Compatible algebras

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Introduction

Roughly speaking, a non-symmetric algebraic operad is an algebraic structure spanned linear operations (not necessarily binary) satisfying certain relations, in which the variables stay in the same order. The canonical example of this type of structure is the operad of the associative algebras, spanned by a binary product $\cdot : a \otimes b \to a \cdot b$, which satisfies the relation

$$(a \cdot b) \cdot c = a \cdot (b \cdot c),$$

for any elements a, b, c in an algebra over this operad.

Associative algebras are naturally related to other type of algebraic structure: Lie algebras. By definition, a Lie algebra over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathfrak{g}$, equipped with a binary operation $[,]$, called *Lie bracket*, satisfying the following relations:

(1) antisymmetry, $[x_1, x_2] = -[x_2, x_1]$,
(2) Jacobi identity, $[x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]] = 0$,

for all $x_1, x_2, x_3 \in \mathfrak{g}$.
Given an associative algebra $(A, \cdot)$, a Lie algebra structure on the underlying vector space of A, is defined by the bracket

$$[a, b] := a \cdot b - b \cdot a,$$

for $a, b \in A$. This Lie algebra is denoted by $(A^{Lie}, [,])$.
The map $(A, \cdot) \to (A^{Lie}, [,])$ determines a functor, denoted by $(-)^{Lie}$, from the category of associative algebras over a field $\mathbb{K}$ to the category of Lie algebras over the same field.

There exists a canonical left adjoint functor $\mathcal{U}$ to $(-)^{Lie}$. Given a Lie algebra $\mathfrak{g}$, the functor $\mathcal{U}$ assigns an associative algebra $\mathcal{U}(\mathfrak{g})$, called the *universal enveloping algebra* of $\mathfrak{g}$. As a vector space $\mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$, that is $\mathfrak{g}$ coincides with the subspace of homogeneous elements of degree one of $\mathcal{U}(\mathfrak{g})$. Note that the vector space $\mathcal{U}(\mathfrak{g})_n$ has positive dimension, for all $n \in \mathbb{N}$. So, one of the main questions about associative algebras is when an associative algebra is the enveloping algebra of a Lie algebra.
The answer of this problem, in characteristic zero, was given in 1965 by J.W. Milnor and J.C. Moore in [28]. Let us describe briefly their result, which involves the notion of bialgebras.

A coassociative coalgebra is the dual notion of an associative algebra, that is, a vector space C over $\mathbb{K}$, equipped with a linear map

$$\Delta : C \to C \otimes_{\mathbb{K}} C,$$

which is coassociative, that is, Δ satisfies that

$$(\Delta \otimes \mathrm{Id}_C) \circ \Delta = (\mathrm{Id}_C \otimes \Delta) \circ \Delta.$$

A unital associative algebra $(A, \cdot, \eta)$ equipped with coassociative counital coproduct (Δ, ϵ) satisfying that the coproduct Δ and the counit ϵ are algebra morphisms, is called a bialgebra.

Our main example of bialgebra is the free associative algebra over a vector space V, with the concatenation product and the coproduct given in terms of shuffle permutations (see [6]).

Milnor-Moore's Theorem(see [28]) states that, over a field of characteristic zero, any conilpotent cocommutative bialgebra H is isomorphic to the enveloping algebra of a Lie algebra. So, there exists an equivalence of categories between conilpotent cocommutative bialgebras and Lie algebras, over a field $\mathbb{K}$ of characteristic zero.

The statement becomes less definite in positive characteristic (see for instance [6],[28] and [29]). But the Milnor-Moore Theorem still holds when the category $\mathrm{Vect}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$, is replaced by other symmetric monoidal categories (see [27]).

In the context of operad theory, J.-L. Loday introduced in [25] the notion of *generalized bialgebra*, which includes, between other cases, the classical notions of bialgebras, Lie bialgebras, infinitesimal bialgebras and dendriform bialgebras.

A type of generalized bialgebra is determined by a coalgebra structure C^c, an algebra structure A and compatibility relations between the operations and the co-operations, called *distributive laws*. In the triple of operads involved in the Milnor-Moore Theorem, we have $C^c = Com^c$ (cocommutative coassociative coalgebra) and $A = \mathrm{As}$ (associative algebra) and the relation between these structures is the Hopf compatibility relation. Loday proved that, under certain conditions, there is a structure theorem for generalized bialgebras, which gives rise to various generalizations of the Milnor-Moore Theorem.

Of particular interest to our work is the notion of *infinitesimal bialgebra*, which was introduced by M. Joni and G.-C. Rota in [20], and modified by J.-L. Loday and M. Ronco in [26], in order to consider infinitesimal bialgebras with unit.

A *unital infinitesimal bialgebra* $(H, \cdot, \Delta)$ is a vector space H over a field $\mathbb{K}$, equipped with a unital associative product $\cdot$ and a counital coassociative coproduct Δ, which are related by the unital infinitesimal relation :

$$\Delta(x \cdot y) = (x \otimes 1) \cdot \Delta(y) + \Delta(x) \cdot (1 \otimes y) - x \otimes y.$$

The main example of infinitesimal bialgebra is the tensor algebra over a vector space V, denoted by $T(V)$, equipped with the concatenation product and the deconcatenation coproduct. The space $T(V)$, with this structure, is denoted by $T^c(V)$.

The structure theorem in this case (see [26]), states that any conilpotent unital infinitesimal bialgebra H is isomorphic to $(T^c(V)$, for some vector space V.

The main object of our work is the study of generalized bialgebra structures for certain non-symmetric operad: the operad of compatible associative algebras.
A *compatible associative algebra* over a field $\mathbb{K}$ is a vector space A equipped with two associative products, $\cdot : A \otimes A \to A$ and $\circ : A \otimes A \to A$, such that the sum

$$x * y := x \cdot y + x \circ y,$$

is an associative product, too. Equivalently, A is a compatible associative algebra if the product $x \star y := \mu x \cdot y + \lambda(x \circ y)$ is associative for all pair of elements $\mu, \lambda \in \mathbb{K}$. When $\mu = 1$, the product $\star$ can be considered as a deformation of the product $\cdot$ in the parameter λ. Following the notation given by H. Strohmayer in [37], the operad associated to this type of algebras is denoted by As^2.

One of the motivations for the study of As^2-algebras is its close relationship with others algebraic structures, as compatible Lie algebras and bi-Hamiltonian algebras. Recall that a *compatible Lie algebra* is a $\mathbb{K}$-vector space A, equipped with two Lie brackets $[,]$ and $\{,\}$, satisfying that their sum is also a Lie bracket.

When the compatible Lie algebra A is equipped with a commutative and associative product $\cdot$ such that the brackets are both derivations for the product $\cdot$, we say that A is a *bi-Hamiltonian algebra*. In particular, a bi-Hamiltonian algebra has two structures of Poisson algebra, so the brackets are called Poisson brackets.

Given an associative algebra $(A, \cdot)$ a natural problem is to find out the possible associative products, defined on the underline vector space of A, which are compatible with the original associative product. In [30], A. Odesskii and V. Sokolov showed that the associative products compatible with the usual matrix product are in one-to-one correspondence with representations of certain algebraic structures, called M-structures. They studied the semisimple case and introduced, when A is finite direct sum of matrix algebras, the PM-structures (see [30]), whose representations describe the compatible associative products on A. The same authors showed, in [30], that the classification of M and PM structures are related with the Cartan matrices of certain affine Dynkin diagrams.

In [5], J. F. Carinena, J. Grabowski and G. Marmo introduced the notion of Nijenhuis tensor for associative algebras, which originates interesting examples of compatible associative algebras, some of which are considered in Subsection 3.1.4.

The operads of Lie compatible algebras and bi-Hamiltonian algebras have been studied in [9]. In this work, V. Dotsenko and A. Khoroshkin computed the dimensions of the components for the operad of the compatible Lie algebras and for the bi-Hamiltonian operad. They also calculated the characters of these spaces as S_n-modules and $S_n \times SL_2$-modules.

As in the case of associative algebras, there exists a functor from the category of As^2 algebras over $\mathbb{K}$ to the category of compatible Lie algebras over the same field. If $(A, \cdot, \circ)$ is a compatible associative algebra, then the Lie brackets given by

$$\begin{aligned} [x, y] &= x \cdot y - y \cdot x, \\ \{x, y\} &= x \circ y - y \circ x, \end{aligned}$$

define a structure of compatible Lie algebra on the underlying vector space of A.

In [37], H. Strohmayer developed the general notion of compatible algebraic structures. He computed the Koszul dual of As^2, denoted by ${}^2\mathrm{As}$, which is a set theoretical operad, by arising an operadic partition poset (see [39]). H. Strohmayer showed, using B. Valette's results, that ${}^2\mathrm{As}$ is a Koszul operad, and therefore As^2 is Koszul operad, too. In [41], Y. Zhang explicitly gave the realization of the homology complex for the compatible associative algebras.

Using that the operad As^2 is Koszul, V. Dotsenko obtained in [8] the dimensions of the operad As^2 and calculated the characters of $\mathrm{As}^2(\mathrm{n})$ as S_n-module and $S_n \times SL_2$-module. In particular, the dimension of $\mathrm{As}^2(\mathrm{n})$ is $c_n \cdot n!$, where c_n is Catalan number.

In our work, we give an explicit construction of free objects in the category of As^2-algebras, using planar rooted trees. As As^2 is a non-symmetric operad, the operad is determined by the free object on one element. We show that any free As^2-algebra admits a coassociative coproduct which satisfies the unital infinitesimal condition with both associative products. This last result motivates the definition of *compatible infinitesimal bialgebra*, as a compatible algebra $(A, \cdot, \circ)$ equipped with a coproduct $\Delta : A \to A \otimes A$, satisfying the unital infinitesimal relation with both associative products.

The study of the subspace of primitive elements of compatible associative bialgebra, gives rise to the notion of $\mathcal{N}$-algebra. We show that operad $\mathcal{N}$ is non-symmetric, and that the dimension of the $\mathbb{K}$-vector space $\mathcal{N}_n$ is the Catalan number c_{n-1}. We obtain a structure theorem for conilpotent compatible associative bialgebras, which gives a new triple of operads $(As, \mathrm{As}^2, \mathcal{N})$, in the sense of [25].

In our work, we consider also a particular type of compatible associative algebras, the so-called *matching dialgebras*, previously studied in [40]. A matching dialgebra over field $\mathbb{K}$ is a vector space A equipped with two associative products $\cdot$ and $\circ$ satisfying the following compatibility conditions:

$$(a \cdot b) \circ c = a \cdot (b \circ c) \text{ and } (a \circ b) \cdot c = a \circ (b \cdot c),$$

for all the elements $a, b, c \in A$. Clearly, any matching dialgebra is a compatible associative algebra.

We give various examples of this type of algebras. In particular, we consider matching dialgebras obtained from semi-homomorphism of associative algebras. Moreover, we describe the free matching dialgebra over a vector space V as a quotient of the free compatible associative algebra $\mathrm{As}^2(\mathrm{V})$. We also develop the notion of compatible infinitesimal bialgebras in a matching dialgebra, which we obtain a triple of operads, as noted J-L. Loday in [25].

As a second step, we consider the operad of algebras equipped with two compatible associative products, satisfying that one of them is commutative. In this case the operad do not is non-symmetric since one of the products is commutative. We obtain a basis for this operad and give a recursive formula that allow us to compute their dimensions.

Finally, we look at the operad of compatible associative algebras, whose two products are commutative, which we denote by Com^2. We show that it is Koszul. Unlike the cases studied in [37], the Koszul dual of Com^2, denoted ${}^2Lie$, is not a set-theoretical operd. So, we cannot associate a operadic partition poset to ${}^2Lie$, and the methods developed in [39] are not applicable in this case. In [18], E. Hoffbeck developed the notion of PBW basis for operads. He showed that an operad $\mathcal{P}$ that admits a PBW basis is Koszul. Moreover, if an operad $\mathcal{P}$ has a PBW basis, then its Koszul dual operad has a PBW basis, which is determined by the PBW basis of $\mathcal{P}$. In our case, the study of the Koszul dual operad ${}^2Lie$ allows us to determine a PBW basis for Com^2, which is Koszul. The PBW basis in this case is described on the vector space spanned by the set of increasing trees. In particular, the dimension of $\mathrm{Com}^2(\mathrm{n})$ is equal to $n!$. Additionally, we obtain recursive formulas that allow us to express the elements of Com^2 as linear combination of elements of this basis.

This work is organized as follows: Section 1 is devoted to recall the basic notion of associative algebras, coalgebras, bialgebras and Hopf algebras. In the Section 2 is devoted to recall the basic concepts of operad theory required to understand the remaining sections. In Section 3 we give some examples of compatible associative algebras that arise in the study of Nijenhuis tensor for associative algebras, and construct the free compatible associative algebra over a vector space V. In Section 4 we develop the notion of compatible infinitesimal bialgebra, and introduce the notion of $\mathcal{N}$-algebra. In Section 5 we consider a particular case of compatible associative algebras: the matching dialgebras. We show some examples that arise from semi-homomorphisms of associative algebras. We also describe some notions of coalgebras compatible with this type of structure. In Section 6 we study the operad with two compatible associative products with the additional condition that one of them is commutative, while Section 7 is devoted to the study of compatible commutative algebras, that is compatible associative algebras satisfying that both products are associative and commutative.

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Notations

All vector spaces and algebras considered in the manuscript are over a field $\mathbb{K}$. Given a set X, we denote by $\mathbb{K}[X]$ the vector space spanned by X. For any vector space V, we denote by $V^{\otimes n}$ the tensor product of $V \otimes \cdots \otimes V$, n times, over $\mathbb{K}$. In order to simplify notation, we shall denote an element of $V^{\otimes n}$ indistinctly by $x_1 \otimes \cdots \otimes x_n$ or simply $x_1 \cdots x_n$.
If n is a positive integer, we denote by $[n]$ the set $\{1, \ldots, n\}$. The symmetric group of permutations of $[n]$ is denoted by S_n. Given a permutation $\sigma \in S_n$, we write $\sigma = (\sigma(1), \ldots, \sigma(n))$, identifying σ with its image.

1. Bialgebras

This section is dedicated to developed basic notions of associative algebras, coalgebras, bialgebras and Hopf algebras.
The concepts and examples described here have been chosen to make easier the reading of the manuscript. Our aim is to help the reader to understand a classical result in the study of Hopf algebras: the Milnor-Moore Theorem. This Theorem states that, in characteristic zero, a conilpotent cocommutative Hopf algebra H is isomorphic to the universal enveloping algebra of its Lie algebra of primitive elements, $H \cong \mathcal{U}(\mathrm{Prim}(H))$. Although the development of this section is standard in Hopf algebra theory, our text is essentiality based on the description given by J.-L. Loday and B. Vallette in [22].

1.1. **Associative algebras.**

1.1.1. **Definition.** An *associative algebra* is a vector space A equipped with a bilinear map $\mu : A \otimes A \to A$, which satisfies that $\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id)$. The associativity of the product μ can be represented by the following commutative diagram:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

An algebra A is unitary if there exist a linear application $\eta : \mathbb{K} \to A$, called unit, such that the following diagram commute :

$$\begin{tikzcd} \mathbb{K} \otimes A \arrow[r, "\eta\otimes id"] \arrow[dr, "\simeq"'] & A \otimes A \arrow[d, "\mu"] & A \otimes \mathbb{K} \arrow[l, "id\otimes\eta"'] \arrow[dl, "\simeq"] \\ & A & \end{tikzcd}$$

1.1.2. **Notation.** For any pair of elements x and y in an associative algebra A, we denote by $x \cdot y$ the element $\mu(x \otimes y)$. The associativity condition satisfied by μ implies that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $\eta(\lambda) \cdot x = \lambda x = x \cdot \eta(\lambda)$, for all $x, y, z \in A$ and $\lambda \in \mathbb{K}$.

Note that if $(A, \cdot)$ and $(B, \circ)$ are two associative algebras, then the tensor product $A \otimes B$ is also an associative algebra with the product defined by

$$(x_1 \otimes y_1) * (x_2 \otimes y_2) := (x_1 \cdot x_2) \otimes (y_1 \circ y_2),$$

for all $x_1, x_2 \in A$, $y_1, y_2 \in B$.
Let us now recall the notion of *graded vector spaces*.

1.1.3. **Definition.** A vector space V is *graded* if there exist a collection of subspaces $\{V_n\}_{n\in\mathbb{Z}}$ such that

$$V = \bigoplus_{n\in\mathbb{Z}} V_n.$$

An element $x \in V_n$ is homogeneous of degree n and we write $|x| = n$. We assume that n will run over the integer non negative only, with which we will write

$$V = \bigoplus_{n\geq} V_n.$$

1.1.4. **Definition.** An associative unital algebra A is *graded* if its underlying vector space is graded, that is $A = \bigoplus_{n\geq 0} A_n$, and it satisfies:

(1) $A_n \cdot A_m \subset A_{n+m},$, for all integers $n, m \geq 0$ and
(2) $\eta(\mathbb{K}) \subset A_0$.

An associative algebra $(A, \cdot)$ is *commutative* if the product $\cdot$ satisfies that $a \cdot b = b \cdot a$, for all $a, b \in A$. From now on, for us a commutative product is an associative and commutative product and a *commutative algebra* $(A, \cdot)$ is a vector space A with a commutative product $\cdot : A \otimes A \to A$.

1.1.5. **Example.** The polynomial ring $A = \mathbb{K}[x_1, x_2, \ldots, x_n]$ in n variables over the field $\mathbb{K}$ is a graded commutative algebra. In this case, $A_0 = \mathbb{K}$, while the homogeneous component of degree $i \geq 1$ is the subgroup of all $\mathbb{K}$-linear combinations of monomials of degree i.

1.1.6. *Tensor algebra.* Let V be a vector space over $\mathbb{K}$. The tensor algebra over V is the graded vector space

$$T(V) = \bigoplus_{n\geq 0} V^{\otimes n} = \mathbb{K}1 \oplus V \oplus \cdots V^{\otimes n} \oplus \cdots,$$

equipped with the concatenation product $T(V) \otimes T(V) \to T(V)$ given by

$$\lambda \cdot (v_1 \cdots v_n) = (v_1 \cdots v_n) \cdot \lambda = \lambda v_1 \cdots v_n, \text{ if } \lambda \in \mathbb{K}$$

and

$$(v_1 \cdots v_n) \cdot (v_{n+1} \cdots v_{n+m}) = v_1 \cdots v_n v_{n+1} \cdots v_{n+m}.$$

The concatenation product is associative. The tensor algebra $T(V)$ is graded, where the component of degree zero is the field $\mathbb{K} = T^0(V)$ and $T^n(V) := V^{\otimes n}$ is the component of degree n. Its unit is given by $1_{\mathbb{K}} \in \mathbb{K} = T^0(V)$. The tensor algebra is the free associative algebra over V, which means that $T(V)$ satisfies the following universal property:
given a associative unital algebra A and a linear application $f : V \to A$, there exists an unique algebra morphism $\widetilde{f} : T(V) \to A$ which extends f,

that is, the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "i"'] & A \\ T(V) \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

where $i : V \to T(V)$ is the inclusion of $V = T^1(V)$ in $T(V)$.
In fact, if $f : V \to A$ is a linear application, we define $\widetilde{f} : T(V) \to A$ by:

(1) $\widetilde{f}(1) = 1$,
(2) $\widetilde{f}(v) = f(v)$, if $v \in V$,
(3) $\widetilde{f}(v_1 \cdots v_n) = f(v_1) \cdot \ldots \cdot f(v_n)$, where $\cdot$ denote the associative product of A.

It is immediate to verify that $\widetilde{f}$ is morphism of algebras. Moreover, it is clear that this morphism is unique. So, $T(V)$ is free on V.

1.1.7. **Proposition.** *Let $A = \bigoplus_{n \geq 0} A_n$ be a graded algebra and consider a two-sided ideal $\mathcal{I}$ of A generated by homogeneous elements. We have that*

$$\mathcal{I} = \bigoplus_{n \geq 0} \mathcal{I} \cap A_n$$

and the quotient algebra $A/\mathcal{I}$ is graded with $(A/\mathcal{I})_n = A_n/(\mathcal{I} \cap A_n)$, for all $n \geq 0$.

Proof. The ideal $\mathcal{I}$ is generated by homogeneous elements x_i of degree d_i. So, if $x \in \mathcal{I}$ then we have that

$$x = \sum_i a_i \cdot x_i \cdot b_i,$$

for some $a_i, b_i \in A$. Since A is a graded algebra, we can write $a_i = \sum_j a_{ij}$ and $b_i = \sum_k b_{ij}$, where a_{ij} and b_{ij} are homogeneous elements of degree j. So, $x = \sum_{i,j,k} \cdot a_{ij} \cdot x_i b_{ik}$ is a sum of homogeneous elements of degree $d_j + j + k$ in $\mathcal{I}$, which implies that $\mathcal{I} \subseteq \bigoplus_{n \geq 0} \mathcal{I} \cap A_n$. The other inclusion is obvious. The graduation of the quotient $A/\mathcal{I}$ follows from the previous result. $\square$

1.1.8. *Symmetric algebra.* Let V be a vector space. The *symmetric algebra* $S(V)$ over V is the quotient of the tensor algebra $T(V)$ by the two-sided ideal $\mathcal{I}(V)$, generated by all the elements of the form

$$v_1 v_2 - v_2 v_1, \text{ for all } v_1, v_2 \in V.$$

As a free associative unital algebra $T(V)$ is generated by V, so the commutativity of the elements of degree one in the quotient $S(V)$, implies that $S(V)$ is a commutative algebra. On the other hand, since the ideal $\mathcal{I}(V)$ is generated by homogeneous elements of degree two, $S(V)$ is a graded algebra. If we denote by $S^n(V)$ the image of $T^n(V)$ under the projection of $T(V)$

onto $S(V)$, we get that $S^n(V)$ is the subspace of homogeneous elements of degree n.

The subspace $S^n(V)$ can be characterized as the quotient of $T^n(V)$ by the subspace generated by all the elements of the form

$$(v_1 v_2 \cdots v_n) - (v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(n)}),$$

for all $v_i \in V$ and all permutation $\sigma \in S_n$.

1.1.9. **Remark.** (1) The symmetric algebra $S(V)$ satisfies the following universal property:

For any commutative unital algebra A and any $\mathbb{K}$-linear map $f : V \to A$, there exists an unique algebra morphism $\widetilde{f} : T(V) \to A$ which extends f, that is, the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "i"'] & A \\ S(V) \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

where $i : V \to S(V)$ is the inclusion of V in $S(V)$. The construction of the extension $\widetilde{f}$ is similar to the associative case (see 1.1.6).

(2) If V is finite dimensional, then $S(V)$ is isomorphic to the algebra of polynomials in n variables $\mathbb{K}[x_1, \ldots, x_n]$ ([22], 1.1.10).

For the next example, we need to recall the definition of a Lie algebra.

1.1.10. **Definition.** A *Lie algebra* over $\mathbb{K}$ is a vector space $\mathfrak{g}$ together with a binary operation $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, called *bracket*, which satisfies the following relations:

(1) antisymmetry, $[x_1, x_2] = -[x_2, x_1]$,
(2) Jacobi identity, $[x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]] = 0$,

for all $x_1, x_2, x_3 \in \mathfrak{g}$.

The underlying vector space of any associative algebra A has a natural structure of Lie algebra with the bracket

$$[x, y] := x \cdot y - y \cdot x,$$

for $x, y \in A$. We denote by A^{Lie} the underlying vector space of A, equipped with the Lie algebra structure given by $[-,-]$. This construction defines a forgetful functor from the category of associative algebras to the category of Lie algebras,

$$(-)^{Lie} : As\text{-alg} \to Lie\text{-alg}.$$

1.1.11. *Universal enveloping algebra.* Let $\mathfrak{g}$ be a Lie algebra. The tensor algebra $T(\mathfrak{g})$ is the free unital associative algebra over the underlying vector space of $\mathfrak{g}$.

1.1.12. **Definition.** The *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ is the quotient of $T(\mathfrak{g})$ by the two-sided ideal generated by the elements

$$x \otimes y - y \otimes x - [x, y], \text{ for all } x, y \in \mathfrak{g}.$$

Note that there exists a Lie algebra homomorphism $\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ given by the composition $\iota := \pi \circ i$, where $i : \mathfrak{g} \to T(\mathfrak{g})$ is the inclusion of $\mathfrak{g} = T(\mathfrak{g})_1$ in $T(\mathfrak{g})$ and $\pi : T(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is the projection of $T(\mathfrak{g})$ onto the quotient $\mathcal{U}(\mathfrak{g})$.

1.1.13. **Proposition.** *Let $\mathfrak{g}$ be a Lie algebra. The universal enveloping algebra satisfies the following universal property:*
if A is an associative algebra and $f : \mathfrak{g} \to A^{Lie}$ is a Lie algebra morphism, then there exist an algebra morphism $\widetilde{f} : \mathcal{U}(\mathfrak{g}) \to A$ such that the following diagram commute:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f"] \arrow[d, "\iota"'] & A \\ \mathcal{U}(\mathfrak{g}) \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

Proof. Let A be an associative algebra and consider a Lie algebra morphism $f : \mathfrak{g} \to A^{Lie}$. As f is a linear application, by the universal property of $T(\mathfrak{g})$, there exists an unique algebra morphism $\widetilde{f} : T(\mathfrak{g}) \to A$ which extend f. So, we get that:

$$\widetilde{f}(x \otimes y - y \otimes x) = f(x)f(y) - f(y)f(x) = [f(x), f(y)] = f([x, y]),$$

which implies that $\widetilde{f}(x \otimes y - y \otimes x - [x, y]) = 0$, consequently the algebra morphism $\widetilde{f}$ is defined on the quotient $\mathcal{U}(\mathfrak{g})$.

□

1.1.14. **Remark.** Proposition 1.1.13 shows that the functor $\mathcal{U} : Lie\text{-alg} \to As\text{-alg}$ is left adjoint to the functor $(-)^{Lie} : As\text{-alg} \to Lie\text{-alg}$, that is, for any Lie algebra $\mathfrak{g}$ and any associative algebra A there is an isomorphism

$$\mathrm{Hom}_{\mathrm{As}}(\mathcal{U}(\mathfrak{g}), A) \cong \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, A^{Lie}).$$

1.2. **Coalgebras.** The notion of a *coalgebra* is dual to the notion of algebra, and is obtained by reversing all arrows in the diagrams defining an associative algebra.

1.2.1. **Definition.** A *coassociative coalgebra* is a $\mathbb{K}$-vector space C with a coproduct $\Delta : C \to C \otimes C$ such that the following diagram commute:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes\mathrm{Id}"] \\ C \otimes C \arrow[r, "\mathrm{Id}\otimes\Delta"] & C \otimes C \otimes C \end{tikzcd}$$

Additionally, a coalgebra C is *counitary* if there exist $\epsilon : C \to \mathbb{K}$, called counit, such that the following diagram commute:

$$\begin{tikzcd} & C \arrow[dl, "\simeq"'] \arrow[d, "\Delta"] \arrow[dr, "\simeq"] & \\ \mathbb{K} \otimes C & C \otimes C \arrow[l, "\mathrm{Id}\otimes\epsilon"'] \arrow[r, "\epsilon\otimes\mathrm{Id}"] & A \otimes \mathbb{K} \end{tikzcd}$$

A coalgebra (C, Δ) is *cocommutative* when the coproduct satisfies $\Delta = \tau \circ \Delta$, where $\tau : C \otimes C \to C \otimes C$ is given $\tau(x \otimes y) = y \otimes x$.
Given two coalgebras (C, Δ, ϵ) and (C', Δ', ϵ'), a morphism of coalgebras is a linear map $f : C \to C'$ such that

$$(f \otimes f) \circ \Delta = \Delta' \circ f \text{ and } \epsilon = \epsilon' \circ f.$$

1.2.2. **Remark.**
(1) The field $\mathbb{K}$ is a coalgebra with the coproduct given by $\Delta(1) = 1 \otimes 1$.
(2) A coalgebra is *coaugmented* if there exists a morphism of coalgebras $u : \mathbb{K} \to C$. In particular, $\epsilon \circ u = \mathrm{Id}_{\mathbb{K}}$. If C is coaugmented, then C is isomorphic, as vector space, to $\overline{C} \oplus \mathbb{K}1$, where $\overline{C} = \ker \epsilon$. In such case, we define the *reduced coproduct* $\bar{\Delta} : \overline{C} \to \overline{C} \otimes \overline{C}$ by

$$\bar{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x,$$

which is also coassociative.
(3) Let $(C, \Delta_A, \epsilon_A)$ and $(D, \Delta_D, \epsilon_B)$ be two counital coalgebras. The tensor product $C \otimes D$ has a natural structure of coalgebra with the coproduct given by $\Delta_{C\otimes D} := (\mathrm{Id} \otimes \tau_{C,D} \otimes \mathrm{Id}) \circ (\Delta_C \otimes \Delta_D)$ and the counit $\epsilon_{C\otimes D} := \epsilon_C \otimes \epsilon_D$.

1.2.3. **Definition.** A co-algebra (C, Δ, ϵ) is *graded* if the vector space $C = \bigoplus_{n\geq 0} C_n$ is graded, and the coproduct Δ satisfies:

(1) $\Delta(C_n) \subseteq \bigoplus_{i\leq n} C_i \otimes C_{n-i}$, for all $n \geq 0$ and
(2) $\epsilon(C_n) = 0$, for $n \neq 0$.

1.2.4. **Notation.** Let (C, Δ, ϵ) be a coalgebra. Given $x \in C$ we denote the image of x under Δ using the Sweedler's notation,

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \text{ or simply } \Delta(x) = x_{(1)} \otimes x_{(2)}.$$

Using Sweedler's notation, the coassocitivity of Δ is expressed as:

$$(x_{(1)(1)} \otimes x_{(1)(2)}) \otimes x_{(2)} = x_{(1)} \otimes (x_{(2)(1)} \otimes x_{(2)(2)}),$$

for all $x \in C$. So, we may write

$$(\Delta \otimes \mathrm{Id})\Delta(x) = (\mathrm{Id} \otimes \Delta)\Delta(x) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)}.$$

The counit condition of ϵ may be reformulated as:

$$\epsilon(x_{(1)})x_{(2)} = x = x_{(1)}\epsilon(x_{(2)}),$$

for all $x \in C$. On the other hand, C is cocommutative if $x_{(1)} \otimes x_{(2)} = x_{(2)} \otimes x_{(1)}$, for all $x \in C$.

The coassociativity of the coproduct Δ allow us to define n-ary cooperations on a coalgebra C.

1.2.5. **Definition.** Given a coalgebra (C, Δ) (not necessarily counital), we define recursively n-ary co-operations, $\Delta^{(n)} : C \to C^{\otimes n}$ as follow:

- $\Delta^{(1)} = \mathrm{Id}$, $\Delta^{(2)} = \Delta$
- $\Delta^{(3)} := (\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$
- $\Delta^{(n)} := (\Delta \otimes \mathrm{Id}^{\otimes(n-2)}) \circ \Delta^{(n-1)}$

1.2.6. **Remark.** By the coassociativity of Δ, we get

$$\Delta^{(n)} := (\mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \Delta \otimes \mathrm{Id} \cdots \otimes \mathrm{Id}) \circ \Delta^{(n-1)}.$$

In particular,

$$\begin{aligned} \Delta^{(n)}(x) &= \Delta^{(n-1)}(x_{(1)}) \otimes x_{(2)} \\ &= x_{(1)} \otimes \Delta^{(n-1)}(x_{(2)}). \end{aligned}$$

Extending Sweedler's notation, we write

$$\Delta^{(n)}(x) = x_{(1)} \otimes \cdots \otimes x_{(n)}.$$

Given a coaugmented coalgebra C, there exists a natural filtration on $\overline{C}$ given by:

- $F_1(C) := \{x \in \overline{C} \mid \bar{\Delta}(x) = 0\}$,
- $F_n(C) := \{x \in \overline{C} \mid \bar{\Delta}^{(r)}(x) = 0, \text{ for any } r \geq n\}$ for a given $n \geq 1$, where $\bar{\Delta}(x) := \Delta(x) - 1 \otimes x - x \otimes 1$.

The previous filtration give place to the notion of conilpotent coalgebra.

1.2.7. **Definition.** The counital coalgebra C is said to be *conilpotent* if

$$\overline{C} = \bigcup_{n \geq 1} F_n(C) \ .$$

1.2.8. **Remark.** Note that if (C, Δ) is a conilpotent coalgebra, any element in $\overline{C}$ is *conilpotent*, that is, for any $x \in \overline{C}$ there exists n such that $\bar{\Delta}^{(m)}(x) = 0$, for any $m \geq n$. We say that the *conilpotency degree* of $x \in \overline{C}$ is the smallest positive integer n that $\bar{\Delta}^{(n)}(x) = 0$. Moreover, by Remark 1.2.6, if $\bar{\Delta}(x) = x_{(1)} \otimes x_{(2)}$, the conilpotency degree of the elements $x_{(1)}$ and $x_{(2)}$ is strictly smaller than the conilpotency degree of x.

1.2.9. *Tensor Coalgebra.* Let V be a vector space. We denote by $T^c(V)$ the vector space

$$T^c(V) = \bigoplus_{n\geq 0} V^{\otimes n} = \mathbb{K} \oplus V \oplus \cdots V^{\otimes n} \oplus \cdots ,$$

equipped with the deconcatenation coproduct $\Delta^c : T^c(V) \to T^c(V) \otimes T^c(V)$ given by

$$\Delta^c(x_1 \cdots x_n) := \sum_{i=0}^{n} (x_1 \cdots x_i) \otimes (x_{i+1} \cdots x_n) \text{ and } \Delta^c(1) = 1 \otimes 1,$$

for $x_1, \ldots, x_n \in V$.

The counit $\epsilon : T^c(V) \to \mathbb{K}$ is the projection on $\mathbb{K}$, that is, it is the identity on $\mathbb{K}$ and 0 otherwise.

The coproduct Δ is coassociative and counital. Note that $T^c(V)$ is coaugmented by the inclusion $i : \mathbb{K} \to T(V)$. Moreover, $T^c(V)$ is a graded and conilpotent coalgebra. In this case, the filtration is given by

$$F_n(T^c(V)) = \bigoplus_{r\leq n} V^{\otimes r} .$$

The reduced coproduct is given by

$$\bar{\Delta}^c(v_1 \cdots v_n) := \sum_{i=1}^{n-1} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n.$$

1.2.10. **Remark.** The coalgebra $T^c(V)$ satisfies the following universal condition:

any linear map $\varphi : C \to V$, where C is a conilpotent coassociative coalgebra, satisfying $\varphi(1) = 0$, extends uniquely into a coaugmented coalgebra morphism $\tilde{\varphi} : C \to T^c(V)$:

$$\begin{tikzcd} C \arrow[d, "\tilde{\varphi}"'] \arrow[dr, "\varphi"] & \\ T^c(V) \arrow[r, "p_V"] & V \end{tikzcd}$$

where $p_V : T^c(V) \to V$ is the projection map, which is the identity on V and 0 otherwise. We say then that $T^c(V)$ is the *cofree conilpotent coalgebra* over V (see [22], Proposition 1.2.7).

1.2.11. **Definition.** Let (C, Δ, ϵ, u) be a coaugmented coalgebra. An element $x \in \overline{C}$ is *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$ or, equivalently, $x \in F_1(C)$. The subspace of primitive elements of C is denoted by $\mathrm{Prim}(C)$.

1.3. **Bialgebras.**

1.3.1. **Definition.** A *bialgebra* $(H, \mu, \eta, \Delta, \epsilon)$ is $\mathbb{K}$-vector space H equipped with a unital algebra structure (H, μ, η) and a counital coalgebra structure (H, Δ, ϵ), satisfying that Δ y ϵ are algebra homomorphisms,

$$\Delta(x \cdot y) = (\mu \otimes \mu) \circ (Id_H \otimes \tau \mathrm{Id}_H) \circ (\Delta(x) \otimes \Delta(y)) \text{ and } \epsilon(x \cdot y) = \epsilon(x)\epsilon(y),$$

for any pair of elements $x, y \in H$.

1.3.2. **Remark.**

(1) The compatibility condition between the algebra structure and the coalgebra structure in H is equivalent to require that μ and η are coalgebra homomorphisms.

(2) When we consider the non-unital bialgebra $\overline{H} := \ker(\epsilon)$ with the reduced coproduct $\bar{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$, the compatibility relation is given by:

$$\begin{aligned} \bar{\Delta}(xy) &= x \otimes y + y \otimes x + x_{(1)} \otimes x_{(2)}y + x_{(1)}y \otimes x_{(2)} + \\ & \quad xy_{(1)} \otimes y_{(2)} + y_{(1)} \otimes xy_{(2)} + x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}, \end{aligned}$$

where $\bar{\Delta}(x) = x_{(1)} \otimes x_{(2)}$ and $\bar{\Delta}(y) = y_{(1)} \otimes y_{(2)}$.

