GRADED REPRESENTATION THEORY OF TEMPERLEY-LIEB ALGEBRAS OF TYPE A AND B

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A thesis submitted in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

Institute of Mathematics and Physics
University of Talca

JANUARY 2013

Contents

ACKNOWLEDGEMENTS

En primer lugar quiero agradecer a mi supervisor Steen Ryom-Hansen por su apoyo e infinita paciencia durante estos cuatro años, probablemente nada de lo que está escrito en esta tesis existiría sin su ayuda. Le doy las gracias también a todos los profesores y amigos del IMAFI por su apoyo y simpatía, y hacer de estos cuatro años una gran etapa de mi vida.

Agradezco a mis padres por apoyarme cada vez que lo necesité y por incentivarme siempre a seguir estudiando. Aunque ya no esté en este mundo le doy todas las gracias a mi abuelita, Marta, se que estaría muy orgullosa de ver lo lejos que he llegado. No puedo terminar estos agradecimientos sin mencionar a mi (algún día cercano) esposa, Lilian, por su cariño e incondicionalidad, y por haberme seguido a Talca. Espero que me sigas por muchos años más. A mis pequeños hijos, Lautaro y Mateo, les agradezco por seguir dandole sentido a todo esto.

Finalmente, me gustaría agradecer a la Comisión Nacional de Ciencia y Tecnología CONICYT por haberme financiado durante mi doctorado con la Beca de Doctorado Nacional.

To Lilian, of course.

CHAPTER 1

Introduction

Let S_n be the symmetric group on n letters. Studying the representation theory of the symmetric groups (more generally, the representation theory of any group or algebra) means understanding how they interact with vector spaces. The main goal of representation theory is to understand the structure of the irreducible representations. The representation theory of S_n over a field k of characteristic zero was treated around one century ago, in the works of Young, Schur and Frobenius. They found that the irreducible representations, the Specht modules, are parametrized by partitions λ of n, i.e. weakly decreasing sequences of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ that sum to n. It has also be known for a long time that there are several ways of constructing these Specht modules, some of which are of combinatorial nature while other rely on geometrical or other methods. Similarly, the dimension of the Specht module $S(\lambda)$ associated with λ can be expressed in different ways.

The situation becomes much more difficult when k is assumed to be a field of characteristic p, for example the finite field $\mathbb{F}_p$. Although the Specht module construction still works in this setting, $S(\lambda)$ is no longer irreducible in general. Still the $S(\lambda)$ are useful object in characteristic p as well. Indeed, it is known that the irreducible representations $D(\lambda)$ are parametrized by the socalled p-regular partitions λ of n, i.e. those λ that do not contain p equal λ_i. For each such λ, one obtains $D(\lambda)$ as the unique irreducible quotient of $S(\lambda)$. On the other hand, in spite of this very concrete realization of $D(\lambda)$ the basic problem of determining the dimension of $D(\lambda)$ is still unsolved. Indeed, this is considered by many as the main unsolved problem of representation theory in positive characteristic and has been at the center of much research activity over the last 30 years.

A main conjecture in this setting is the James/Lusztig conjecture stating that for $n < p^2$ one should have

$$[S(\lambda) : D(\mu)] = d_{\lambda,\mu}(1)$$

where $[S(\lambda) : D(\mu)]$ denotes composition factor multiplicity and $d_{\lambda,\mu}(1)$ the evaluation at 1 of a certain parabolic Kazhdan-Lusztig polynomial $d_{\lambda,\mu}(q) \in \mathbb{Z}[q]$. It is calculated most efficiently using the LLT-algorithm. All evidence known so far, computational as well as theoretical, supports this conjecture, but a proof is still missing.

The appearance of polynomials in the conjecture might indicate that the group algebra $\mathbb{F}_pS_n$ may be a graded algebra. To be precise, it indicates the existence of an isomorphism of algebras $\varphi : \mathbb{F}_pS_n \to \mathcal{R}_n$ where $\mathcal{R}_n = \bigoplus_i R_{n,i}$ is a graded algebra.

