tx$ *for all* $x \in X, t \in \mathbb{T}_0^+$.

b) *Let* $x \in D(A)$, $\mathrm{Re}_\mu(c)(t) \in D\{S\}$ *and suppose that* $A\widehat{S}(\lambda) = \widehat{S}(\lambda)A$ *when* $\mathrm{Re}_\mu(\lambda^2)(t) > \mathrm{Re}_\mu(c)(t)$, *then,* $S(t)x \in D(A)$ *and* $AS(t)x = S(t)Ax$ *for all* $t \in \mathbb{T}_0^+$.

c) *Let* $x, y \in X$, *then* $x \in D(A)$ *and* $Ax = y$, *if and only if,* $\int_0^t (t-\sigma(s))S(s)y\,\Delta s = S(t)x - tx$ *for all* $t \in \mathbb{T}_0^+$.

Proof. a) Notice that the abstract sine function satisfies the equation (5.14) for the case that $x = 0$. Therefore, by Corollary 5.18, we have that $S(\cdot)x$ is a mild solution of the problem (5.1) when $x = 0$, obtaining the desired result.

b) For $x \in X$ and $\mathrm{Re}_\mu(\lambda)(t) > \mathrm{Re}_\mu(c)(t)$ for $t \in \mathbb{T}_0^+$, we have

$$\int_0^\infty e_{\ominus\lambda}^\sigma(t,0)S(t)Ax\,\Delta t = \widehat{S}(\lambda)Ax = A\widehat{S}(\lambda)x = \int_0^\infty e_{\ominus\lambda}^\sigma(t,0)AS(t)x\,\Delta t,$$

which implies by the uniqueness of Laplace transform [**41**, Theorem 3.14], that $AS(t)x = S(t)Ax$ for all $t \in \mathbb{T}_0^+$.

c) Define $S_1(t) = \int_0^t S(s)\,\nabla s$ and $S_2(t) = \int_0^t S_1(s)\,\Delta s$. By hypothesis and by Corollary 5.5, we get:

$$\widehat{S_1}(\lambda) = \frac{\widehat{S}(\lambda)}{\lambda}, \quad \widehat{S_2}(\lambda) = \frac{\widehat{S_1}(\lambda)}{\lambda}, \quad \text{and} \quad \widehat{S_2}(\lambda) = \frac{\widehat{S}(\lambda)}{\lambda^2}. \tag{5.27}$$

Assume that $\int_0^t (t-s)S(s)y\,\Delta s = S(t)x - tx$. Taking Laplace transform, we get

$$\begin{aligned}
\int_0^\infty e_{\ominus\lambda}^\sigma(t,0)\int_0^t (t-\sigma(s))S(s)y\,\Delta s\,\Delta t &= \int_0^\infty e_{\ominus\lambda}^\sigma(t,0)\int_0^t\int_0^s S(r)y\,\Delta r\,\Delta s\,\Delta t \\
&= \int_0^\infty e_{\ominus\lambda}^\sigma(t,0)\int_0^t S_1(s)y\,\Delta s\,\Delta t \\
&= \int_0^\infty e_{\ominus\lambda}^\sigma(t,0)S_2(t)y\,\Delta t \\
&= \frac{1}{\lambda^2}(\lambda^2 - A)^{-1}y,
\end{aligned}$$

by (5.27) and by definition. Also, by definition and (5.10), we have

$$\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\big(S(t)x - tx\big)\,\Delta t = (\lambda^2 - A)^{-1}x - \frac{x}{\lambda^2},$$

which implies

$$\begin{aligned}
\frac{1}{\lambda^2}(\lambda^2 - A)^{-1}y &= (\lambda^2 - A)^{-1}x - \frac{x}{\lambda^2}\\
(\lambda^2 - A)^{-1}y &= \lambda^2(\lambda^2 - A)^{-1}x - x\\
(\lambda^2 - A)(\lambda^2 - A)^{-1}y &= (\lambda^2 - A)\lambda^2(\lambda^2 - A)^{-1}x - (\lambda^2 - A)x\\
y &= \lambda^2 x - (\lambda^2 - A)x \;=\; Ax
\end{aligned}$$

for $\mathrm{Re}_\mu(\lambda^2)(t) > \mathrm{Re}_\mu(\omega)(t)$.