(3) A bialgebra $(H, \mu, \eta, \Delta, \epsilon)$ is graded, if H is a graded algebra and a graded coalgebra, with the same graduation.

1.3.3. **Proposition.** *If H is a bialgebra, then the subspace of primitive elements* $\mathrm{Prim}(H)$ *is a Lie subalgebra of H.*

Proof. Suppose that x and y are primitive elements of H, we get that

$$\begin{aligned} \Delta(x \cdot y) = (\mu \otimes \mu) \circ (Id_H \otimes \tau \otimes Id_H)(x \otimes 1_{\mathbb{K}} \otimes y \otimes 1_{\mathbb{K}} + x \otimes 1_{\mathbb{K}} \otimes 1_{\mathbb{K}} \otimes y + 1_{\mathbb{K}} \otimes x \otimes y \otimes 1_{\mathbb{K}} \\ + 1_{\mathbb{K}} \otimes x \otimes 1_{\mathbb{K}} \otimes y) = x \cdot y \otimes 1_{\mathbb{K}} + x \otimes y + y \otimes x + 1_{\mathbb{K}} \otimes x \cdot y, \end{aligned}$$

which implies that

$$\begin{aligned} \Delta([x, y]) &= \Delta(x \cdot y) - \Delta(y \cdot x) \\ &= (x \cdot y - y \cdot x) \otimes 1_{\mathbb{K}} + 1_{\mathbb{K}} \otimes (x \cdot y - y \cdot x) \\ &= [x, y] \otimes 1_{\mathbb{K}} + 1_{\mathbb{K}} \otimes [x, y]. \end{aligned}$$

So, $\mathrm{Prim}(H)$ is a Lie subalgebra of H. □

1.4. **The tensor bialgebra.**

Let V be a vector space. As $T(V)$ is the free associative unital algebra over V, there exists an unique algebra morphism $\Delta_{sh} : T(V) \to T(V) \otimes T(V)$ such that

$$\Delta_{sh}(v) = v \otimes 1 + 1 \otimes v,$$

for any $v \in V$.

By the universal property of $T(V)$, the coproduct Δ_{sh} is coassociative. Therefore, $T(V)$ is a bialgebra, which is graded, cocommutative and conilpotent. The data $(T(V), \cdot, \Delta_{sh}, \iota, \epsilon)$ is called the *tensor bialgebra* over V.

Let us give an explicit formula for the coproduct Δ_{sh}.

1.4.1. **Definition.** A (n, m)-*shuffle* is a permutation $\sigma \in S_{n+m}$ such that $\sigma(1) < \cdots < \sigma(n)$ and $\sigma(n+1) < \cdots < \sigma(n+m)$. We denote the set all (n, m)-shuffle by $\mathrm{Sh}(n, m)$.

1.4.2. **Proposition.** *Let V be a vector space. The coproduct Δ_{sh} on $T(V)$ is given by*

$$\Delta_{sh}(v_1 \cdots v_n) = \sum_{i=0}^{n} \sum_{\sigma \in Sh(i,n-i)} v_{\sigma(1)} \cdots v_{\sigma(i)} \otimes v_{\sigma(i+1)} \cdots v_{\sigma(n)}.$$

1.4.3. **Remark.** In an analogous way, the symmetric algebra $S(V)$ has a natural structure of bialgebra by defining

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \text{for } x \in V$$

and extending this map to $\Delta : S(V) \to S(V) \otimes S(V)$, using the universal property of $S(V)$.

Let us describe the formula for the coproduct Δ in this case. For an homogeneous element $x = v_1 \cdots v_n \in S^n(V)$ (see 1.1.8), let I be a subset of $[n]$. If $I = \emptyset$, then we define $x_I := 1$. If $I = \{i_1, \ldots, i_k\}$, with $I \neq \emptyset$, then we define $x_I := v_{i_1} \cdots v_{i_k}$. So, $x_I \in S^k(V)$. In this way, we have that

$$\Delta(x) = \sum_{I \cup J = [n]} x_I \otimes x_J.$$

The symmetric algebra over V with this coalgebra structure is denoted by $S^c(V)$.

1.5. **Bialgebra structure for $\mathcal{U}(\mathfrak{g})$.** Let $\mathfrak{g}$ be a Lie algebra. As $\mathcal{U}(\mathfrak{g})$ is a quotient of $T(\mathfrak{g})$, it suffices to verify that the bialgebra structure is compatible with the quotient map to get a bialgebra structure on $\mathcal{U}(\mathfrak{g})$. The map Δ is Lie algebra homomorphism. Indeed, if $x, y \in \mathfrak{g}$ then:

$$\begin{aligned}
\Delta([x,y]) &= [x,y] \otimes 1 + 1 \otimes [x,y] \\
&= (xy - yx) \otimes 1 + 1 \otimes (xy - yx) \\
&= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x) \\
&= \Delta(x)\Delta(y) - \Delta(y)\Delta(x) \\
&= [\Delta(x), \Delta(y)].
\end{aligned}$$

By the universal property of $\mathcal{U}(\mathfrak{g})$, there exists a unique algebra homomorphism $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$, which extend $\Delta(x) = x \otimes 1 + 1 \otimes x$, for $x \in \mathfrak{g}$. Therefore, the universal enveloping $\mathcal{U}(\mathfrak{g})$ is a bialgebra.

Note that, in particular, if H is a bialgebra, then $\mathrm{Prim}(H)$ is a Lie algebra (1.3.3). So, $\mathcal{U}(\mathrm{Prim}(H))$ is a biagebra with the structure previously described.

To define a *Hopf algebra*, we need previously to introduce the notion of *antipode*.

1.5.1. **Definition.** Let $(H, \mu, \Delta, \eta, \epsilon)$ be a bialgebra. Given a pair of $\mathbb{K}$-linear maps $f, g \in \mathrm{End}_{\mathbb{K}}(H)$, the *convolution product* of f and g is the linear map $f \star g : H \to H$ given by the composition

$$f \star g := \mu \circ (f \otimes g) \circ \Delta.$$

It is immediate to verify that the endomorphism $\eta \circ \epsilon$ is a unit for $\star$ and that $\star$ is associative. So, the data $(\mathrm{Hom}_{\mathbb{K}}(H), \star, \eta \circ \epsilon)$ is a unital associative algebra.

1.5.2. **Definition.** A *Hopf algebra* is a bialgebra $(H, \mu, \Delta, \eta, \epsilon)$ equipped with a linear map $S : H \to H$ such that S is the inverse of $\mathrm{id} : H \to H$ with respect to the convolution product, that is

$$S \star \mathrm{id} = \mathrm{id} \star S = \eta \circ \epsilon.$$

The linear endomorphism S is called an *antipode* for H.

1.5.3. **Remark.** If H conilpotent bialgebra, then H is equipped with an antipode map S, given by the formula

$$S(x) := -x + \sum_{n \geq 1} (-1)^{n+1} \mu^n \circ \bar{\Delta}^{n-1}(x).$$

In particular, for any vector space V, the tensor bialgebra $T(V)$ is a Hopf algebra. Similarly, if $\mathfrak{g}$ is a Lie algebra, the universal enveloping $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra, too.

1.6. **Dual Hopf algebra.** Let V be a vector space. Consider $V^* = \mathrm{Hom}_{\mathbb{K}}(H, \mathbb{K})$ the dual space of V. The linear map $\varphi : V^* \otimes V^* \to (V \otimes V)^*$, given by $\varphi(f \otimes g)(x \otimes y) = f(x)g(y)$, is a isomorphism whenever V is finite dimensional.

On the other hand, if V and W are vector spaces and $F : V \to W$ is a linear map, then F induce a linear map $F^* : W^* \to V^*$, given by $F^*(f) = f \circ F$.

If (C, Δ) is a coalgebra, then $(C^*, \Delta^* \circ \varphi)$ is an associative algebra (in this case, we do not need that C has finite dimension).

Respectively, if (A, μ) is associative algebra of finite dimension, then $(A^*, \varphi^{-1} \circ \mu)$ is a coassociative coalgebra.

Suppose now that (H, μ, Δ) is a Hopf algebra of dimension $n < \infty$. Let $B = \{e_i\}_{i \in [n]}$, respectively $B^* = \{\delta_i\}_{i \in [n]}$, be a basis the underlying vector space of H, respectively its dual basis. There exists a canonical isomorphism of vector spaces $H^* \cong H$, given by $\delta_i \leftrightarrow e_i$. The last isomorphism, which strongly depends on the basis, gives another Hopf algebra structure on H, induced by (H^*, μ^*, Δ^*), that we denote (H, μ', Δ').

The product μ' and co-product Δ' are given, in terms of the basis B, by the formulas:

(1) $\mu'(e_i \otimes e_j) = \sum_r \lambda^r_{i,j} e_r$, where $\Delta(e_r) = \sum_{l,k} \lambda^r_{l,k} e_l \otimes e_k$ and

(2) $\Delta'(e_i) = \sum_{j,l} a^i_{jl} e_j \otimes e_l$, where $\mu(e_j \otimes e_l) = \sum_i a^r_{jl} e_r$.

The last construction may be extended to any graded Hopf algebra. Let $V = \bigoplus_{n\geq 0} V_n$ be graded vector space such that its homogeneous components are of finite dimension.

(1) The graded dual V^* is the graded vector space $\bigoplus_{n\geq 0} V_n^*$. Moreover, as its homogeneous components are of finite dimension, $V \cong V^*$.
(2) The tensor product $V \otimes V$ is also a graded vector space, with graduation given by $(V \otimes V)_n = \bigoplus_{i=0}^{n} V_i \otimes V_{n-i}$, for all $n \geq 0$. Moreover, under the finite dimensional hypothesis, $(V \otimes V)^* \cong V^* \otimes V^*$.
(3) Consider V and W are two graded vector space, both with its homogenous components of finite dimension, and $F : V \to W$, homogeneous of degree d, that is, $F(V_n) \subseteq W_{n+d}$, for all $n \geq 0$. Then, there exist an unique $F^* : W^* \to V^*$ given by $F^*(f) = f \circ F$.

The previous considerations imply that if (H, μ, Δ) is a graded Hopf algebra, then the graded dual H^* is also a Hopf algebra. Moreover, as $H \cong H^*$, it induces another Hopf algebra structure on H.

1.6.1. **Example.** Consider the tensor bialgebra $(T(V), \mu, \Delta)$, where μ is the concatenation product and Δ is the shuffle co-product. In this case, the dual coalgebra structure is given by the tensor coalgebra $T^c(V)$ considered in 1.2.9. The coproduct is given by the deconcatenation coproduct,

$$\Delta^c(x_1 \cdots x_n) := \sum_{i=0}^{n} (x_1 \cdots x_i) \otimes (x_{i+1} \cdots x_n).$$

Conversely, it is not difficult to see that the dual product is given by

$$(v_1 \cdots v_n) * (v_{n+1} \cdots v_{n+m}) = \sum_{\sigma \in Sh(n,m)} v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n+m)},$$

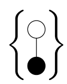
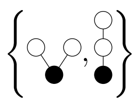
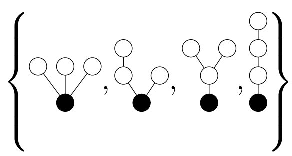
which is called *the shuffle product*. The bialgebra $(T^c(V), *, \Delta^c)$ is known as *the shuffle bialgebra*.

1.7. **Cartier-Milnor-Moore theorem.** Let us describe one of the main structure theorems for cocommutative bialgebras, due to J. Milnor and J. Moore. The version included in the present manuscript is due to J.-L. Loday and B. Vallette in [25], Theorem 1.3.6, which contains two classical results: the Cartier-Milnor-Moore theorem and the Poincaré-Birkhoff-Witt theorem [25]. For more details about this theorem we refer to [25], while the original version of the result, we refer to [28] and [35].

1.7.1. **Theorem.** *Let $\mathbb{K}$ be a characteristic zero field. For any cocommutative bialgebra H over $\mathbb{K}$, the following are equivalents:*

(1) H is conilpotent,
(2) $H \cong \mathcal{U}(\mathrm{Prim}(H))$ as bialgebra.
(3) $H \cong S^c(\mathrm{Prim}(H))$ as a conilpotent coalgebra.

1.7.2. *The Grossman-Larson's Hopf algebra.* In [17], R. Grossman and R.G. Larson described Hopf algebras which are associated with certain families of trees. Let $\mathcal{T}$ be the set of all non-planar rooted trees and $\mathcal{T}_n$ the subset of trees in $\mathcal{T}$ with $n+1$ vertices. For instance,

- $\mathcal{T}_0 = \{\bullet\}$,
- $\mathcal{T}_1 =$ ,
- $\mathcal{T}_2 =$ ,
- $\mathcal{T}_3 =$ ,

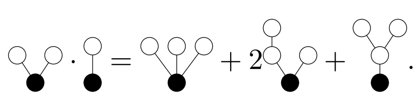
where the root of the tree is painted in black. The tree with only one vertex is denoted by e, and belongs to $\mathcal{T}_0$. Given a tree t and a vertex $v \in t$, we say that a vertex $v' \in t$ is a child of v if v' is directly connected to the vertex v.

Let $H = \mathbb{K}[\mathcal{T}]$ be the vector space generated by the set $\mathcal{T}$. The vector space H is naturally graded by

$$H = \bigoplus_{n \geq 0} \mathbb{K}[\mathcal{T}_n],$$

where $\mathbb{K}[\mathcal{T}_n]$ is the vector space spanned by $\mathcal{T}_n$.

The associative product on H is defined as follows. If t_1 and t_2 are two rooted trees, the product $t_1 \cdot t_2$ is the sum of the trees obtained by attaching the children of the root of t_1 to the vertices of t_2 in all possible ways. For example:



Note that this product is not commutative and the unit is given by the tree e.

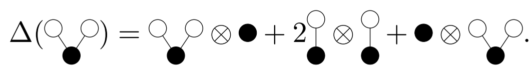
Let us describe the coalgebra structure for H. Let t be a tree and $v_1, \ldots, v_r$ the children of the root of t. Let $X = \{t_1, \ldots, t_r\}$ be the set of subtrees of t whose roots are given by $v_1, \ldots, v_r$, respectively.

Consider Y a subset of X. If $Y = \emptyset$, then define $t_Y := e$. If $Y = \{t_{i_1}, \ldots, t_{i_k}\}$, then define t_Y as the tree that is obtained by attaching of

trees $t_{i_1}, \ldots, t_{i_k}$ to a new the root.
The coproduct $\Delta(t)$ is given by

$$\Delta(t) = \sum_{Y \subseteq X} t_Y \otimes t_{X \setminus Y}.$$

So, the coproduct $\Delta(t)$ is the sum of the 2^r terms $t' \otimes t''$, where the children of the root of t' and the children of the root t'' range over all 2^r possible partitions of the children of the root of t into two subsets. For instance,



Note that, if $t = e$, then $\Delta(e) = e \otimes e$. The counit ϵ is given by $\epsilon(e) = 1$, $\epsilon(t) = 0$ if $t \neq e$.

The coproduct Δ is cocommutative and $(H, \cdot, \Delta)$ is a conilpotent bialgebra. So, by Remark 1.5.3, H is a Hopf algebra.

By Cartier-Milnor-Mooree theorem, H is isomorphic, as bialgebra, to the universal enveloping algebra of its primitive elements, $H \cong \mathcal{U}(\mathrm{Prim}(H))$. Explicitly, $\mathrm{Prim}(H)$ has as basis the set all rooted trees whose root has exactly one child (for more details, see [17])

1.8. **Chain complex and homology.** We give the basic notions of chain complexes and homology.

For two graded vector spaces V and W, a *morphism of degree* r , $f : V \to W$, is a family of linear maps $f_n : V_n \to W_{n+r}$, for all $n \in \mathbb{Z}$. The integer r is called *the degree* of f and we write $|f| = r$. Note that $|f(v)| = |f| + |v|$.

The tensor product $V \otimes W$ (in the graded framework) is defined as:

$$(V \otimes W)_n := \bigoplus_{i+j=n} V_i \otimes W_j.$$

The *suspension* of the graded vector space V is sV, where we identify $(sV)_i$ with V_{i-1}. So, sV is the graded vector space V with the degree shifted in 1. Respectively, the *desuspension* of V is given by $s^{-1}V$. In this case, it is the graded vector space V with V_{i+1} considered in degree i.

1.8.1. *Chain complex.* A *chain complex*, also called *differential graded vector space*, is a graded vector space (C_*, d) equipped with a linear map

$$d : C_* \to C_{*-1},$$

of degree -1, called *differential o boundary map*, satisfying the condition $d \circ d = 0$. We assume that the graduation of C is non negative and we represent the chain complex by

$$\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0.$$

Given a differential vector space (C, d), a *cycle* of degree n is an element of the kernel $\mathrm{Ker} d_n$, while a *boundary* in C_n is an element of $\mathrm{Im} d_{n+1}$. The condition $d \circ d = 0$ implies that $\mathrm{Im}(d_{n+1}) \subseteq \mathrm{Ker}(d_n)$.

The *n-th. homology group* of a chain complex (C, d) is the quotient $H_n(C_*) := \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1})$.

A *morphism of complexes* $f : C \to C'$ is a map of graded vector spaces of degree 0, such that the following diagram commute for any n,

$$\begin{tikzcd} C_n \arrow[r, "d"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "d"] & C'_{n-1} \end{tikzcd}$$

The commutativity of the diagram implies that f induces a morphism $f_* : H_n(C) \to H_n(C')$.

A homomorphism of complexes $f : C_* \to C'_*$ is a *quasi-isomorphism* if f_* is an isomorphism, for all integer n.

Given two differential graded vector spaces (chain complexes) (V, d_V) and (W, d_W), their *tensor product* is the graded vector space $V \otimes W$, whose subspace of homogeneous elements of degree n is $(V \otimes W)_n := \bigoplus_{i+j=n} V_i \otimes W_j$, equipped with the differential map

$$d_{V\otimes W} := d_V \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes d_W.$$

If $v \otimes w \in V_i \otimes W_j$, then

$$d_{V\otimes W}(v \otimes w) = d_V(v) \otimes w + (-1)^i v \otimes d_W(w).$$

1.8.2. *Differential graded algebra and differential graded coalgebra.* A *differential graded associative algebra* $(A, \cdot, d)$ is a graded algebra equipped with a differential map $d : A \to A$ of degree -1 satisfying that d is a derivation for the product, that is:

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b).$$

The last identity can be expressed as:

$$d \circ \cdot = \cdot \circ (d \otimes \mathrm{id} + \mathrm{id} \otimes d),$$

where $\cdot$ is the product in A. So, $\cdot : A \otimes A \to A$ is a morphism of chain complexes.

Respectively, a *differential graded coassociative coalgebra* (C, Δ, d) is a graded coalgebra equipped with a differential map $d : C \to C$ of degree -1, which is a *coderivation* for the coproduct Δ, that is:

$$\Delta \circ d = (d \otimes \mathrm{id} + \mathrm{id} \otimes d) \circ \Delta.$$

Equivalently, $\Delta : C \to C \otimes C$ is a morphism of chain complexes.

1.8.3. *Bicomplex.* A *bicomplex* (also called *double complex*) is a bigraded vector space $C = \{C_{pq}\}_{p\geq 0, q\geq 0}$ together with a horizontal differential d^h : $C_{pq} \to C_{(p-1)q}$ and a vertical differential $d_v : C_{pq} \to C_{p(q-1)}$ satisfying that:

$$d_h \circ d_v + d_v \circ d_h = 0.$$

Graphically, we have:

$$\begin{tikzcd}
{} \arrow[d, "d^v"'] & {} \arrow[d, "d^v"'] & {} \arrow[d, "d^v"'] & \\
C_{20} \arrow[d, "d^v"'] & C_{21} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & C_{22} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & {} \arrow[l, "d^h"'] \\
C_{10} \arrow[d, "d^v"'] & C_{11} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & C_{12} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & {} \arrow[l, "d^h"'] \\
C_{00} & C_{01} \arrow[l, "d^h"'] & C_{02} \arrow[l, "d^h"'] & {} \arrow[l, "d^h"']
\end{tikzcd}$$

The *total complex* associated to a bicomplex (C, d_h, d_v) is defined by

$$(\mathrm{Tot}C)_n := \bigoplus_{p+q=n} C_{pq} \text{ and } d = d^h + d^v.$$

We can easily verify that the conditions satisfied by the differential maps d^v and d^h imply that $((\mathrm{Tot}V)_n, d)$ is a chain complex. The homology groups $H_n(\mathrm{Tot}C)$ are called the *homology groups of the bicomplex* C_{**}.

Other chain complex may be defined from a bicomplex (C, d_h, d_v). First, consider the homology groups of vertical complex $H_p(C_{*,q})$, for a fixed q. The horizontal differential induces a map

$$(d^h)_* : H_p(C_{*,q}) \to H_p(C_{*,q-1}),$$

for all fixed p. The homology groups of the last complex are denoted by $H^h_q H^v_p(C)$. Similarly, we can define other complex considering first the horizontal homology and then the vertical homology. In such case, we denote the homology groups by $H^v_p H^h_q(C)$.

The following proposition establishes a relationship between these homology groups and the total complex homology groups.

1.8.4. **Proposition.** *Let (C, d_h, d_v) be a bicomplex. Suppose that, for all integer q, the vertical homology groups $H_p(C_{*,q})$ are 0, for $p > 0$. Let $K_n := H_0(C_{*,n})$, be the unique non trivial homology group . We have that $H_n(\mathrm{Tot}C_{**}) = H_n(K_*, d^v)$, for $n \geq 0$.*

Proof. See [23], Proposition 1.0.12. □

1.9. **Unital infinitesimal bialgebras.** In [20], M. Joni and G.-C. Rota introduced the notion of *infinitesimal bialgebra*. This notion was modified by J.-L. Loday and M. Ronco [26] in order to consider infinitesimal bialgebras with units. We give a brief description of the last type of structure, for a more complete definition and basic results we refer to [26].

1.9.1. **Definition.** A *unital infinitesimal bialgebra* $(H, \cdot, \Delta)$ is a vector space H equipped with a unital associative product $\cdot : H \otimes H \to H$ and a counital coproduct $\Delta : H \to H \otimes H$ which satisfy the following relation:

$$\Delta(x \cdot y) = x \cdot y_{(1)} \otimes y_{(2)} + x_{(1)} \otimes x_{(2)} \cdot y - x \otimes y,$$

for $x, y \in H$, where $\Delta(x) = x_{(1)} \otimes x_{(2)}$ and $\Delta(y) = y_{(1)} \otimes y_{(2)}$.

1.9.2. **Remark.** Note that the unital infinitesimal relation in a unital infinitesimal bialgebra $(H, \cdot, \Delta)$ is equivalent to require that the reduced coproduct $\bar{\Delta}$ verifies the relation

$$\bar{\Delta}(x \cdot y) = x \cdot y_{(1)} \otimes y_{(2)} + x_{(1)} \otimes x_{(2)} \cdot y + x \otimes y,$$

for $x, y \in \overline{H}$, where $\bar{\Delta}(x) = x_{(1)} \cdot x_{(2)}$ and $\bar{\Delta}(y) = y_{(1)} \otimes y_{(2)}$.

The last equality implies that

$$\begin{aligned}\bar{\Delta}^{(n)}(x \cdot y) &= \sum_{i=1}^{n} x_{(1)} \otimes \cdots \otimes (x_{(i)} \cdot y_{(1)}) \otimes \cdots \otimes y_{(n-i)} \\ &\quad + \sum_{i=1}^{n-1} x_{(1)} \otimes \cdots \otimes x_{(i)} \otimes y_{(1)} \otimes \cdots \otimes y_{(n-i)},\end{aligned}$$

where $\bar{\Delta}^{(i)}(x) = x_{(1)} \otimes \cdots \otimes x_{(i)}$, for all $1 \leq i \leq n$.

1.9.3. **Example.** Consider $H = \mathbb{K}[x]$, the polynomial algebra in the variable x with its usual product. Define $\Delta : H \to H \otimes H$ as

$$\Delta(x^n) = \sum_{i=0}^{n} x^i \otimes x^{n-i},$$

for $n \geq 0$. The polynomial algebra $\mathbb{K}[x]$, equipped with Δ, is a unital infinitesimal bialgebra.

1.9.4. **Remark.** The unital infinitesimal relation differs from the infinitesimal relation used by S. Joni and G.-C. Rota in [20] by the presence of term $-x \otimes y$, which implies that $\Delta(1) = 1 \otimes 1$.

The main example of example of unital infinitesimal bialgebra is given by the tensor algebra $T(V)$ over a vector space V, equipped with coassociative coalgebra structure given by deconcatenation coproduct.

1.9.5. **Proposition.** *The tensor algebra $T(V)$ over V equipped with the deconcatenation coproduct Δ is a unital infinitesimal bialgebra.*

Proof. Consider the tensors $x = v_1 \cdots v_p$ and $y = v_{p+1} \cdots v_{p+n}$. Computing $\Delta(x \cdot y)$, we obtain:

$$\begin{aligned}
\Delta(x \cdot y) &= \Delta(v_1 \cdots v_p v_{p+1} \cdots v_{p+n}) \\
&= \sum_{i=0}^{p+n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_{p+n} \\
&= \sum_{i=0}^{p} v_1 \cdots v_i \otimes v_{i+1} \cdots v_p v_{p+1} \cdots v_{p+n} \\
&\qquad + \sum_{i=p+1}^{p+n} v_1 \cdots v_p v_{p+1} \cdots v_i \otimes v_{i+1} \cdots v_{p+n} \\
&= \sum_{i=0}^{p} v_1 \cdots v_i \otimes (v_{i+1} \cdots v_p) \cdot y \\
&\qquad + \sum_{j=0}^{p+n} x \cdot (v_{p+1} \cdots v_i) \otimes v_{i+1} \cdots v_{p+n} - x \otimes y \\
&= x_{(1)} \otimes x_{(2)} \cdot y + x \cdot y_{(1)} \otimes y_{(2)} - x \otimes y.
\end{aligned}$$

□

Let us recall the linear operator e, originally defined by J.-L. Loday and M. Ronco in [26], for a conilpotent unital infinitesimal bialgebra $(H, \cdot, \Delta)$.

1.9.6. **Definition.** Let $(H, \cdot, \Delta)$ be a conilpotent unital infinitesimal bialgebra, with unit η and counit ϵ. The linear operator $e : H \to H$ is defined by

$$e := J - J \star J + \cdots + (-1)^{n-1} J^{\star n} + \cdots ,$$

where $J := \mathrm{Id} - \eta \circ \epsilon$.

1.9.7. **Remark.** Note that the operator e is well defined. In fact, if $x = 1_H$, then $e(x) = 0$ because $\Delta(1_H) = 1_H \otimes 1_H$. If $x \in \overline{H}$, we can write

$$(*) \quad e(x) = x - x_{(1)} \cdot x_{(2)} + x_{(1)} \cdot x_{(2)} \cdot x_{(3)} - \cdots + (-1)^{n-1} x_{(1)} \cdot x_{(2)} \cdots x_{(n)} + \cdots ,$$

where $\bar{\Delta}^{(n)}(x) = x_{(1)} \otimes \cdots \otimes x_{(n)}$, and we have omitted the sum symbol. Since $(H, \cdot, \Delta)$ is a conilpotent coalgebra, there exists a positive integer n such that $\bar{\Delta}^{(n)}(x) = 0$. So, the sum in $(*)$ is finite. Therefore, e is well defined.

The following proposition was proved in [26].

1.9.8. **Proposition.** *Let $(H, \cdot, \Delta)$ be a conilpotent unital infinitesimal bialgebra. The linear operator $e : H \to H$ defined in 1.9.6, has the following properties:*

(1) $\mathrm{Im}(e) = \mathrm{Prim}(H)$,
(2) *if $x, y \in \overline{H}$ then $e(x \cdot y) = 0$,*
(3) *the operator e is an idempotent,*
(4) *for $(T(V), \cdot, \Delta)$, where $\cdot$ is the concatenation and Δ the deconcatenation, the operator is the identity on V and 0 on the other components.*

Proof. First, note that Remark 1.9.7 implies that if $x \in \overline{H}$, then

$$(*) \quad e(x) = x - x_{(1)} \cdot e(x_{(2)}).$$

(1) By $(*)$, we have that if $x \in \mathrm{Prim}(H)$, then $e(x) = x$, because $\bar{\Delta}(x) = 0$. So, $\mathrm{Prim}(H) \subseteq \mathrm{Im}(e)$ and $e(x) \in \mathrm{Prim}(H)$, if $x \in \mathrm{Prim}(H)$.

Let x be a element of H. We want to see that $e(x)$ is a primitive element. If $x = 1$ then $e(x) = 0$, because $\Delta(1) = 1 \otimes 1$. For $x \in \overline{H}$, we proceed by induction on $n =$ the conilpotency degree of x.

If $n = 1$ then x is primitive element and $e(x) = x$. Consider now $n > 1$ and suppose that $e(y)$ is a primitive element if the degree of y is smaller than n. Let x be a element in (H) whose conilpotency degree is n. By $(*)$, we have that

$$\begin{array}{rcl} \bar{\Delta}(e(x)) & = & \bar{\Delta}(x - x_{(1)}e(x_{(2)})) \\ & = & x_{(1)} \otimes x_{(2)} - \bar{\Delta}(x_{(1)} \cdot e(x_{(2)})) \\ & = & x_{(1)} \otimes x_{(2)} - x_{(1)} \cdot \bar{\Delta}(e(x_{(2)})) \\ & & -x_{(1)(1)} \otimes x_{(1)(2)} \cdot e(x_{(2)}) - x_{(1)} \otimes e(x_{(2)}). \end{array}$$

By Remark 1.2.8, the conilpotency degree of $x_{(1)}$ and $x_{(2)}$ is strictly less than the conilpotency degree of x for all $x_{(1)} \otimes x_{(1)}$ in the sum $\bar{\Delta}(x) = x_{(1)} \otimes x_{(2)}$. By induction, we have that $\bar{\Delta}(e(x_{(2)})) = 0$.

On the other hand, by the coassociativity of $\bar{\Delta}$, we have that
$x_{(1)(1)} \otimes x_{(1)(2)} \cdot e(x_{(2)}) = x_{(1)} \otimes x_{(2)(1)} \cdot e(x_{(2)(2)})$. Therefore,

$$\begin{array}{rcl} \bar{\Delta}(e(x)) & = & x_{(1)} \otimes x_{(2)} - x_{(1)} \otimes x_{(2)(1)} \cdot e(x_{(2)(2)}) - x_{(1)} \otimes e(x_{(2)}) \\ & = & x_{(1)} \otimes (x_{(2)} - x_{(2)(1)} \cdot e(x_{(2)(2)})) - x_{(1)} \otimes e(x_{(2)}) \\ & = & x_{(1)} \otimes e(x_{(2)}) - x_{(1)} \otimes e(x_{(2)}) \\ & = & 0, \end{array}$$

which proves that $e(x)$ is a primitive element.

(2) We proceed by induction on the conilpotency degree of the product $x \cdot y$, with $x, y \in \overline{H}$. If x and y are both primitives, we have that

$$\bar{\Delta}(x \cdot y) = x \otimes y \text{ and } \bar{\Delta}^{(2)}(x \cdot y) = 0.$$

So, we obtain that $e(x \cdot y) = x \cdot y - (x \cdot y)_{(1)} e((x \cdot y)_{(2)}) = x \cdot y - x \cdot y = 0$.

Suppose now that the conilpotency degree of $x \cdot y$ is $n > 2$ and suppose the assertion is true for degree strictly less than n. Computing $e(x \cdot y)$, we have that:

$$\begin{aligned} e(x \cdot y) &= x \cdot y - x_{(1)} \cdot e(x_{(2)} \cdot y) - x \cdot y_{(1)} \cdot e(y_{(2)}) - x \cdot e(y) \\ &= x \cdot (y - y_{(1)} \cdot e(y_{(2)})) - x \cdot e(y), \text{ by induction} \\ &= x \cdot e(y) - x \cdot e(y), \text{ by } (*) \\ &= 0, \end{aligned}$$

which concludes the proof.

(3) The assertion follows immediately from (1), because $e(x) = x$, if x is primitive and $e(x)$ is primitive for all $x \in H$.

(4) The proof of this statement is by direct inspection.

□

The following theorem was stated by J.-L. Loday and M. Ronco in [26].

1.9.9. **Theorem.** *Any conilpotent unital infinitesimal bialgebra H is isomorphic to $T^c(\mathrm{Prim}H) := (T(\mathrm{Prim}H), \nu, \Delta)$, where ν is the concatenation product and Δ is the deconcatenation coproduct.*

Proof. Let $(H, \cdot, \Delta)$ be a conilpotent unital infinitesimal bialgebra. We denote by $V := \mathrm{Prim}(H)$ and a tensor in $V^{\otimes n}$ is denoted by $x_1 \cdots x_n$. We define $G : \overline{H} \to \overline{T}(V)$ by the formula

$$G(x) := \sum_{n \geq 1} e^{\otimes n} \circ \bar{\Delta}^{(n)}(x).$$

Note that as H is conilpotent and $\mathrm{Im}(e) = \mathrm{Prim}(H)$, by Proposition 1.9.8, then G is well defined. Moreover, we can write

$$G(x) = e(x) + e(x_{(1)})e(x_{(2)}) + e(x_{(1)})e(x_{(2)})e(x_{(3)}) + \cdots .$$

By Remark 1.9.2 and Proposition 1.9.8, part (2), we have that:

$$\begin{aligned} G(x \cdot y) &= \sum_{n \geq 1} \sum_{i=1}^{n} e(x_{(1)}) \cdots e(x_{(i)}) e(y_{(1)}) \cdots e(y_{(n-i)}) \\ &= \sum_{n \geq 1} \sum_{i=1}^{n} (e^{\otimes i} \circ \bar{\Delta}^{(i)}(x))(e^{\otimes (n-i)} \circ \bar{\Delta}^{(n-i)}(y)) \\ &= G(x)G(y). \end{aligned}$$

So, G is an algebra morphism.

Let us show that G is also a coalgebra morphism.
We must prove that $(G \otimes G) \circ \bar{\Delta} = \bar{\Delta} \circ G$, where we use $\bar{\Delta}$ indistinctly for the reduced of H and of $H \otimes H$. As $G(x) = \sum_{n \geq 1} e(x_{(1)}) \cdots e(x_{(n)})$, we have

that

$$\begin{aligned}\bar{\Delta}(G(x)) &= \sum_{n\geq 1}\bar{\Delta}(e(x_{(1)})\cdots e(x_{(n)}))\\ &= \sum_{n\geq 1}\sum_{i=1}^{n-1}(e(x_{(1)})\cdots e(x_{(i)}))\otimes(e(x_{(i+1)})\cdots e(x_{(n)})).\end{aligned}$$

On the other hand, we know that

$$\begin{aligned}(G\otimes G)(\bar{\Delta}(x)) &= G(x_{(1)})\otimes G(x_{(2)})\\ &= (\sum_{l\geq 1} e(x_{(1)(1)})\cdots e(x_{(1)(l)}))\otimes(\sum_{m\geq 1} e(x_{(2)(1)})\cdots e(x_{(2)(m)})).\end{aligned}$$

Distributing and using the coassociativity of $\bar{\Delta}$, we obtain

$$\begin{aligned}(G\otimes G)(\bar{\Delta}(x)) &= \sum_{n\geq 1}\sum_{i=1}^{n-1}(e(x_{(1)})\cdots e(x_{(i)}))\otimes(e(x_{(i+1)})\cdots e(x_{(n)}))\\ &= \bar{\Delta}(G(x)).\end{aligned}$$

Therefore, G is bialgebra morphism.

To see that it is a isomorphism, define $F:\overline{T}(V)\to\overline{H}$ by

$$F(v_1\cdots v_n) := v_1\cdot\ldots\cdot v_n,\ \text{for } n\geq 1.$$

It is clear that F is a bialgebra morphism. Let us now see that $F(G(x)) = x$, for all $x\in\overline{H}$. The proof is by induction on the conilpotency degree of x.

Since $\bar{\Delta}^{(n)} = \bar{\Delta}^{(n-1)}(x_{(1)})\otimes x_{(2)}$, we can write

$$\begin{aligned}G(x) &= e(x)+e(x_{(1)})e(x_{(2)})+e(x_{(1)})e(x_{(2)})e(x_{(3)})+\cdots\\ &= e(x)+G(x_{(1)})e(x_{(2)}).\end{aligned}$$

Let x be a element in $\overline{H}$ and let n be degree of x. If $n=1$, then x is primitive. So, we have that $\bar{\Delta}(x)=0$ and $e(x)=x$, which implies that $F(G(x))=e(x)=x$.