Such an isomorphism φ has been constructed by Brundan and Kleshchev in a recent important paper. The algebra $\mathcal{R}_n$ turns out to be a cyclotomic version of an algebra introduced by Khovanov-Lauda and Rouquier. It is given by generators

$$\{e(\mathbf{i}) | \mathbf{i} \in \mathbb{F}_p^n\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}$$

and a long list of relations between them. Brundan and Kleshchev obtain φ by constructing concrete elements in $\mathbb{F}_pS_n$ satisfying these relations and checking that the associated map induces an isomorphism. This involves lengthy calculations, but still their proof works even for cyclotomic Hecke algebras with the parameter q specialized at an e'th root of unity. It should here be noted that the resulting gradings are far from being 'visible' to the naked eye and depend heavily on p or the multiplicative order of q in the Hecke algebra setting.

The goal of this thesis is to study the graded representation theory of certain quotients of the cyclotomic Hecke algebras of level 1 and 2, which admitting nice diagrammatical presentations, the Temperley-Lieb algebra $Tl_n(q)$ and the blob algebra, respectively. The first was introduced by Temperley and Lieb in [33] as complex associative algebras that arose in their study of transfer matrix approaches to (planar) lattice models. But has since turned out to be related to many topics of mathematics as well, including knot theory, operator theory, algebraic combinatorics and algebraic Lie theory. For instance, these algebras were subsequently rediscovered by Jones in [15] who used them to define what is now known as the

Jones polynomial in knot theory. As of today, it is an object well known to a general audience in physics as well as mathematics and at the same time it remains at the center of a big number of research articles being published each year in both areas.

Our main emphasis lies on a two-parameter generalization $b_n(q,m)$ of the Temperley-Lieb algebra that was introduced by P. Martin and H. Saleur in [22], as a way of introducing periodicity in the physical model defining $Tl_n(q)$. An important feature of both $Tl_n(q)$ and $b_n(q,m)$ is the fact that they are diagram algebras, that is they have bases parameterized by certain planar diagrams, such that the multiplications are given by concatenation of these diagrams. In the case of $Tl_n(q)$ these diagrams are the socalled *bridges* or *Temperley-Lieb diagrams*, in the case of $b_n(q,m)$ the diagrams are certain marked Temperley-Lieb diagrams and for this reason $b_n(q,m)$ was called the blob algebra in [22].

We are interested in the non-semisimple representation theory of $Tl_n(q)$ and $b_n(q,m)$, which is the case where q is specialized at a root of unity. The $Tl_n(q)$-case is connected via Schur-Weyl duality to the representation theory of the quantum group associated with SL_2. The $b_n(q,m)$-case is more intriguing and has received quite a lot of attention over the last decade. It has been shown to share a surprisingly number of properties with objects that normally arise in Lie theory. In particular, it was shown in [24] that the decomposition numbers are given by evaluations at 1 of certain Kazhdan-Lusztig polynomials associated with an infinite dihedral Weyl group.

The fact that the decomposition numbers for $b_n(q,m)$ come from polynomials (as in the group algebra of the symmetric group case) gives a first indication of the existence of a $\mathbb{Z}$-graded structure on $b_n(q,m)$ and on its standard modules, and indeed a main goal of our thesis is to construct such a graded structure on $b_n(q,m)$.

A main input to our thesis comes from the seminal work of Brundan and Kleshchev that constructs isomorphisms between cyclotomic Hecke algebras and Khovanov-Lauda-Rouquier (KLR) algebras (of type A), [3]. Since the KLR algebras are $\mathbb{Z}$-graded, the various Hecke algebras become $\mathbb{Z}$-graded in this way as well. On the other hand, $b_n(q,m)$ is known to be a quotient of the cyclotomic Hecke algebra $\mathcal{H}_n(q,Q)$ of type $G(2,1,n)$, and our basic idea is now to exploit this quotient construction.

A big step towards our goal is taken already in Chapter 4 of our thesis, where we show that the ideal $\mathcal{J}_n \subset \mathcal{H}_n(q,Q)$, defining $b_n(q,m)$, is homogeneous, thus making $b_n(q,m)$ a $\mathbb{Z}$-graded algebra. This result relies on a realization of $\mathcal{J}_n$ due to P. Martin and D. Woodcock in [23], in terms of certain explicitly given idempotents that turn our to be well behaved with respect to the KLR-relations.