Conversely, let $x, y \in X$ such that $x \in D(A)$ and $Ax = y$. By part a), we have:

$$\begin{aligned}
S(t)x - tx &= A\int_0^t (t - \sigma(s))S(s)x\,\Delta s \;=\; \int_0^t (t - \sigma(s))AS(s)x\,\Delta s\\
&= \int_0^t (t - \sigma(s))S(s)Ax\,\Delta s \;=\; \int_0^t (t - \sigma(s))S(s)y\,\Delta s
\end{aligned}$$

where we used part b), finishing the proof. □

Our definitions of abstract sine and cosine functions on time scales are directly related with the existence of mild solutions to the problem (5.1). Because of this, we now discuss the existence of mild solutions to problem (5.1).

Theorem 5.34. *Let $A \in \mathcal{B}(X)$. Then, the problem* (5.1) *has a unique classical solution.*

Proof. We fix $a \in \mathbb{T}_0^+$, and define the operator $\Gamma : \mathcal{C}_{rd}([0,a], X) \to \mathcal{C}_{rd}([0,a], X)$ by

$$(\Gamma u)(t) = x + ty + \int_0^t (t - \sigma(s))u(s)\,\Delta s, \quad t \in \mathbb{T}_0^+.$$

For $t \le a \in \mathbb{T}_0^+$, it is not difficult to prove that Γ has a unique fixed point $u(\cdot)$, which is the solution to (5.1). Moreover, since

$$u(t) = x + ty + \int_0^t (t - \sigma(s))u(s)\,\Delta s, \quad t \in \mathbb{T}_0^+$$

for such fixed point $u(\cdot)$, we have that $u^{\Delta\Delta}(t) = Au(t)$ and $u(0) = x$. Therefore, $u(\cdot)$ is a classical solution of (5.1). □

Remark 5.35. For the case $\mu(0) = 0$, the group property combined with the Lemma 5.23, show that the only time scale that satisfies such conditions is $\mathbb{T} = \mathbb{R}$ (see Lemma 5.24); consequently, $C(\cdot)$ is a cosine function in the classical sense. Also, this result shows what strong the semigroup property is. For instance, this result does not allow include the hybrid continuous-discrete time scales such that $0 \in \mathbb{T}$ and 0 is right-dense, and the quantum time scale $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}, q > 1$, because this time scale does not satisfy the property.

We finish this section with an example. Let us assume that A generates a cosine family $C : \mathbb{T}_0^+ \to \mathcal{B}(X)$ with associated sine family $S : \mathbb{T}_0^+ \to \mathcal{B}(X)$, and let $\lambda \in \mathbb{R}$ be an eigenvalue of A.

Example 5.36. If $\lambda \in \mathbb{R}$, then

$$C(t)x = \cosh_{\sqrt{\lambda}}(t,0)x, \quad S(t)y = \frac{1}{\sqrt{\lambda}}\sinh_{\sqrt{\lambda}}(t,0)y, \quad t \in \mathbb{T}_0^+.$$

Indeed, let $u(t) = \cosh_{\sqrt{\lambda}}(t,0)x + \frac{1}{\sqrt{\lambda}}\sinh_{\sqrt{\lambda}}(t,0)y$. From the definition of hyperbolic functions on time scales [**17**, Definition 3.17], it is immediate that $u(0) = x$. Applying delta derivative, we get

$$u^{\Delta}(t) = \sqrt{\lambda}\sinh_{\sqrt{\lambda}}(t,0)x + \cosh_{\sqrt{\lambda}}(t,0)y,$$

and from this formula we obtain $u^{\Delta}(0) = y$. Applying delta derivative again, we get

$$\begin{aligned} u^{\Delta\Delta}(t) &= \lambda\cosh_{\sqrt{\lambda}}(t,0)x + \sqrt{\lambda}\sinh_{\sqrt{\lambda}}(t,0)y \\ &= \lambda u(t). \end{aligned}$$

Therefore, from the uniqueness of solutions of the problem (5.1) for the case $A := \lambda$, we conclude that $u(t) = C(t)x + S(t)y$, and the result is proved.