Consider now $n>1$ and suppose that $F(G(y))=y$, when the degree of y is strictly less than n. Computing $F(G(x))$, we obtain that:

$$\begin{aligned}F(G(x)) &= F(e(x))+F(G(x_{(1)}))F(e(x_{(2)}))\\ &= e(x)+x_{(1)}e(x_{(2)}),\ \text{by induction}\\ &= x-x_{(1)}e(x_{(2)})+x_{(1)}e(x_{(2)}),\ \text{by 1.9.8}\\ &= x.\end{aligned}$$

On the other hand, we have that,

$$\begin{aligned}G(F(v_1\cdots v_n)) &= G(v_1\cdot\ldots\cdot v_n)\\ &= G(v_1)\cdots G(v_n)\\ &= e(v_1)\cdots e(v_n)\\ &= v_1\cdots v_n,\end{aligned}$$

where we have used the fact that $e(v_i)=v_i$, for all $1\leq i\leq n$, because v_i is primitive. Therefore, $\overline{H}$ is isomorphic to $\overline{T}(V)$. $\square$

1.9.10. **Remark.** Note that Theorem 1.9.9 is still valid if we consider a non-unital associative algebra $(H, \cdot, \Delta)$ together with a coassociative coproduct $\Delta : H \to H \otimes H$ satisfying

$$\Delta(x \cdot y) = x \cdot y_{(1)} \otimes y_{(2)} + x_{(1)} \otimes x_{(2)} \cdot y + x \otimes y.$$

2. Algebraic Operads

The present section is dedicated to introduce the basics concepts of operad theory used in the next sections. For a more details about operads, we refer to [22] and [13].

2.0.1. **Definition.** A $\mathbb{S}$-*module* is a collection $M = \{M(n), n \geq 1\}$ of vector spaces such that each $M(n)$ is a right S_n-module.

Given two $\mathbb{S}$-modules M and N, a morphism of $\mathbb{S}$-modules, $\varphi : M \to N$ is a collection of $\mathbb{K}[S_n]$-morphisms $\varphi(n) : M(n) \to N(n)$.

2.0.2. **Notation.** The category of S-module is denoted by $\mathbb{S}$-mod.

To give the definition of an *algebraic operad*, we previously need to describe certain type of permutations. Let n be a positive integer and consider an ordered collection of positive integer $m = (m_1, \dots, m_n)$:

(1) Given a permutation $\sigma \in S_n$, we can define a permutation $\sigma_m \in S_{m_1+\cdots+m_n}$, called *block permutation*, as follows. First, we denote by
$$\overline{m}_i := (m_1 + \cdots + m_{i-1} + 1, \dots, m_1 + \cdots + m_{i-1} + m_i),$$
for each $1 \leq i \leq n$. So, we define $\sigma_m := (\overline{m}_{\sigma(1)}, \dots, \overline{m}_{\sigma(n)})$. For instance, if $m = (3, 2, 2)$ and $\sigma = (3, 2, 1)$, then $\sigma_m = (6, 7, 4, 5, 1, 2, 3)$.

(2) Given permutations $\sigma_i \in S_{m_i}$, for $1 \leq i \leq n$, the permutation $\sigma_1 \times \cdots \times \sigma_n \in S_{m_1+\cdots+m_n}$ is the permutation that is obtained by concatenating the $\sigma_i' s$. For example, if $\sigma_1 = (2, 1)$, $\sigma_2 = (3, 2, 1)$ and $\sigma_3 = (1, 3, 2)$, then
$$\sigma_1 \times \sigma_2 \times \sigma_3 = (2, 1, 5, 4, 3, 6, 8, 7).$$

2.0.3. **Definition.** An *algebraic operad*, or simply an *operad*, is a $\mathbb{S}$-module $P = \{P(n), n \geq 1\}$ together with a collection of linear maps, called *compositions*,

$$\gamma : P(n) \otimes P(m_1) \otimes \cdots \otimes P(m_n) \to P(m_1 + \cdots + m_n),$$

for all collection positive integer $\{m_1, \dots, m_n\}$. We write $\mu(\nu_1, \dots, \nu_n)$ instead $\gamma(\mu \otimes \nu_1 \otimes \cdots \otimes \nu_n)$. The compositions γ satisfy the following conditions:

(1) There exist an element $1 \in P(1)$, called the unit, such that $\mu(1, \dots, 1) = \mu$ for any l and for any $\mu \in P(l)$.

(2) The composition of operations is associative, that is,
$$\mu(\mu_1(\nu_1^1, \dots, \nu_{m_1}^1), \dots, \mu_n(\nu_1^n, \dots, \nu_{m_n}^n)) = \mu(\mu_1, \dots, \mu_n)(\nu_1^1, \dots, \nu_{m_n}^n).$$

(3) Compatibility of γ with the action of S_n. Consider $\mu \in P(n)$ and $\mu_i \in P(m_i)$, for $1 \leq i \leq n$. Then:

(a) For all permutation $\sigma \in S_n$,

$$(\mu \cdot \sigma)(\mu_1, \ldots, \mu_n) = \mu(\mu_{\sigma(1)}, \ldots, \mu_{\sigma(1)}) \cdot \sigma_{(m_1, \ldots, m_n)}.$$

(b) Given permutations $\sigma_i \in S_{m_i}$, for $1 \leq i \leq n$,

$$\mu(\mu_1, \ldots, \mu_n) \cdot (\sigma_1 \times \cdots \times \sigma_n) = \mu(\mu_1 \cdot \sigma_1, \ldots, \mu_n \cdot \sigma_n).$$

2.0.4. **Example.** Let V be a vector space. Its operad of endomorphisms, E_V is the element

$$E_V(n) := \mathrm{Hom}(V^{\otimes n}, V),$$

with the compositions defined as follows: consider $f : V^{\otimes n} \to V$ and $f_i : V^{\otimes m_i} \to V$, for $1 \leq i \leq n$. The element $\gamma(f, f_1, \ldots f_n)$ is

$$\gamma(f, f_1, \ldots f_n) := f \circ (f_{m_1} \otimes \cdots \otimes f_{m_n}).$$

The action of S_n on $E_V(n)$ is induced by the left action of the symmetric group on $V^{\otimes n}$,

$$(f \cdot \sigma)(v_1 \cdots v_n) := f(v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)}).$$

We have that $E_V(1) = \mathrm{End}(V)$ and $1 = 1_V$, where $1_V : V \to V$ is the identity map.

2.0.5. **Remark.** The notion of operad can be developed over any symmetric monoidal category, equipped with a symmetric monoidal operation $\otimes$. For instance, *a set operad* is an operad in the category of sets, with the set product $\times$ as monoidal operation. There exist several ways to define an operad, which are equivalent to Definition 2.0.3.

An operad P is a monad in category of $\mathbb{S}$-modules, equipped with another monoidal structure, which is not symmetric. Denote by $\mathrm{Vect}_{\mathbb{K}}$ the category of vector spaces over a fiel $\mathbb{K}$. Let P be a $\mathbb{S}$-module, its *Schur functor* , which is also denoted by P, $P : \mathrm{Vect} \to \mathrm{Vect}$ is defined by

$$P(V) := \bigoplus_{n \geq 0} P(n) \otimes_{S_n} V^{\otimes n}.$$

If P and Q are two $\mathbb{S}$-modules, then the composition $P \circ Q$ is again a Schun functor of some $\mathbb{S}$-module, which is also denoted by $P \circ Q$. The description of $(P \circ Q)(n)$ involves sums, tensor products and induced representations of the representations of $P(i)$ and $Q(i)$ for all $i \leq n$. Explicitly, we have that

$$(P \circ Q)(n) = \bigoplus_{k=1}^{n} \bigoplus_{i_1 + \cdots + i_k = n} P(k) \otimes_{S_k} ((Q(i_1) \otimes \cdots \otimes Q(i_k)) \otimes_{S_{i_1} \otimes \cdots \otimes S_{i_k}} \mathbb{K}[S_k]).$$

The category of $\mathbb{S}$-modules, equipped with this composition is a monoidal category ([25], 1.1.2). An operad P is an $\mathbb{S}$-module together with two maps $\eta : \mathrm{Id}_{\mathrm{Vect}} \to P$ and $\gamma : P \circ P \to P$, which make P into a monad.

2.0.6. **Definition.** Given two operads P and Q, a morphism of operads $\varphi : P \to Q$ is a morphism of $\mathbb{S}$-modules, satisfying the following conditions:

(1) $\varphi(1)(1_P) = 1_Q$ and
(2) for all $\mu \in P(n)$, $\mu_i \in P(m_i)$, with $1 \le i \le n$, we have that

$$\varphi(m_1 + \cdots + m_n)(\mu(\mu_1, \ldots, \mu_n)) = \varphi(n)(\mu)(\varphi(m_1)(\mu_1), \ldots, \varphi(m_n)(\mu_n)).$$

Let us now develop the idea of algebraic operad as a type of algebra. Let P be an algebraic theory defined by operations and relations. In this context, the vector space $P(n)$ is spanned by the n-ary operations of the theory. Given an operation $\mu \in P(n)$, the right S_n-module action is given by

$$(\mu \cdot \sigma)(x_1, \ldots, x_n) = \mu(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

for $\sigma \in S_n$ and variables $x_1, \ldots, x_n$. The element $1 \in P(1)$ is the 1-ary identity map. The composition γ correspond to the natural compositions of operations. For variables $x_1, \ldots, x_{m_1}, \ldots, x_{m_1+\cdots+m_n}$, and operations $\mu_i \in P(m_i)$ and $\mu \in P(n)$, the composition is given by

$$\begin{aligned} &\gamma(\mu, \mu_1, \ldots, \mu_n)(x_1, \ldots, x_{m_1}, \ldots, x_{m_1+\cdots+m_n}) \\ &= \mu(\mu_1(x_1, \ldots, x_{m_1}), \ldots, \mu_n(x_{m_1+\cdots+m_{n-1}+1}, \ldots, x_{m_1+\cdots+m_n})). \end{aligned}$$

2.0.7. *The operad* As. The vector space $\mathrm{As}(n)$ is generated by all possible operations in n variables which we can define using a unique associative product $\cdot$. The unique operations that we can perform using $\cdot$ on n different variables $x_1, \ldots, x_n$ are the linear combinations of the products $x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n)}$, for any $\sigma \in S_n$. So, $\mathrm{As}(n)$ is generated by $n!$ operations, one for each permutation $\sigma \in S_n$. Moreover, the action of S_n on $\mathrm{As}(n)$ coincides with the product of the elements in S_n. So, as representation of S_n, $\mathrm{As}(n)$ is $\mathbb{K}[S_n]$, the regular representation of S_n.

To describe the compositions in the operad As, consider permutations $\sigma \in S_n$ and $\sigma_i \in S_{m_i}$, with $1 \le i \le n$. The compositions of these operations is given by (see [22]):

$$\sigma(\sigma_1, \ldots, \sigma_n) = (\sigma_1 \times \cdots \times \sigma_n)\sigma_{(m_1,\ldots,m_n)}.$$

For instance, for $\sigma = (3,2,1)$, $\sigma_1 = (2,3,1)$, $\sigma_2 = (1,2)$ and $\sigma_3 = (2,1)$, we get that $(\sigma_1 \times \sigma_2 \times \sigma_3)\sigma_{(3,2,2)} = (7,6,4,5,2,3,2,1)$ and

$$\begin{aligned} \sigma(\sigma_1, \sigma_2, \sigma_3)(x_1, \ldots, x_7) &= \sigma(\sigma_1(x_1, x_2, x_3), \sigma_2(x_4, x_5), \sigma_3(x_6, x_7)) \\ &= \sigma(x_2 \cdot x_3 \cdot x_1, x_4 \cdot x_5, x_7 \cdot x_6) \\ &= x_7 \cdot x_6 \cdot x_4 \cdot x_5 \cdot x_2 \cdot x_3 \cdot x_1 \\ &= (\sigma_1 \times \sigma_2 \times \sigma_3)\sigma_{(3,2,2)}(x_1, \ldots, x_7). \end{aligned}$$

2.0.8. *The operad* Com. The commutative algebras are defined by a binary operation $\cdot$, which is associative and commutative. Since the product is associative, using the previous paragraph, we get that the n-ary operations of the theory are linear combinations of expressions of type $x_{\sigma(1)} \cdot \cdots \cdot x_{\sigma(n)}$, for any permutation $\sigma \in S_n$.

As $\cdot$ is commutative, we get that:

$$x_1 \cdot \ldots \cdot x_n = x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n)},$$

for any permutation $\sigma \in S_n$.

So, the unique n-ary operation in $\mathrm{Com}(n)$ is the product $x_1 \cdot \cdots \cdot x_n$, and $\mathrm{Com}(n) = \mathbb{K}$ is trivial representation of S_n. Denoting the product of n elements by $1_n := x_1 \cdot \ldots \cdot x_n$, the composition of the operad Com is given by

$$1_n(1_{m_1}, \ldots, 1_{m_n}) = 1_{m_1+\cdots+m_n}.$$

2.0.9. *The operad Lie.* The description of the operad *Lie*, describing Lie algebras, is more complicated than the operads As and Com. In two variables, $Lie(2)$ is generated by a single operation $[x_1, x_2]$, with $[x_2, x_1] = -[x_1, x_2]$. So, $Lie(2) = \mathbb{K}$ is the sign representation of S_2.

For $n \geq 3$, the space of operations $Lie(n)$ is more complicated to describe. Given three variables, we may apply to them two types of bracketings, $[x_{\sigma(1)}, [x_{\sigma(2)}, x_{\sigma(3)}]]$ and $[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}]$, for any $\sigma \in S_3$. The antisymmetry of the bracket implies that

$$[x_{\sigma(1)}, [x_{\sigma(2)}, x_{\sigma(3)}]] = [[x_{\sigma(2)}, x_{\sigma(3)}], x_{\sigma(1)}].$$

So, the elements of the type $[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}]$, with $\sigma \in S_n$, generate $Lie(n)$. Again, by antisymmetry, we have that the elements $[[x_1, x_2], x_3]$, $[[x_1, x_3], x_2]$ and $[[x_2, x_3], x_1]$ generate $Lie(3)$. Now, by Jacoby identity,

$$[[x_2, x_3], x_1] = [[x_1, x_3], x_2] - [[x_1, x_2], x_3].$$

So, the set $\{[[x_1, x_2], x_3], [[x_1, x_3], x_2]\}$ is a basis for $Lie(3)$.

It is well-known (see, for instance, [21] and for an operadic proof [9]) that the dimension of the space $Lie(n)$ is $(n-1)!$. As a complex representation of S_n, it can be shown that $Lie(n)$ is isomorphic to the induced representation $\mathrm{Ind}_{C_n}^{S_n}(\rho)$, where C_n is the cyclic group of order n and ρ is the representation of C_n of dimension one given by an irreducible nth root of unity (see [21]).

Denote the element $[[\ldots[x_1, x_2], \ldots, x_{n-1}], x_n]$ by

$$[x_1, x_2, \ldots, x_n] := [[\ldots[x_1, x_2], \ldots, x_{n-1}], x_n].$$

The proof of the following proposition is similar to the proof given in [3], Lemma 4.1, where it is considered another basis for $Lie(n)$.

2.0.10. **Proposition.** *The set of elements $[x_1, x_{\sigma(2)}, \ldots, x_{\sigma(n)}]$ with $\sigma \in S_n$ such that $\sigma(1) = 1$, form a basis for $Lie(n)$.*

Proof. Let us denote B the set of these elements. Since the dimension of $Lie(n)$ is $(n-1)!$, it suffices to show that B generates $Lie(n)$. First, let us prove that any monomial in $Lie(n)$ can be express as linear combination of elements of the form $[X, x_i]$, where $2 \leq i \leq n$ and X is a monomial of degree $(n-1)$ in the remaining variables.

The Jacoby identity may be rewritten as:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

If $n \geq 2$, any monomial in $Lie(n)$ can be write as $[X, Y]$, where X and Y are monomials of degree strictly smaller than n. By antisymmetry of the bracket, we can suppose that that the variable x_1 is in X. If Y is of degree one our assertion is true. If Y is of degree bigger than one, we can write $[X,Y] = [X,[Y_1,Y_2]]$. So, by Jacoby identity, we have that

$$[X,[Y_1,Y_2]] = [[X,Y_1],Y_2] - [[X,Y_2],Y_1].$$

If Y_1 and Y_2 are of degree one, we have proved the assertion. If Y_1 or Y_2 are of degree bigger than one, we can apply again the Jacoby identity. Iterating this process, which is finite, we arrive to the expected result.

Let us now see that any monomial in $Lie(n)$ can be write as linear combination of elements of B. The proof is by induction on n. The assertion is obvious if $n = 2$. Consider $n > 2$. By the previous result, any monomial can be write as linear combination of elements of the form $[X, x_i]$, where $2 \leq i \leq n$ and X is a monomial of degree $(n-1)$ in the remaining variables. So, Applying the inductive hypothesis on X, we have the assertion and we can conclude that B is a basis for $Lie(n)$. $\square$

2.1. **Algebras over an operad.**

2.1.1. **Definition.** Let P be an algebraic operad. A *P-algebra* is a vector space A equipped with a morphism of operads $\rho : P \to E_A$, where E_A is the operad of endomorphisms of A, defined in 2.0.4.

Note that A is a P algebra if, and only if, there exists a collection of linear maps

$$\rho_n : P(n) \otimes A^{\otimes n} \to A,$$

which is compatible with the compositions γ_P, and such that, for any permutation $\sigma \in S_n$ and any operation $\mu \in P(n)$,

$$\rho_n((\sigma \cdot \mu) \otimes (a_1 \cdots a_n)) = \rho_n(\mu \otimes (a_{\sigma(1)} \cdots a_{\sigma(n)})).$$

It is easy to verify that an As-algebra is nothing but an associative algebra in the usual sense.

2.1.2. *Free P-algebra.* A P-algebra $F(V)$, equipped with a linear map $\iota : V \to F(V)$ is said to be free over the vector space V if it satisfies the following universal condition:
for any P-algebra A and any linear map $f : V \to A$,there exists a unique P-algebra morphism $\widetilde{f} : F(V) \to A$, which extends f:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\iota"'] & A \\ F(V) \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

Note that a free algebra is unique up to a unique isomorphism. For example, the free associative algebra is the tensor algebra $T(V)$, while the free commutative algebra is the symmetric algebra $S(V)$.

2.1.3. **Remark.** Let V be a vector space and let P be an algebraic operad. The graded vector space $P(V)$ is defined by

$$P(V) := \bigoplus_{n \geq 1} P(n) \otimes_{\mathbb{K}[S_n]} V^{\otimes n},$$

where the left action of S_n on $V^{\otimes n}$ is given by $\sigma \cdot (v_1 \cdots v_n) = v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)}$.

The compositions in P induce linear maps $P(n) \otimes P(V)^{\otimes n} \to P(V)$, which make of $P(V)$ a free P-algebra over the vector space V.

2.2. **Non-symmetric operads.**

2.2.1. **Definition.** A non-symmetric operad is a graded vector space P equipped with a family of linear maps

$$\gamma : P_n \otimes P_{m_1} \otimes \cdots \otimes P_{m_n} \to P_{m_1 + \cdots + m_n},$$

for each positive integer n and each ordered collection of positive integer $m_1, \ldots, m_n$, which satisfy the conditions (1) and (2) in the Definition 2.0.3.

Sometimes, we will refer to an operad P as a symmetric operad to clearly distinguish it from a non-symmetric operad.

2.2.2. **Remark.** Given a non-symmetric operad P, we can associate a symmetric operad, which we also denote by P, as follows:

(1) The vector space $P(n) = P_n \otimes \mathbb{K}[S_n]$.
(2) The action of S_n on $P(n)$ is given by the action of S_n on the regular representation $\mathbb{K}[S_n]$.
(3) The composition map in the symmetric operad P is given by the tensor product of the composition map γ in the non-symmetric operad P with the composition map of the symmetric operad As given by

$$\sigma(\sigma_1, \ldots, \sigma_n) = (\sigma_1 \times \cdots \times \sigma_n)\sigma_{(m_1, \ldots, m_n)}.$$

Considered as symmetric operad, P is sometimes called a *regular operad*. In particular, if P is a types of algebraic structures in whose relations the variables stay in the same order, then P is a regular operad.

2.2.3. **Example.** The classical example of a regular operad is As. Using the same notation for non-symmetric operad associated, we have that As_n is generated by only one operation,

$$\mu_(x_1, \ldots, x_n) = x_1 \cdot \ldots \cdot x_n.$$

The composition map

$$\gamma : \mathrm{As}_n \otimes \mathrm{As}_{m_1} \otimes \cdots \otimes \mathrm{As}_{m_n} \to \mathrm{As}_{m_1 + \cdots + m_n}$$

is given by $\mu_n(\mu_1, \ldots, \mu_n) := \mu_{m_1 + \cdots + m_n}$.

2.2.4. **Remark.** By Remark 2.2.2, if P is a regular operad, then the free P-algebra on a vector space is given by

$$\begin{aligned} P(V) &= \bigoplus_{n\geq 1} P(n) \otimes_{\mathbb{K}[S_n]} V^{\otimes n} \\ &= \bigoplus_{n\geq 1} (P_n \otimes \mathbb{K}[S_n]) \otimes_{\mathbb{K}[S_n]} V^{\otimes n} \\ &= \bigoplus_{n\geq 1} P_n \otimes V^{\otimes n} \end{aligned}$$

2.2.5. **Proposition.** *A non-symmetric operad P is completely determined by the free P-algebra on one generator.*

Proof. We have that in a non-symmetric operad P the free algebra on one generator is given by

$$P(\mathbb{K}) := \bigoplus_{n\geq 1} P_n \otimes \mathbb{K}^{\otimes n} = \bigoplus_{n\geq 1} P_n.$$

So, each P_n correspond to the n-multilinear part of graded vector space $P(\mathbb{K})$. Since $P(\mathbb{K})$ is a P-algebra, for each positive integer n, we have linear maps

$$\rho_n : P_n \otimes P(\mathbb{K})^{\otimes n} \to P(\mathbb{K}).$$

So, we can define composition maps

$$\gamma : P_n \otimes P_{m_1} \otimes \cdots \otimes P_{m_n} \to P_{m_1+\cdots+m_n}$$

by $\gamma(\mu \otimes \mu_1 \otimes \cdots \otimes \mu_n) := \rho_n(\mu \otimes \mu_1 \otimes \cdots \otimes \mu_n)$. □

2.3. **Binary quadratic operads.** There exists a close relationship between the description of algebraic operads and spaces spanned by colored trees. More precisely, the free operad on an $\mathbb{S}$-module E may be described on the vector space spanned by non-planar trees with the internal vertices colored by the elements of E. As any operad is a quotient of a free operad, it may be described in terms of a quotient of a vector space spanned by a collection of trees. In particular, when the operad is generated by binary operations and these operations are subject uniquely to quadratic relations (that is, relations which involve three arguments), such an operad is called a *binary quadratic operad*. In particular, the operads As, Com and *Lie* are examples of binary quadratic operads.

2.3.1. *The free operad.* The forgetful functor from operads to $\mathbb{S}$-modules admit a left adjoint functor, which gives place to a *free operad* over an $\mathbb{S}$-module. If E is an $\mathbb{S}$-module, the free operad over E is denoted by $\mathcal{F}(E)$ and it is characterized by the following universal property: any $\mathbb{S}$-module morphism $f : E \to P$, where P is an operad, extends uniquely into an operad morphism $\widetilde{f} : \mathcal{F}(E) \to P$.

Let us describe free operad over a $\mathbb{S}$-module E. For this, we use non-planar rooted trees.

If t is a non-planar rooted tree, we will refer to it simply as a tree. Given a tree t and a vertex $v \in t$, we denote by $\mathrm{In}(v)$ the set of input edges at v. Given a positive integer n, if I is a set such that $|I| = n$, a *I-tree* is a tree t with n leaves, which are decorated by the elements of the set I. In particular, if $I = [n] = \{1, \ldots, n\}$, we say that t is a n-tree. Consider a I-tree t such that I is a set of positive integers, with $|I| = n$. There is a natural bijection between $\sigma_I : I \to [n]$ which preserves the order(from lowest to highest) in the set I.

We say that is *normalized* if the leaves of t are redecorated by the elements of $[n]$ according to bijection σ_I, which makes of t a n-tree. A tree t is *binary* if each vertex of t has exactly two inputs. It is known that the number of n-binary trees is given by

$$(2n-3)!! = 1 \cdot 3 \cdot 5 \cdots (2n-3).$$

We have a natural composition between labelled trees as follows. Given a n-tree t and $t_1, \ldots, t_n$, m_i-trees, with $1 \leq i \leq n$, the tree $t(t_1, \ldots, t_n)$ is the $(m_1 + \cdots + m_n)$-tree that is obtained by grafting the roots of the trees $t_1, \ldots, t_n$ in the n leaves of t, with the natural re-enumeration of the leaves of the trees $t_1, \ldots, t_n$.

Consider now an $\mathbb{S}$-module $E = \{E(n) | n \geq 1\}$, which will represent the set of operations in certain algebraic theory. We assume that $E(1) = 0$ and for each $n > 1$, $E(n)$ is generated by a finite number of n-ary operations. Given a tree t, we define

$$E(t) := \bigotimes_{v \in t} E(\mathrm{In}(v)),$$

where $v \in t$ is a vertex of t. For any positive integer n we define

$$\mathcal{F}(E)(n) := \bigotimes_{n-\text{tree } t} E(t).$$

In this way, $\mathcal{F}(E)(n)$ is the space of all the n-ary operations that we can construct from the operations that generate E. The composition in $\mathcal{F}(E)$ is induced by the composition between trees.
In low dimension, we have

$$\begin{array}{ll} \mathcal{F}(E)(1) & = E(1) = 0, \\ \mathcal{F}(E)(2) & = E(2), \\ \mathcal{F}(E)(3) & = E(3) \oplus (3E(2) \otimes E(2)), \end{array}$$

where the copies of $E(2) \otimes E(2)$ is determined by the 3-binary trees, which allow us to see the action S_3 on $3E(2) \otimes E(2)$ from the action of S_2 on $E(2)$:

$$3E(2) \otimes E(2) = \mathrm{Ind}_{S_2}^{S_3}(E(2) \otimes E(2)) = (E(2) \otimes E(2)) \otimes_{S_2} \mathbb{K}[S_3],$$

where the action of S_2 on $E(2) \otimes E(2)$ is on the second factor only.

From the description that we have given, we can think $\mathcal{F}(E)(n)$ as the space generated by all n-trees, whose vertices are labelled by the operations that generate E, where an vertex with k inputs is labelled by an element of $E(k)$. In particular, if E is generated only by binary operations, that is, $E(n) = 0$, if $n \neq 2$, we have that $\mathcal{F}(E)(n)$ consists of n-binary trees whose vertices are labelled by the binary operations that generate E.

2.3.2. **Definition.** Let P be an operad. An *ideal* of P is an $\mathbb{S}$-submodule I of P such that for any family of operations $\{\mu, \mu_1, \ldots, \mu_n\}$, if one of them is in I, then $\mu(\mu_1, \ldots, \mu_n)$ is also in I. Given a collection of sets $R = \{R(n)\}_{n \geq 1}$ such that $R(n) \subseteq P(n)$, the ideal of generated by R is the smallest ideal of P which has R as a subset. The ideal generated by R is denoted by (R).

If I is an ideal P, then the *quotient operad* is defined by:

$$(P/I)(n) = \frac{P(n)}{I(n)},$$

with the composition maps induced by the composition maps defined in the operad P.

2.3.3. **Definition.** An operad $P = \mathcal{F}(E)/(R)$ is a *binary quadratic operad* if E is generated by binary operations and $R \subseteq \mathcal{F}(E)(3)$. In such case, we write $P = P(E, R)$.

A binary quadratic operad is an operad which codifies an algebraic theory whose generating operations are binary, under quadratic relations. Examples of this type of operads are As, Com and *Lie*.

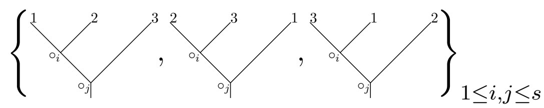
2.3.4. **Example.** If $P =$ As, we have that E is generated by two operations $x_1 \cdot x_2$ and $x_2 \cdot x_1$. So, $E = \mathbb{K}[S_2]$, the regular representation. In this case,

$$\begin{aligned} \mathcal{F}(E)(3) &= \mathrm{Ind}_{S_2}^{S_3}(E \otimes E) \\ &= (\mathbb{K}[S_2] \otimes \mathbb{K}[S_2]) \otimes_{S_2} \mathbb{K}[S_3] \\ &= \mathbb{K}[S_2] \otimes \mathbb{K}[S_3] \\ &= \mathbb{K}[S_3] \oplus \mathbb{K}[S_3]. \end{aligned}$$

So, the dimension of $\mathcal{F}(E)(3)$ is 12. We can think the generators of first summand as the operations of the type $x_{\sigma(1)} \cdot (x_{\sigma(2)} \cdot x_{\sigma(3)})$ and the generators of the second summand as the operations of the type $(x_{\sigma(1)} \cdot x_{\sigma(2)}) \cdot x_{\sigma(3)}$, where $\sigma \in S_3$. The space of relations R is generated by 6 elements, the set

$$\{x_{\sigma(1)} \cdot (x_{\sigma(2)} \cdot x_{\sigma(3)}) - (x_{\sigma(1)} \cdot x_{\sigma(2)}) \cdot x_{\sigma(3)} | \ \sigma \in S_3\}.$$

2.3.5. **Remark.** Let $P = P(E, R)$ be a binary quadratic operad. Since the $\mathbb{S}$-module E is generated by binary operations, we have that the free operad $\mathcal{F}(E)$ consists of linear combinations of rooted trees whose internal vertices has exactly two inputs, which are labelled by these binary operations. More specifically, if $B_E = \{\circ_1, \ldots, \circ_s\}$ is a basis for $E(2)$, we have that the space $\mathcal{F}(E)(3)$ is generated by the set of trees:



So, since that $1 \leq i, j \leq s$, the dimension of $\mathcal{F}(E)(3)$ is equal to $3 \cdot s^2$. We can also represent this basis by means of operations in three variables, we have that

$$B = \{(x_1 \circ_i x_2) \circ_j x_3, (x_2 \circ_i x_3) \circ_j x_1, (x_3 \circ_i x_1) \circ_j x_2\}_{1 \leq i,j \leq s}.$$

In the binary quadratic operad $P = P(E, R)$, the set of relations R consists of t linearly independent relations, which are linear combinations of elements in the basis B_E, where $t \leq 3 \cdot s^2$.

2.3.6. **Example.** Consider the Lie operad $P = Lie$. It is generated by a single binary operation: the Lie bracket $[x_1, x_2]$. By antisymmetry of the bracket $[-,-]$, we have that $[x_1, x_2] \cdot (2, 1) = -[x_1, x_2]$. So, $E(2)$ is the sign representation of S_2. The space $\mathcal{F}(E)(3)$ is generated by three elements $[[x_1, x_2], x_3]$, $[[x_2, x_3], x_1]$ and $[[x_3, x_1], x_2]$. Applying the Jacoby identity, the space of relations is generated by a single element in $\mathcal{F}(E)(3)$. So,

$$R = \{[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2]\}.$$

2.4. **Coalgebras over a operad.** Given an operad P, a P algebra is a vector space with some n-ary operations, for $n \geq 1$. A *P-coalgebra* is a vector space C, equipped with an operation $\delta_\mu : C \longrightarrow C^{\otimes n}$, for any n-ary operation $\mu \in P$, which satisfies the relations obtained by reversing all the arrows in the relations defining a P algebra. To simplify the exposition, we shall only consider binary quadratic operads. For more details about coalgebras over an operad and its relationship with the concept of *cooperad* we refer to [22].

Given a binary quadratic operad $\mathcal{P}$ with $P(1) = 0$, which is generated by binary operations $\mu_1, \ldots, \mu_n$ and relations of the form

$$\sum_{i,j} \alpha_{ij} \mu_i(\mu_j \otimes \mathrm{id}) = \sum_{i,j} \beta_{ij} \mu_i(\mathrm{id} \otimes \mu_j), \quad \alpha_{ij}, \beta_{ij} \in \mathbb{K},$$

a *coalgebra* C over P is defined by cooperations $\delta_{\mu_i} : C \to C \otimes C$ satisfying the relations:

$$\sum_{i,j} \alpha_{ij} (\delta_{\mu_j} \otimes \mathrm{id}) \delta_{\mu_i} = \sum_{i,j} \beta_{ij} (\mathrm{id} \otimes \delta_{\mu_j}) \delta_{\mu_i}, \quad \alpha_{ij}, \beta_{ij} \in \mathbb{K}$$

(see [25], 1.3.6). Thus, for instance, when $P = \mathrm{As}$, the notion of coalgebra over As coincides with the notion of coassociative coalgebra given in 1.2.1.

2.4.1. *Primitive part, conilpotency.* Let P be an binary quadratic operad and consider C a P-coalgebra. We define a filtration on C as follows:

$$F_1C = \mathrm{Prim}C = \{x \in C | \delta_\mu(x) = 0 \text{ for any operation } \mu \in P\}.$$

The elements of Prim$C$ are called *primitive elements* of C and the subspace Prim$C$ is called the *subspace of primitive elements* of C. We define a filtration by

$$F_rC := \{x \in C | \delta_\mu(x) = 0 \text{ for any } \mu \in P(n), \text{ with } n > r\}.$$

We say that the coalgebra C is *conilpotent* or *connected*, if $C = \bigcup_{r\geq 1} F_rC$.

2.4.2. *Cofree coalgebra.* We define a *cofree* coalgebra over a operad P. We say that a P-coalgebra C_0 is cofree over the vector space V if it is conilpotent and it is equipped with a map $s : C_0 \to V$ such that C_0 satisfies the following universal property: any linear map $\varphi : C \to V$, where C is a conilpotent P-coalgebra, extends uniquely to a P-coalgebra homomorphism $\tilde{\varphi} : C \to C_0$:

$$\begin{tikzcd} C \arrow[d, "\tilde{\varphi}"'] \arrow[dr, "\varphi"] & \\ C_0 \arrow[r, "s"] & V \end{tikzcd}$$

The cofree coalgebra over V is well-defined up to isomorphisms. For instance, when $P = \mathrm{As}$, the cofree coalgebra over V is given by $\overline{T}(V)$ equipped with the deconcatenation coproduct (see 1.2.10).

2.5. **Koszul duality for operads.** The Koszul duality for algebras was introduced by S. B. Priddy in [34]. This notion was generalized to binary quadratic operads by V. Ginzburg and M. Kapranov in [13]. Later, the Koszul dualitity was extended to quadratic operads (not necessarily binary) by E. Getzler in [12], other versions of Koszul duality are due to B. Vallette (see [38]) and R. Berger, M. Dubois-Violette and M. Wambst (see [2] and [1]). In our work, we will only consider Koszul duality for binary quadratic operads. Let us recall some definitions needed to introduce the notion of a Koszul operad.

2.5.1. **Definition.** Let V be a right S_n-module. We denote by $V^\vee$ the right S_n-module $V^* \otimes \mathrm{sgn}_n$, where sgn_n is the sign representation of S_n.

Let us describe the right S_n-module $V^\vee$. By definition, $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ and the right action of a permutation $\sigma \in S_n$ on an element $f \in V^*$ is given $f_\sigma(x) = f(x \cdot \sigma^{-1})$. So, the action of S_n on $V^\vee$ is given by

$$f^\sigma(x) = \mathrm{sgn}(\sigma) f(x \cdot \sigma^{-1}).$$

Moreover, if V is finite dimensional and $B_V = \{e_1, \dots, e_s\}$ is a basis for V, then there is a natural isomorphism of vector spaces $V \cong V^\vee$. The image of e_i under this isomorphism is denoted by $e_i^\vee$, for all $1 \leq i \leq s$.

2.5.2. **Definition.** Let E be a $\mathbb{S}$-module such that $E(n)$ is finite dimensional for all n. Its *Czech dual* is the $\mathbb{S}$-module $E^\vee$ given by $E^\vee(n)$.

Consider now $P = P(E, R)$ a binary quadratic operad and let $B_E = \{\circ_1, \ldots, \circ_s\}$ be a basis for $E(2)$. Considering the basis B of $\mathcal{F}(E)(3)$ as in Remark 2.3.5, we have a natural pairing $\langle , \rangle : \mathcal{F}(E^\vee)(3) \otimes \mathcal{F}(E)(3) \to \mathbb{K}$, given by

$$\langle (x_a \circ_i x_b) \circ_j x_c, (x_d \circ_k^\vee x_e) \circ_l^\vee x_f \rangle := \delta_{(a,b,c),(d,e,f)} \delta_{i,k} \delta_{j,l}.$$

We denote by $R^\perp$ the relations orthogonal to R with respect to this pairing.