On the other hand, this does not immediately imply a $\mathbb{Z}$-grading on the standard modules for $b_n(q,m)$ and indeed a major part of our thesis is dedicated to this point. An important ingredient to this comes from the recent paper by Hu and Mathas, [14], that introduces the concept of a graded cellular algebra and shows that the cyclotomic Hecke algebras are graded cellular with respect to the $\mathbb{Z}$-grading given by Brundan and Kleshchev's work. We then achieve our goal in the Chapter 5 by showing that $b_n(q,m)$ is a graded cellular algebra.

A main difficulty in applying [14], is due to the fact that the cell structure on $\mathcal{H}_n(q,Q)$ considered in [14] is related to the dominance order on bipartitions, which is known to be incompatible with the natural order for the category of $b_n(q,m)$-modules, see [30] and [31]. We overcome this problem by showing that $b_n(q,m)$ is an algebra endowed with a family of Jucys-Murphy elements, in the sense of Mathas [26], with respect to a natural order that we introduce in Chapter 3. This involves delicate arguments involving the diagram basis for $b_n(q,m)$.

It should be mentioned that our results are also valid in the Temperley-Lieb algebra case where the relevant Hecke algebra $\mathcal{H}_n(q)$ this time is of type A, and even in this case our results seem to be new. On the other hand, in the Temperley-Lieb algebra case there is actually a simpler way to show that the ideal of $\mathcal{H}_n(q)$ defining $Tl_n(q)$ is graded. It is based on certain properties of Murphy's standard basis that were proved by M. Härterich in [13].

Let us sketch the layout of the thesis. In Chapter 2 we introduce the main objects of our study, the Temperley-Lieb and the blob algebra. We also define the cyclotomic Hecke algebras of level 1 and 2, and recall the relevant results from the literature involving them. In Chapter 3 we recall the diagrammatic realization of the Temperley-Lieb algebra and the blob algebra. We also setup all of the combinatorics which we will need to understand the representation theory of $Tl_n(q)$ and b_n. Specifically, we define partitions, bipartitions, Young diagrams, tableaux, bitableaux and the Bratteli diagram. We use this combinatorics to parameterizing the diagrammatic basis of $Tl_n(q)$ and b_n. The set of all of two-columns standard tableaux shall be ordered by dominance, whereas the set of all one-line standard bitableaux shall be ordered in a non-conventional way. We end this chapter by relating these two orders.

In Chapter 4 we introduce the basic notions of graded representation theory. Our main emphasis will be the concept of graded cellular algebra, this concept plays a central role in the remainder of this thesis. Actually, all algebras considered in this thesis are graded cellular algebras. At the end of this chapter we show that the $Tl_n(q)$ and b_n are $\mathbb{Z}$-graded algebras. In Chapter 5 we show that the images in b_n of the Jucys-Murphy elements of $\mathcal{H}_n(q, Q)$ make the blob algebra into an algebra with a family of Jucys-Murphy elements, in the sense of Mathas [26]. As we explain in the beginning of that Section 5.1, this is quite surprising. Using this we prove that the blob algebra is a graded cellular algebra. The analogous result for $Tl_n(q)$ follows in a simpler way. We end this chapter by giving two examples that illustrate our results. Finally, in Chapter 6 we find the graded decomposition numbers for the blob algebra. In order to obtain these numbers, we first need to understand the degree function on the set of all one-line standard bitableaux defined in Chapter 5 to make the blob algebra a graded cellular algebra. We shall give a characterization of the degree function in terms of the Bratteli diagram. With this at hand, we can prove the existence of a family of positively graded cellular subalgebras of b_n. Then, we reduce the problem of finding the graded decomposition numbers for b_n to the problem of finding the graded decomposition numbers for the positive graded cellular algebras mentioned above.

CHAPTER 2

Preliminaries

In this chapter we fix the notation that shall be used throughout the thesis. We introduce the algebras to be studied, the Temperley-Lieb algebra, the blob algebra, the corresponding Hecke and Khovanov-Lauda-Rouquier algebras and recall the relevant results from the literature involving them. The important diagrammatic realizations of the Temperley-Lieb algebra and the blob algebra shall be postponed to the forthcoming Chapter 3.
Throughout the thesis the ground field shall be the complex field $\mathbb{C}$ although some of our results hold in greater generality. For $q \in \mathbb{C}^{\times}$ and an integer k we define

$$[k] = [k]_q := q^{k-1} + q^{k-3} + \ldots + q^{-k+1} \in \mathbb{C}$$

the usual Gaussian coefficient. All our algebras are associative and unital.