5. Inhomogeneous second order abstract Cauchy problem

In this section, we investigate the existence of solutions of the following inhomogeneous abstract Cauchy problem on time scales:

$$\begin{cases} u^{\Delta\Delta}(t) &= Au(t) + f(t), \quad t \in \mathbb{T}_0^+, \\ u(0) &= x, \\ u^{\Delta}(0) &= y, \end{cases} \tag{5.28}$$

where $x, y \in X$ and $\mathbb{T}_0^+$. We assume that the values $u(t) \in X$ and $f : \mathbb{T}_0^+ \to X$ is an rd-continuous function. Also, we assume that A generates a cosine function $C : \mathbb{T}_0^+ \to X$ and a sine function $S : \mathbb{T}_0^+ \to X$, and that there exists an rd-continuous function g such that:

$$(\lambda^2 I - A)\widehat{g}(\lambda) = \widehat{f}(\lambda), \quad \lambda > \omega.$$

We start by introducing the definition of a mild solution of problem (5.28).

Definition 5.37. We say that an rd-continuous function $u : \mathbb{T}_0^+ \to X$ is a ***mild solution of*** (5.28) if

$$u(t) = x + ty + A\int_0^t (t-\sigma(s))u(s)\,\Delta s + \int_0^t (t-\sigma(s))f(s)\,\Delta s \tag{5.29}$$

for all $t \in \mathbb{T}_0^+$.

We restrict us to consider the operator $A \in \mathcal{B}(X)$. In this case, for each $s \in \mathbb{T}_0^+$, we consider the abstract Cauchy problem given by:

$$\begin{cases} u^{\Delta\Delta}(t) &= Au(t), \quad t \in \mathbb{T}_0^+, \\ u(s) &= x, \\ u^{\Delta}(s) &= y. \end{cases} \tag{5.30}$$

Definition 5.38. We say that an rd-continuous function $u : [s,\infty)_{\mathbb{T}} \to X$ is a ***mild solution of*** (5.30) if

$$u(t) = x + ty + A\int_s^t (t-\sigma(r))u(r)\,\Delta r,$$

for all $t \geq s \in \mathbb{T}_0^+$.

We can show that problem (5.30) has a unique solution $u(t,s)$ for all $x, y \in X$. We define $C(t,s)x + S(t,s)y = u(t,s)$. Also, $C : \{(t,s) : t \geq s, t, s \in \mathbb{T}_0^+\} \to \mathcal{B}(X)$ and $S : \{(t,s) : t \geq s, t, s \in \mathbb{T}_0^+\} \to \mathcal{B}(X)$ are strongly rd-continuous maps.

Theorem 5.39. *Let $A \in \mathcal{B}(X)$ and assume that $f : \mathbb{T}_0^+ \to X$ is an rd-continuous function. Then, the mild solution $u(t)$ of the problem* (5.28) *is given by*

$$u(t) = C(t,t_0)x + S(t,t_0)y + \int_{t_0}^t S(t,\sigma(r))f(r)\,\Delta r. \tag{5.31}$$

Remark 5.40. In the following proof, we only seek the solution for the case $t \geq t_0$. Hence, $C(t,t_0)$ and $S(t,t_0)$ are only defined for $t \geq t_0$. On the other hand, by the definition of abstract sine function, let us consider

$$S(t) := S(t,0) = \int_0^t C(s)\,\Delta s. \tag{5.32}$$

Based on (5.32), we can denote it by

$$S(t,s) = \int_s^t C(r)\,\Delta r. \tag{5.33}$$

By (5.33), it is clear that $S(t,s) = 0$ whenever $t = s$.