2.5.3. **Definition.** Let $P = P(E, R)$ be a binary quadratic operad, with $E(2)$ finite dimensional. We define its *Koszul dual operad* by $P^\vee = \mathcal{F}(E^\vee)/(R^\perp)$.

2.5.4. **Example.** It is well-known that $Lie^! = \mathrm{Com}$ and $\mathrm{Com}^! = Lie$. Let us compute the Koszul dual of the operad Lie. In the presentation of the operad Lie, we have that E is generated by $[x_1, x_2]$, with $[x_1, x_2] \cdot (2, 1) = -[x_1, x_2]$, and

$$R = \{[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2]\}.$$

Let us denote by $\circ$ the dual of $[,]$. So, $E^\vee(2)$ is generated by $\circ$. Since $[x_1, x_2] \cdot (2, 1) = -[x_1, x_2]$, we have that $(x_1 \circ x_2) \cdot (2, 1) = x_1 \circ x_2$. So, the product $\circ$ is commutative and $E^\vee(2)$ is the trivial representation. Using the formula of Remark 2.3.5, a direct compute shows that

$$R^\perp = \{(x_1 \circ x_2) \circ x_3 - (x_2 \circ x_3) \circ x_1, (x_1 \circ x_2) \circ x_3 - (x_3 \circ x_1) \circ x_2\}.$$

So, the product is associative, which implies that $Lie^! = \mathrm{Com}$.

2.5.5. *Koszul dual: Non-symmetric case.* Let us now consider the case of a binary quadratic operad $P = P(E, R)$ which is additionally regular. So, the operad P is generated by binary operations which do not satisfy any symmetry property. Since P is a regular operad, we have that $E = E' \otimes \mathbb{K}[S_2]$, for some vector space E'. So, the space of operations in three variables is

$$\begin{aligned}
\mathcal{F}(E)(3) &= \mathrm{Ind}_{S_2}^{S_3}(E \otimes E) \\
&= (E \otimes E) \otimes_{S_2} \mathbb{K}[S_3] \\
&= [(E' \otimes \mathbb{K}[S_2]) \otimes (E' \otimes \mathbb{K}[S_2])] \otimes_{S_2} \mathbb{K}[S_3] \\
&= [(E' \otimes \mathbb{K}[S_2]) \otimes E'] \otimes \mathbb{K}[S_3] \\
&= (E'^{\otimes 2} \oplus E'^{\otimes 2}) \otimes \mathbb{K}[S_3].
\end{aligned}$$

We can represent the generators of the first summand as operations of the type $x_1 \cdot (x_2 \circ x_3)$ and the generators of the second summand as operations of the type $(x_1 \cdot x_2) \circ x_3$, where $(\cdot, \circ)$ is any pair of generators elements of E'. Let us denote by V_P the vector space generates by the operations described above. So, we have that

$$\mathcal{F}(E)(3) = V_P \otimes \mathbb{K}[S_3].$$

The following proposition is proved in [24], Appendix B, Proposition 3.

2.5.6. **Proposition.** *Let $P = P(E, R)$ be a binary quadratic operad is additionally regular with $E = E' \otimes \mathbb{K}[S_2]$. The Koszul dual operad of P is a regular operad and*

$$P^! = P(E, R^\perp),$$

where $R^\perp$ is the annihilator of R for the scalar product on V_P given by

$$\begin{array}{rcl} \langle x_1 \cdot (x_2 \circ x_3), x_1 \cdot (x_2 \circ x_3) \rangle & = & 1, \\ \langle (x_1 \cdot x_2) \circ x_3, (x_1 \cdot x_2) \circ x_3 \rangle & = & -1, \end{array}$$

and 0 *in other case, where $(\cdot, \circ)$ is any pair of generator elements of E'.*

2.5.7. **Example.** Consider the regular operad $P = \mathrm{As}$. In this case, E' is generated by a single element $\cdot$. So, the vector space V_P is generated by two elements, $x_1 \cdot (x_2 \cdot x_3)$ and $(x_1 \cdot x_2) \cdot x_3$. In this case, the subspace of relations is generated by

$$R = \{x_1 \cdot (x_2 \cdot x_3) - (x_1 \cdot x_2) \cdot x_3\}.$$

So, the annihilator of R with respect to scalar product described in Proposition 2.5.6 is generated by a single element. Since

$$\langle x_1 \cdot (x_2 \cdot x_3) - (x_1 \cdot x_2) \cdot x_3, x_1 \cdot (x_2 \cdot x_3) - (x_1 \cdot x_2) \cdot x_3 \rangle = 0,$$

we have that R is its own annihilator. So, $\mathrm{As}^! = \mathrm{As}$.

2.5.8. *Homology and Koszul duality.* Let $P = P(E, R)$ be a binary quadratic operad and let $P^!$ be its dual operad. V. Ginzburg and M. Kapranov have showed in [13], that to any P-algebra A there is an associated chain-complex C_*^P given by

$$\cdots \to P^!(n)^\vee \otimes_{S_n} A^{\otimes n} \to P^!(n-1)^\vee \otimes_{S_{n-1}} A^{\otimes (n-1)} \to \cdots \to P^!(1)^\vee \otimes A,$$

whose differential d agrees, in low dimension, with the P-algebra structure of A,

$$\gamma_A(2) : P(2) \otimes A^{\otimes 2} \to A.$$

The differential d is characterized by the condition above and the fact that, on the cofree coalgebra $P^{!*}(sA)$, d is a graded coderivation([24], appendix B.4). The associated homology groups are denoted $H_n^P(A)$, for $n \geq 1$. We say that an operad P is *Koszul* when, for any vector space V, the groups $H_n^P(P(V))$ are trivial for $n > 1$. This definition is equivalent to the first one given by V. Ginzburg and M. Kapranov in [13], the equivalence of both definitions was stated by them in the same work. It is well-known that the operads As, $P = \mathrm{Com}$ and *Lie* are Koszul operads (see [13]). Let us give a brief description of the chain-complexes associated to these operads.

Let us first consider the case $P = \mathrm{As}$, the operad associated to the associative algebras. Since $\mathrm{As}^! = \mathrm{As}$, we have that

$$C_n^{\mathrm{As}}(A) = \mathrm{As}^!(n)^\vee \otimes_{S_n} A^{\otimes n} = \mathbb{K}[S_n] \otimes_{S_n} A^{\otimes n} = A^{\otimes n},$$

for any associative algebra A.

So, the complex given the homology of A, with coefficients in the field $\mathbb{K}$, is of the form
$$\cdots \to A^{\otimes n} \to A^{\otimes(n-1)} \to \cdots \to A,$$
where the elements of A have degree 1. In order to simplify notation, we denote by $(a_1, \ldots, a_n)$ the n-tensor $a_1 \otimes \cdots \otimes a_n$ in $A^{\otimes n}$.

As the differential d_2 coincides with the product on A, we have that $d_2(a, b) = a \cdot b$. In the case $C_*^{\mathrm{As}}(A)$, the coalgebra structure is given by deconcatenation coproduct,
$$\Delta(a_1, \ldots, a_n) = \sum_{n-1}^{i=1} (a_1, \ldots, a_i) \otimes (a_{i+1}, \ldots, a_n).$$
Since d is a graded coderivation with respect to Δ, we obtain that
$$d_n(a_1, \ldots, a_n) = \sum_{i=1}^{n-1} (-1)^{i+1} (a_1, \ldots, a_i \cdot a_{i+1}, \ldots, a_n).$$
When $P = \mathrm{Com}$, we have that for a commutative associative algebra A, the chain-complex $C_*^{\mathrm{Com}}(A)$ is given by the *Harrison complex* (see [22], 13.1.10).

For the case $P = Lie$, we have that $Lie^!(n)^\vee = \mathrm{Com}(n) \otimes \mathrm{sgn}_n$. Since $\mathrm{Com}(n)$ is the trivial representation of S_n, we obtain that $\mathrm{Com}(n) \otimes \mathrm{sgn}_n$ is the sign representation. So, for a Lie algebra $\mathfrak{g}$, we have that
$$C_n^{Lie}(\mathfrak{g}) = \mathrm{sgn}_n \otimes_{S_n} \mathfrak{g}^{\otimes n} = \Lambda^n(\mathfrak{g}),$$
the nth exterior power of the space $\mathfrak{g}$. The differential d is given by
$$d(x_1 \wedge \cdots \wedge x_n) = \sum_{i<j} (-1)^{i+j-1} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_n.$$

Note that the antisymmetry property of the bracket $[,]$ implies that the differential d is well defined and the Jacobi identity implies that $d^2 = 0$.

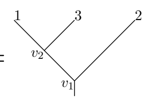
2.6. **PBW basis for operads.** In [18], E. Hoffbeck introduced the notion of Poincaré-Birkhoff-Witt (for short, we write PBW) basis for operads and he showed that, if an operad P admits a PBW basis, then P is a Koszul operad ([18], Theorem 3.10), generalizing the criterion given by S. Priddy for algebras in [34]. To simplify the exposition, we only will consider binary quadratic operads.

To describe the method developed by E. Hoffbeck, we need to introduce some conventions and definitions. Recall that a n-binary tree t is a binary non-planar rooted tree with n leaves which are labelled by the set $[n]$. We identify each leaf of t with the element in $[n]$ that labelled it. Given a n-binary tree t, we denote by $V(t)$ the set of vertices of t and by $E(t)$ the set of edges of t. An edge $e \in E(t)$ is oriented from a $S(e) \in V(t) \cup [n]$ to a $F(e) \in V(t) \cup \{0\}$, where 0 denote the root of t. Sometimes, to specify the tree t to which we refer, if $e \in E(t)$ we will write $S_t(e)$ and $F_t(e)$. If $v \in V(t)$, we denote set of the inputs of v by

$$I_v = \{S(e), e \in E(t) \text{ such that } F(e) = v\}.$$

We fix a planar representation of t as follows. We say that a leaf i is *linked to a vertex* $v \in V(t)$ if there is a monotonic path of edges between i and v. We assume also that a leaf i is linked to itself. Thus, if $a \in I_v$, we denote by $m(a) \in [n]$ the minimum of the leaves that are linked to a.

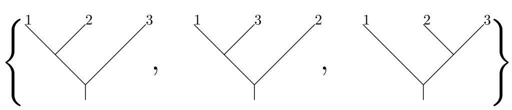
For instance, if t is the tree given by

$t =$ 

then $I_{v_1} = \{v_2, 2\}$, $m(v_2) = 1$ and $m(2) = 2$.

Given a binary tree t and a vertex $v \in t$, the application m induces an order on I_v. In this way, the planar representation of t is obtained to order the inputs of each vertex $v \in t$ in an increasing order from left to right.

2.6.1. **Example.** The planar representation of the 3-binary trees is given by the set



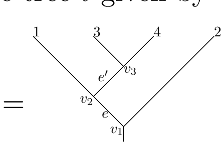
In what follows, when we speak of a tree t, we will be considering the planar representation of t described above.

A *subtree* t' of a n-tree t is a tree such that $V(t') \subseteq V(t)$ and $E(t') \subseteq E(t)$ satisfying the following conditions.

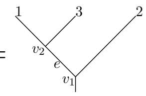
(1) For all $v \in V(t')$ and for all $e \in E(t)$, we have that if $S(e) = v$ or $F(e)$, then $e \in E(t')$.
(2) For all $e \in E(t')$ if e is an internal edge, then $S_{t'}(e) = S_t(e)$ and $F_{t'}(e) = F_t(e)$. Moreover, if $S_{t'}(e)$ is a leaf in the tree t' then it is labelled by $m(S_t(e))$. This makes of t' a I-tree, where I is some subset of $[n]$, with $|I| = m \leq n$. We consider the tree t' normalized, which t' is a m-tree.

The subtree t' of a tree t generated by an internal edge $e \in E(t)$ is the tree t_e such that $V(t_e) = \{S(e), F(e)\}$ and the only internal edge of t_e is e. The leaves are labelled as specified above.

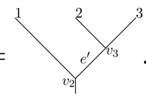
2.6.2. **Example.** Consider the tree t given by

$t =$ 

The subtree generated by the edge e is

$t_e =$ 

and the subtree generated by the edge e' is

$t_{e'} =$  .

Given a binary quadratic operad $P = P(E, R)$, we denote by $\mathcal{B}_n^{\mathcal{F}(E)}$ the basis for $\mathcal{F}(E)(n)$ given by the n-binary trees whose vertices are labelled by the binary operations that generate E, which are considered with the planar representation described above. We refer to these trees as labelled trees. A basis for the free operad $\mathcal{F}(E)$ is given by

$$\mathcal{B}^{\mathcal{F}(E)} = \bigcup_{n\geq 1} \mathcal{B}_n^{\mathcal{F}(E)}.$$

Given a n-binary tree t, we denote by $B_t^{\mathcal{F}(E)}$ the set of all the elements of $\mathcal{B}^{\mathcal{F}(E)}$ whose underlying tree is t.

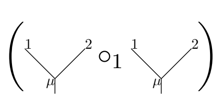
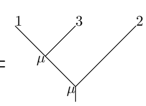
The description of a PBW basis requires the partial compositions on the free operad $\mathcal{F}(E)$. Given positive integers m and n, we define

$$\circ_i : \mathcal{F}(E)(m) \otimes \mathcal{F}(E)(m) \to \mathcal{F}(E)(m + n - 1),$$

for $1 \leq i \leq m$, as follows: for any pair of labelled trees $\alpha \in \mathcal{F}(E)(m)$ and $\beta \in \mathcal{F}(E)(n)$, the composition $\alpha \circ_i \beta$ is the labelled tree obtained by grafting the root of β to the ith. leaf of α.

2.6.3. **Definition.** Given two labelled trees α and β with m and n leaves, respectively, a permutation $w \in S_{m+n-1}$ is a *pointed shuffle* of the composition $\alpha \circ_i \beta$ if the order the entries of α and β coincides with the order in the composition $(\alpha \circ_i \beta) \cdot w$, and the minimal leaf of β in the composition is i.

Note that this definition implies that the leaves labelled from 1 to $i - 1$ are not modified. For example,

 $\cdot (1, 3, 2) =$ 

For the construction of a PBW basis, we need to fix an order on the basis of $\mathcal{F}(E)(n)$, for each $n \in \mathbb{N}$, which must satisfy the following compatibility condition:

For two pairs (α, β) and (α', β') of labelled trees, with m leaves the first pair and n leaves the second one, we have that if $\alpha \leq \beta$ and $\alpha' \leq \beta'$, then for all i, $(\alpha \circ_i \alpha') \cdot w \leq (\beta \circ_i \beta') \cdot w$, for all pointed shuffle w.

In such case, we say that $\leq$ is a *suitable order* for $\mathcal{F}(E)$.

2.6.4. **Example.** A example of a suitable order is the so-called *lexicographical order*, considered in [18], Example 3.4. Let us describe this order. We can associate to each element of the basis $\mathcal{B}^{\mathcal{F}(E)}$ a sequence of words. Given a labelled trees α with n leaves, we associate a sequence of n words $(a_1, \ldots, a_n)$ to α as follows. For any $1 \leq i \leq n$, there exists a unique monotonic path of vertices from the root to i-th. leaf. The word a_i is composed of the labels of the vertices, which are the generating operations of E, read from bottom to top.

For an ordered basis of E, let a and b be two words on the elements of the basis. We compare the length of the words, that is $a < b$, whenever $l(a) < l(b)$. If the lengths of the words are equal, then we compare them lexicographically.

Given two labelled trees α and β with n leaves such that $\alpha \neq \beta$, we can compare them as follows. We have associated to α and β two sequences of n words $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, respectively. So, we compare first a_1 and b_1 lexicographically. For instance, if $a_1 < b_1$, then $\alpha < \beta$. If $a_1 = b_1$, we compare a_2 and b_2, etc. This define a strict relation of order.

In [18], Proposition 3.5, it is proved that it is a suitable order for $\mathcal{F}(E)$.

Analogously, we can define the *reverse-length order.* If a and b are two words, we first compare the length of them and $a > b$ if $l(a) < l(b)$. If $l(a) = l(b)$, then we compare them lexicographically. So, to define the order on $\mathrm{F}(n)$, we proceed in a similar way to the previous case.

2.6.5. *Poincaré-Birkhoff-Witt basis.* We can introduce now the definition of PBW basis.

Let $P = \mathcal{F}(E)/(R)$ be a binary quadratic operad. Consider a set $B^P \subset B^{\mathcal{F}(E)}$ of elements representing a basis of the underlying vector space of P. Given a tree $t \in \mathcal{F}$, we denote by B_t^P the set of all the elements of B^P whose underlying tree is t. We say that B^P is a *PBW basis* for P if $1 \in B^P$, $B^E \subset B^P$ and for each $t \in \mathcal{F}$ we have that:

(1) For $\alpha \in B_t^P$, $\beta \in B_{t'}^P$ and any pointed shuffle w, either $(\alpha \circ_i \beta) \cdot w$ is in $B^P_{(t \circ_i t')\cdot w}$, or the elements of the basis $\delta \in B^P$, which appear in the unique decomposition $(\alpha \circ_i \beta) \cdot w \equiv \sum_\delta c_\delta \delta$, satisfy that $\delta > (\alpha \circ_i \beta) \cdot w$ in $\mathcal{F}(E)$.
(2) A element α is in B_t^P if, and only if, for every internal edge e of t, the labelled tree $\alpha_{|t_e}$ belongs to $B^P_{t_e}$.

2.6.6. **Remark.** As a consequence of condition (2), we only need to specify the quadratic part of the basis to determine the basis completely. As E. Hoffbeck has described in [18], there are two main methods to find PBW basis:

- We can star with a basis and find an order on E such that it is a PBW basis, that is, to verify that it satisfies the condition (1) and (2).
- We can start with an ordered basis of E. The quadratic part is chosen in such a way that it satisfies the conditions (1) and (2). The quadratic elements determine the choice of elements of the basis in higher degree. Finally, we must check that effectively this set is a basis for P.

E. Hoffbeck also proved in [18], Theorem 5.1, that if P has a PBW basis, then its Koszul dual $P^!$ also has a PBW basis, whose quadratic elements are determined by the quadratic elements of P. Let us describe how the basis of $P^!$ is obtained from the PBW basis of P.

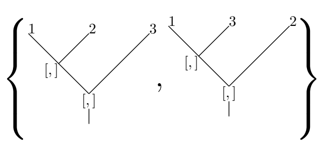
We have that in $\mathcal{F}(E)(3)$ its basis $B^{\mathcal{F}(E)(3)}$ can be described as the disjoint union:

$$B^{\mathcal{F}(E)(3)} = \{\alpha \quad | \alpha \notin B^P\} \cup \{\alpha \quad | \alpha \in B^P\}.$$

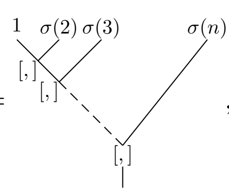
The elements of first set, labelled now with the respective dual elements, form a basis of the subspace of quadratic elements of the PBW basis of $P^!$.

In higher degrees, the labelled trees are constructed using the previous choice and applying condition (2). The set obtained in that way is a PBW basis of $P^!$, with respect to the opposite order originally defined. In Section 7 we use this result to find a PBW basis for the operad Com^2, generated by two associative and commutative compatible products.

2.6.7. **Example.** Consider the operad *Lie*, generated by a single binary operation $[-,-]$. Using a lexicographical order, we get that a PBW basis for the subspace of quadratic elements is given by the set of trees of the form:



By the PBW conditions, in higher degrees the PBW basis consists of trees of the form

$t =$ 

,

where σ is any permutation of the set $\{2,\dots,n\}$. This PBW basis for $Lie(n)$ is just the basis described in 2.0.10.

3. Compatible associative algebras

3.0.1. **Definition.** A *compatible associative algebra* is a vector space A together with two associative products $\cdot : A \otimes A \to A$ and $\circ : A \otimes A \to A$, satisfying that their sum $* := \cdot + \circ$ is an associative product, too.

Note that the if the product $* = \cdot + \circ$ is associative, then all linear combinations $\lambda \cdot \ + \mu \circ$ are associative products, for any coefficients λ and μ in $\mathbb{K}$.

3.0.2. **Remark.** The condition $\cdot + \circ$ is an associative product, is equivalent to:

$$x \circ (y \cdot z) + x \cdot (y \circ z) = (x \circ y) \cdot z + (x \cdot y) \circ z,$$

for all elements $x, y, z \in A$.

3.1. Examples of associative compatible algebras.

3.1.1. *Matrix case.* Let $(A, \cdot)$ be an associative algebra. An associative product $\circ : A \otimes A \to A$, compatible with the original product, is obtained by setting $x \circ y = x \cdot a \cdot y$, for some fixed element $a \in A$.

An interesting problem is to find all the possible associative products $\circ$ on A compatible with the original product $\cdot$.

When $A = Mat_n$ is the vector space of square matrices with coefficients in the field $\mathbb{K}$, in [30], the authors describe all the associative products $\circ$ on A compatible with usual matrix product. In this case, there exists a linear map $R : A \to A$, such that any associative product $\circ$ has the form

$$x \circ y = R(x) \cdot y + x \cdot R(y) - R(x \cdot y).$$

Note that in our first example, the map $R : A \to A$ is given by $R(x) := a \cdot x$, for any $a \in A$.

3.1.2. **Example.** Consider $A = Mat_n$ with usual matrix product, and let a and b be two elements of A. Define $R : A \to A$ as $R(x) := a(xb - bx)$. The new product $\circ$ is given by

$$x \circ y = (ax - xa)(by - yb),$$

which is associative whenever $a^2 = b^2 = 1$ and $ab = -ba$.

In general, if $(A, \cdot)$ is a unital associative algebra and a and b are elements of A satisfying the conditions of the previous paragraph, then $(A, \cdot, \circ)$ is a compatible associative algebra.

3.1.3. **Remark.** Note that if $(A, \cdot)$ is an associative algebra and $R : A \to A$ is a linear map, then the product $\circ$ defined as above is not necessarily an associative product. However, the products $\cdot$ and $\circ$ verify the compatibility condition

$$x \circ (y \cdot z) + x \cdot (y \circ z) = (x \circ y) \cdot z + (x \cdot y) \circ z.$$

So, if $\circ$ is associative, then $(A, \cdot, \circ)$ is an associative compatible algebra.

3.1.4. *Nijenhuis algebras.* In [5], J.F. Carinena, J. Grabowski and G. Marmo introduced the notion of *Nijenhuis tensor* for an associative algebra. Given an associative algebra $(A, \cdot)$ and a linear map $R : A \to A$, the map R is a *Nijenhuis tensor* if it satisfies the following condition:

$$R(R(x) \cdot y + x \cdot R(y) - R(x \cdot y)) = R(x) \cdot R(y), \text{ for all } x \text{ and } y \text{ in } A .$$

Thus, an *associative Nijenhuis algebra* is an associative algebra $(A, \cdot)$ together with a Nijenhuis tensor $R : A \to A$. Defining $x \circ y = R(x) \cdot y + x \cdot R(y) - R(x \cdot y)$, we get that $R(x \circ y) = R(x) \cdot R(y)$. In this case, $\circ$ is an associative product.

So, any associative Nijenhuis algebra defines a compatible associative algebra. For more details about Nijenhuis tensor, we refer to [5].

3.1.5. *Modified Quasi-Shuffle Algebra.* In [10], for the construction of the free commutative unital associative Nijenhuis algebra over a commutative unital associative algebra, K. Ebrahimi-Fard introduced the *modified quasi-shuffle product.*

Let $(A, \cdot)$ a unital associative algebra with unit e and consider the tensor algebra $T(A) = \bigoplus_{n \geq 0} A^{\otimes^n}$ over the vector space A.

The *quasi-shuffle product* [19] on $T(A)$ is defined inductively by

$$(aU) * (bV) = a(U * bV) + b(aU * V) - (a \cdot b)(U * V),$$

with $U * 1 = 1 * U = U$, $U \in T(A)$. This product is associative and it is commutative if $(A, \cdot)$ is a commutative algebra (see [19]).
The *modified quasi-shuffle product* [10] on $T(A)$ is defined inductively using the algebra unit $e \in A$ by :

$$(aU) \circledast (bV) = a(U \circledast bV) + b(aU \circledast V) - e(a \cdot b)(U \circledast V),$$

with $U \circledast 1 = 1 \circledast U = U$, $U \in T(A)$. Again, this product is associative and it is commutative whenever $(A, \cdot)$ is a commutative algebra.

The operator $B_e^+ : T(A) \to T(A)$ defined by $B_e^+(a_1 \cdots a_n) = ea_1 \cdots a_n$, is a Nijenhuis tensor with respect to the product $\circledast$, and therefore $T(A)$ is an associative compatible algebra, whose products are both commutative when A is a commutative algebra (see [10]).

3.1.6. **Remark.** The free commutative unital associative Nijenhuis algebra over a commutative unital associative $\mathbb{K}$-algebra A (see [10]) is the data $(\overline{T}(A), \boxtimes, B_e^+)$, where the product $\boxtimes$ is defined by:

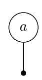
$$aU \boxtimes bV = (a \cdot b)(U \circledast V).$$

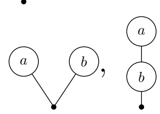
The product $\boxtimes$ is associative and commutative, and $e \boxtimes U = U \boxtimes e = U$, for all $U \in \overline{T}(A)$.

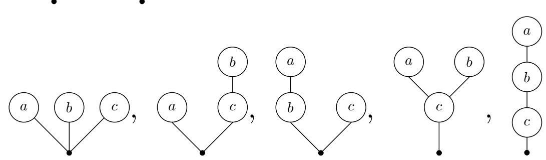
3.2. **The free associative compatible algebra.** Free objects in the category of associative compatible algebras were studied by V. Dotsenko in [8], where he determined the structures of S_n-module and $S_n \times SL_2$-module of $\mathrm{As}^2(n)$. In particular, he showed that the dimension of $\mathrm{As}^2(n)$ is $c_n \cdot n!$, where c_n is the nth. Catalan number. He constructed associative products on vector spaces spanned by trees using R. Grossman and R.G. Larson's constructions (see [17]), but the associative products are compatible only in certain cases and quite difficult to deal with.

We give a different construction of the free associative compatible algebra, applying Dotsenko's results on its dimensions, by means of planar rooted trees.

Let V be a vector space with basis $X = \{a_i\}_{i \in I}$. Denote by T_n^X the set of planar rooted trees with $(n+1)$ vertices, whose vertices different from the root are colored by the elements of X. For instance:

$T_1^X = \{$  $: a \in X\}$

$T_2^X = \{$  $: a, b \in X\}$

$T_3^X = \{$  $: a, b, c \in X\}.$

Consider the vector space $\mathrm{As}^2(V) = \mathbb{K}[\bigcup_{n\geq 1} T_n^X] = \bigoplus_{n \geq 1} \mathbb{K}[T_n^X]$, whose basis is the set $\bigcup_{n \geq 1} T_n^X$ of all planar rooted colored trees.

3.2.1. **Remark.** For any tree t in T_n^X, we say that t has degree n and we write $|t| = n$. We consider the tree t oriented from bottom to top.

Given a vertex $v \in t$, we say that a vertex $v' \in t$ is a child of v if v' is directly connected to the vertex v.

3.2.2. **Notation.** Given a tree t, the set of vertices of t is denoted by $\mathrm{Vert}(t)$ and the root of t by $\mathrm{root}(t)$. The subset $\mathrm{Vert}(t) \setminus \{\mathrm{root}(t)\}$ of $\mathrm{Vert}(t)$ is denoted by $\mathrm{Vert}^*(t)$.

We define two associative products in $\mathrm{As}^2(V)$.

3.2.3. **Definition.** Let t, w be trees in $\mathrm{As}^2(V)$. Define $t \cdot w$ as the tree obtained by identifying the roots of t and w. Extending this binary operation by linearity, we get an associative product

$$\cdot : \mathrm{As}^2(V) \otimes \mathrm{As}^2(V) \to \mathrm{As}^2(V).$$

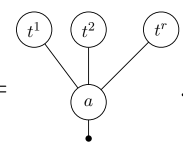
3.2.4. **Remark.** Note that any tree t in $\mathrm{As}^2(V)$ may be written in a unique way as $t = t^1 \cdot \ldots \cdot t^r$, where $r \geq 1$ and the root of each t^i has only one child, for each $i \in \{1, \ldots, r\}$. Clearly, we have that $|t| = \sum_{i=1}^{r} |t^i|$.

When the root of a tree $t \in T_n^X$ has a unique child, we say that t is *irreducible*. We identify the elements of the basis X with the trees of degree one (which are irreducible).

Denoting by Irr the vector space spanned by the set of all irreducible trees in $\mathrm{As}^2(V)$, we have that $(\mathrm{As}^2(V), \cdot)$ is free over Irr as an associative algebra. The set of irreducible trees of degree n is denoted by Irr_n.

3.3. **Second product.** Let t and w be trees in $\mathrm{As}^2(V)$, with $t = t^1 \cdot \ldots \cdot t^r$ as described in Remark 3.2.4. A second product $t \circ w$ is defined proceeding by induction on the degree n of w.

If $n = 1$, then $w = a$, for some $a \in X$. In this case, the element $t \circ w$ is the tree obtained by replacing the root of t by the vertex, colored with a, and adding a new root.

$t \circ w =$  .

Assume now that $n > 1$, and that the product $t \circ w$ has been defined for any $|w| < n$.

Let $w = w^1 \cdot \ldots \cdot w^m$ be the unique decomposition of w as a product of irreducible trees.

If $m = 1$, then $w = w^1 = u \circ a$, where u is a tree such that $|u| = n - 1$ and a is an element of the basis X. Applying a recursive argument, we may suppose that $t \circ u$ is already defined. The product $t \circ w$ is the element

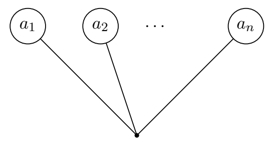
$$t \circ w := t \circ (u \circ a) = (t \circ u) \circ a.$$

For $m > 1$, the element $t \circ w$ is defined by the following formula:

$$\begin{aligned} t \circ w = & \sum_{i=1}^{m} ((t \cdot w^1 \cdot \ldots \cdot w^{i-1}) \circ w^i) \cdot \ldots \cdot w^m \\ & - \sum_{i=2}^{m} t \cdot ((w^1 \cdot \ldots \cdot w^{i-1}) \circ w^i) \cdot w^{i+1} \cdot \ldots \cdot w^m \end{aligned}$$

Note that the recursive hypothesis states that each term of the previous formula is well defined.

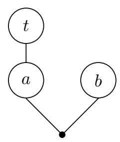
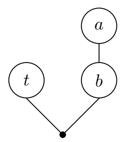
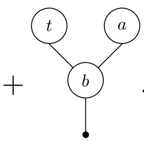
3.3.1. **Example.** Let $a_1, \ldots, a_n$ be elements of the basis X, with $n \geq 2$. Consider the tree w given by

$w =$  $= a_1 \cdot \ldots \cdot a_n$

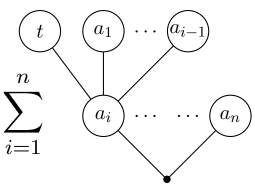
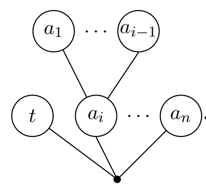
If $n = 2$, then we have that

$$t \circ w = t \circ (a_1 \cdot a_2) = (t \circ a_1) \cdot a_2 - t \cdot (a_1 \circ a_2) + (t \cdot a_1) \circ a_2,$$

therefore we get

$t \circ w =$  $-$  $+$  .

In general, for $w = a_1 \cdot \ldots \cdot a_n$, we get the formula:

$t \circ w = \sum_{i=1}^{n}$  $- \sum_{i=2}^{n}$  .

3.3.2. **Proposition.** *The vector space* $\mathrm{As}^2(V)$*, equipped with the products* $\cdot$ *and* $\circ$ *defined in 3.2.3 and 3.3, is the free associative compatible algebra on* V*.*

Proof. Note that the definition of the product $\circ$ implies that the products $\cdot$ and $\circ$ satisfy the compatibility condition.

Let us prove that the product $\circ$ is associative. Let t_1, t_2, t_3 be trees in $\mathrm{As}^2(V)$. To see that $t_1 \circ (t_2 \circ t_3) = (t_1 \circ t_2) \circ t_3$, we proceed by induction on the degree of the third term t_3.

If the degree of t_3 is one, the assertion is follows easily from Definition 3.3.

Suppose that $n = |t_3| > 1$. If t_3 is an irreducible tree, then $t_3 = w \circ a$, where w is a tree with $|w| = n - 1$, and a is an element of the basis X. Applying a recursive argument, we get the following identities:

$$\begin{aligned} t_1 \circ (t_2 \circ t_3) &= t_1 \circ (t_2 \circ (w \circ a)) \\ &= t_1 \circ ((t_2 \circ w) \circ a) \\ &= (t_1 \circ (t_2 \circ w)) \circ a \\ &= ((t_1 \circ t_2) \circ w) \circ a \\ &= (t_1 \circ t_2) \circ (w \circ a) \\ &= (t_1 \circ t_2) \circ t_3, \end{aligned}$$

which imply the result.

Suppose now that $t_3 = w \cdot z$, where w and z are trees of degree smaller than $|t_3|$.

By the compatibility condition and a recursive argument, we get that:

$$\begin{aligned} t_1 \circ (t_2 \circ t_3) &= t_1 \circ (t_2 \circ (w \cdot z)) \\ &= t_1 \circ ((t_2 \circ w) \cdot z - t_2 \cdot (w \circ z) + (t_2 \cdot w) \circ z) \\ &= (t_1 \circ (t_2 \circ w)) \cdot z - t_1 \cdot ((t_2 \circ w) \circ z) + (t_1 \cdot (t_2 \circ w)) \circ z \\ &\quad -(t_1 \circ t_2) \cdot (w \circ z) + t_1 \cdot (t_2 \circ (w \circ z)) - (t_1 \cdot t_2) \circ (w \circ z) \\ &\quad +t_1 \circ ((t_2 \cdot w) \circ z) \\ &= ((t_1 \circ t_2) \circ w) \cdot z - (t_1 \circ t_2) \cdot (w \circ z) \\ &\quad +(t_1 \circ (t_2 \cdot w) - (t_1 \cdot t_2) \circ w + t_1 \cdot (t_2 \circ w)) \circ z. \end{aligned}$$

As $(t_1 \circ t_2) \cdot w = t_1 \circ (t_2 \cdot w) - (t_1 \cdot t_2) \circ w + t_1 \cdot (t_2 \circ w)$, we conclude that

$$\begin{aligned} t_1 \circ (t_2 \circ t_3) &= ((t_1 \circ t_2) \circ w) \cdot z - (t_1 \circ t_2) \cdot (w \circ z) + ((t_1 \circ t_2) \cdot w) \circ z \\ &= (t_1 \circ t_2) \circ (w \cdot z) \\ &= (t_1 \circ t_2) \circ t_3. \end{aligned}$$

To end the proof, we need to see that $\mathrm{As}^2(V)$ is free as associative compatible algebra. Let A be an associative compatible algebra and let $f : V \to A$ be a linear map. The homomorphism $\widetilde{f} : \mathrm{As}^2(V) \to A$ is defined in a recursive way.

Let t be a tree in $\mathrm{As}^2(V)$. If $|t| = 1$ then $t = a$ with $a \in X$ and therefore $\widetilde{f}(t) = f(a)$.

Suppose that $|t| > 1$. If $t = t' \circ a$, for some $a \in X$, is irreducible, we define

$$\widetilde{f}(t) = \widetilde{f}(t') \circ f(a),$$

which is well defined by a recursive argument.

If $t = t^1 \cdot \ldots \cdot t^r$ for some $r > 1$, then we can assume that $\widetilde{f}(t^i)$ is defined, for $1 \leq i \leq r$, and set $\widetilde{f}(t) = \widetilde{f}(t^1) \cdot \ldots \cdot \widetilde{f}(t^r)$.

To see that $\widetilde{f}$ is unique, consider $g : \mathrm{As}^2(V) \to A$, a homomorphism of compatible associative algebras such that $g(a) = f(a)$, for $a \in V$. Let t be a tree in $\mathrm{As}^2(V)$. If $|t| = 1$, then $t = a$, with $a \in X$. So, by definition of $\widetilde{f}$, $g(t) = \widetilde{f}(t)$. Suppose that $|t| > 1$. We have that $t = t' \circ a$, for some $a \in X$, or $t = t^1 \cdot \ldots \cdot t^r$, for some $r > 1$. Applying a recursive argument, we have that $g(t) = \widetilde{f}(t)$. This show that $\widetilde{f}$ is unique, which ends the proof.