2.1 The Temperley-Lieb algebra, the blob algebra, the Hecke algebras

Definition 2.1.1. Let $q \in \mathbb{C}^{\times}$. The Temperley-Lieb algebra $Tl_n(q)$ is the $\mathbb{C}$-algebra on the generators $U_1, ..., U_{n-1}$ subject to the relations

$$\begin{aligned} U_i^2 &= -[2]U_i && \text{if } 1 \leq i \leq n-1 \\ U_iU_jU_i &= U_i && \text{if } |i-j| = 1 \\ U_iU_j &= U_jU_i && \text{if } |i-j| > 1. \end{aligned}$$

The main object of the thesis is the blob algebra, introduced in [22] by P. Martin and H. Saleur as a generalization of the Temperley-Lieb algebra. It is usually defined in terms of a basis of blobbed Temperley-Lieb diagrams and their compositions, from which it derives its name. Such that for the Temperley-Lieb algebra, the blob algebra can be defined as an algebra with generators and relations. Let y_e be an invertible element of $\mathbb{C}$.

Definition 2.1.2. The blob algebra $b_n = b_n(q, y_e)$ is the $\mathbb{C}$-algebra on the generators $e, U_1, ..., U_{n-1}$ subject to the relations

$$\begin{aligned}
U_i^2 &= -[2]U_i && \text{if } 1 \leq i \leq n-1 \\
U_iU_jU_i &= U_i && \text{if } |i-j| = 1 \\
U_iU_j &= U_jU_i && \text{if } |i-j| > 1 \\
U_1eU_1 &= y_eU_1 && \\
e^2 &= e && \\
U_ie &= eU_i && \text{if } 2 \leq i \leq n-1.
\end{aligned}$$

Assume that $[m] \neq 0$. The parametrization of b_n through $y_e = \frac{-[m-1]}{[m]}$ includes the non-semisimple cases, see [24, Section 2]. Under this choice of y_e we denote $b_n(q, y_e)$ by $b_n(m)$ and replace e by the rescaled generator $U_0 := -[m]e$. When we are working with $b_n(m)$ we always assume that q is a l-th root of unity.

The Temperley-Lieb algebra and the blob algebra were introduced from motivations in statistical mechanics. An important feature, that we postpone to the next chapter, is that they both have diagrammatic realizations by planar diagrams. Another significant feature of these algebras is that $Tl_n(q)$ and $b_n(m)$ can be realized as quotients of Hecke algebras of type A and B, respectively. This explains why the blob algebra is sometimes referred as the Temperley-Lieb algebra of type B.

We next define the related Hecke algebras.

Definition 2.1.3. Let $q \in \mathbb{C}$ and assume that $q \neq 0, 1$. The Hecke algebra $\mathcal{H}_n(q)$ of type A_{n-1} is the $\mathbb{C}$-algebra with generators $T_1, \ldots, T_{n-1}$, subject to the relations

$$(T_i - q)(T_i + 1) = 0 \qquad \text{for } 1 \leq i \leq n-1 \tag{2.1.1}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \qquad \text{for } 1 \leq i \leq n-2 \tag{2.1.2}$$

$$T_iT_j = T_jT_i \qquad \text{for } |i-j| > 1 \tag{2.1.3}$$

It follows easily from the relations that T_r is an invertible element in $\mathcal{H}_n(q)$, with $T_r^{-1} = q^{-1}(T_r - q + 1)$. We may then define elements $L_1, \ldots, L_n \in \mathcal{H}_n(q)$ by $L_1 := 1$ and recursively $L_{r+1} = q^{-1}T_rL_rT_r$ for all admissible r. They are the first examples of Jucys-Murphy elements that play an important role in our thesis.