Proof of Theorem 6.35. We define the following function:

$$u(t) := C(t)x + S(t)y + \int_0^t S(t,\sigma(r))f(r)\,\Delta r, \quad t \in \mathbb{T}_0^+. \tag{5.34}$$

On the other hand, by Proposition 5.33 a), we have

$$\begin{aligned} A\int_0^t (t-s)\int_0^s S(s,\sigma(r))f(r)\,\Delta r\,\Delta s &= A\int_0^t\int_0^s (t-\sigma(s))S(s,\sigma(r))f(r)\,\Delta r\,\Delta s \\ &= A\int_0^t\int_r^t (t-\sigma(s))S(s,\sigma(r))f(r)\,\Delta s\,\Delta r \\ &= \int_0^t A\int_r^t (t-\sigma(s))S(s,\sigma(r))f(r)\,\Delta s\,\Delta r \\ &= \int_0^t \Big(S(t,\sigma(r)) - t\Big)f(r)\,\Delta r - \\ &\quad \int_0^t \Big(S(\sigma(r),\sigma(r)) - \sigma(r)\Big)f(r)\,\Delta r \\ &= \int_0^t \Big(S(t,\sigma(r)) - (t-\sigma(r))\Big)f(r)\,\Delta r. \end{aligned} \tag{5.35}$$

where in the second equality, we used the change of order of the integration (see [**16**] for details). Combining (5.34) with (5.35), and replacing in (5.29), we obtain

$$\begin{aligned} u(t) &= x + ty + A\int_0^t (t-\sigma(r))u(r)\,\Delta r + \int_0^t (t-\sigma(r))f(r)\,\Delta r \\ &= x + ty + A\int_0^t (t-\sigma(s))C(s)x\,\Delta s + A\int_0^t (t-\sigma(s))S(s)y\,\Delta s \\ &\quad + A\int_0^t (t-\sigma(s))\int_0^s S(s,\sigma(r))f(r)\,\Delta r\,\Delta s + \int_0^t (t-\sigma(r))f(r)\,\Delta r \\ &= C(t)x + S(t)y + A\int_0^t (t-\sigma(s))\int_0^s S(s,\sigma(r))f(r)\,\Delta r\,\Delta s + \int_0^t (t-\sigma(r))f(r)\,\Delta r \\ &= C(t)x + S(t)y + \int_0^t \Big(S(t,\sigma(r)) - (t-\sigma(r))\Big)f(r)\,\Delta r + \int_0^t (t-\sigma(r))f(r)\,\Delta r \\ &= C(t)x + S(t)y + \int_0^t S(t,\sigma(r))f(r)\,\Delta r, \end{aligned}$$

concluding the proof. $\square$

6. Nonlinear second order abstract Cauchy problem

In this section, we investigate the existence of solutions of the following nonlinear abstract Cauchy problem on time scales

$$\begin{cases} u^{\Delta\Delta}(t) &= Au(t) + f(t, u(t)), \quad t \in \mathbb{T}_0^+, \\ u(0) &= x, \\ u^{\Delta}(0) &= y, \end{cases} \tag{5.36}$$

where $x, y \in X$ and $\mathbb{T}_0^+ = \mathbb{T}_0 \cap \mathbb{R}_+$. We assume that the values $u(t) \in X$ and $f : \mathbb{T}_0^+ \times X \to X$ is an rd-continuous function with respect to the first variable. Also, we assume that A generates a cosine function $C : \mathbb{T}_0^+ \to X$ and a sine function $S : \mathbb{T}_0^+ \to X$.

We start by introducing the definition of a mild solution of problem (5.36).