□

3.4. **The operad As^2.** As the operad As^2 is generated by two associative products which do not satisfy any symmetry, the operad As^2 is regular. So, As^2 is determined by a non-symmetric operad, with $\mathrm{As}^2 = \{\mathrm{As}^2_n\}_{n \geq 1}$, such that

$$\mathrm{As}^2(n) = \mathrm{As}^2_n \otimes \mathbb{K}[S_n],$$

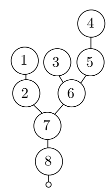
where $\mathbb{K}[S_n]$ is the regular representation of S_n.

By Proposition 2.2.5, the non-symmetric operad As^2 is completely determined by the free As^2-algebra on one generator.

So, As^2_n is the vector space spanned by the set of planar trees with $(n+1)$ vertices. As a consequence of the previous assertion, the dimension of As^2_n is the Catalan number c_n, and therefore the dimension of $\mathrm{As}^2(n)$ is $n!c_n$, as showed by V. Dotsenko in [8].

Let T_n be the set of planar trees with $(n+1)$ vertices. We can identify each tree t in T_n with an operation in n variables, $x_1, \dots, x_n$. There is only one way to label the non-root vertices of t by the set $\{1, \dots, n\}$ so that the labels are in an increasing order, seen from left to right, and an increasing order on each path from of a vertex to the root. Using the description of the products $\cdot$ and $\circ$ in $\mathrm{As}^2(V)$, this rule determine a unique operation μ_t in n variables $x_1, \dots, x_n$.

For example, when t is the tree

$t =$ 

we get $\mu_t = ((x_1 \circ x_2) \cdot ((x_3 \cdot (x_4 \circ x_5)) \circ x_6)) \circ x_7 \circ x_8$.

To describe the composition γ of the operad, we use the structure of compatible associative algebra of $\mathrm{As}^2(\mathbb{K}) = \bigoplus_{n \geq 1} T_n$. Explicitly, given a collection of trees $t, t_1, \dots, t_n$, where $t \in T_n$ and $t_i \in T_{m_i}$, for $1 \leq i \leq n$ and considering the identification of t with the operation μ_t, we define

$$\gamma(t, t_1, \dots, t_n) := \mu_t(t_1, \dots, t_n),$$

which can be determined inductively by the formulas given in 3.2 for the products $\cdot$ and $\circ$.

3.5. **Koszul dual of** As^2**.** The Koszul dual of As^2 was originally described by H. Strohmayer in [37]. In his work, H. Strohmayer established the general notion of operads of compatible structures.

He proved, among other examples, that $(\mathrm{As}^2)^!$ is a Koszul operad. As a consequence of this result, the operad As^2 is Koszul operad, too. Following the notation of [37], we denote the Koszul dual of As^2 by ${}^2\mathrm{As}$.
The operad ${}^2\mathrm{As}$ is generated by two associative products, denoted by $\cdot$ and $\circ$, respectively, satisfying the relations

$$(x_1 \cdot x_2) \circ x_3 = x_1 \cdot (x_2 \circ x_3) = (x_1 \circ x_2) \cdot x_3 = x_1 \circ (x_2 \cdot x_3).$$

The operad ${}^2\mathrm{As}$ was studied in detail by Y. Zhang, C. Bai and L.Guo in [42].

Let us describe a basis for ${}^2\mathrm{As}_n$, given by Y. Zhang in [41]. For each $0 \leq i \leq n-1$, the operations μ_i are defined by

$$\mu_i(x_1, \dots, x_n) = x_1 \cdot \dots \cdot x_{i+1} \circ \dots \circ x_n,$$

with i copies of the product $\cdot$ and $(n-i)$ copies of the product $\circ$ (note that, by the relations present in ${}^2\mathrm{As}$, the position of the products $\cdot$ and $\circ$ in μ_i is irrelevant).

The set $\{\mu_1, \dots, \mu_n\}$ is a basis for ${}^2\mathrm{As}_n$ and the composition in the operad ${}^2\mathrm{As}$ is given by

$$\mu_n(\mu_{m_1}, \dots, \mu_{m_n}) = \mu_{n+m_1+\dots+m_n},$$

as it was showed in [41], Proposition 3.2. So, ${}^2\mathrm{As}_n$ is n-dimensional and ${}^2\mathrm{As}_n = \mathrm{As}_n \oplus \dots \oplus \mathrm{As}_n$, it is direct sum of n copies of As_n.

3.6. **Operadic homology for As^2-algebras.** The operadic homology for As^2 was studied by Y. Zhang in [41]. We give an explicit realization of this chain-complex by means of a bicomplex, which is defined by Hochschild complexes induced by the two associative products defined in a compatible associative algebra.

Let us recall some facts about non-unital Hochschild complex. Given a non-unital associative algebra $(A, \cdot)$ and a right A-module M, the Hochschild complex of A with coefficients in M is given by:

$$C_*(A, M): \quad \dots \to M \otimes A^{\otimes n} \to M \otimes A^{\otimes (n-1)} \to \dots \to M.$$

Denoting an element of $M \otimes A^{\otimes n}$ by $(a_0, a_1, \dots, a_n)$, the differential d is given by

$$d(a_0, \dots, a_n) := \sum_{i=0}^{n-1} (-1)^i (a_0, \dots, a_i \cdot a_{i+1}, \dots, a_n).$$

Defining $d^i(a_0, \dots, a_n) = (a_0, \dots, a_i a_{i+1}, \dots, a_n)$, for $0 \leq i \leq n-1$, we can express the boundary d as:

$$d = \sum_{i=0}^{n-1} (-1)^i d^i.$$

The homology groups are denoted by $H_*(A, M)$. If A is a free associative algebra, that is, $A = \overline{T}(W)$ for some vector space W, then $H_0(A, M) = M/MA$ and $H_n(A, M) = 0$, for $n \geq 1$(see for instance [23]).

In particular, if $M = A$, the Hochschilds complex is denoted by $C_*(A)$ and the homology groups by $H_*(A)$.

This chain complex defines the operadic homology of associative algebras (see 2.5.8).The differential is described by the formula

$$d(a_1, \dots, a_n) := \sum_{i=1}^{n-1} (-1)^{i+1} (a_1, \dots, a_i a_{i+1}, \dots, a_n).$$

Consider now a compatible associative algebra $(A, \cdot, \circ)$. By definition, its operadic chain complex is given by

$$\cdots \to C_n^{\mathrm{As}^2}(A) \to C_{(n-1)}^{\mathrm{As}^2}(A) \to \cdots \to C_1^{\mathrm{As}^2}(A),$$

where $C_n^{\mathrm{As}^2}(A) := (\mathrm{As}^2)^!(n)^\vee \otimes_{S_n} A^{\otimes n}$, whose differential d is described in 2.5.8. Since the Koszul dual of the operad As^2 is given by ${}^2\mathrm{As}$ and, in the symmetric framework,

$$(*) \quad {}^2\mathrm{As}(n) = \mathbb{K}[S_n] \oplus \cdots \oplus \mathbb{K}[S_n]$$

is the direct sum of n-copies $\mathbb{K}[S_n]$, the regular representation of S_n, we get

$$\begin{array}{rcl} C_n^{\mathrm{As}^2}(A) & = & (\mathrm{As}^2)^!(n)^\vee \otimes_{S_n} A^{\otimes n} \\ & = & (\mathrm{sgn}_n \otimes (\mathbb{K}[S_n] \oplus \cdots \oplus \mathbb{K}[S_n])) \otimes_{S_n} A^{\otimes n} \\ & = & (\mathrm{sgn}_n \otimes A^{\otimes n}) \oplus \cdots \oplus (\mathrm{sgn}_n \otimes A^{\otimes n}) \\ & = & A^{\otimes n} \oplus \cdots \oplus A^{\otimes n}. \end{array}$$

We can reorganize the chain-complex $C_*^{\mathrm{As}^2}(A)$ as a bicomplex. Explicitly, we have:

3.6.1. **Proposition.** *Let $(A, \cdot, \circ)$ be a compatible associative algebra. The Hochschild differentials $d_\cdot$, $d_\circ$, induce a bi-complex:*

$$\begin{tikzcd}
{} \arrow[d, "d_\cdot"'] & {} \arrow[d, "d_\cdot"'] & {} \arrow[d, "d_\cdot"'] & \\
A^{\otimes 3} \arrow[d, "d_\cdot"'] & A^{\otimes 4} \arrow[l, "d_\circ"'] \arrow[d, "d_\cdot"'] & A^{\otimes 5} \arrow[l, "d_\circ"'] \arrow[d, "d_\cdot"'] & {} \arrow[l, "d_\circ"'] \\
A^{\otimes 2} \arrow[d, "d_\cdot"'] & A^{\otimes 3} \arrow[l, "d_\circ"'] \arrow[d, "d_\cdot"'] & A^{\otimes 4} \arrow[l, "d_\circ"'] \arrow[d, "d_\cdot"'] & {} \arrow[l, "d_\circ"'] \\
A & A^{\otimes 2} \arrow[l, "d_\circ"'] & A^{\otimes 3} \arrow[l, "d_\circ"'] & {} \arrow[l, "d_\circ"']
\end{tikzcd}$$

Proof. We must prove that $d_\circ d_\cdot + d_\cdot d_\circ = 0$. Using the characterization of the Hochschild boundary, we have that

$$d_\cdot d_\circ = \sum_{i,j} (-1)^{i+j+2} d_\cdot^j d_\circ^i,$$

where the sum runs all the pairs (i,j), with $1 \leq i \leq n-1$ and $1 \leq j \leq n-2$. Analogously,

$$d_\circ d_\cdot = \sum_{i,j} (-1)^{i+j+2} d_\circ^j d_\cdot^i,$$

with $1 \leq i \leq n-1$ and $1 \leq j \leq n-2$.

A direct compute shows that:

(1) $d_\circ^j d_\cdot^1 = d_\cdot^1 d_\circ^{j+1}$, if $1 < j \leq n-2$.
(2) $d_\circ^j d_\cdot^i = d_\cdot^{i-1} d_\circ^j$, if $3 \leq i \leq n-1$ and $1 \leq j \leq i-2$.
(3) $d_\circ^j d_\cdot^i = d_\cdot^i d_\circ^{j+1}$, if $3 \leq i \leq n-3$ and $i+2 \leq j \leq n-2$.

On the other hand, the compatibility condition between the products $\cdot$ and $\circ$ implies that

$$d_\cdot^i d_\circ^i + d_\circ^i d_\cdot^i = d_\circ^i d_\cdot^{i+1} + d_\cdot^i d_\circ^{i+1}.$$

All these facts together imply that $d_\circ d_\cdot + d_\cdot d_\circ = 0$. $\square$

Note that the total complex of the bicomplex described in 3.6.1, is the operadic chain complex of a compatible associative algebra A because $C_n^{\mathrm{As}^2}(A)$ is the direct sum of n-copies of $A^{\otimes n}$ and the operadic differential can be identified with the differential $d = d_\cdot + d_\circ$. In particular, since the operad As^2 is a Koszul operad, if $A = \mathrm{As}^2(V)$ is the free compatible associative algebra over V, the complex $C_*^{\mathrm{As}^2}(A)$ is acyclic. We give a direct proof of this fact without to use the Koszulity of As^2.

3.6.2. **Proposition.** *Let $A = \mathrm{As}^2(V)$ be the free compatible associative algebra over the vector space V. The chain complex*

$$\cdots \to C_n^{\mathrm{As}^2}(A) \to C_{(n-1)}^{\mathrm{As}^2}(A) \to \cdots \to C_1^{\mathrm{As}^2}(A)$$

is acyclic.

Proof. For the proof, we use the description of the free compatible associative algebra $A = \mathrm{As}^2(V)$ given in 3.2 and the results obtained in Section 5. By Theorem 4.2.14, A is free as associative algebra over the free N-algebra $N(V)$(described in 4.2.13) for both products. For each $p \geq 1$, we have that, in the bicomplex, the pth column correspond to the chain complex

$$C_*^p(A): \quad \cdots \to A^{\otimes(n+1)} \to A^n \to \cdots \to A^{\otimes p},$$

where $A^{\otimes p}$ is in degree 1 and the differential is $d_\cdot$. We denote the homology groups of this complex by $H_*^p(A)$. Since the groups of homology are the same that in $C_*(A)$, for $n > p$, and the associative algebra $(A, \cdot)$ is free over $N(V)$, we have that $C_*^p(A)$ is acyclic and that

$$H_1^p(A) = A^{\otimes p}/\mathrm{Im} d_\cdot = A^{\otimes(p-1)} \otimes N(V).$$

Denoting $K_p := H_1^p(A) = A^{\otimes(p-1)} \otimes W$, we get a well defined chain complex

$$\cdots \to K_{p+1} \to K_p \to \cdots \to K_1,$$

whose differential is induced by horizontal differential $d_\circ$(see for instance [23], 1.0.11). Since in the bicomplex the columns are acyclic, we have that $H_n(K_*, d_\circ) = H_n^{\mathrm{As}^2}(A)$, for all $n \geq 1$ (see [23], Proposition 1.0.12). So, as $K_p = A^{\otimes(p-1)} \otimes \mathcal{N}(V)$ and the differential is induced by the Hochschild differential $d_\circ$, $(K_*, d_\circ)$ is the Hochschild complex of $(A, \circ)$ with coefficients in $\mathcal{N}(V)$. Since $(A, \circ)$ is free over $\mathcal{N}(V)$, the complex $(K_*, d_\circ)$ is acyclic and

$$H_1(K_*, d_\circ) = \mathcal{N}(V)/\mathcal{N}(V)A = V.$$

Therefore, $H_1^{\mathrm{As}^2}(A) = V$ and $H_n^{\mathrm{As}^2}(A) = 0$ for $n > 1$, which ends the proof. □

4. Compatible infinitesimal bialgebras

In this section, we introduce *compatible infinitesimal bialgebras*, which uses the notion of unital infinitesimal bialgebra introduced in Section 2. To work in the more general context, we do not assume the existence of unity. So, an infinitesimal bialgebra is an associative algebra $(H, \cdot)$ equipped with a coassociative coproduct $\Delta : H \longrightarrow H \otimes H$ satisfying

$$\Delta(x \cdot y) = x_{(1)} \otimes (x_{(2)} \cdot y) + (x \cdot y_{(1)}) \otimes y_{(2)} + x \otimes y,$$

for $x, y \in H$, with $\Delta(x) = x_{(1)} \otimes x_{(2)}$ and $\Delta(y) = y_{(1)} \otimes y_{(2)}$ for $x, y \in H$.

In this context, an element $x \in H$ is called primitive when $\Delta(x) = 0$.

4.0.1. **Definition.** A *compatible infinitesimal bialgebra* over $\mathbb{K}$ is an associative compatible algebra $(H, \cdot, \circ)$ equipped with a coassociative coproduct $\Delta : H \longrightarrow H \otimes H$ such that $(H, \cdot, \Delta)$ and $(H, \circ, \Delta)$ are both unital infinitesimal bialgebras.

4.0.2. **Lemma.** *The notion of compatible infinitesimal bialgebra is well-defined.*

Proof. A direct computation shows that:

$$\begin{array}{lll}
(1) \quad \Delta((x \cdot y) \circ z) & = & x_{(1)} \otimes (x_{(2)} \cdot y) \circ z + x \cdot y_{(1)} \otimes y_{(2)} \circ z + x \otimes (y \circ z) \\
& & +(x \cdot y) \circ z_{(1)} \otimes z_{(2)} + (x \cdot y) \otimes z, \\
(2) \quad \Delta((x \circ y) \cdot z) & = & x_{(1)} \otimes (x_{(2)} \circ y) \cdot z + x \circ y_{(1)} \otimes y_{(2)} \cdot z + x \otimes (y \cdot z) \\
& & +(x \circ y) \cdot z_{(1)} \otimes z_{(2)} + (x \circ y) \otimes z, \\
(3) \quad \Delta(x \cdot (y \circ z)) & = & x_{(1)} \otimes x_{(2)} \cdot (y \circ z) + x \cdot y_{(1)} \otimes y_{(2)} \circ z + x \otimes (y \circ z) \\
& & +x \cdot (y \circ z_{(1)}) \otimes z_{(2)} + (x \cdot y) \otimes z, \\
(4) \quad \Delta(x \circ (y \cdot z)) & = & x_{(1)} \otimes x_{(2)} \circ (y \cdot z) + x \circ y_{(1)} \otimes y_{(2)} \cdot z + x \otimes (y \cdot z) \\
& & +x \circ (y \cdot z_{(1)}) \otimes z_{(2)} + (x \circ y) \otimes z.
\end{array}$$

Using the compatibility condition between the products $\cdot$ and $\circ$, we get that:

(1) $x_{(1)} \otimes (x_{(2)} \cdot y) \circ z + x_{(1)} \otimes (x_{(2)} \circ y) \cdot z = x_{(1)} \otimes x_{(2)} \cdot (y \circ z) + x_{(1)} \otimes x_{(2)} \circ (y \cdot z)$,

(2) $(x \cdot y) \circ z_{(1)} \otimes z_{(2)} + (x \circ y) \cdot z_{(1)} \otimes z_{(2)} = x \cdot (y \circ z_{(1)}) \otimes z_{(2)} + x \circ (y \cdot z_{(1)}) \otimes z_{(2)}$,

which implies that

$$\Delta((x \cdot y) \circ z + (x \circ y) \cdot z) = \Delta(x \cdot (y \circ z) + x \circ (y \cdot z)).$$

□

4.0.3. **Proposition.** *Let V be a vector space, the free associative compatible algebra* $\mathrm{As}^2(\mathrm{V})$ *has a natural structure of compatible infinitesimal bialgebra.*

Proof. The coproduct $\Delta : \mathrm{As}^2(V) \to \mathrm{As}^2(V) \otimes \mathrm{As}^2(V)$ is defined by induction on the degree of a tree t in $\mathrm{As}^2(V)$.

For $t = a \in X$, its image is $\Delta(t) = 0$. When $|t| > 1$, we consider two cases:

(1) for $t = t' \circ a$, with $a \in X$, we define

$$\Delta(t) = t'_{(1)} \otimes t'_{(1)} \circ a + t' \otimes a.$$

(2) for $t = t' \cdot t''$ with $|t'| < |t|$ and $|t''| < |t|$, we have that

$$\Delta(t) = t'_{(1)} \otimes t'_{(2)} \cdot t'' + t' \cdot t''_{(1)} \otimes t''_{(2)} + t' \otimes t''.$$

Lemma 4.0.2 and the inductive hypothesis, state that Δ is well defined. Note that if $\Delta(t) = t_{(1)} \otimes t_{(2)}$ then $|t_{(1)}| < |t|$ and $|t_{(2)}| < |t|$.

To see that Δ is coassociative, we proceed by induction on degree of t. Let t be a tree. For $|t| = 1$ the result is immediate.
For $|t| > 1$, we consider two case:

First, if t is an irreducible tree, then $t = t' \circ a$, with $a \in X$. So, we have that:

$$\begin{aligned}
(\Delta \otimes \mathrm{Id})\Delta(t) &= (\Delta \otimes \mathrm{Id})(t'_{(1)} \otimes t'_{(2)} \circ a + t' \otimes a) \\
&= \Delta(t'_{(1)}) \otimes t'_{(2)} \circ a + \Delta(t') \otimes a \\
&= t'_{(1)(1)} \otimes t'_{(1)(2)} \otimes t'_{(2)} \circ a + t'_{(1)} \otimes t'_{(2)} \otimes a.
\end{aligned}$$

Applying the recursive hypothesis to t', we write

$$(\Delta \otimes \mathrm{Id})\Delta(t) = t'_{(1)} \otimes t'_{(2)} \otimes t'_{(3)} \circ a + t'_{(1)} \otimes t'_{(2)} \otimes a.$$

On the other hand, using a similar argument to computer $(\mathrm{Id} \otimes \Delta)\Delta(t)$, we have that

$$\begin{aligned}
(\mathrm{Id} \otimes \Delta)\Delta(t) &= (\mathrm{Id} \otimes \Delta)(t'_{(1)} \otimes t'_{(2)} \circ a + t' \otimes a \\
&= t'_{(1)} \otimes \Delta(t'_{(2)} \circ a) + t' \otimes \Delta(a) \\
&= t'_{(1)} \otimes t'_{(2)(1)} \otimes t'_{(2)(2)} \circ a + t'_{(1)} \otimes t'_{(2)} \otimes a \\
&= t'_{(1)} \otimes t'_{(2)} \otimes t'_{(3)} \circ a + t'_{(1)} \otimes t'_{(2)} \otimes a,
\end{aligned}$$

which gives the expected result.

Second, if t is reducible tree, then $t = t' \cdot t''$, with $|t'| < |t|$ and $|t''| < |t|$. Applying the recursive hypothesis to t' and t'', we have that

$$\begin{aligned}
(\Delta \otimes \mathrm{Id})\Delta(t) \quad &= \quad (\Delta \otimes \mathrm{Id})(t'_{(1)} \otimes t'_{(2)} \cdot t'' + t' \cdot t''_{(1)} \otimes t''_{(2)} + t' \otimes t'') \\
&= \quad t'_{(1)(1)} \otimes t'_{(1)(2)} \otimes t'_{(2)} \cdot t'' + t'_{(1)} \otimes t'_{(2)} \cdot t''_{(1)} \otimes t''_{(2)} \\
&+ t' \cdot t''_{(1)(1)} \otimes t''_{(1)(2)} \otimes t''_{(2)} + t' \otimes t''_{(1)} \otimes t''_{(2)} + t'_{(1)} \otimes t'_{(2)} \otimes t'' \\
&= \quad t'_{(1)} \otimes t'_{(2)} \otimes t'_{(3)} \cdot t'' + t'_{(1)} \otimes t'_{(2)} \cdot t''_{(1)} \otimes t''_{(2)} \\
&+ t' \cdot t''_{(1)} \otimes t''_{(2)} \otimes t''_{(3)} + t' \otimes t''_{(1)} \otimes t''_{(2)} + t'_{(1)} \otimes t'_{(2)} \otimes t''
\end{aligned}$$

and,

$$\begin{aligned}
(\mathrm{Id}\otimes\Delta)\Delta(t) &= (\mathrm{Id}\otimes\Delta)(t'_{(1)}\otimes t'_{(2)}\cdot t''+t'\cdot t''_{(1)}\otimes t''_{(2)}+t'\otimes t'') \\
&= t'_{(1)}\otimes t'_{(2)(1)}\otimes t'_{(2)(2)}\cdot t''+t'_{(1)}\otimes t'_{(2)}\cdot t''_{(1)}\otimes t''_{(2)} \\
&\quad +t'_{(1)}\otimes t'_{(2)}\otimes t''+t'\cdot t''_{(1)}\otimes t''_{(2)(1)}\otimes t''_{(2)(2)}+t'\otimes t''_{(2)}\otimes t''_{(2)} \\
&= t'_{(1)}\otimes t'_{(2)}\otimes t'_{(3)}\cdot t''+t'_{(1)}\otimes t'_{(2)}\cdot t''_{(1)}\otimes t''_{(2)} \\
&\quad +t'_{(1)}\otimes t'_{(2)}\otimes t''+t'\cdot t''_{(1)}\otimes t''_{(2)}\otimes t''_{(3)}+t'\otimes t''_{(2)}\otimes t''_{(2)}.
\end{aligned}$$

So, we conclude that

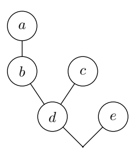
$$(\Delta\otimes\mathrm{Id})\Delta(t)=(\mathrm{Id}\otimes\Delta)\Delta(t),$$

which ends the proof.

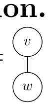
□

4.1. **Formula for the coproduct Δ.** We want to give an explicit formula for the coproduct Δ, for which we previously describe an order on the vertices of a tree.

Given a tree t, we consider the set $\mathrm{Vert}(t)$ ordered by the *level order*, that is, the vertices of t are ordered by reading the vertices of t from left to right and from top to bottom. For instance, if t is the tree

$t =$  ,

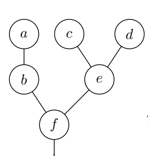
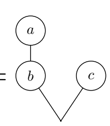
then $\mathrm{Vert}(t)$ is ordered by $a<b<c<d<e<\mathrm{root}(t)$.

4.1.1. **Notation.** Given a tree t and a vertex $v\in\mathrm{Vert}^*(t)$, we denote by e_v the edge $e_v =$  of t, with initial vertex w and final vertex v.

4.1.2. **Definition.** Let $A=\{v_1,\dots,v_l\}$ be a ordered subset of $\mathrm{Vert}^*(t)$ with $v_1<\cdots<v_l$ respect to the level order.

We define t_A as the tree of the degree l such that $\mathrm{Vert}^*(t_A)=A$, which is obtained from t by deletions of the vertices that are not in A and by successive contractions of the edges of which these are the final vertex.

4.1.3. **Example.** When t is the tree

$t =$  , $t_{\{a,b,c\}} =$  .

4.1.4. **Proposition.** *Let t be a tree of degree n, with $\mathrm{Vert}^*(t)$ given by the ordered vertices $a_1<\cdots<a_n$. The coproduct $\Delta(t)$ is given by the formula*

$$\Delta(t)=\sum_{i=1}^{n-1} t_{\{a_1,\dots,a_i\}}\otimes t_{\{a_{i+1},\dots,a_n\}}.$$

Proof. Let t be a tree of degree n and $a_1 < \cdots < a_n$ its vertices, different of the root, ordered by the level order.

We prove the assertion by induction on n. For $n = 1$, $t = a \in V$. So, $\Delta(a) = 0$ and the assertion is true.

For $n > 1$, we consider two cases. First, if t is an irreducible, then $t = t' \circ a_n$, where t' is a tree of degree $(n-1)$, with vertices $a_1, \ldots, a_{n-1}$. In this case, note that $t' = t_{\{a_1,\ldots,a_{n-1}\}}$ and the vertices $a_1, \ldots, a_{n-1}$ preserve the order that they originally had in t. Moreover, if $1 \leq k \leq l \leq n-1$, then $t'_{\{a_k,a_{k+1},\ldots,a_l\}} = t_{\{a_k,a_{k+1},\ldots,a_l\}}$.

By definition of coproduct Δ and by a recursive argument, we have that

$$\begin{aligned}
\Delta(t) &= \Delta(t' \circ a_n) \\
&= \Delta(t') \circ a_n + t' \circ \Delta(a_n) + t' \otimes a_n \\
&= \sum_{i=1}^{n-2} t'_{\{a_1,\ldots,a_i\}} \otimes t'_{\{a_{i+1},\ldots,a_{n-1}\}} \circ a_n + t' \otimes a_n \\
&= \sum_{i=1}^{n-2} t_{\{a_1,\ldots,a_i\}} \otimes t_{\{a_{i+1},\ldots,a_{n-1}\}} \circ a_n + t_{\{a_1,\ldots,a_{n-1}\}} \otimes a_n \\
&= \sum_{i=1}^{n-1} t_{\{a_1,\ldots,a_i\}} \otimes t_{\{a_{i+1},\ldots,a_n\}},
\end{aligned}$$

because $t_{\{a_{i+1},\ldots,a_{n-1}\}} \circ a_n = t_{\{a_{i+1},\ldots,a_n\}}$, for $i = 1, \ldots, n-2$.

If t is a reducible tree, then we may write $t = t' \cdot t''$, where t' and t'' are trees of degree smaller than n. If t' is of degree l, then $\mathrm{Ver}(t') = \{a_1, \ldots, a_l\}$ and $\mathrm{Ver}(t'') = \{a_{l+1}, \ldots, a_{l+m}\}$, where $n = l + m$. Note that the vertices of t' and t'' preserve the order that they had in t. Moreover, $t' = t_{\{a_1,\ldots,a_l\}}$ and $t'' = t_{\{a_{l+1},\ldots,a_{l+m}\}}$.

By definition of the coproduct Δ and by a recursive argument, we obtain that:

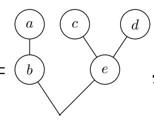
$$\begin{aligned}
\Delta(t) &= \Delta(t' \cdot t'') \\
&= \Delta(t') \cdot t'' + t' \cdot \Delta(t'') + t' \otimes t'' \\
&= \sum_{i=1}^{l-1} t'_{\{a_1,\ldots,a_i\}} \otimes t'_{\{a_{i+1},\ldots,a_l\}} \cdot t'' \\
&\quad + \sum_{j=1}^{m-1} t' \cdot t''_{\{a_{l+1},\ldots,a_{l+j}\}} \otimes t''_{\{a_{l+j+1},\ldots,a_{l+m}\}} + t' \otimes t'' \\
&= \sum_{i=1}^{l-1} t_{\{a_1,\ldots,a_i\}} \otimes t'_{\{a_{i+1},\ldots,a_l\}} \cdot t'' \\
&\quad + \sum_{j=1}^{m-1} t' \cdot t''_{\{a_{l+1},\ldots,a_{l+j}\}} \otimes t_{\{a_{l+j+1},\ldots,a_{l+m}\}} + t' \otimes t''.
\end{aligned}$$

As $t'_{\{a_{i+1},\dots,a_l\}} \cdot t'' = t_{\{a_{i+1},\dots,a_n\}}$, for $i = 1, \dots, l-1$, and $t' \cdot t''_{\{a_{l+1},\dots,a_{l+j}\}} = t_{\{a_1,\dots,a_{l+j}\}}$, for $j = 1, \dots, m-1$, we get

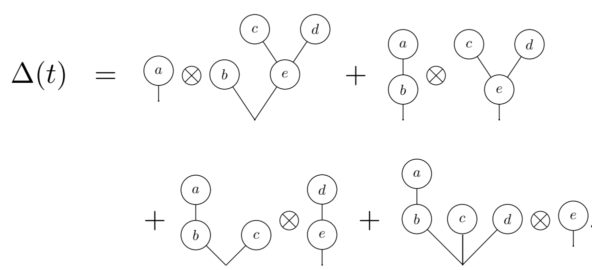
$$\Delta(t) = \sum_{i=1}^{n-1} t_{\{a_1,\dots,a_i\}} \otimes t_{\{a_{i+1},\dots,a_n\}},$$

which ends the proof. $\square$

4.1.5. **Example.** When t is the tree

$t =$  ,

the coproduct $\Delta(t)$ is given by



4.1.6. **Remark.** Consider $H_V = \mathrm{As}^2(V)$ with the coproduct Δ as in Proposition 4.0.3. The triples $(H_V, \cdot, \Delta)$ and $(H_V, \circ, \Delta)$ are graded infinitesimal bialgebras, with the natural graduation of H_V. In particular, the vector space V is the component of degree one. By definition, $\Delta(v) = 0$, for all $v \in V$. So, $(H, \cdot, \Delta)$ and $(H, \circ, \Delta)$ are conilpotent infinitesimal bialgebras.

Applying the result obtained by J.-L. Loday and M. Ronco in [26], Theorem 2.6 , we obtain that H is isomorphism to $\overline{T}(\mathrm{Prim}H)$.

In particular, the associative algebras $(\mathrm{As}^2(V), \cdot)$ and $(\mathrm{As}^2(V), \circ)$ are free as associative algebras. This result is an alternative proof to that obtained by Dotsenko in [8], using operad theory .

4.2. **Structure theorem for compatible infinitesimal bialgebras.** Our aim is to prove that any conilpotent compatible infinitesimal bialgebra can be reconstructed from the subspace of its primitive elements.

We introduce the notion of $\mathcal{N}$-algebra, which describes the algebraic structure of the subspace of primitive elements of any compatible infinitesimal bialgebra.

4.2.1. **Definition.** A *$\mathcal{N}$-algebra* is a vector space V equipped with n-ary operations $N_n : V^{\otimes n} \longrightarrow V$, for $n \geq 2$, which satisfy the following conditions:

(1) $N_n(x_1, \ldots, N_2(x_n, x_{n+1})) = \sum_{i=1}^{n-1} N_{i+1}(x_1, \ldots, N_{n-i+1}(x_i, \ldots, x_n), x_{n+1}),$

(2) for $n \geq 3$

$$N_2(x_1, N_n(x_2, \ldots, x_{n+1})) = N_n(N_2(x_1, x_2), x_3, \ldots, x_{n+1}) - \sum_{i=3}^{n} N_i(x_1, N_{n+2-i}(x_2, \ldots, x_{n+3-i}), x_{n+3-i+1}, \ldots, x_{n+1}),$$

(3) for $r, n \geq 3$

$$\begin{aligned} N_n(x, y_1 \ldots, y_{n-2}, N_r(z, t_1, \ldots, t_{r-2}, w)) = \\ & N_r(N_n(x, y_1, \ldots, y_{n-2}, z), t_1, \ldots, t_{r-2}, w) \\ & + \sum_{i=1}^{n-2} N_{r+i}(x, y_1, \ldots, y_{i-1}, N_{n-i}(y_i, \ldots, y_{n-2}, z), t_1, \ldots, t_{r-2}, w) \\ & - \sum_{i=1}^{r-2} N_{n+r-i-1}(x, y_1, \ldots, y_{n-2}, N_{i+1}(z, t_1, \ldots, t_i), t_{i+1}, \ldots, t_{r-2}, w), \end{aligned}$$

For instance, the relations in low degrees give:

(1) N_2 is an associative product.
(2) $N_3(x, y, N_2(z, t)) = N_2(N_3(x, y, z), t) + N_3(x, N_2(y, z), t)$.
(3) $N_2(x, N_3(y, z, t)) = N_3(N_2(x, y), z, t) - N_3(x, N_2(y, z), t)$.
(4) $N_3(x, y, N_3(z, t, w)) = N_3(N_3(x, y, z), t, w) + N_4(x, N_2(y, z), t, w) - N_4(x, y, N_2(z, t), w)$.

4.2.2. **Remark.** Let $\mathcal{N}$ be the algebraic operad of $\mathcal{N}$-algebras. It is clear that the operad $\mathcal{N}$ is regular. So, the S_n-module $\mathcal{N}(n)$ is of the form $\mathcal{N}(n) = \mathcal{N}_n \otimes \mathbb{K}[S_n]$ for some vector space $\mathcal{N}_n$, where $\mathbb{K}[S_n]$ is the regular representation of S_n.

4.2.3. **Proposition.** *The dimension of the vector space $\mathcal{N}_n$ is equal to the Catalan number c_{n-1}.*

Proof. Denote by $|\mathcal{N}_n|$ the dimension of $\mathcal{N}_n$, as a $\mathbb{K}$-vector space. We know that $|\mathcal{N}_n|$ is the dimension of the subspace of homogeneous elements of degree n of the free $\mathcal{N}$ algebra on one generator x. From the Definition 4.2.1, it is clear that the vector space $\mathcal{N}_n$ has a basis formed by all the elements of type:

$$N_r(M_1(\ldots), \ldots, M_{r-1}(\ldots), x),$$

where each M_i is an element in the basis of $\mathcal{N}_{m_i}$ and $m_1 + \cdots + m_{r-1} = n - 1$. So, we get that

$$|\mathcal{N}_n| = \sum |\mathcal{N}_{m_1}| \cdot \ldots \cdot |\mathcal{N}_{m_{r-1}}|$$

where the sum is taken over all the families $\{m_i\}_{1 \leq i \leq r-1}$ such that $m_1 + \cdots + m_{r-1} = n - 1$.

In particular, $|\mathcal{N}_1| = 1$ and, for $n > 1$, we have that:

$$\begin{aligned} |\mathcal{N}_n| &= \sum c_{m_1} \cdot \ldots \cdot c_{m_{r-1}} \\ &= c_{n-1}, \end{aligned}$$

by a recursive hypothesis, which implies that $|\mathcal{N}_1| = c_0$ and that the integers $|\mathcal{N}_n|$ are defined by same equation than the Catalan numbers. We may conclude that $|\mathcal{N}_n| = c_{n-1}$, for $n \geq 1$. □

4.2.4. **Definition.** Given a compatible associative algebra $(A, \cdot, \circ)$, the n-ary operations $N_n : A^{\otimes n} \longrightarrow A$ on A are defined as follows:

$$N_n(x_1, \ldots, x_n) = (x_1 \cdot \ldots \cdot x_{n-1}) \circ x_n - x_1 \cdot ((x_2 \cdot \ldots \cdot x_{n-1}) \circ x_n)$$

4.2.5. **Remark.** The operations N_n satisfy the following relations:

(1) $N_2(x, y) = x \circ y - x \cdot y$.
(2) $N_n(x_1, \ldots, x_n) = N_3(x_1, x_2 \cdot \ldots \cdot x_{n-1}, x_n)$, for any $n \geq 4$.

4.2.6. **Proposition.** *Let $(A, \cdot, \circ)$ be a compatible associative algebra. For any family of elements $x_1, x_2, x_3, x_4, x_5 \in A$ we have that:*

(1) $N_2(N_2(x_1, x_2), x_3) = N_2(x_1, N_2(x_2, x_3))$,
(2) $N_3(x_1, x_2, N_2(x_3, x_4)) = N_2(N_3(x_1, x_2, x_3), x_4) + N_3(x_1, N_2(x_2, x_3), x_4)$,
(3) $N_3(x_1, x_2, N_3(x_3, x_4, x_5)) = N_3(N_3(x_1, x_2, x_3), x_4, x_5)$
$\quad +N_4(x_1, N_2(x_2, x_3), x_4, x_5) - N_4(x_1, x_2, N_2(x_3, x_4), x_5)$

Proof. The first relation states that N_2 is an associative product, which is true because N_2 is linear combination of the products $\cdot$ and $\circ$.