Definition 2.1.4. Let $q, \lambda_1, \lambda_2 \in \mathbb{C}$ and suppose that $q \neq 0, 1$. The cyclotomic Hecke algebra $\mathcal{H}_n(q; \lambda_1, \lambda_2)$ of type $G(2,1,n)$ is the $\mathbb{C}$-algebra with generators $L_1, \ldots, L_n$, $T_1, \ldots, T_{n-1}$ and relations

$$\begin{aligned} (L_1 - \lambda_1)(L_1 - \lambda_2) &= 0, & L_r L_s &= L_s L_r, \\ (T_r + 1)(T_r - q) &= 0, & T_r L_r &= L_{r+1}(T_r - q + 1), \\ T_s T_{s+1} T_s &= T_{s+1} T_s T_{s+1}, & & \\ T_r L_s &= L_s T_r, & &\text{if } |r - s| > 1, \\ T_r T_s &= T_s T_r, & &\text{if } s \neq r, r+1 \end{aligned}$$

for all admissible r, s.

Once again, T_r is invertible with $T_r^{-1} = q^{-1}(T_r - q + 1)$. From this one gets that $L_{r+1} = q^{-1} T_r L_r T_r$. Moreover, it follows from the relations that $f(L_1, \ldots, L_n)$ is a central element of $\mathcal{H}_n(q; \lambda_1, \lambda_2)$ for $f(x_1, \ldots, x_n)$ a symmetric polynomial. These L_i are also called Jucys-Murphy elements.

We now explain the relations mentioned above between the algebras that we have defined.

Theorem 2.1.5. *The are surjections Φ_1 and Φ_2 given by*

$$\begin{array}{llllll} \Phi_1: & \mathcal{H}_n(q^2) & \longrightarrow & Tl_n(q), & T_i & \mapsto & qU_i + q^2 \\ \Phi_2: & \mathcal{H}_n(q^2) & \longrightarrow & Tl_n(q), & T_i & \mapsto & -qU_i - 1. \end{array}$$

The kernel of Φ_1 is the ideal generated by

$$q^{-6} T_1 T_2 T_1 - q^{-4} T_1 T_2 - q^{-4} T_2 T_1 + q^{-2} T_1 + q^{-2} T_2 - 1$$

and the kernel of Φ_2 is the ideal generated by

$$T_1 T_2 T_1 + T_1 T_2 + T_2 T_1 + T_1 + T_2 + 1.$$

Proof: This is well known. $\square$

There are two, not obviously equivalent, ways to generalize this Theorem to the blob algebra case. One is given in [12], but for our purposes it is more convenient to work with the second one, that appears in [23]. Set $Q := q^m$ and define $\mathcal{H}_n(m) = \mathcal{H}_n(q^2; Q, Q^{-1})$. Assume

$$q^4 \neq 1, \qquad Q \neq Q^{-1}, \qquad Q \neq q^2 Q^{-1}, \qquad Q^{-1} \neq q^2 Q. \tag{2.1.4}$$

With the above conditions, one can define elements $E_1, E_2 \in \mathcal{H}_2(m)$ by the formulas

$$E_1 = \frac{(T_1 - q^2)(L_1 - Q^{-1})(L_2 - Q^{-1})}{(1+q^2)(Q-Q^{-1})(Q^{-1} - q^{-2}Q)}$$

$$E_2 = \frac{(T_1 - q^2)(L_1 - Q)(L_2 - Q)}{(1+q^2)(Q^{-1}-Q)(Q - q^{-2}Q^{-1})}.$$

The factors of E_1 and E_2 commute with each other. Using this and $L_2 = q^{-2}T_1L_1T_1$, one finds that they verify the following equations

$$(T_1 + 1)E_1 = 0, \qquad (T_1 + 1)E_2 = 0, \tag{2.1.5}$$

$$(L_1 - Q)E_1 = 0, \qquad (L_1 - Q^{-1})E_2 = 0, \tag{2.1.6}$$

$$(L_2 - Qq^{-2})E_1 = 0, \qquad (L_2 - Q^{-1}q^{-2})E_2 = 0 \tag{2.1.7}$$

and from this it follows that E_1 and E_2 are idempotents associated with irreducible representations of $\mathcal{H}_2(m)$ of dimension one. Note that E_1 and E_2 are the unique idempotents satisfying (2.1.5) and (2.1.6). They are denoted e_2^{-1} and e_2^{-2} in [23]. For all n there is a canonical embedding $\mathcal{H}_n(m) \hookrightarrow \mathcal{H}_{n+1}(m)$. Using it repeatedly we consider E_1 and E_2 as elements of $\mathcal{H}_n(m)$ and denote by $\mathcal{J}_n$ the ideal of $\mathcal{H}_n(m)$ generated by them.