Definition 5.41. We say that an rd-continuous function $u : \mathbb{T}_0^+ \to X$ is a ***mild solution of*** (5.36) if

$$u(t) = x + ty + A\int_0^t (t - \sigma(s))u(s)\,\Delta s + \int_0^t (t - \sigma(s))f(s, u(s))\,\Delta s \tag{5.37}$$

for all $t \in \mathbb{T}_0^+$.

The next theorem is the main result of this section and it will be very important to study nonlinear second order abstract Cauchy problem on time scales. The proof of the next theorem follows very similar to the proof of Theorem 5.39, but we will repeat it here for reader's convenience.

Theorem 5.42. *Let $A \in \mathcal{B}(X)$ and assume that $f : \mathbb{T}_0^+ \times X \to X$ is an rd-continuous function. Then, the mild solution $u(t)$ of the problem* (5.36) *is given by*

$$u(t) = C(t, t_0)x + S(t, t_0)y + \int_{t_0}^t S(t, \sigma(r))f(r, u(r))\,\Delta r. \tag{5.38}$$

Proof. We define the following function

$$u(t) := C(t)x + S(t)y + \int_0^t S(t, \sigma(r))f(r, u(r))\,\Delta r, \quad t \in \mathbb{T}_0^+. \tag{5.39}$$

On the other hand, by Proposition 5.33 a), we have

$$\begin{aligned} A\int_0^t (t - \sigma(s))\int_0^s S(s, \sigma(r))f(r, u(r))\,\Delta r\,\Delta s &= A\int_0^t \int_0^s (t - \sigma(s))S(s, \sigma(r))f(r, u(r))\,\Delta r\,\Delta s \\ &= A\int_0^t \int_{\sigma(r)}^t (t - \sigma(s))S(s, \sigma(r))f(r, u(r))\,\Delta s\,\Delta r \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t A \int_{\sigma(r)}^t (t-\sigma(s))S(s,\sigma(r))f(r,u(r))\,\Delta s\,\Delta r \\
&= \int_0^t \Big(S(t,\sigma(r))-t\Big)f(r,u(r))\,\Delta r \\
&\quad -\int_0^t \Big(S(\sigma(r),\sigma(r))-\sigma(r)\Big)f(r,u(r))\,\Delta r \\
&= \int_0^t \Big(S(t,\sigma(r))-(t-\sigma(r))\Big)f(r,u(r))\,\Delta r. \qquad (5.40)
\end{aligned}
$$

where in the second equality, we used the change of order of the integration (see [**16**]). Combining (5.39) with (5.40), and replacing in (5.37), we obtain

$$
\begin{aligned}
u(t) &= x+ty+A\int_0^t (t-\sigma(r))u(r)\,\Delta r+\int_0^t (t-\sigma(r))f(r,u(r))\,\Delta r \\
&= x+ty+A\int_0^t (t-\sigma(s))C(s)x\,\Delta s+A\int_0^t (t-\sigma(s))S(s)y\,\Delta s \\
&\quad +A\int_0^t (t-\sigma(s))\int_0^s S(s,\sigma(r))f(r,u(r))\,\Delta r\,\Delta s+\int_0^t (t-\sigma(r))f(r,u(r))\,\Delta r \\
&= C(t)x+S(t)y+A\int_0^t (t-\sigma(s))\int_0^s S(s,\sigma(r))f(r,u(r))\,\Delta r\,\Delta s \\
&\quad +\int_0^t (t-\sigma(r))f(r,u(r))\,\Delta r \\
&= C(t)x+S(t)y+\int_0^t \Big(S(t,\sigma(r))-(t-\sigma(r))\Big)f(r,u(r))\,\Delta r \\
&\quad +\int_0^t (t-\sigma(r))f(r,u(r))\,\Delta r \\
&= C(t)x+S(t)y+\int_0^t S(t,\sigma(r))f(r,u(r))\,\Delta r,
\end{aligned}
$$

concluding the proof. $\square$

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