Let us prove the second statement. The proof of the other ones is obtained in an analogous way.

Let us denote the element $N_2(x, y)$ as $x * y = x \circ y - x \cdot y$. We have that,

$$\begin{aligned} N_3(x_1, x_2, N_2(x_3, x_4) &= (x_1 \cdot x_2) \circ (x_3 * x_4) - x_1 \cdot (x_2 \circ (x_3 * x_4)) \\ &= (x_1 \cdot x_2) \circ x_3 \circ x_4 - (x_1 \cdot x_2) \circ (x_3 \cdot x_4) \\ &\quad -x_1 \cdot (x_2 \circ x_3 \circ x_4) + x_1 \cdot (x_2 \circ (x_3 \cdot x_4)). \end{aligned}$$

Applying the compatibility condition between $\cdot$ and $\circ$, we get:

(1) $(x_1 \cdot x_3) \circ (x_3 \cdot x_4) = ((x_1 \cdot x_2) \circ x_3) \cdot x_4 - x_1 \cdot x_2 \cdot (x_3 \circ x_4) + (x_1 \cdot x_2 \cdot x_3) \circ x_4$,
(2) $x_1 \cdot (x_2 \circ (x_3 \cdot x_4)) = x_1 \cdot (x_2 \circ x_3) \cdot x_4 - x_1 \cdot x_2 \cdot (x_3 \circ x_4) + x_1 \cdot ((x_2 \cdot x_3) \circ x_4)$.

So, we obtain that

$$\begin{aligned} N_3(x_1, x_2, N_2(x_3, x_4) &= (x_1 \cdot x_2) \circ x_3 \circ x_4 - ((x_1 \cdot x_2) \circ x_3) \cdot x_4 \\ &\quad -(x_1 \cdot x_2 \cdot x_3) \circ x_4 - x_1 \cdot (x_2 \circ x_3 \circ x_4) \\ &\quad +x_1 \cdot (x_2 \circ x_3) \cdot x_4 + x_1 \cdot ((x_2 \cdot x_3) \circ x_4). \end{aligned}$$

Regrouping the terms, we get:

$$\begin{aligned} N_3(x_1, x_2, N_2(x_3, x_4) &= N_3(x_1, x_2, x_3) \circ x_4 - N(x_1, x_2, x_2) \cdot x_4 \\ &\quad +(x_1 \cdot (x_2 * x_3)) \circ x_4 - x_1 \cdot ((x_2 * x_3) \circ x_4 \\ &= N_2(N_3(x_1, x_2, x_3), x_4) + N_3(x_1, N_2(x_2, x_3), x_4), \end{aligned}$$

which proves the equality. □

4.2.7. **Lemma.** *Let $(A, \cdot, \circ)$ be an compatible associative algebra and let $\{N_n\}_{n\geq 2}$ be the family of products introduced in Definition 4.2.4. For elements $x, y, z \in A$, we have that:*

(1) $N_2(x \cdot y, z) = N_3(x, y, z) + x \cdot N_2(y, z)$.

(2) $N_2(x, y \cdot z) = N_3(x, y, z) + N_2(x, y) \cdot z$.

Proof. The formulas are obtained by a straightforward computation, using the definition of the operations N_ns. □

The following result is immediate to prove.

4.2.8. **Proposition.** *Let A be a compatible associative algebra A. For any family of elements $x_1, \dots, x_n \in A$, we have that:*

(1) $N_2(x_1 \cdot \ldots \cdot x_{n-1}, x_n) = N_n(x_1, \dots, x_n) + x_1 \cdot N_{n-1}(x_2, \dots, x_n) + \dots + x_1 \cdot \ldots \cdot x_{n-2} \cdot N_2(x_{n-1}, x_n)$,

(2) $N_2(x_1, x_2 \cdot \ldots \cdot x_n) = N_n(x_1, \dots, x_n) + N_{n-1}(x_1, \dots, x_{n-1}) \cdot x_n + \dots + N_2(x_1, x_2) \cdot x_3 \cdot \ldots \cdot x_n$,

4.2.9. **Theorem.** *Let $(A, \cdot, \circ)$ be a compatible associative algebra with n-ary operations N_n introduced in Definition 4.2.4. The data $(A, \{N_n\})$ is a $\mathcal{N}$-algebra.*

Proof. We apply Remark 4.2.5 together with Proposition 4.2.6 and Proposition 4.2.8.

Let us prove the relation (1) of Definition 4.2.1. The proofs of the remaining relations follow by similar arguments.

Let A be a compatible associative algebra and consider $x_1, \dots, x_n, x_{n+1} \in A$, with $n \geq 3$. The equality was proved in Proposition 4.2.6 for $n = 3$. Let $n > 3$, by Remark 4.2.5, we have that :

$$N_n(x_1, x_2, \dots, x_{n-1}, N_2(x_n, x_{n+1})) = N_3(x_1, x_2 \cdot \ldots \cdot x_{n-1}, N_2(x_n, x_{n+1})).$$

So, from Proposition 4.2.6, we obtain that

$$\begin{aligned} N_3(x_1, x_2 \cdot \ldots \cdot x_{n-1}, N_2(x_n, x_{n+1})) &= N_2(N_3(x_1, x_2 \cdot \ldots \cdot x_{n-1}, x_n), x_{n+1}) \\ &\quad + N_3(x_1, N_2(x_2 \cdot \ldots \cdot x_{n-1}, x_n), x_{n+1}). \end{aligned}$$

Applying Proposition 4.2.6 and Remark 4.2.5 to the second term, we get:

$$\begin{aligned} &N_3(x_1, N_2(x_2 \cdot \ldots \cdot x_{n-1}, x_n), x_{n+1}) \\ &\qquad = \sum_{i=2}^{n-1} N_3(x_1, x_2 \cdot \ldots \cdot x_{i-1} \cdot N_{n-i+1}(x_i, \dots, x_n), x_{n+1}) \\ &\qquad = \sum_{i=2}^{n-1} N_{i+1}(x_1, x_2, \dots, x_{i-1}, N_{n-i+1}(x_i, \dots, x_n), x_{n+1}). \end{aligned}$$

To end the proof it suffices to apply Remark 4.2.5 to first term, which implies that

$$N_n(x_1, \dots, N_2(x_n, x_{n+1})) = \sum_{i=1}^{n-1} N_{i+1}(x_1, \dots, N_{n-i+1}(x_i, \dots, x_n), x_{n+1}).$$

□

4.2.10. **Theorem.** *Let H be a compatible infinitesimal bialgebra with coproduct Δ. If the elements $x_1, \dots, x_n$ in H are primitive, then $N_n(x_1, \dots, x_n)$ is primitive, too. Therefore, we have that Prim(H) is a $\mathcal{N}$-subalgebra of $(H, \{N_n\})$.*

Proof. The cases $n = 2$ and $n = 3$ are obvious.

Suppose that $n \geq 4$ and that the elements $x_1, \dots, x_n$ are primitive elements in H. Recall that

$$(*)\quad N_n(x_1, \dots, x_n) = N_3(x_1, x_2 \cdot \dots \cdot x_{n-1}, x_n) \\ = (x_1 \cdot x_2 \cdot \dots \cdot x_{n-1}) \circ x_n - x_1 \cdot ((x_2 \cdot \dots \cdot x_{n-1}) \circ x_n).$$

Applying a recursive argument on $n \geq 2$, it is immediate to verify that

$$\Delta(x_1 \cdot \dots \cdot x_n) = \sum_{i=1}^{n-1} (x_1 \cdot \dots \cdot x_i) \otimes (x_{i+1} \cdot \dots \cdot x_n).$$

Applying the formula above to $(*)$ we obtain that :

$$\begin{aligned} \Delta((x_1 \cdot \dots \cdot x_{n-1}) \circ x_n) &= \sum_{i=1}^{n-2} (x_1 \cdot \dots \cdot x_i) \otimes ((x_{i+1} \cdot \dots \cdot x_{n-1}) \circ x_n) \\ &+ (x_1 \cdot \dots \cdot x_{n-1}) \otimes x_n, \end{aligned}$$

and

$$\begin{aligned} \Delta((x_2 \cdot \dots \cdot x_{r-1}) \circ x_r) &= \sum_{i=1}^{n-2} (x_2 \cdot \dots \cdot x_i) \otimes ((x_{i+1} \cdot \dots \cdot x_{n-1}) \circ x_n) \\ &+ (x_2 \cdot \dots \cdot x_{n-1}) \otimes x_n, \end{aligned}$$

for $n \geq 3$.

Therefore, we may conclude that

$$\Delta(x_1 \cdot ((x_2 \cdot \dots \cdot x_{r-1}) \circ x_r)) = \sum_{i=2}^{n-2} (x_1 \cdot \dots \cdot x_i) \otimes ((x_{i+1} \cdot \dots \cdot x_{n-1}) \circ x_n) \\ + (x_1 \cdot \dots \cdot x_{n-1}) \otimes x_n + x_1 \otimes ((x_2 \cdot \dots \cdot x_{n-1}) \circ x_n),$$

and thus $\Delta(N_n(x_1, \dots, x_n)) = 0$, which ends the proof. □

4.2.11. **Remark.** Let V be a vector space. Denote by H_V the free compatible associative algebra $\mathrm{As}^2(V)$ with the compatible infinitesimal bialgebra structure given in Proposition 4.0.3. By remark 4.1.6, $(H_V, \Delta, \cdot)$ is isomorphic as bialgebra to $\overline{T}(\mathrm{Prim}(H_V)$. By identifying the product $\cdot$ with the concatenation product in $\overline{T}(\mathrm{Prim}(H_V)$, we have tha any element $x \in H_V$ is written in a unique way as a linear combination of elements of the type $x_1 \cdot \dots \cdot x_r$, where $x_i \in \mathrm{Prim}(H_V)$, for each $1 \leq i \leq r$.

4.2.12. **Lemma.** *Let V be a vector space and H_V the compatible infinitesimal algebra as in Remark 4.2.11. As $\mathcal{N}$-algebra,* Prim(H_V) *is generated by V*

Proof. Let N_V be the sub-$\mathcal{N}$-algebra of H_V generated by V. Since any element $x \in V$ is a primitive element, $N_V \subseteq \mathrm{Prim}(H_V)$.

Let us see that any element $x \in H_V$ can written as a linear combination of elements of the type $x_1 \cdot \ldots \cdot x_r$, where $x_i \in N_V$, for each $1 \leq i \leq r$. It is sufficient to verify the assertion for any tree $t \in H_V$. We prove the statement by induction on degree of t.

If $|t| = 1$, then $t = a$, for some element a in the basis X of H_V. Suppose $|t| = n$, with $n > 1$.

If t is a reducible tree, then $t = t_1 \cdot \ldots \cdot t_r$, where $r > 1$ and t_i is a tree of degree smaller than n, for each $1 \leq i \leq r$. By a recursive argument, the assertion is true for each t_i, which implies the result for t.

Now, if t is an irreducible tree, then $t = t' \circ a$, where t' is a tree of degree $(n-1)$ and a is an element in the basis X. Applying a recursive argument, t' is a linear combination of elements of the type $x_1 \cdot \ldots \cdot x_r$, where $x_1, \ldots, x_r \in N_V$, with $1 \leq r \leq n-1$. So, $t = t' \circ a$ is a linear combination of elements of the type $(x_1 \cdot \ldots \cdot x_r) \circ a$, where $x_1, \ldots, x_r \in N_V$, with $1 \leq r \leq n-1$ and a an element of the basis X.

Now, consider the operation

$$N_{r+1}(x_1, \ldots, x_r, a) = (x_1 \cdot \ldots \cdot x_r) \circ a - x_1 \cdot ((x_2 \cdot \ldots \cdot x_r) \circ a),$$

applied on the elements $x_1, \ldots, x_r, a$. We have that

$$(x_1 \cdot \ldots \cdot x_r) \circ a = N_{r+1}(x_1, \ldots, x_r, a) + x_1 \cdot ((x_2 \cdot \ldots \cdot x_r) \circ a).$$

Applying a recursive argument to x_1 and $(x_2 \cdot \ldots \cdot x_r) \circ a$, in the right side of the previous equality, we get the result.

By Remark 4.2.11, this implies that $\mathrm{Prim}(H_V)$ is generated, as $\mathcal{N}$-algebra by N_V, which implies that it is generated by V. This ends the proof.

$\square$

4.2.13. **Proposition.** *Let V be vector space and let H_V be the free associative compatible algebra $\mathrm{As}^2(V)$, spanned by V. The $\mathcal{N}$-algebra $\mathrm{Prim}(H_V)$, of primitive elements of H_V, is the free $\mathcal{N}$-algebra on V.*

Proof. Note that, by Lemma 4.2.12, as $\mathcal{N}$-algebra, $\mathrm{Prim}(H_V)$ is graded and generated by V. Denote by $\mathrm{Prim}(H_V)_n$ the subspace of homogeneous elements of degree n of $\mathrm{Prim}(H_V)$.

By Proposition 4.2.3, to see that $\mathrm{Prim}(H_V)$ is the free N-algebra on V it suffices to show that the dimension of $\mathrm{Prim}(H_V)_n$ is equal to $(\mathrm{dim}V)^n c_{n-1}$.

Let us compute the dimension of $\mathrm{Prim}(H_V)_n$. Recall from [26] the linear operator e. Since $(H_V, \cdot, \Delta)$ is a conilpotent infinitesimal bialgebra, we can define $e : H \to H$ given by

$$e(x) = x - x_{(1)} \cdot x_{(2)} - x_{(2)} \cdot x_{(2)} \cdot x_{(3)} + \cdots,$$

where $\Delta(x) = x_{(1)} \otimes \cdots \otimes x_{(n)}$, for all $x \in H_V$. Consider the set

$$B_n = \{e(t) | t \text{ is an irreducible tree of degree } n \text{ in } H_V\}.$$

Let us prove that the set B_n is a basis of $\mathrm{Prim}(H_V)_n$.

From [26], Proposition 2.5, we have that $B_n \subseteq e(H_V) = \mathrm{Prim}(H_V)$ and for any reducible tree $t = t_1 \cdot \ldots \cdot t_r$,

$$e(t) = e(t_1 \cdot \ldots \cdot t_r) = 0.$$

So, $e(\mathrm{Irr}) = e(H_V) = \mathrm{Prim}(H_V)$, because all element $x \in H_V$ can be written as a linear combination of elements in $\bigcup_{n\geq 1} \mathrm{Irr}_n$.

On the other hand, the same result asserts that, if t is a irreducible tree of degree n, then

$$e(t) = t - t_{(1)} \cdot e(t_{(2)}).$$

So, if t_1 and t_2 are different irreducible trees in H_V, then $e(t_1) \neq e(t_2)$. In particular, since the number of irreducible trees of degree n is equal to $(\mathrm{dim} V)^n c_{n-1}$, we have that $|B_n| = (\mathrm{dim} V)^n c_{n-1}$.

Let us see that the set B_n is linearly independent. Note that in particular $|B_n| = c_{n-1}$. To simplify the notation, denote $l = |B_n|$ and let $\{t^1, \ldots, t^l\}$ be the set irreducible trees of degree n in H_V. We have that $B_n = \{e(t^1), \ldots, e(t^l)\}$.

Let $\{\alpha_1, \ldots, \alpha_l\}$ be a family of elements in the field $\mathbb{K}$ and suppose that

$$\alpha_1 e(t^1) + \ldots + \alpha_l e(t^l) = 0.$$

Since $e(t^i) = t^i - t^i_{(1)} \cdot e(t^i_{(2)})$, for any $1 \leq i \leq l$, we have that

$$\alpha_1 t^1 + \ldots + \alpha_l t^l = \sum_{i=1}^{l} \alpha_i (t^i_{(1)} \cdot e(t^i_{(2)})).$$

But this is possible only if $\alpha_i = 0$, for all $1 \leq i \leq l$, because the right side is linear combination of reducible trees. So, B_n is linearly independent and we may conclude that it is a basis of $\mathrm{Prim}(H_V)_n$. The dimension of $\mathrm{Prim}(H_V)_n$ is equal to $(\mathrm{dim} V)^n c_{n-1}$, which ends the proof.

□

Proposition 4.2.13 and Proposition 4.1.6 imply the following structure theorem.

4.2.14. **Theorem.** *Let V be vector space and let $\mathcal{N}(V)$ be the free $\mathcal{N}$-algebra generated by V. The free associative compatible algebra* $\mathrm{As}^2(V)$ *is isomorphic to $T^c(\mathcal{N}(V))$.*

4.2.15. **Remark.** Let $(A, \cdot, \circ, \Delta)$ be a compatible infinitesimal bialgebra. Consider $x * y = \alpha(x \cdot y) + \beta(x \circ y)$ a linear combination of the products $\cdot$ and $\circ$, where α and β are elements in the field $\mathbb{K}$.

A direct compute shows that

$$\Delta(x * y) = x_{(1)} \otimes x_{(2)} * y + x * y_{(1)} \otimes y_{(2)} + (\alpha + \beta) x \otimes y.$$

In particular, when $x*y = x\cdot y - x\circ y$, $(A, \cdot, *, \Delta)$ is a compatible associative algebra with coalgebra structure satisfying:

(1) $\Delta(x \cdot y) = x_{(1)} \otimes x_{(2)} \cdot y + x \cdot y_{(1)} \otimes y_{(2)} + x \otimes y$,
(2) $\Delta(x * y) = x_{(1)} \otimes x_{(2)} * y + x * y_{(1)} \otimes y_{(2)}$.

So, Δ is infinitesimal unitary with respect to the product $\cdot$, in the Loday-Ronco's sense, and infinitesimal with respect to the product $*$, in the Joni-Rota's sense.

This notion of bialgebras is equivalent to the notion of bialgebras that we have given in Definition 4.0.1.

5. Matching Dialgebras

In this section, we consider a particular case of compatible associative algebras, the *matching dialgebras*. In [40], Y. Zhang, Ch. Bai and L. Guo studied the operad of matching dialgebras. They constructed the free matching dialgebras on a vector space V by defining a matching dialgebra structure on the double tensor space $\overline{T}(\overline{T}(V))$.

In the same work, the authors proved that the operad of matching dialgebras is Koszul and compute the complex which gives the homology groups.

The aim of the present section is to study the notion of bialgebras in matching dialgebras.

Motivated by the *path Hopf algebra* $P(S)$ described by A.B. Goncharov in [16], we introduced *bi-matching dialgebras*. We show that the Goncharov's Hopf algebras is part of a family of bi-matching dialgebras, which can be constructed from a bialgebra $(H, \cdot, \Delta)$(in the usual sense) and a *right semi-homomorphism* $R : H \to H$, which is a coderivation with respect to the coproduct Δ.

We also develop the notion of compatible infinitesimal bialgebra in a matching dialgebras. In particular, a free matching dialgebra is a compatible infinitesimal bialgebra, which we obtain another example of a Loday's good triple of operads (see [25]).

5.0.1. **Definition.** A *matching dialgebra* is a vector space A with two associative products $\cdot$ and $\circ$ such that

$$(x \cdot y) \circ z = x \cdot (y \circ z),\ (x \circ y) \cdot z = x \circ (y \cdot z)$$

for all $x, y, z \in A$.

We recall from [40] the notion of *right semi-homomorphism* of algebra.

This type of linear map gives an interesting family of examples of matching dialgebras.

5.0.2. **Definition.** Let $(A, \cdot)$ be an associative algebra. A $\mathbb{K}$-linear map $R : A \to A$ is a *right semi-homomorphism* if it satisfies the condition

$$R(x \cdot y) = R(x) \cdot y,\ \text{for all } x, y \in A.$$

5.0.3. **Remark.** Note that if $(A, \cdot)$ is an associative algebra and $R : A \to A$ is a right semi-homomorphism, then $(A, \cdot, \circ)$ is a matching dialgebra with the product $\circ : A \otimes A \to A$ given by $x \circ y := x \cdot R(y)$ (see [40]).

5.0.4. **Example.** If $(A, \cdot)$ is an associative algebra and a is an element in A, then the map $R : A \to A$ defined as $R(x) := a \cdot x$, for $x \in A$, is a right semi-homomorphism.

In particular, when $(A, \cdot)$ is a unital associative algebra with unit $e \in A$ and $R : A \to A$ is a right semi-homomorphism, then the linear map R is completely determined by the action of R on the unit e. Indeed, if $x \in A$, then $R(x) = R(e \cdot x) = R(e) \cdot x$. So, for the case of unital associative algebra A, any right semi-homomorphism $R : A \to A$ is given by $R(x) := a \cdot x$, where a is some element in A.

5.1. **The free matching dialgebra.** The free matching dialgebra over a vector space V is a quotient of the free compatible associative algebra $\mathrm{As}^2(V)$. In particular, we may define an explicit compatible infinitesimal bialgebra structure on the free objects of the category of the matching dialgebras.

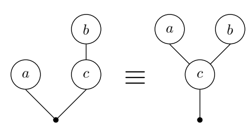
Given a vector space V, with basis X, let T_n^X be the set of planar rooted trees with $(n+1)$ vertices, whose non-root vertices are colored by the elements of X.

In the Subsection 3.2, we define a compatible associative algebra structure on the vector space spanned $\bigcup_{n\geq 1} T_n^X$ of colored planar rooted trees, where X is a basis of V, and proved that $\mathrm{As}^2(V)$ is the free compatible associative algebra over V.

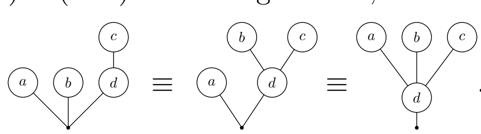
The free matching dialgebra over V may be obtained as the quotient $\mathrm{As}^2(V)$, by the ideal spanned by the elements $(x \cdot y) \circ z = x \cdot (y \circ z)$ and $(x \circ y) \cdot z = x \circ (y \cdot z)$, for x, y and z in V.

We want to find a set of trees which gives a set of representatives of the classes of $\mathrm{As}^2(V)$ modulo these relations.

5.1.1. **Examples.** In low degree, we identify the trees:



because $a \cdot (b \circ c) = (a \cdot b) \circ c$. In degree four, we have that



In the general case, we have the following result.

5.1.2. **Proposition.** *Any tree $t \in T_n^X$, for $n \geq 1$, is equivalent to a tree of the type*

$$t = t^1 \cdot \ldots \cdot t^r,$$

where each t^k is a tree of the form $t^k = a_1^k \circ \ldots \circ a_{n_k}^k$, with $1 \leq k \leq r$ and $n_1 + \ldots + n_r = n$.

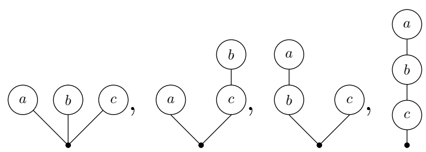
Proof. For $n = 3$, the result was proved in 5.1.1. For $n > 3$, suppose that the assertion is true for any tree of degree strictly less than n. If t is an irreducible tree, then $t = t' \circ a$, with $|t'| = n - 1$ and a an element of degree one. Applying a recursive argument to t', we get

$$t = (t'_1 \cdot \ldots \cdot t'_r) \circ a = t'_1 \cdot \ldots \cdot (t'_r \circ a).$$

If t is a reducible tree, then $t = t' \cdot t''$, where t' and t'' are trees of degree strictly less than n. So, applying the inductive hypothesis to t' and t'', we obtain the assertion for t, which ends the proof. $\square$

5.1.3. **Notation.** We denote by D_n^X the set of all trees of degree n, described in Proposition 5.1.2 and by D^X the set $\bigcup_{n \geq 1} D_n^X$.

For instance, in degree three we have that:

$D_3^X = \{$  $: a, b, c \in X\}.$

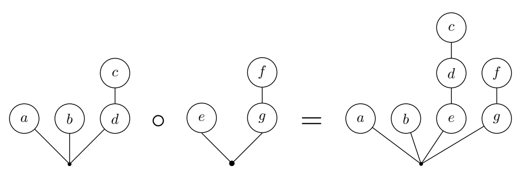
The free matching dialgebras is the vector space $\mathrm{As}_2(V) = \bigoplus_{n \geq 1} \mathbb{K}[D_n^X]$, whose basis is the set D^X.

Given two elements $t = t^1 \cdot \ldots \cdot t^r$ and $w = w^1 \cdot \ldots \cdot w^s$ in D^X of degree n and m, respectively, we have that $t \cdot w$ is the tree in D_{n+m}^X that is obtained by identifying the roots of t and w, while that $t \circ w$ is the tree

$$t \circ w = t^1 \cdot \ldots \cdot t^{r-1}(t^r \circ w^1) \cdot w^2 \ldots \cdot w^s,$$

where $t^r \circ w^1$ is the tree that is obtained by identify the root of t^r with the only leaf of the tree w^1.

For instance,



5.1.4. **Remark.** In [40], Y. Zhang, C. Bai and L. Guo defined the free matching dialgebra over the vector space V as the double tensor space $\overline{T}(\overline{T}(V))$.

For a vector space W, $\overline{T}_*(W) = \bigoplus_{n \geq 1} W^{\otimes n}$ denotes the non-unitary tensor algebra, where the tensor product is denoted by $\otimes_*$.

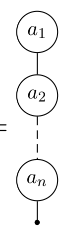
Under this notation, the double tensor space is $\overline{T}_{*_1}(\overline{T}_{*_2}(V))$. The products $\cdot$ and $\cdot$ are defined as follows:

For $u = u_1 \otimes_{*_1} \cdots \otimes_{*_1} u_m$ and $v = v_1 \otimes_{*_1} \cdots \otimes_{*_1} v_n$ in $\overline{T}_{*_1}(\overline{T}_{*_2}(V))$ with $u_i, v_j \in \overline{T}_{*_2}(V)$, for $1 \leq i \leq m$, $1 \leq j \leq n$, define:

(1) $u \cdot v = u_1 \otimes_{*_1} \cdots \otimes_{*_1} u_m \otimes_{*_1} v_1 \otimes_{*_1} \cdots \otimes_{*_1} v_n$, the tensor product $\otimes_{*_1}$.
(2) $u \circ v = u_1 \otimes_{*_1} \cdots \otimes_{*_1} (u_m \otimes_{*_2} v_1) \otimes_{*_1} \cdots \otimes_{*_1} v_n$.

In [40], it is showed that $(\overline{T}_{*_1}(\overline{T}_{*_2}(V)), \cdot, \circ)$ is a matching dialgebra, which is free on the vector space V.

The identification between both versions of the free matching dialgebras is clear. In our description, the tensors of first type are the trees of the type:

$t =$ 

,

where $a_i \in X$, for $1 \leq i \leq n$, are element in the basis X. In this context, we will call *irreducible tree* to the trees of this type.

5.1.5. **Remark.** For $V = \mathbb{K}$, we identify the free matching dialgebra $\mathrm{As}_2(\mathbb{K})$ with the partition algebra

$$C = \bigoplus_{n \geq 1} C^n,$$

where C^n is vector space generated by all the ordered partition of a positive integer n. We denote by $c_{(n_1,\ldots,n_l)}$ the ordered partition $n = n_1 + \cdots + n_l$ of n. The products $\cdot$ and $\circ$ are given respectively by

$$c_{(n_1,\ldots,n_l)} \cdot c_{(m_1,\ldots,m_k)} := c_{(n_1,\ldots,n_l,m_1,\ldots,m_k)}$$

and

$$c_{(n_1,\ldots,n_l)} \circ c_{(m_1,\ldots,m_k)} := c_{(n_1,\ldots,n_l+m_1,\ldots,m_k)}.$$

Since the non-symmetric operad As_2 is completely described by the free matching algebra on $V = \mathbb{K}$, we have that $(\mathrm{As}_2)_n = C^n$. In particular, the dimension of $(\mathrm{As}_2)_n$ is 2^n.

5.2. **Bi-matching dialgebras.** We introduce the notion bi-matching dialgebras. For this, we requires the following proposition, which shows that the operad of matching dialgebras is a Hopf operad.

5.2.1. **Proposition.** *If $(A, \cdot, \circ)$ is a matching dialgebras, then $A \otimes A$ is a matching dialgebras with the products defined by:*

(1) $(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = a_1 \cdot b_1 \otimes a_2 \cdot b_2$ *and*
(2) $(a_1 \otimes a_2) * (b_1 \otimes b_2) = a_1 \cdot b_1 \otimes a_2 \circ b_2 + a_1 \circ b_1 \otimes a_2 \cdot b_2$.

Proof. The proof follows by direct computation. $\square$

5.2.2. **Remark.** Note that in Proposition 5.2.1, the associativity of the product $*$ requires the compatibility condition between the products $\cdot$ and $\circ$.

The following notion of bialgebra was originally introduced by A.B. Goncharov in [16].

5.2.3. **Definition.** A bi-matching dialgebra is a matching dialgebra $(H, \cdot, \circ)$ equipped with a coassociative coproduct $\Delta : H \to H \otimes H$ such that Δ is morphism of matching dialgebras with respect to the matching dialgebra structure of $H \otimes H$ defined in Proposition 5.2.1.

5.2.4. **Proposition.** *Let $(H, \cdot, \Delta)$ be a bialgebra and let $R : H \to H$ be a right semi-homomorphism. If R is a coderivation for the product $\cdot$, then*

$$\Delta(x \circ y) = \Delta(x) * \Delta(y),$$

for any $x, y \in H$, where $x \circ y = x \cdot R(y)$ is the product defined in Remark 5.0.3.

Proof. By a straightforward calculation, we get:

$$\begin{aligned}
\Delta(x \circ y) &= \Delta(x \cdot R(y)) \\
&= \Delta(x) \cdot \Delta(R(y)) \\
&= x_{(1)} \otimes x_{(2)} \cdot (R(y_{(1)}) \otimes y_{(2)} + y_{(1)} \otimes R(y_{(2)})) \\
&= x_{(1)} \cdot R(y_{(1)}) \otimes x_{(2)} \cdot y_{(2)} + x_{(1)} \cdot y_{(1)} \otimes x_{(2)} \cdot R(y_{(2)}) \\
&= x_{(1)} \circ y_{(1)} \otimes x_{(2)} \cdot y_{(2)} + x_{(1)} \cdot y_{(1)} \otimes x_{(2)} \circ y_{(2)} \\
&= \Delta(x) * \Delta(y),
\end{aligned}$$

which proves the formula. $\square$

5.2.5. **Example.** Consider the Grossman-Larson's Hopf algebra $H = \mathbb{K}[\mathcal{T}]$ with basis the set of all non-planar rooted trees $\mathcal{T}$ described in [17]. Recall that the tree e with one vertex is the unit for the product defined in H. Consider the linear map $R : H \to H$ such that, for any rooted tree t, $R(t)$ is the sum of trees obtained from t by attaching one more outgoing edge and vertex to each vertex of t, which is originally defined on the Connes-Kreimer's Hopf algebra in [7].

In [33], Proposition 2.2, F. Panaite showed the linear map R is a right semi-homomorphism for H. In fact, $R(x) = R(e) \cdot x$, for all $x \in H$. In his work F. Panaite showed that R is a coderivation for the coproduct Δ defined in H. Indeed, since $R(e)$ is a primitive element, we have that

$$\begin{aligned}
\Delta(R(x)) &= \Delta(R(e) \cdot x) \\
&= \Delta(R(e)) \cdot \Delta(x) \\
&= (R(e) \otimes e + e \otimes R(e)) \cdot x_{(1)} \otimes x_{(2)} \\
&= R(e) \cdot x_{(1)} \otimes x_{(2)} + x_{(1)} \otimes R(e) \cdot x_{(2)} \\
&= R(x_{(1)}) \otimes x_{(2)} + x_{(1)} \otimes R(x_{(2)}),
\end{aligned}$$

and R is a coderivation. So, $(H, \cdot, \circ, \Delta)$ is a bialgebra, where $\circ$ is the associative product induced by R and the compatibility condition between the products $\cdot$ and $\circ$ with the coproduct Δ is as in Remark 5.2.2.

5.2.6. **Remark.** The previous result obtained by F. Panaite may be generalized to any bialgebra $(H, \cdot, \Delta)$ with unit $e \in H$, that is, if $R : H \to H$ is a right semi-homomorphism and $R(e)$ is a primitive element of H, then R is a coderivation. The proof is similar to that given in the Example 5.2.5.

5.2.7. **Example.** Let $H = \mathbb{K}[X]$ be the $\mathbb{K}$-algebra of polynomial in one variable, with the usual product and the coproduct given by:

$$\Delta(X^n) := \sum_{i=0}^{n} \binom{n}{i} X^{n-i} \otimes X^i,$$

with the homomorphism R defined by $R(X^n) = X^{n+1}$. As $R(1) = X$ is a primitive element, we get that R is a coderivation.

5.3. **The Goncharov's Hopf algebra.** Let us describe the path algebra $P(S)$, introduced by A. B. Goncharov in [16], which motivates our notion of bialgebra, described in Remark 5.2.2.

Let S be a finite set. Denote by $P(S)$ the $\mathbb{K}$-vector space with basis

$$p_{s_0,\ldots,s_n}, \quad \text{for } n \geq 1, \text{ and } s_k \in S, \text{ for } k = 0, \ldots, n.$$

The associative product $\cdot : P(S) \otimes P(S) \to P(S)$ is defined as follows:

$$p_{a,X,b} \cdot p_{c,Y,d} = \begin{cases} p_{a,X,Y,d} & \text{, for } b = c, \\ 0 & \text{, for } b \neq c, \end{cases}$$

where the letters a, b, c, d denote elements, and X and Y denote sequences, possibly empty, of elements of the set S. In particular, $p_{a,b} = p_{a,x} \cdot p_{x,b}$, for $x \in S$, and the unit for this product is the element $e = \sum_{i \in S} p_{i,i}$.

The coproduct $\Delta : P(S) \to P(S) \otimes P(S)$ is given by:

$$\Delta(p_{a,x_1,\ldots,x_n,b}) = \sum_{k=0}^{n} \sum_{\sigma \in Sh(k,n-k)} p_{a,x_{\sigma(1)},\ldots,x_{\sigma(k)},b} \otimes p_{a,x_{\sigma(k+1)},\ldots,x_{\sigma(n)},b}.$$

For instance, $\Delta(p_{a,b}) = p_{a,b} \otimes p_{a,b}$, for $a, b \in S$, and

$$\Delta(e) = \sum_{i \in S} p_{i,i} \otimes p_{i,i} \neq e \otimes e.$$

The linear map $R : P(S) \to P(S)$, given by:

$$R(e) = \sum_{i \in S} p_{i,i,i}$$

is a right semi-homomorphism.

With the definition above, we get that $R(p_{a,X,b}) = p_{a,a,X,b}$, for any element $p_{a,X,b}$ of the basis. So, R induces a new associative product $\circ : P(S) \otimes P(S) \to P(S)$ by setting $x \circ y = x \cdot R(y)$, that is:

$$p_{X,b} \circ p_{c,Y} = \begin{cases} p_{X,b,Y}, & \text{for } b = c, \\ 0, & \text{for } b \neq c, \end{cases}$$

where $b, c \in S$, and X and Y are sequences of elements of S.

5.3.1. **Proposition.** *The right semi-homomorphism $R : P(S) \to P(S)$ is a coderivation.*

Proof. Note that for any element $i \in S$, we have that $\Delta(p_{i,i,i}) = p_{i,i,i} \otimes p_{i,i} + p_{i,i} \otimes p_{i,i,i}$, therefore:

$$\Delta(R(e)) = \sum_{i \in S} p_{i,i,i} \otimes p_{i,i} + p_{i,i} \otimes p_{i,i,i}.$$

Let $x = p_{a,X,b}$ be an element of the basis of $P(S)$. By definition of the coproduct Δ, we have that the element $\Delta(x)$ is a sum of tensors of type

$$p_{a,X',b} \otimes p_{a,X'',b},$$

where X' and X'' are (possibly empty) ordered subsequences of X.