Theorem 2.1.6. *The map Φ given by*

$$\begin{array}{rccl} \Phi : & \mathcal{H}_n(m) & \longrightarrow & b_n(m) \\ & T_i - q^2 & \mapsto & qU_i \\ & L_1 - q^m & \mapsto & (q - q^{-1})U_0 \end{array}$$

induces a $\mathbb{C}$-algebra isomorphism between $\mathcal{H}_n(m)/\mathcal{J}_n$ and $b_n(m)$.

Proof: See [23, Proposition 4.2]. □

We would like to have an integral version of the last result, but want also to avoid those choices of the parameters that correspond to the conditions (2.1.4). This can for example be achieved by localizing $\mathbb{C}[q, q^{-1}, Q, Q^{-1}]$ conveniently. To be precise, we choose for R the localization of the Laurent polynomial ring $\mathbb{C}[q, q^{-1}, Q, Q^{-1}]$ at S, defined as the multiplicatively closed subset of $\mathbb{C}[q, q^{-1}, Q, Q^{-1}]$ generated by the polynomials $1, q^4 - 1, Q - Q^{-1}, Q - Q^{-1}q^2$ and $Q^{-1} - Qq^2$. For integers l and m we denote by $\mathfrak{m}$ the ideal $\langle q - e^{2\pi i/l}, Q - q^m \rangle$ of R. Then we have that either $\mathfrak{m} = R$ or else $\mathfrak{m}$ is a maximal ideal in R. In the last case we define $\mathcal{O} := R_\mathfrak{m}$ and get that $\mathcal{O}$ is a discrete valuation ring with maximal ideal $\mathfrak{m}$, quotient field $K := \mathbb{C}(q, Q)$ and residue field $\mathcal{O}/\mathfrak{m} = \mathbb{C}$ containing the l'th root of unity q.

Throughout the thesis we assume that $\mathcal{O}, K$ and $\mathbb{C}$ are chosen as above, and furthermore, in order to simplify notation, that l is odd. In the next section we recall the $\mathbb{Z}$-grading on $\mathcal{H}_n(q^2)$ and $\mathcal{H}_n(m)$ given by Brundan and Kleshchev in [3]. Note that since l is assumed to be odd, the condition from *loc. cit.* that q^m be a power of q^2, or equivalently, that the congruence $2k \equiv m \mod l$ be solvable, is always fulfilled.

We now define $b_n^{\mathcal{O}}(m)$ as the $\mathcal{O}$-algebra on generators $e, U_1, ..., U_{n-1}$ subject to the same relations as for b_n. Then $b_n^{\mathcal{O}}(m)$ is free over $\mathcal{O}$ as can be seen using the results of the appendix of [7], note that they are valid over any commutative ring. The rational blob algebra $b_n^K(m)$ is defined the same way, and we have base changes isomorphisms $b_n^{\mathcal{O}}(m) \otimes_{\mathcal{O}} \mathbb{C} = b_n(m)$ and $b_n^{\mathcal{O}}(m) \otimes_{\mathcal{O}} K = b_n^K(m)$. Finally we define

$\mathcal{H}_n^{\mathcal{O}}(m)$ as the $\mathcal{O}$-algebra on generators $L_1, \ldots, L_n$, $T_1, \ldots, T_{n-1}$ subject to the same relations as for $\mathcal{H}_n(m)$, but using parameters $\lambda_1 = Q$ and $\lambda_2 = Q^{-1}$. Similarly, we define $\mathcal{H}_n^K(m)$ and we have base changes isomorphisms as above.

Theorem 2.1.7. *There is a surjection* $\Phi : \mathcal{H}_n^{\mathcal{O}}(m) \longrightarrow b_n^{\mathcal{O}}(m)$.

Proof: The argument given in [23, Proposition 4.2] involves checking blob relations and therefore gives a surjection $\mathcal{H}_n^{\mathcal{O}}(m) \longrightarrow b_n^{\mathcal{O}}(m)$, as claimed. □

2.2 The Khovanov-Lauda-Rouquier algebra

Recall that the quantum characteristic of an element q of a field F is the smallest positive integer j such that $1 + q + \ldots + q^{j-1} = 0$, setting $j = 0$ if no such integer exists. With our choice of $q \in \mathbb{C}$ the quantum characteristic is l. Recall that we assume that l is odd. Note that without loss of generality we can assume that $0 \leq m < l$. We set $I = \mathbb{Z}/l\mathbb{Z}$ and refer to I^n as the residue sequences of length n. Note that in order to apply [3], we should actually use the quantum characteristic of q^2 in the definition of I, but since l is assumed to be odd, the two definitions coincide. In the following $\mathcal{H}$ refers to either $\mathcal{H}_n(q^2)$ or $\mathcal{H}_n(m)$ (with $q \in \mathbb{C}$ chosen as above). Let M be a finite dimensional $\mathcal{H}$-module. By [17, Lemma 7.1.2] the eigenvalues of each L_r on M are of the form q^{2i} for $i \in I$. So M decomposes as the direct sum $M = \bigoplus_{\boldsymbol{i} \in I^n} M_{\boldsymbol{i}}$ of its generalized weight spaces

$$M_{\boldsymbol{i}} := \{v \in M \mid (L_r - q^{2i_r})^k v = 0 \text{ for } r = 1, \ldots, n \text{ and } k \gg 0\}.$$

In particular, taking M to be the regular left module $\mathcal{H}$, we obtain a system $\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I^n\}$ of mutually orthogonal idempotents in $\mathcal{H}$ such that $e(\boldsymbol{i})M = M_{\boldsymbol{i}}$

for each M as above.

We can now define nilpotent elements $y_1, \dots, y_n \in \mathcal{H}$ via the formula

$$y_r = \sum_{\boldsymbol{i} \in I^n} (1 - q^{-2i_r} L_r) e(\boldsymbol{i}). \tag{2.2.1}$$

For $1 \leq r < n$ and $\boldsymbol{i} \in I^n$, Brundan and Kleshchev define in [3] certain formal power series, $P_r(\boldsymbol{i}), Q_r(\boldsymbol{i}) \in \mathbb{C}[[y_r, y_{r+1}]]$, such that $Q_r(\boldsymbol{i})$ has non-zero constant term, see [3, (4.27) and (4.36)] for the explicit formulas. Since each y_r is nilpotent in $\mathcal{H}$, we can consider $P_r(\boldsymbol{i})$ and $Q_r(\boldsymbol{i})$ as elements of $\mathcal{H}$, with $Q_r(\boldsymbol{i})$ invertible. We then set

$$\psi_r = \sum_{\boldsymbol{i} \in I^n} (T_r + P_r(\boldsymbol{i})) Q_r(\mathbf{i})^{-1} e(\boldsymbol{i}). \tag{2.2.2}$$

The main theorem in [3] gives a presentation of $\mathcal{H}$ in terms of the elements

$$\{\psi_1, \cdots, \psi_{n-1}\} \cup \{y_1, \cdots, y_n\} \cup \{e(\mathbf{i}) \mid \mathbf{i} \in I^n\}$$

and a series of relations between them that we describe shortly. An important point of these relations is that they are homogeneous with respect to a nontrivial $\mathbb{Z}$-grading on $\mathcal{H}$. To describe the $\mathbb{Z}$-grading it is convenient to introduce the matrix $(a_{ij})_{i,j \in I}$, given by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 0 & \text{if } i \neq j \pm 1 \\ -1 & \text{if } i = j \pm 1. \end{cases}$$

With this at hand, we are now able to state [3, Main Theorem]. The Theorem holds in greater generality than shown here, namely for all cyclotomic Hecke algebras, including the degenerate algebras, but for our purpose the following version is enough.

Theorem 2.2.1. *The algebra $\mathcal{H}$ is isomorphic to a cyclotomic Khovanov-Lauda-Rouquier algebra of type A. To be precise, it is isomorphic to the $\mathbb{C}$-algebra generated by*

$$\{\psi_1, \cdots, \psi_{n-1}\} \cup \{y_1, \cdots, y_n\} \cup \{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I^n\}$$

subject to the following relations for $\boldsymbol{i}, \boldsymbol{j} \in I^n$ and all admissible r, s