Using the Sweedler' notation, we write

$$\Delta(x) = x_{(1)} \otimes x_{(2)} = p_{a,X_{(1)},b} \otimes p_{a,X_{(2)},b}.$$

Computing $\Delta(R(x))$, we obtain that:

$$\begin{aligned} \Delta(R(x)) &= \Delta(R(e) \cdot x) \\ &= (\textstyle\sum_{i \in S} p_{i,i,i} \otimes p_{i,i} + p_{i,i} \otimes p_{i,i,i}) \cdot x_{(1)} \otimes x_{(2)} \\ &= (\textstyle\sum_{i \in S} p_{i,i,i} \otimes p_{i,i} + p_{i,i} \otimes p_{i,i,i}) \cdot p_{a,X_{(1)},b} \otimes p_{a,X_{(2)},b} \\ &= p_{a,a,a} \cdot p_{a,X_{(1)},b} \otimes p_{a,a} \cdot p_{a,X_{(2)},b} + p_{a,a} \cdot p_{a,X_{(1)},b} \otimes p_{a,a,a} \cdot p_{a,X_{(2)},b} \\ &= p_{a,a,X_{(1)},b} \otimes p_{a,X_{(2)},b} + p_{a,X_{(1)},b} \otimes p_{a,a,X_{(2)},b} \\ &= R(x_{(1)}) \otimes x_{(2)} + x_{(1)} \otimes R(x_{(2)}), \end{aligned}$$

which ends the proof. □

5.4. **Notion of compatible infinitesimal bialgebra in matching dialgebras.** We consider the notion of compatible infinitesimal bialgebra in matching dialgebras. A direct compute shows that this notion of bialgebra is well-defined in a matching dialgebra.

Let $(A, \circ, \Delta)$ be an infinitesimal bialgebra. The product $\circ$ and the coproduct Δ may be extended to $\overline{T}(A) = \bigoplus_{n \geq 1} A^{\otimes n}$ as follows:

(1) $(a_1 \dots a_n) \circ (b_1 \dots b_m) = a_1 \dots a_{n-1}(a_n \circ b_1) b_2 \dots b_m$ and

(2) $\Delta(a_1 \dots a_n) = \sum_{i=1}^{n-1} a_1 \dots a_{i-1} \Delta(a_i) a_{i+1} \dots a_n + \sum_{i=1}^{n-1} a_1 \dots a_i \otimes a_{i+1} \dots a_n$

If we denoted by $\cdot$ the concatenation product in $\overline{T}(A)$, then $(\overline{T}(A), \cdot, \circ)$ is a matching dialgebra, and Δ is infinitesimal for both products.

In particular, consider the free matching dialgebra $\mathrm{As}_2(V) = \overline{T}(\overline{T}(V))$. In this case, $A = \overline{T}(V)$ is an infinitesimal bialgebra with the concatenation product and the deconcatenation coproduct. Identifying the tree $a_1 \circ \ldots \circ a_n$ with a tensor in $\overline{T}(V)$ and the product $\circ$ with the concatenation product, we get:

$$\Delta(a_1 \circ \ldots \circ a_n) := \sum_{i=1}^{n-1} (a_1 \circ \ldots \circ a_i) \otimes (a_{i+1} \circ \ldots \circ a_n).$$

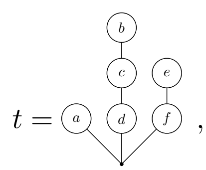
Thus, extending Δ to $\overline{T}(\overline{T}(V))$, we have that $\mathrm{As}_2(V)$ is a compatible infinitesimal bialgebra.

The explicit formula for the coproduct Δ is given by:

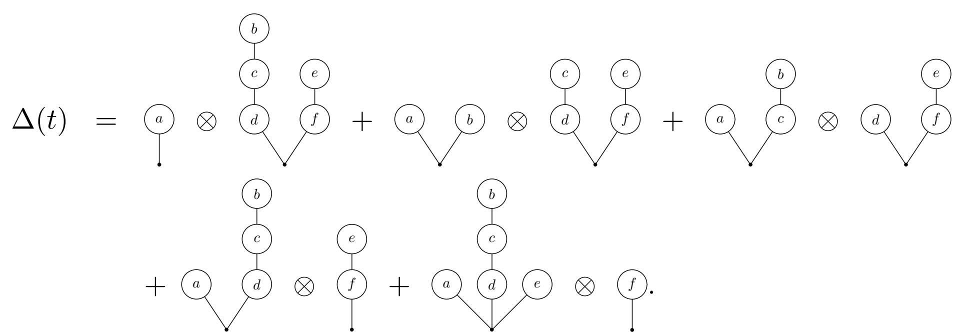
$$\Delta(t) = \sum_{i=1}^{n-1} t_{\{a_1,\ldots,a_i\}} \otimes t_{\{a_{i+1},\ldots,a_n\}},$$

described in Proposition 4.1.4, which extends the deconcatenation coproduct of $\overline{T}(V)$.

5.4.1. **Example.** When t is the tree



the coproduct $\Delta(t)$ is given by:



5.4.2. **Remark.** The primitive part of the compatible infinitesimal bialgebra $\mathrm{As}_2(V)$ is generated by the associative product $*$ given by $x * y = x \circ y - x \cdot y$ together with the $\mathcal{N}$-operations of superior degree. By the compatibility conditions defining a matching dialgebras, the $\mathcal{N}$-algebra structure of $\mathrm{As}_2(V)$ is reduced only associative product $*$. Thus, we get a good triple of operads $(\mathrm{As}, \mathrm{As}_2, \mathrm{As})$, in the Loday's sense (see [25]).

5.5. As-Com$_2$**-operad.** Let us consider the matching dialgebras where one of the associative products is commutative. We denote the associated operad by As-Com$_2$. The associative product is denoted by $\circ$ and the commutative product by $\cdot$.

The following lemma is obtained by a straightforward computation, applying the compatibility condition between the products and the commutativity of the product $\cdot$.

5.5.1. **Lemma.** *If $x_1, x_2, x_3 \in$ As-Com$_2$ and $\sigma \in S_3$, then*

$$(x_{\sigma(1)} \circ x_{\sigma(2)}) \cdot x_{\sigma(3)} = (x_1 \circ x_2) \cdot x_3 = (x_{\sigma(1)} \cdot x_{\sigma(2)}) \circ x_{\sigma(3)}.$$

5.5.2. **Remark.** As an immediate consequence of Lemma 5.5.1, we get that: if $x_1, \ldots, x_n, y \in$ As-Com$_2$, then

$$(x_1 \circ \ldots \circ x_n) \cdot y = (x_1 \circ \ldots \circ x_{i-1} \circ y \circ x_{i+1} \circ \ldots \circ x_n) \cdot x_i,$$

for all $i \in \{1, \ldots, n\}$.

In this way, if $x \in$ As-Com$_2(n)$ is a monomial in the variables $x_1, \ldots, x_n$ of the form $x = (x_{\sigma(1)} \circ \ldots \circ x_{\sigma(i)}) \cdot x_{\sigma(i+1)} \cdot \ldots \cdot x_{\sigma(n)}$ for some $\sigma \in S_n$ and $1 \leq i \leq n-1$, then

$$x = (x_1 \circ \ldots \circ x_i) \cdot x_{i+1} \cdot \ldots \cdot x_n.$$

5.5.3. **Proposition.** *Let $x \in$ As-Com$_2(n)$ be a monomial constructed with the elements $x_1, \ldots, x_n$ and the products $\circ$ and $\cdot$, in such a way that $\circ$ appears k times, for $0 \leq k \leq n-1$:*

(1) If $0 \leq k \leq n-2$, then $x = (x_1 \circ \ldots \circ x_{k+1}) \cdot x_{k+2} \cdot \ldots \cdot x_n$.
(2) If $k = n-1$, then $x = x_{\sigma(1)} \circ \ldots \circ x_{\sigma(n)}$, for some $\sigma \in S_n$.

Proof. For $k = n-1$, the result is immediate.

For $0 \leq k \leq n-2$, x can be written as $x = y_1 \cdot \ldots \cdot y_{n-k}$, where each y_j is a monomial of the type $y_j = x_{i_1} \circ \ldots \circ x_{i_l}$, where $\{i_1, \ldots, i_l\}$ is subset of $\{1, \ldots, n\}$.

Applying Lemma 5.5.2, we may permute the products $\cdot$ and $\circ$, until we obtain that:

$$x = (x_{\sigma(1)} \circ \ldots \circ x_{\sigma(i)}) \cdot x_{\sigma(i+1)} \cdot \ldots \cdot x_{\sigma(n)},$$

for some $\sigma \in S_n$. Therefore, we may conclude that

$$x = (x_1 \circ \ldots \circ x_i) \cdot x_{i+1} \cdot \ldots \cdot x_n.$$

$\square$

5.5.4. **Remark.** Consider $n > 1$ and a monomial $x \in$ As-Com$_2(n)$ constructed with the elements $x_1, \ldots, x_n$ and the products $\circ$ and $\cdot$, in such a way that $\circ$ appears k times, for $0 \leq k \leq n-1$. From Proposition 5.5.3,if we permute the variables or we permute the products $\cdot$ and $\circ$ in the monomial x, we obtain the same element. So, the action of symmetric group S_n is trivial on this element.

5.5.5. **Proposition.**

$$\text{As-Com}_2(n) = 1_n \oplus \cdots \oplus 1_n \oplus \mathbb{K}[S_n],$$

where the sum consist of $(n-1)$ *copies of the trivial representation and one copy of regular representation of* S_n.

Proof. The proof follows from Proposition 5.5.3 and Remark 5.5.4. □

As a consequence of the previous result, we have that the dimension of $\text{As-Com}_2(n)$ is $(n-1)+n!$.

6. The operad As-Com^2

This section is devoted to study the operad of associative compatible algebras such that one of products is commutative. We denote this operad by As-Com^2. So, As-Com^2 is generated by a commutative product $\cdot$ and by an associative product $\circ$, which satisfy compatibility condition of Remark 3.0.2.

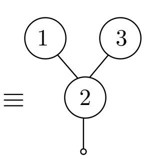
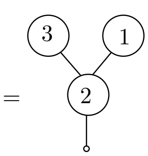
6.1. **Basis for** As-Com^2**.** We want to find a basis for As-Com^2. In order to do this, we previously introduce the notion of n-tree.

6.1.1. **Definition.** Let n be a positive integer. A n-rooted tree, or simply n-tree, is a non-planar rooted tree t with $(n+1)$-vertices, whose vertices different from the root are colored by the set $[n]$, while the root is not colored. We denote the set all the n-trees by T_n and by T the union $\bigcup_{n\geq 1} \mathrm{T}_n$.

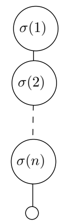
Given a n-tree t, we identify each vertex of t, different of the root, with the element of the set $[n]$ which colors it. The *degree* of a vertex a in the n-tree t, different of the root, is the number of inputs of this vertex and it is denoted by $|a|$.

Given $X = \{a_1, \ldots, a_n\}$ a set of n positive integers such that $a_1 < \cdots < a_n$, a n-tree in the set of vertices X is a non-planar rooted tree t with $(n+1)$ vertices, whose vertices, which are not the root, are colored by the set X. Given a n-tree t, we denote by t_X the n-tree in the set of vertices X such that the vertex i in t is colored by the element a_i in t_X, for $1 \leq i \leq n$.

6.1.2. **Remark.** Since the product $\cdot$ is commutative and the product $\circ$ is associative, we identity each n-tree t with a monomial m_t in As-Com$^2(n)$. For example:

$(x_1 \cdot x_3) \circ x_2 \equiv$  $=$  $\equiv (x_3 \cdot x_1) \circ x_2.$

In particular, for any permutation $\sigma \in S_n$, we denote by t_σ the n-tree,

$t_\sigma =$ 

The n-tree t_σ is associated to the element $x_{\sigma(1)} \circ \ldots \circ x_{\sigma(n)}$ in $\text{As-Com}^2(\text{n})$.

6.1.3. **Lemma.** *Let x, y and z be elements of* As-Com^2, *we get obtain that*

$$(x \cdot y) \circ z = z \circ (x \cdot y) + (x \circ z) \cdot y - x \cdot (z \circ y) - (z \circ x) \cdot y + x \cdot (y \circ z).$$

Proof. The compatibility condition between the products states that :

$$x \circ (y \cdot z) = (x \circ y) \cdot z - x \cdot (y \circ z) + (x \cdot y) \circ z$$

and

$$x \circ (z \cdot y) = (x \circ z) \cdot y - x \cdot (z \circ y) + (x \cdot z) \circ y.$$

While the commutativity of $\cdot$ implies that:

$$(x \cdot y) \circ z = (x \circ z) \cdot y - x \cdot (z \circ y) + (x \cdot z) \circ y + x \cdot (y \circ z) - (x \circ y) \cdot z.$$

As $(z \cdot x) \circ y = z \circ (x \cdot y) - (z \circ x) \cdot y + z \cdot (x \circ y)$, we get that :

$$(x \cdot y) \circ z = z \circ (x \cdot y) + (x \circ z) \cdot y - x \cdot (z \circ y) - (z \circ x) \cdot y + x \cdot (y \circ z),$$

which ends the proof. □

6.1.4. **Proposition.** *If $x, y_1, \ldots, y_n$ are elements in* As-Com^2, *then*

$$\begin{aligned} x \circ (y_1 \cdot \ldots \cdot y_n) = & \sum_{i=1}^{n} ((x \cdot y_1 \cdot \ldots \cdot y_{i-1}) \circ y_i) \cdot y_{i+1} \ldots \cdot y_n \\ & - \sum_{i=2}^{n} x \cdot ((y_1 \cdot \ldots \cdot y_{i-1}) \circ y_i) \cdot y_{i+1} \cdot \ldots \cdot y_n. \end{aligned}$$

Proof. The proof follows by induction on n and by the compatibility condition between the associative products $\cdot$ and $\circ$. □

From the previous reduction formula, we have that $\text{As-Com}^2(n)$ is, as vector space, generated by T_n.

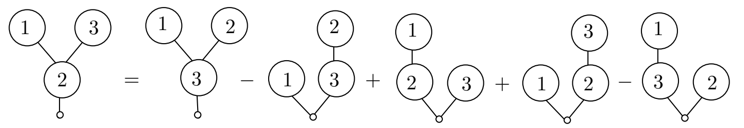
However, the set T_n is not a basis for $\text{As-Com}^2(n)$.

6.1.5. **Example.** Consider the case $n = 3$. $\text{As-Com}^2(3)$ is generated by the elements $(x_1 \cdot x_2) \circ x_3$, $(x_1 \cdot x_2) \circ x_3$, $(x_1 \cdot x_2) \circ x_3$, $(x_{\sigma(1)} \circ x_{\sigma(2)}) \cdot x_{\sigma(3)}$, $x_{\sigma(1)} \circ x_{\sigma(2)} \circ x_{\sigma(3)}$ and $x_1 \cdot x_2 \circ x_3$, where $\sigma \in S_3$.

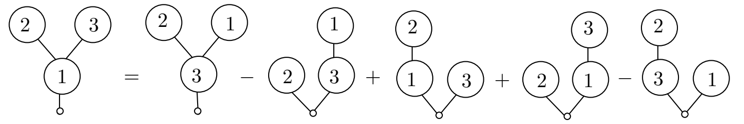
By Lemma 6.1.3, we have that:

$$\begin{aligned} (x_1 \cdot x_3) \circ x_2 &= x_2 \circ (x_1 \cdot x_3) + (x_1 \circ x_2) \cdot x_3 - x_1 \cdot (x_2 \circ x_3) \\ &\quad -(x_2 \circ x_1) \cdot x_3 + x_1 \cdot (x_3 \circ x_2) \\ &= (x_1 \cdot x_2) \circ x_3 - x_1 \cdot (x_2 \circ x_3) + (x_1 \circ x_2) \cdot x_3 \\ &\quad +x_1 \cdot (x_3 \circ x_2) - (x_1 \circ x_3) \cdot x_2. \end{aligned}$$

In terms of trees, we get



and



Moreover, it is clear that the element $(x_1 \cdot x_2) \circ x_3 = (x_2 \cdot x_1) \circ x_3$ can not be reduced. So, the set

$$\{(x_1 \cdot x_2) \circ x_3,\ \ x_1 \cdot x_2 \cdot x_3,\ \ (x_{\sigma(1)} \circ x_{\sigma(2)}) \cdot x_{\sigma(3)},\ \ x_{\sigma(1)} \circ x_{\sigma(2)} \circ x_{\sigma(3)}\},$$

where $\sigma \in S_3$, is a basis for As-Com$^2(3)$. Therefore, the dimension of As-Com$^2(3)$ is equal to 14.

6.1.6. **Remark.** Via the identification described in Remark 6.1.2, each n-tree t can be expressed as $t = t_1 \cdot \ldots \cdot t_r$, where the root of each t_i has exactly one child and $\sum_{i=1}^{r} |t_i| = n$. When $r > 1$, we say that t is a *reducible* n-tree.

The tree t is an *irreducible* n-tree if $r = 1$, that is, the root of t has only one child. In such case, if $n > 1$, $t = t' \circ a$, where $a \in [n]$ and t' is a $(n-1)$-tree in the set of vertices $[n] \setminus \{a\}$.

We denote by Red-T the set of all the reducible n- trees.

6.1.7. **Definition.** Let X, Y be elements in As-Com^2. We say that X is equivalent to Y if $X - Y$ belongs to $\mathbb{K}$[Red-T]. We denote the previous relation by $X \sim Y$.

6.1.8. **Example.** In As-Com$^2(3)$, we have that

$$(x_1 \cdot x_2) \circ x_3 \sim (x_{\sigma(1)} \cdot x_{\sigma(2)}) \circ x_{\sigma(3)},$$

for any permutation $\sigma \in S_3$. More in general, in As-Com$^2(n)$, we have that

$$(x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n-1)}) \circ x_{\sigma(n)} \sim (x_1 \cdot \ldots \cdot x_{n-1}) \circ x_n,$$

for any $\sigma \in S_n$. In fact, if $\sigma(n) = n$, then by the commutativity of the product $\cdot$, we get:

$$(x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n-1)}) \circ x_{\sigma(n)} = (x_1 \cdot \ldots \cdot x_{n-1}) \circ x_n.$$

Suppose that $\sigma(n) = k$, with $k \neq n$. By commutativity of the product $\cdot$, we may assume that $\sigma(n-1) = n$. By Lemma 6.1.3, we have that

$$(x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n-2)} \cdot x_n) \circ x_k \sim x_k \circ (x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n-2)} \cdot x_n).$$

Applying the reduction formula 6.1.4 to the element $x_k \circ (x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n-2)} \cdot x_n)$ and using the commutativity of $\cdot$, we obtain

$$(x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n-1)}) \circ x_{\sigma(n)} \sim (x_1 \cdot \ldots \cdot x_{n-1}) \circ x_n.$$

6.1.9. **Remark.** The relation $\sim$ satisfies the following conditions:

(1) if $X \sim Y$, then $Y \sim X$,
(2) if $X \sim Y$ and $Y \sim Z$, then $X \sim Z$,
(3) if $X \sim Y$ and $Z \sim T$, then $X + Z \sim Y + T$,

for any elements X, Y, Z and T.

Let us describe the method that we will use to construct a basis B_n for As-Com$^2(n)$, using n-trees.

Suppose that we have constructed a basis B_i for As-Com$^2(i)$, which consists of i-trees, for $1 \leq i \leq n-1$. The reducible n-trees in B_n are constructed from the irreducible trees present in the basis $B_1, \dots, B_{n-1}$. Specifically, an reducible n-tree in the basis B_n is given by $t = t^1_{X_1} \cdot \ldots \cdot t^r_{X_r}$, where $[n] = \bigcup_{i=1}^{r} X_r$ is a partition of $[n]$, with $r \geq 2$, and $t^i \in B_{|X_i|}$ is an irreducible $|X_i|$-tree, for $1 \leq i \leq r$.

The set of n-trees constructed in this way form a linearly independent set in As-Com$^2(n)$. To complete the basis B_n, we consider a linearly independent set of irreducible n-tree such that if t is an irreducible n-tree, then t is equivalent to a linear combination of these trees under the relation $\sim$ defined in 6.1.7. Since these elements do not belong to the subspace generated by the reducible n-trees in B_n, the set B_n is linearly independent.

Moreover, B_n will generate As-Com$^2(n)$. In fact, by induction, if $t = t_1 \cdot \ldots \cdot t_r$ is a reducible n-tree, then t can be expressed as a linear combination of n-trees of the type $t = t^1_{X_1} \cdot \ldots \cdot t^r_{X_r}$, where X_i is the set of vertices of t_i and $t^i \in B_{|X_i|}$ is an irreducible $|X_i|$-tree, for $1 \leq i \leq r$.

So, we can generate all the reducible n-trees by mean of elements of B_n. Now, if t is an irreducible n-tree, then by construction, t is equivalent to a linear combination of irreducibles n-trees in B_n. So, since each reducible n-tree is generated by B_n, we have that t is generated by B_n. This shows that B_n is a basis for As-Com$^2(n)$.

6.1.10. **Lemma.** *Let x, y, z and w be elements of* As-Com^2. *We have that*

$$(x \cdot y) \circ z \circ w \sim w \circ (x \cdot y) \circ z$$

.

Proof. Note that, by the compatibility condition, we have:

$$z \circ (y \cdot w) \circ x = ((z \circ y) \cdot w) \circ x - (z \cdot (y \circ w)) \circ x + (z \cdot y) \circ w \circ x$$

and

$$z \circ (w \cdot y) \circ x = ((z \circ w) \cdot y) \circ x - (z \cdot (w \circ y)) \circ x + (z \cdot w) \circ y \circ x.$$

Therefore, we get that:

$$((z \circ y) \cdot w) \circ x - (z \cdot (y \circ w)) \circ x + (z \cdot y) \circ w \circ x =$$
$$((z \circ w) \cdot y) \circ x - (z \cdot (w \circ y)) \circ x + (z \cdot w) \circ y \circ x.$$

On the other hand, consider the following equivalences :

- $((z \circ y) \cdot w) \circ x \sim (x \cdot w) \circ z \circ y$,
- $(z \cdot (y \circ w)) \circ x \sim (x \cdot (y \circ w)) \circ z$,
- $(z \cdot y) \circ (w \circ x) \sim ((w \circ x) \cdot y) \circ z$,
- $((z \circ w) \cdot y) \circ x \sim (x \cdot y) \circ z \circ w$,
- $(z \cdot (w \circ y)) \circ x \sim (x \cdot (w \circ y)) \circ z$,
- $(z \cdot w) \circ (y \circ x) \sim ((y \circ x) \cdot w) \circ z$,

they imply that

$$\begin{aligned} (x \cdot w) \circ z \circ y \quad &\sim \quad (x \cdot y) \circ z \circ w - (x \cdot (w \circ y)) \circ z \\ &\quad +((y \circ x) \cdot w) \circ z + (x \cdot (y \circ w)) \circ z - ((w \circ x) \cdot y) \circ z. \end{aligned}$$

In a similar way, given that $x \circ (z \cdot w) \circ y = x \circ (w \cdot z) \circ y$, we have that:

$$\begin{aligned} (x \cdot w) \circ z \circ y \quad &\sim \quad (y \cdot w) \circ x \circ z - (x \cdot y) \circ z \circ w \\ &\quad +(x \cdot (w \circ y)) \circ z + (x \cdot y) \circ w \circ z - ((x \circ w) \cdot y) \circ z. \end{aligned}$$

Using the equivalences above, we obtain

$$\begin{aligned} 2(x \cdot y) \circ z \circ w \quad &\sim \quad (y \cdot w) \circ x \circ z + 2(x \cdot (w \circ y)) \circ z - ((x \circ w) \cdot y) \circ z \\ &\quad +(x \cdot y) \circ w \circ z - ((y \circ x) \cdot w) \circ z - (x \cdot (y \circ w)) \circ z \\ &\quad +((w \circ x) \cdot y) \circ z. \end{aligned}$$

But, as

$$\begin{aligned} (x \cdot y) \circ w \circ z = {} & (x \cdot w) \circ y \circ z + ((x \circ w) \cdot y) \circ z \\ & +((y \circ w) \cdot x) \circ z - ((x \circ y) \cdot w) \circ z - ((w \circ y) \cdot x) \circ z \end{aligned}$$

and

$$\begin{aligned} (y \cdot w) \circ x \circ z \quad &= \quad (x \cdot w) \circ y \circ z + ((y \circ x) \cdot w) \circ z \\ &\quad +((w \circ x) \cdot y) \circ z - ((w \circ y) \cdot x) \circ z - ((x \circ y) \cdot w) \circ z, \end{aligned}$$

we may conclude that

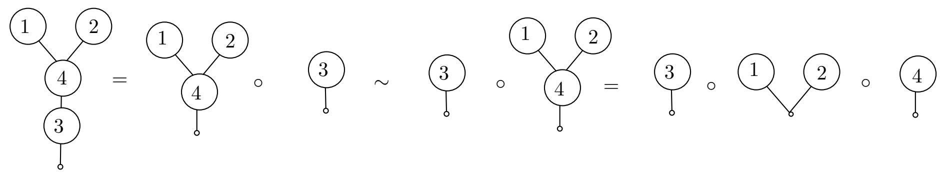
$$2(x \cdot y) \circ z \circ w \sim 2(x \cdot w) \circ y \circ z - 2((x \circ y) \cdot w) \circ z + 2((w \circ x) \cdot y) \circ z.$$

and, we get the expected result

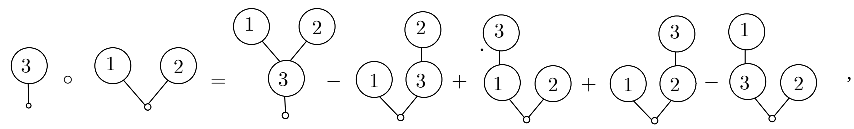
$$\begin{aligned} (x \cdot y) \circ z \circ w \quad &\sim \quad ((w \circ x) \cdot y - w \cdot (x \circ y) + (w \cdot x) \circ y) \circ z \\ &\sim \quad w \circ (x \cdot y) \circ z. \end{aligned}$$

$\square$

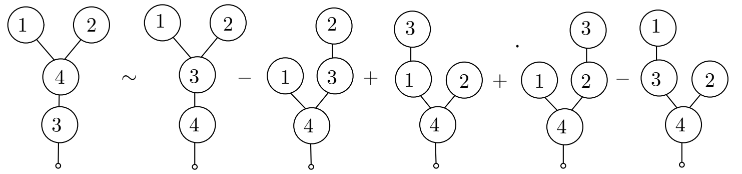
6.1.11. **Example.** In degree 4, we have that:



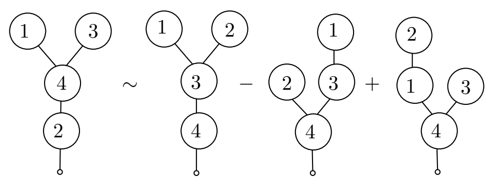
As



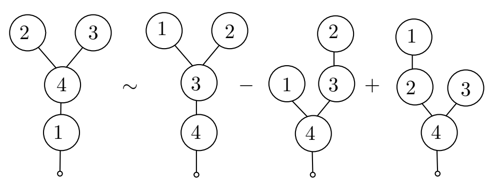
we obtain that:



In a similar way, as $(1 \cdot 3) \circ 4 \circ 2 \sim 2 \circ (1 \cdot 3) \circ 4$, we have that:



and, as $(2 \cdot 3) \circ 4 \circ 1 \sim 1 \circ (2 \cdot 3) \circ 4$, we get



6.1.12. **Definition.** A n-tree t is a B-tree whenever it satisfies one of the following conditions:

(1) $t = t_\sigma$, for some $\sigma \in S_n$, or
(2) the vertices of t satisfy the following condition. If a is a vertex of t, different of the root, such that $|a| \geq 2$, then:
  (a) all the vertices that are above the vertex a are smaller than a, and
  (b) the direct path from the vertex a to the root is increasing.

6.1.13. **Notation.** We denote the set B-trees of degree n by B_n.

For example, B_3 is the basis for As-Com$^2(3)$ described in Example 6.1.5.

6.1.14. **Remark.** Suppose that $t = t_1 \cdot \ldots \cdot t_r$ is a tree in B_n. Denote by $X = [n]$ and by X_i the set vertices of the tree t_i, for each $i \in \{1, \ldots, r\}$.

We have that t_i is an irreducible B-tree in the set of vertices X_i, for each $i \in \{1, \ldots, r\}$. So, we can see each B-tree t as a forest of irreducible B-trees.

In particular, for $r = 1$ and $n > 1$, we have that $t = t' \circ a$. By the conditions on the vertices in a B-tree, for $a \neq n$, necessarily $t = t_\sigma$, for some $\sigma \in S_n$.

On the other hand, if $a = n$, then $t' \in B_{n-1}$. Thus, if $t \in \mathrm{B}_n$ is an irreducible n-tree, then $t = t_\sigma$, for some $\sigma \in S_n$, or $t = t' \circ n$, for some $t' \in \mathrm{B}_{n-1}$.

Moreover, note that, in the second case, if $t = t' \circ n$, then there exists $k \in [n]$ such that $t = (s_1 \cdot \ldots \cdot s_l) \circ k \circ (k+1) \circ \ldots \circ n$, where $k < k+1 < \cdots < n$ are consecutive integers and $s_1 \cdot \ldots \cdot s_l \in \mathrm{B}_{k-1}$.

6.1.15. **Proposition.** *If t is a n-tree, then t can be reduced to a linear combination of trees in* B_n.

Proof. The proof is by induction on the degree of the tree t.

Let n be the degree of t. If $n = 1$ or $n = 2$ the assertion is obvious.

For $n = 3$, B_3 is a basis for As-Com$^2(3)$.

Consider $n > 3$ and suppose the result is true for $1 \leq r < n$. Let t be an irreducible n-tree. In such case, $t = t' \circ a$, where $a \in [n]$ and t' is a $(n-1)$-tree in the set of vertices $[n] \setminus \{a\}$.
Suppose that t is not a B-tree. So, $a \neq n$ and $t \neq t_\sigma$, for all $\sigma \in S_n$. By inductive hypothesis, t' can be reduced to a linear combination of B-trees in the set of vertices $[n] \setminus \{a\}$, whose reduction is derived from the relations between the products $\cdot$ and $\circ$.

So, t can be expressed as a linear combination of elements of the type $s \circ a$, where s is B-tree in the set of vertices $[n] \setminus \{a\}$. Let us show that these elements may be written as a linear combination of elements in B_n.

If s is an irreducible tree, then $s = (w_1 \cdot w_2) \circ w_3$, where $w_1 \cdot w_2$ is a B-tree in some subset of $[n] \setminus \{a\}$, and $w = a_1 \circ \ldots \circ a_k \circ n$, with $a_1 < \cdots < a_k < n$. By Lemma 6.1.10, we have that

$$s \circ a \sim a \circ (w_1 \cdot w_2) \circ w_3 = (a \circ (w_1 \cdot w_2) \circ (a_1 \circ \ldots \circ a_k)) \circ n.$$

So, $s \circ a = (a \circ (w_1 \cdot w_2) \circ (a_1 \circ \ldots \circ a_k)) \circ n +$some linear combination of reducible trees. Since the monomial $a \circ (w_1 \cdot w_2) \circ (a_1 \circ \ldots \circ a_k)$ is a linear combination of $(n-1)$-trees, we apply the inductive hypothesis of this element.

In similar way, we apply the inductive hypothesis to each child of the reducible trees present in the decomposition of $s \circ a$. So, we have that $s \circ a$ is linear combination of elements of B_n.

Assume now that s is a reducible tree and write $s = s_1 \cdot \ldots \cdot s_l$, where $l > 1$ and each s_i is an irreducible B-tree in some subset of $[n] \setminus \{a\}$, with $1 \leq i \leq l$. By the commutativity of $\cdot$, we may assume that n is a vertex of

s_l. By Lemma 6.1.3, we have that:

$$(s_1 \cdot \ldots \cdot s_l) \circ a = ((s_1 \cdot \ldots \cdot s_{l-1}) \cdot s_l) \circ a \sim a \circ ((s_1 \cdot \ldots \cdot s_{l-1}) \cdot s_l).$$

Using the compatibility condition between the products $\cdot$ and $\circ$, we get

$$a \circ ((s_1 \cdot \ldots \cdot s_{l-1}) \cdot s_l) = (a \circ (s_1 \cdot \ldots \cdot s_{l-1})) \cdot s_l - a \cdot ((s_1 \cdot \ldots \cdot s_{l-1}) \circ s_l) \\ +(a \cdot s_1 \cdot \ldots \cdot s_{l-1}) \circ s_l.$$

For the term $(a \cdot s_1 \cdot \ldots \cdot s_{l-1}) \circ s_l$, when $s_l = s' \circ n$, we may apply a recursive argument.
If s_l is of the form $s_l = a_1 \circ \ldots \circ n \circ \ldots \circ a_k$, we apply Lemma 6.1.10, and afterwards a recursive argument, as in the previous case.

For the other terms, we use a similar recursive argument. Therefore, t is a linear combination of elements of B_n.

Consider a reducible n-tree $t = t_1 \cdot \ldots \cdot t_r$, where t_i is an irreducible tree, for $1 \leq i \leq r$. Applying the result obtained in the previous paragraph to each t_i, we get that t is a linear combination of reducible n-trees in B_n, which ends the proof.

□

6.1.16. **Theorem.** *The set* B_n *is a basis for* As-Com$^2(n)$.

Proof. By Proposition 6.1.15, each n-tree may be written as a linear combination of n-trees in B_n. So, since As-Com$^2(n)$ is generated by T_n, B_n generates As-Com$^2(n)$.

Let us show that the set B_n is linearly independent for each positive integer n. The proof is by induction on n.

The case $n = 1$ and $n = 2$ are obvious and in Example 6.1.5 we have seen that B_3 is basis of As-Com$^2(3)$. Consider $n > 3$ and suppose that the assertion is true for all $k < n$.

First, let us see that the set of reducible n-trees in B_n is linearly independent. Suppose that such set is not linearly independent. In this case, there exists a reducible n-tree

$$t = t_1 \cdot \ldots \cdot t_r \in \mathrm{B}_n,$$

which is a linear combination of the others elements of B_n. But in this case, at least one t_i must be reducible, for some $1 \leq i \leq r$, which contradicts our hypothesis.

By other hand, if $\mathrm{B}_{n-1} = \{t_1, \ldots, t_l\}$, then the set of trees $\{t_1 \circ n, \ldots, t_l \circ n\}$ is linearly independent because the set B_{n-1} is linearly independent.

By Remark 6.1.14, the set of irreducible n-trees in B_n is described by:

$$\mathrm{A}_n = \{t \circ n | t \in \mathrm{B}_{n-1}\} \cup \{t = t_\sigma | \sigma \in S_n, \sigma(n) \neq n\},$$

we have that A_n is linearly independent. An element of the last cannot be a linear combination of reducible n-trees in B_n. So, B_n is linearly independent, which ends the proof.

□

6.2. **Dimension of** $\mathrm{As\text{-}Com}^2(n)$**.** Let us denote by A_n the set of the irreducible n-trees in B_n, and by b_n and a_n the number of elements of B_n and A_n, respectively .

From Remark 6.1.14, we have that if $t \in \mathrm{A}_n$, then $t = t_\sigma$, for some $\sigma \in S_n$, or $t = t' \circ n$, for some $t' \in B_{n-1}$. Let us denote by A_n^1 and A_n^2 the subsets of A_n given by

$$\mathrm{A}_n^1 = \{t \in \mathrm{T}_n | t = t_\sigma, \text{ for some } \sigma \in S_n, \sigma(n) \neq n\}$$

and

$$\mathrm{A}_n^2 = \{t \in \mathrm{T}_n | t = t' \circ n, t' \in \mathrm{B}_{n-1}\}.$$

The set A_n is the disjoint union of the sets A_n^1 and A_n^2. So, we get the equation

$$(*) \quad a_{n+1} = b_n + n \cdot n!.$$

Let us denote by $A(x)$ and $B(x)$ the exponential generating functions, associated to the sequences a_n and b_n, respectively.

As any element of B_n is a forest of B-trees (Remark 6.1.14), we have that $B(x) = e^{A(x)}$. Thus, the equation $(*)$ is equivalent, in terms of generating functions, to

$$A'(x) = B(x) + C(x),$$

where $B(x) = e^{A(x)}$ and $C(x) = \dfrac{x}{(1-x)^2}$.

As $B'(x) = e^{A(x)} A'(x)$, we have that

$$B'(x) = B(x)^2 + B(x)C(x).$$

Given that the sequences associated to exponential generating functions $B(x)^2$ and $B(x)C(x)$ are, respectively,

$$d_n = \sum_{k=0}^{n} \binom{n}{k} b_k b_{n-k}, \quad e_n = \sum_{k=0}^{n} \binom{n}{k} b_k (n-k)(n-k)!,$$

we obtain an explicit recursive formula to determine b_n, which is

$$b_{n+1} = \sum_{k=0}^{n} \binom{n}{k} b_k [b_{n-k} + (n-k)(n-k)!],$$

where $b_0 = 1$. In particular, the dimensions of $\mathrm{As\text{-}Com}^2(n)$ in low degree are given by the sequence $1, 3, 14, 85, 632, 5559, 56444, 649557, 8353352, 118712191$.

7. Compatible commutative algebras

In this section, we study the operad generated by two commutative products, which are compatible. The algebras over this operad will be called *compatible commutative algebras*. Following the notation given by Strohmayer in [37], we denote this operad by Com^2.

7.1. **The dual of** Com^2**.** Let us compute the Koszul dual operad of Com^2. The operad Com^2 is generated by two commutative products, which are denoted by $\cdot$ and $\circ$, respectively, which are compatible. From the compatibility condition given in Remark 3.0.2 and the commutativity of the products, we have that:

$$\begin{aligned}(x_1 \cdot x_2) \circ x_3 + (x_1 \circ x_2) \cdot x_3 &= (x_1 \cdot x_3) \circ x_2 + (x_1 \circ x_3) \cdot x_2 \\ &= (x_2 \cdot x_3) \circ x_1 + (x_2 \circ x_3) \cdot x_1.\end{aligned}$$

So, the space of relations is generated by :

$$\begin{aligned}\mu_1 &= (x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3), \\ \mu_2 &= (x_1 \circ x_2) \circ x_3 - x_1 \circ (x_2 \circ x_3), \\ \mu_3 &= (x_1 \cdot x_2) \circ x_3 + (x_1 \circ x_2) \cdot x_3) - (x_1 \cdot x_3) \circ x_2 - (x_1 \circ x_3) \cdot x_2), \\ \mu_4 &= (x_1 \cdot x_2) \circ x_3 + (x_1 \circ x_2) \cdot x_3) - (x_2 \cdot x_3) \circ x_1 - (x_2 \circ x_3) \cdot x_1).\end{aligned}$$

From the commutativity of the products, the Koszul dual of Com^2 is generated by two Lie brackets $\cdot_\vee$ and $\circ_\vee$, which are determined by the products $\cdot$ and $\circ$, respectively. To determine the rest of relations, we consider the natural pairing between the space of relations. Note that it is sufficient to consider elements of the type $(x \cdot_\vee y) \circ_\vee z$ and $(x \circ_\vee y) \cdot_\vee z$.

The inner product with respect to the relations μ_3 and μ_4 is given, for each element of the previous type, by:

$$\begin{array}{lcl} < (x_1 \cdot_\vee x_2) \circ_\vee x_3, \mu_3 >= 1 & , & < (x_1 \cdot_\vee x_2) \circ_\vee x_3, \mu_4 >= 1, \\ < (x_1 \cdot_\vee x_3) \circ_\vee x_2, \mu_3 >= 1 & , & < (x_1 \cdot_\vee x_3) \circ_\vee x_2, \mu_4 >= 0, \\ < (x_2 \cdot_\vee x_3) \circ_\vee x_1, \mu_3 >= 0 & , & < (x_3 \cdot_\vee x_2) \circ_\vee x_1, \mu_4 >= 1, \\ < (x_1 \circ_\vee x_2) \cdot_\vee x_3, \mu_3 >= 1 & , & < (x_1 \circ_\vee x_2) \cdot_\vee x_3, \mu_4 >= 1, \\ < (x_1 \circ_\vee x_3) \cdot_\vee x_2, \mu_3 >= 1 & , & < (x_1 \circ_\vee x_3) \cdot_\vee x_2, \mu_4 >= 0, \\ < (x_2 \circ_\vee x_3) \cdot_\vee x_1, \mu_3 >= 0 & , & < (x_3 \circ_\vee x_2) \cdot_\vee x_1, \mu_4 >= 1. \end{array}$$

Therefore, the space of relations for the Koszul dual operad of Com^2 is given by the relations of the Lie brackets $\cdot_\vee$ and $\circ_\vee$, together with elements of the type: $(x \cdot_\vee y) \circ_\vee z - (x \circ_\vee y) \cdot_\vee z$ and $(x \cdot_\vee y) \circ_\vee z + (y \cdot_\vee z) \circ_\vee x + (z \cdot_\vee x) \circ_\vee y$.

The Koszul dual of Com^2 as an operad generated by two Lie brackets $[,]$ and $\{,\}$ whose relations are given by:

(1) $\{[x, y], z\} = [\{x, y\}, z]$.
(2) $\{[x, y], z\} + \{[y, z], x\} + \{[z, x], y\} = 0$.

7.1.1. **Notation.** Following the notation of H. Strohmayer, we denote the Koszul dual of Com^2 by ${}^2\mathcal{L}ie$.

7.1.2. **Proposition.** *${}^2\mathcal{L}ie(n) = \mathcal{L}ie(n) \oplus \cdots \oplus \mathcal{L}ie(n)$, where the sum consists of n terms.*

Proof. Let us denote by $L(i, n-i)$ the subspace of ${}^2\mathcal{L}ie(n)$ of the operations in n variables with i brackets of type $[,]$ and $(n-i)$ brackets of type $\{,\}$, where i is an integer $0 \leq i \leq n-1$.

This subspace is generated by binary trees with i vertices decorated by the brackets $[,]$ and $(n-i)$ vertices decorated by the brackets $\{,\}$.

From the relation $\{[x,y],z\} = [\{x,y\},z]$, if we have a tree of the previous type and we interchange the place of the brackets, the operation obtained is equivalent to the original one. The previous remark together with the relation $\{[x,y],z\}+\{[y,z],x\}+\{[z,x],y\} = 0$ shows that $L(i, n-i) = \mathcal{L}ie(n)$ for each $0 \leq i \leq n-1$.

Therefore, as ${}^2\mathcal{L}ie(n) = \bigoplus_{i=1}^{n-1} L(i, n-i)$, we get the expected result. $\square$

As an immediate consequence from the previous proposition, we have the following corollary .

7.1.3. **Corollary.** *The dimension of* ${}^2\mathcal{L}ie(n)$ *is equal to* $n!$.

Proof. It is immediate, since the dimension of $Lie(n)$ is equal to $(n-1)!$. $\square$

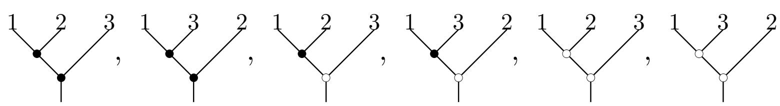
7.2. **A PBW basis for** ${}^2\mathcal{L}ie$**.** We want to construct a PBW basis, in the sense given by E. Hoffbeck in [18], for ${}^2\mathcal{L}ie$.

To construct this basis, we start by defining an order on the brackets $[,]$ and $\{,\}$. Using this order, it is easy to choose a basis in the quadratic part such that it satisfies the conditions required for a PBW basis.

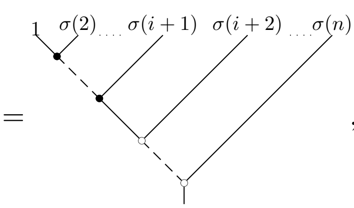
The elements of the basis in superior degree will be determined by the condition that any subtree generated by internal edge is a element of the basis in the quadratic part. Finally, we must check that this set is effectively a basis (for more details, see [18]).

We use the lexicographical order and define $[,] > \{,\}$. If t is a tree, a vertex of t decorated by black color represents the bracket $[,]$ while a vertex decorated by white color represents the bracket $\{,\}$.

We check easily that, in the quadratic part, the following set is a basis and it satisfies the PBW conditions stated in [18]:



In higher degree, the elements are given by trees of the type:

$t_i^\sigma =$ 

,

where $0 \leq i \leq n-1$ and σ is any permutation of the set $\{2, \ldots, n\}$.

It is clear, from the description of operad ${}^2\mathcal{L}ie$, that this set generates ${}^2\mathcal{L}ie(n)$. Furthermore, as the number of elements is equal to $n!$, it is a basis of ${}^2\mathcal{L}ie(n)$.

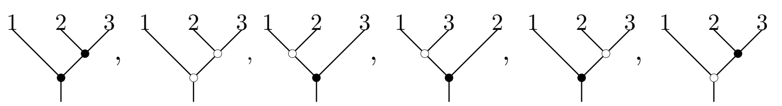
Applying the result obtained for E. Hoffbeck in [18], Theorem 3.10, we get as a consequence the following proposition:

7.2.1. **Proposition.** *The operad* ${}^2\mathcal{L}ie$ *is Koszul. Therefore, the operad* Com^2 *is Koszul, too.*

8. A PBW basis for Com^2

E. Hoffbeck has showed that if a quadratic operad P has a PBW basis, then its Koszul dual operad $P^!$ has a PBW basis, whose quadratic part is determined by the quadratic part of the PBW basis of P (see [18]). So, since Com^2 is the Koszul dual operad of ${}^2\mathcal{L}ie$, we can construct an explicit PBW basis for Com^2, using the PBW basis found for ${}^2\mathcal{L}ie$.

We use the black color to represent the product $\cdot$, dual of the bracket $[,]$, and the white color to represent the product $\circ$, dual of the bracket $\{,\}$. So, the quadratic part of the PBW basis is given by:

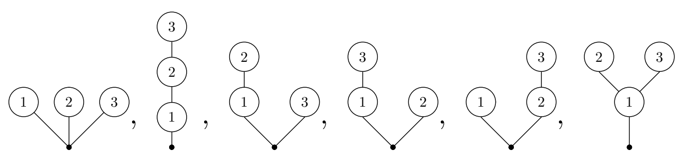


The elements of superior degree will be determined by the condition that any subtree generated by internal edge is in quadratic part of the basis.

8.0.1. **Remark.** Given the variables x_1, x_2, x_3, the previous set of trees represents the set of operations:

$$\{x_1{\cdot}(x_2{\cdot}x_3),\ \ x_1{\circ}(x_2{\circ}x_3),\ \ (x_1{\circ}x_2){\cdot}x_3,\ \ (x_1{\circ}x_3){\cdot}x_2,\ \ x_1{\cdot}(x_2{\circ}x_3),\ \ x_1{\circ}(x_2{\cdot}x_3)\}.$$

Now, since the operad Com^2 is a quotient of the operad As^2, the quadratic elements of the basis can be described by means the following planar rooted trees:

$(*)$ 

In this context, let us describe the PBW condition on the trees of higher degree.

Consider $n > 3$ and let t be a planar rooted tree with n vertices, different to the root, which are colored by the set $[n]$.

A tree t belongs to the PBW basis if any subtree t' of t with three vertices, decorated by set $\{a_1, a_2, a_3\}$, with $1 \leq a_1 < a_2 < a_3 \leq n$, is one of the trees described in $(*)$, where the integer j is replaced by the element a_j, for $j = 1, 2, 3$.

In particular, the vertex of t labelled by n is necessarily a leaf such that it is more to the right with respect to the other vertices that have the same origin that n.

Furthermore, the subtree of t that is obtained by removing the vertex colored by n, is an element of the basis of degree $(n-1)$.

From the previous observations, we construct inductively the PBW basis for $\mathrm{Com}^2(n)$ as follow.

In degree one, the basis has an only one element, the unique tree with two vertices, the root and the vertex colored by 1.

Suppose that $n > 1$ and that we have constructed the basis for $\mathrm{Com}^2(n-1)$. An element in the basis of $\mathrm{Com}^2(n)$ is a tree t, which is obtained from a tree t' in the PBW basis of degree $(n-1)$ by gluing the vertex colored by n with a vertex of t' in such a way that the vertex colored by n is on the right with respect to the other children of the vertex chosen of t' .

8.0.2. **Proposition.** *The dimension of* $\mathrm{Com}^2(n)$ *is equal to* $n!$.

Proof. We prove the result by induction on n. The dimension of $\mathrm{Com}^2(1)$ is one.

Suppose that the dimension of $\mathrm{Com}^2(n)$ is $n!$. From Remark 8.0.1, we have that, if t is a tree in the basis of $\mathrm{Com}^2(n)$,then an element in the basis of $\mathrm{Com}^2(n+1)$ is obtained by gluing a new vertex colored with $(n+1)$ to some vertex of t.

So, each tree t in the basis of $\mathrm{Com}^2(n)$ gives rise to $(n+1)$ different elements in the basis of $\mathrm{Com}^2(n+1)$.

Moreover, note that, if t_1 and t_2 are different trees in the basis of $\mathrm{Com}^2(n)$, then all the trees obtained from them are different.

So, the number of trees in the basis of $\mathrm{Com}^2(n+1)$ is $(n+1)n! = (n+1)!$, which concludes the proof. $\square$

8.0.3. **Remark.** We characterize an element t in the PBW basis of degree n as a rooted planar tree with n vertices, different to the root, which are labelled by the set $[n]$, satisfying the following conditions:

(1) Any path of t starting at the root is increasing.
(2) The vertices of t that have a same origin are in increasing order (read from left to right).

As the product $\cdot$ is commutative, the trees of the PBW basis of Com^2 can be seen as non-planar rooted trees. So, if additionally we considerer the root colored by zero, the elements of PBW basis are known as *increasing Cayley trees* or *recursive trees*. It is known that the number of increasing trees with $(n+1)$ vertices is $n!$ (see, for example, [11]).

8.0.4. **Notation.** We denote the set of elements of the PBW basis of Com^2 with n vertices, different of the root, by I_n.

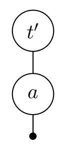
A planar rooted tree t with n vertices, different of the root, colored by a set of n positive integers, $a_1 < \cdots < a_n$, which satisfies the conditions of Remark 8.0.3, is called an *I-tree*. In such case, we say that t is of degree n and we write $|t| = n$.

Given an I-tree t, we identify the set of vertices of t, different of the root, with the set of numbers by which decorate them, denoting this set by $\mathrm{Vert}(t)$.

8.0.5. **Remark.** Note that if t is an I-tree, then t can be written in a unique way as $t = t^1 \cdot \ldots \cdot t^r$, where $r \geq 1$ and the root of each t^i is I-tree which has only one child, for each $i \in \{1, \ldots, r\}$.

In this way, we get that $|t| = \sum_{i=1}^{r} |t_i|$.

When the root of a I-tree t has a unique child, we say that t is an *irreducible I-tree.* In such case, the tree t has the form

$t =$  ,

where a is a positive integer and t' is an I-tree such that all the vertices of t' are bigger than a. Moreover, we have that $t = a \circ t'$.

8.0.6. **Definition.** Let t be a I-tree. We define the *index* of t as the minimal element the set $\mathrm{Vert}(t)$, which we denote by $\mathrm{ind}(t)$.

8.0.7. **Remark.** Let t be an I-tree. If t is an irreducible tree, then $t = a \circ t'$, for some positive integer a and some I-tree t'. So, $\mathrm{ind}(t) = a$.

On the other hand, if $t = t^1 \cdot \ldots \cdot t^r$ is a reducible tree, then $\mathrm{ind}(t) = \mathrm{ind}(t^1)$. In such case, note that $\mathrm{ind}(t^1) < \cdots < \mathrm{ind}(t^r)$.

Given two I-trees t_1 and t_2 such that $\mathrm{Vert}(t_1) \cap \mathrm{Vert}(t_2) = \emptyset$, we say $t_1 < t_2$ if $\mathrm{ind}(t_1) < \mathrm{ind}(t_2)$.

8.1. **Algorithm for PBW basis.** We want to describe an algorithm that to allow us express the elements of Com^2 in terms of the PBW basis found. For this, we consider the following reduction formula.

8.1.1. **Proposition.** *Let n be a positive integer with $n \geq 2$. If $x_1, \ldots, x_n, y$ are elements of* Com^2*, then*

$$\begin{aligned}(x_1 \cdot \ldots \cdot x_n) \circ y = &\sum_{i=1}^{n} x_1 \cdot \ldots \cdot x_{i-1} \cdot (x_i \circ (x_{i+1} \cdot \ldots \cdot x_n \cdot y)) \\ &- \sum_{i=1}^{n-1} x_1 \cdot \ldots \cdot x_{i-1} \cdot (x_i \circ (x_{i+1} \cdot \ldots \cdot x_n)) \cdot y.\end{aligned}$$

Proof. The proof is by induction on n. Consider $n = 2$ and $x_1, x_2, y \in \mathrm{Com}^2$. From the compatibility condition between the products, we have that

$$(x_1 \circ x_2) \cdot x_3 = x_1 \circ (x_2 \cdot y) + x_1 \cdot (x_1 \circ y) - (x_1 \cdot x_2) \circ x_3.$$

Suppose $n > 2$ and consider $x_1, x_2, \ldots, x_n, y \in \mathrm{Com}^2$. By the compatibility condition between the products, we have that

$$\begin{aligned}(*) \quad (x_1 \cdot x_2 \cdot \ldots \cdot x_n) \circ y &= (x_1 \cdot (x_2 \cdot \ldots \cdot x_n)) \circ y \\ &= x_1 \circ (x_2 \cdot \ldots \cdot x_n \cdot y) + x_1 \cdot ((x_2 \cdot \ldots \cdot x_n) \circ y) \\ &\quad -(x_1 \circ (x_2 \cdot \ldots \cdot x_n)) \cdot y.\end{aligned}$$

Applying a recursive argument on the second term of the right side, we obtain that

$$
\begin{aligned}
x_1 \cdot ((x_2 \cdot \ldots \cdot x_n) \circ y) = &\sum_{i=2}^{n} x_1 \cdot x_2 \cdot \ldots \cdot x_{i-1} \cdot (x_i \circ (x_{i+1} \ldots \cdot x_n \cdot y)) \\
&- \sum_{i=2}^{n-1} x_1 \cdot x_2 \cdot \ldots \cdot x_{i-1} \cdot (x_i \circ (x_{i+1} \ldots \cdot x_n)) \cdot y.
\end{aligned}
$$

This together with $(*)$ imply that

$$
\begin{aligned}
(x_1 \cdot \ldots \cdot x_n) \circ y = &\sum_{i=1}^{n} x_1 \cdot \ldots \cdot x_{i-1} \cdot (x_i \circ (x_{i+1} \ldots \cdot x_n \cdot y)) \\
&- \sum_{i=1}^{n-1} x_1 \cdot \ldots \cdot x_{i-1} \cdot (x_i \circ (x_{i+1} \ldots \cdot x_n)) \cdot y,
\end{aligned}
$$

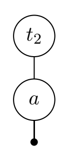
which ends the proof. □

Consider two I-trees t_1 and t_2 with $\mathrm{Vert}(t_1) \cap \mathrm{Vert}(t_2) = \emptyset$. We want to determine an algorithm that allow us to express $t_1 \cdot t_2$ and $t_1 \circ t_2$ as a linear combination of I-trees in the set of vertices $\mathrm{Vert}(t_1) \cup \mathrm{Vert}(t_2)$.

First, $t_1 \cdot t_2$ is the I-tree obtained from by identifying the roots of t_1 and t_2, ordering the children of the root of an increasing way, according to Definition 8.0.6. Note that it is possible because the product $\cdot$ is commutative.

For the product $t_1 \circ t_2$, as $\circ$ is a commutative product, we may assume that $t_1 < t_2$. To write $t_1 \circ t_2$ as a linear combination of I-trees, we proceed by induction on the degree of the tree t_1.

If t_1 is the degree one, then $t_1 = a$, for some positive integer a. In this case, $t_1 \circ t_2$ is the I-tree obtained to identify the root of t_2 with the vertex a ,

$t_1 \circ t_2 =$  .

Suppose that $|t_1| > 1$. We write t_1 in a unique way as $t_1 = t_1^1 \cdot \ldots \cdot t_1^r$, where r is a positive integer and t_1^i are irreducible I-trees, for $i \in \{1, \ldots, r\}$, such that their set of vertices are mutually disjoint.

If $r = 1$, then $t_1 = t_1^1 = a \circ t_1'$, for some positive integer a and I-tree t_1' of degree $(n-1)$. So,

$$t_1 \circ t_2 = (a \circ t_1') \circ t_2 = a \circ (t_1' \circ t_2).$$

Applying a recursive argument, we suppose that $(t_1' \circ t_2)$ is a linear combination of I-trees. Note that each I-tree which appears in the decomposition of $(t_1' \circ t_2)$ is of index greater than a, so $t_1 \circ t_2$ is linear combination of I-trees.

Suppose that $r > 1$. First, we consider $t_1 = t_1^1 \cdot \ldots \cdot t_1^r$ such that $\mathrm{ind}(t_1^r) < \mathrm{ind}(t_2)$. So, $\mathrm{ind}(t_1^i) < \mathrm{ind}(t_2)$, for each $i \in \{1, \ldots, r\}$. To write $t_1 \circ t_2$ as linear combination of I-trees, we apply the formula given in Proposition 8.1.1:

$$\begin{aligned} t_1 \circ t_2 = & \sum_{i=1}^{r} t_1^1 \cdot \ldots \cdot t_1^{i-1} \cdot (t_1^i \circ (t_1^{i+1} \ldots \cdot t_1^r \cdot t_2)) \\ & - \sum_{i=1}^{r-1} t_1^1 \cdot \ldots \cdot t_1^{i-1} \cdot (t_1^i \circ (t_1^{i+1} \ldots \cdot t_1^r)) \cdot t_2. \end{aligned}$$

Now, by a recursive argument, in the previous sum, $t_1^i \circ (t_1^{i+1} \ldots \cdot t_1^r \cdot t_2)$ and $t_1^i \circ (t_1^{i+1} \ldots \cdot t_1^r)$ can be write as linear combination of I-trees, for each $i \in \{1, \ldots, r\}$. So, we can express $t_1 \circ t_2$ as a linear combination in terms of I-trees.

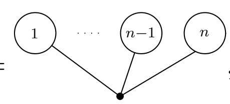
For the general case, consider $k \in \{1, \ldots, r\}$ such that t_1^k is maximal element of the set $\{t_1^i | \quad t_1^i < t_2\}$. Denoting by t_1' and by t_1'' the I-trees given by $t_1' = t_1^1 \cdot \ldots \cdot t_1^i$ and $t_1'' = t_1^{i+1} \cdot \ldots \cdot t_1^r$, we have that $t_1 = t_1' \cdot t_1''$. By the compatibility condition and the commutativity of the products $\cdot$ and $\circ$, we get:

$$\begin{aligned} t_1 \circ t_2 \; & = (t_1' \cdot t_1'') \circ t_2 \\ & = t_1' \circ (t_2 \cdot t_1'') + t_1' \cdot (t_2 \circ t_1'') - (t_1' \circ t_1'') \cdot t_2, \end{aligned}$$

where each term of the right side is linear combination of I-trees, using a recursive argument.

8.1.2. **Example.** Consider the I-tree t_1 given by

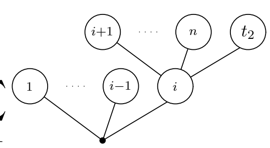
$$t_1 =$$



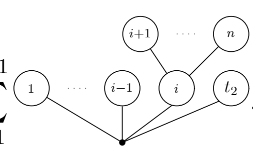
,

where $n \geq 2$. Let t_2 be an I-tree such that $\mathrm{ind}(t_2) > n$. We have that

$$t_1 \circ t_2 = \sum_{i=1}^{n}$$



$$- \sum_{i=1}^{n-1}$$



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8.1.3. **Remark.** Suppose that t_1 is an irreducible I-tree. In such case, there exists a positive integer l, with $l \leq |t_1|$, such that

$$t_1 = a_1 \circ \ldots \circ a_{l-1} \circ a_l,$$

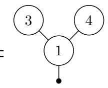
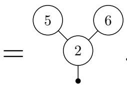
where $\{a_1, \ldots, a_{l-1}\}$ is a set of positive integers and a_l is a positive integer or a reducible I-tree with $a_1 < \cdots < a_{l-1} < \mathrm{Ind}(a_l)$.

Consider an I-tree t_2 such that $\mathrm{Vert}(t_1) \cap \mathrm{Vert}(t_2) = \emptyset$ and $t_1 < t_2$. To compute $t_1 \circ t_2$, we consider $k \in \{1, \ldots, l\}$ such that a_k is the maximal element of the set $\{a_i | \mathrm{Ind}(a_i) < \mathrm{ind}(t_2)\}$.

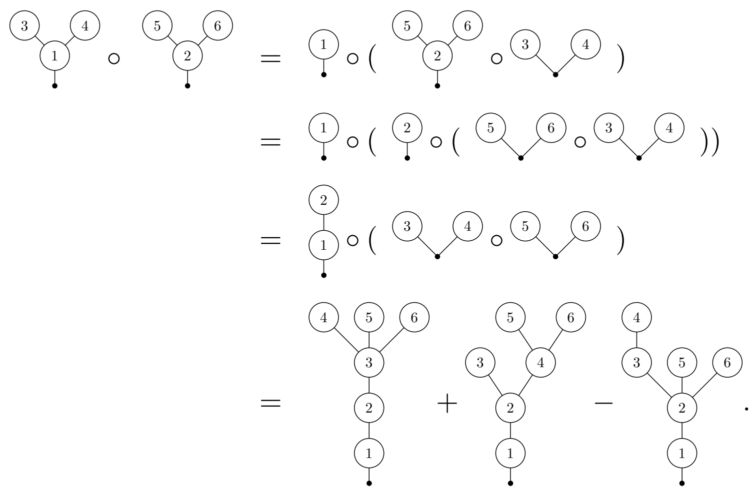
By the commutativity of $\circ$, we have that

$$t_1 \circ t_2 = a_1 \circ \ldots \cdot a_k \circ (t_2 \circ a_{k+1} \circ \ldots \circ a_l).$$

8.1.4. **Example.** Consider the I-trees,

$t_1 =$  and $t_2 =$  .

To compute $t_1 \circ t_2$, we proceed as follows:



8.2. **Bijection between I_n and S_n.** We want to find an explicit bijection between I_n and S_n. To determine it, we consider the usual decomposition of a permutation $\sigma \in S_n$ as a product of disjoint cycles. We denote a cycle of the usual way, with the convention that the smallest element is written more to the right.

For instance, in S_4, we denote the cycle

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix},$$

by $\sigma = (3421)$.

If $\sigma = (a_n \cdots a_1)$ is a cycle, we say that the index of σ is a_1 and we write $\mathrm{ind}(\sigma) = a_1$. Furthermore, given a permutation σ in S_n, we write the decomposition of σ as product of disjoint cycles putting the indices of the cycles in increasing order. Thus, when we write $\sigma = \sigma_1 \cdots \sigma_r$, we have that $\mathrm{ind}(\sigma_1) < \cdots < \mathrm{ind}(\sigma_r)$.

Under these conventions, we assign to each irreducible I-tree a cycle and reciprocally.

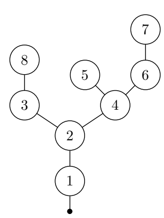
Consider an irreducible I-tree t of degree n whose set of vertices is given by $X = \{a_1, \ldots, a_n\}$, where a_1 is the minimal element of X. We assign to t a cycle on the set X, which we denote by σ_t, writing the elements of σ_t from right to left as we now indicate :

(1) Start with the smallest vertex of t, a_1. We write $\sigma_t = (\cdots a_1)$ and we discard the vertex a_1 in t. We continue with the vertices that are above a_1, reading the vertices of the tree t as follows:
(2) Suppose that a vertex $a \in X$ has been written in σ_t and discarded in the tree t. We choose the next vertex as follows:
  (a) If the vertex a is a leaf, then we continue with the first not discarded vertex of t, which is on the left of a (looking at the tree t from the bottom to up). This vertex is written in σ_t and discarded in t.
  (b) If the vertex a is not a leaf, then we choose between the children of a, the rightmost vertex. This vertex is written in σ_t and discarded in t.
(3) The process ends when all vertices of t haven been discarded.

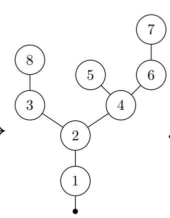
Reciprocally, let $\sigma = (a_n \cdots a_1)$ be a cycle of large n. We denote by $X = \{a_1, \ldots a_n\}$ the set of elements of σ. So, by our convention, a_1 is minimal element of X. We refer to the elements of X as vertices and construct an irreducible I-tree whose set of vertices is X, which we denote by t_σ, as follows:

(1) The first vertex of t_σ is a_1 and we discard this vertex in σ. To draw the remaining vertices of t_σ, we proceed as follows:
(2) Suppose that an element a of σ has been drawn in t_σ and discarded in σ. To draw the following vertex in t_σ, we proceed as follows:
  (a) If all the elements of σ, which are on the left to a, have been marked, then we continue with the smallest element of σ (not discarded), which is on the right of a. Denote this element by b and let v be the vertex colored by the discarded element closest to the right b in σ (seen from left to right). We draw vertex colored by b on the vertex v in such a way that b is the rightmost input of v. We discarded b in σ.
  (b) If, on the left of a, there are elements of σ that have not been discarded, then we continue with the smallest of these elements. We draw this vertex on the vertex a in t_σ and is discarded in σ.
(3) The process ends when all the elements of σ haven been discarded.

8.2.1. **Example.** In degree eight, we have:



$\mapsto (83576421) \mapsto$



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8.2.2. **Remark.** (1) Let t be an irreducible I-tree of degree n, with $n > 1$, and let b be the leftmost leaf of t. Denote by t' the irreducible

I-tree, of degree $(n-1)$, obtained by removing the vertex b of t. If $\sigma_{t'} = (a_{n-1}\cdots a_1)$, then $\sigma_t = (ba_{n-1}\cdots a_1)$.

(2) Let $\sigma = (a_n \cdots a_1)$ be a cycle and let b be a positive integer such that $b \neq a_i$ for all $i \in \{1, \dots, n\}$, with $b > a_1$. Consider t_σ and the cycle $\sigma' = (ba_n \cdots a_1)$. The tree $t_{\sigma'}$ is obtained from t_σ as follows. Reading σ from left to right, let a_k be the first element of σ such that $b > a_k$. We obtain $t_{\sigma'}$ by gluing a vertex decorated by b in the vertex of t_σ decorated by the element a_k, in such a way that b is the leftmost input of the vertex a_k. Moreover, the vertex decorated by b will be the leftmost leaf of the tree $t_{\sigma'}$.

8.2.3. **Definition.** We define the map $\Psi : I_n \to S_n$ by

$$\Psi(t) = \Psi(t_1 \cdot \ldots t_r) = \sigma_{t_1} \cdots \sigma_{t_r},$$

where $t = t_1 \cdot \ldots \cdot t_r$ is the unique decomposition of an I-tree in irreducible I-trees.

Reciprocally, let $\Phi : S_n \to I_n$ be the map

$$\Phi(\sigma) = \Phi(\sigma_1 \cdots \sigma_r) = t_{\sigma_1} \cdots t_{\sigma_r},$$

where $\sigma = \sigma_1 \cdots \sigma_r$ is the unique decomposition of the permutation σ disjoints in cycles with $\mathrm{ind}(\sigma_1) < \cdots < \mathrm{ind}(\sigma_r)$.

We want to see that Ψ is a bijection and Φ is its inverse. We need a previous lemma.

8.2.4. **Lemma.** *If $\sigma = (a_n \cdots a_1)$ is a cycle, then $\Psi(t_\sigma) = \sigma$.*

Proof. The result is obtained by induction on the length of the cycle σ. If the length of the cycle is one, the proof is immediate.

Consider $n > 1$. Write $\sigma = (a_n a_{n-1} \cdots a_1)$ and consider σ' the cycle of length $(n-1)$ given by $\sigma' = (a_{n-1} \cdots a_1)$. By a recursive argument, we have that $\Psi(t_{\sigma'}) = \sigma'$.

Let a_k be the first element of σ such that $a_n > a_k$. By point (2) in Remark 8.2.2, we have that the irreducible I-tree t_σ is obtained by gluing the vertex decorated by a_n to the vertex decorated by a_k in $t_{\sigma'}$, so that a_n is the leftmost input of a_k. So, by point (1) in Remark 8.2.2, $\Psi(t_\sigma) = (a_n a_{n-1} \cdots a_1) = \sigma$, which ends the proof. $\square$

Applying Lemma 8.2.4, it is immediate to see that:

8.2.5. **Proposition.** *The map $\Psi : I_n \to S_n$ is bijective and $\Phi : S_n \to I_n$ is its inverse.*

9. Operadic homology for compatible commutative algebras

Recall that if A is a commutative algebra, the operadic chain-complex of A is given by Harrison complex $C_*^{Harr}(A)$ of A. The Harrison complex $C_*^{Harr}(A)$ is a quotient of the Hochshild complex $C_*(A)$. Explicitly, $C_n^{Harr}(A)$ is the quotient of $C_n(A) = A^{\otimes n}$ by all the non-trivial signed shuffles, that is, by the $(p_1, \ldots, p_r)$-shuffles for $p_i \geq 1$, with $p_1 + \cdots + p_r = n$ and $r \geq 2$ ([22], 13.1.10). For instance,

$$C_2^{Harr}(A) = A^{\otimes 2}/\{\text{ non-trivial shuffles}\} = A^{\otimes 2}/\{a \otimes b - b \otimes a\} = S^2(A).$$

We will show that if $(A, \cdot, \circ)$ is a compatible commutative algebra, then its operadic chain-complex is given by the total complex of a bicomplex whose vertical and horizontal complexes are induced by the Harrison boundary maps of complexes of $(A, \cdot)$ and $(A, \circ)$. As the Harrison complex is a quotient of Hochshild complex, we denote the respective differentials by $d^*_\cdot$ and $d^*_\circ$ (see 3.6.1).

9.0.1. **Proposition.** *Let $(A, \cdot, \circ)$ be a compatible commutative algebra. The total complex of the bicomplex with vertical differential $d^*_\cdot$ and horizontal differential $d^*_\circ$*

$$\begin{tikzcd}
{} \arrow[d, "d^*_\cdot"'] & {} \arrow[d, "d^*_\cdot"'] & {} \arrow[d, "d^*_\cdot"'] & \\
C_3^{Harr}(A) \arrow[d, "d^*_\cdot"'] & C_4^{Harr}(A) \arrow[l, "d^*_\circ"'] \arrow[d, "d^*_\cdot"'] & C_5^{Harr}(A) \arrow[l, "d^*_\circ"'] \arrow[d, "d^*_\cdot"'] & {} \arrow[l, "d^*_\circ"'] \\
C_2^{Harr}(A) \arrow[d, "d^*_\cdot"'] & C_3^{Harr}(A) \arrow[l, "d^*_\circ"'] \arrow[d, "d^*_\cdot"'] & C_4^{Harr}(A) \arrow[l, "d^*_\circ"'] \arrow[d, "d^*_\cdot"'] & {} \arrow[l, "d^*_\circ"'] \\
C_1^{Harr}(A) & C_2^{Harr}(A) \arrow[l, "d^*_\circ"'] & C_3^{Harr}(A) \arrow[l, "d^*_\circ"'] & {} \arrow[l, "d^*_\circ"']
\end{tikzcd}$$

is the Com^2*-operadic complex of A.*

Proof. Let us see that its total complex corresponds to the Com^2-operadic complex of A.

The chain complex $C_*^{\mathrm{Com}^2}(A)$ is given by

$$\cdots \to C_n^{\mathrm{Com}^2}(A) \to C_{(n-1)}^{\mathrm{Com}^2}(A) \to \cdots \to C_1^{\mathrm{Com}^2}(A),$$

where $C_n^{\mathrm{Com}^2}(A) := (\mathrm{Com}^2)^!(n)^\vee \otimes_{S_n} A^{\otimes n}$ and whose differential d is described in 2.5.8. Since the Koszul dual of the operad Com^2 is given by ${}^2Lie$ and

$$(*) \quad {}^2Lie(n) = Lie(n) \oplus \cdots \oplus Lie(n)$$

is the direct sum of n-copies of $Lie(n)$, we have that

$$\begin{aligned} C_n^{\mathrm{Com}^2}(A) &= (\mathrm{Com}^2)^!(n)^\vee \otimes_{S_n} A^{\otimes n} \\ &= ((Lie(n) \oplus \cdots \oplus Lie(n)) \otimes (\mathrm{sgn}_n)) \otimes_{S_n} A^{\otimes n} \\ &= Lie(n)^\vee \otimes_{S_n} A^{\otimes n} \oplus \cdots \oplus Lie(n)^\vee \otimes_{S_n} A^{\otimes n} \\ &= \mathrm{Com}^!(n)^\vee \otimes_{S_n} A^{\otimes n} \oplus \cdots \oplus \mathrm{Com}^!(n)^\vee \otimes_{S_n} A^{\otimes n}. \end{aligned}$$

So, $C_n^{\mathrm{Com}^2}(A)$ is the direct sum of n-copies of $C_n^{\mathrm{Com}}(A)$, the operadic chain-complex associated A as a commutative algebra. Since $C_*^{\mathrm{Com}}(A)$ is the Harrison complex $C_*^{Harr}(A)$, we have that $C_n^{\mathrm{Com}^2}(A)$ is a direct sum of n-copies of $C_n^{Harr}(A)$,

$$C_n^{\mathrm{Com}^2}(A) = C_n^{Harr}(A) \oplus \cdots \oplus C_n^{Harr}(A).$$

In a similar way that for the case As^2 in 3.6.1, we can reorganize the chain-complex $C_*^{\mathrm{Com}^2}(A)$ by means of the bicomplex that we have described, and identify its differential with the total differential $d = d_\cdot^* + d_\circ^*$, which ends the proof. $\square$

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