$$y_1 e(\boldsymbol{i}) = 0 \text{ if } i_1 = \begin{cases} \pm k \mod l & \text{if } \mathcal{H} = \mathcal{H}_n(m) \\ 0 \mod l & \text{if } \mathcal{H} = \mathcal{H}_n(q^2) \end{cases} \tag{2.2.3}$$

$$e(\boldsymbol{i}) = 0 \text{ if } i_1 \neq \begin{cases} \pm k \mod l & \text{if } \mathcal{H} = \mathcal{H}_n(m) \\ 0 \mod l & \text{if } \mathcal{H} = \mathcal{H}_n(q^2) \end{cases} \tag{2.2.4}$$

$$e(\boldsymbol{i})e(\boldsymbol{j}) = \delta_{\boldsymbol{i},\boldsymbol{j}} e(\boldsymbol{i}), \tag{2.2.5}$$

$$\sum_{\boldsymbol{i} \in I^n} e(\boldsymbol{i}) = 1, \tag{2.2.6}$$

$$y_r e(\boldsymbol{i}) = e(\boldsymbol{i}) y_r, \tag{2.2.7}$$

$$\psi_r e(\boldsymbol{i}) = e(s_r \boldsymbol{i}) \psi_r, \tag{2.2.8}$$

$$y_r y_s = y_s y_r, \tag{2.2.9}$$

$$\psi_r y_s = y_s \psi_r, \qquad \text{if } s \neq r, r+1 \tag{2.2.10}$$

$$\psi_r \psi_s = \psi_s \psi_r, \qquad \text{if } |s - r| > 1 \tag{2.2.11}$$

$$\psi_r y_{r+1} e(\boldsymbol{i}) = \begin{cases} (y_r \psi_r + 1) e(\boldsymbol{i}) & \text{if } i_r = i_{r+1} \\ y_r \psi_r e(\boldsymbol{i}) & \text{if } i_r \neq i_{r+1} \end{cases} \tag{2.2.12}$$

$$y_{r+1} \psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_r y_r + 1) e(\boldsymbol{i}) & \text{if } i_r = i_{r+1} \\ \psi_r y_r e(\boldsymbol{i}) & \text{if } i_r \neq i_{r+1} \end{cases} \tag{2.2.13}$$

$$\psi_r^2 e(\boldsymbol{i}) = \begin{cases} 0 & \text{if } i_r = i_{r+1} \\ e(\boldsymbol{i}) & \text{if } i_r \neq i_{r+1} \pm 1 \\ (y_{r+1} - y_r) e(\boldsymbol{i}) & \text{if } i_{r+1} = i_r + 1 \\ (y_r - y_{r+1}) e(\boldsymbol{i}) & \text{if } i_{r+1} = i_r - 1 \end{cases} \tag{2.2.14}$$

$$\psi_r \psi_{r+1} \psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + 1) e(\boldsymbol{i}) & \text{if } i_{r+2} = i_r = i_{r+1} - 1 \\ (\psi_{r+1} \psi_r \psi_{r+1} - 1) e(\boldsymbol{i}) & \text{if } i_{r+2} = i_r = i_{r+1} + 1 \\ (\psi_{r+1} \psi_r \psi_{r+1}) e(\boldsymbol{i}) & \text{otherwise} \end{cases} \tag{2.2.15}$$

where $s_r := (r, r+1)$ is the simple transposition acting in I^n by permutation of the coordinates $r, r+1$ and $k \in \mathbb{Z}$ such that $2k \equiv m \mod l$. The isomorphism maps each of the generators to the element of $\mathcal{H}$ that has the same name. The conditions

$$deg\ e(\boldsymbol{i}) = 0, \qquad deg\ y_r = 2, \qquad deg\ \psi_s e(\boldsymbol{i}) = -a_{i_s, i_{s+1}}$$

for $1 \leq r \leq n$, $1 \leq s \leq n-1$ and $\boldsymbol{i} \in I^n$ define a unique $\mathbb{Z}$-grading on $\mathcal{H}$ with degree function deg.

Following [14] we shall refer to the $e(\boldsymbol{i})$ as the KLR-idempotents. In the following, all statements involving a grading on $\mathcal{H}$ refer to the above Theorem. Note that although the elements L_r and T_r are not homogeneous in $\mathcal{H}$, they can be expressed in terms of homogeneous generators in the following way, see equations (4.42) and (4.43) of [3]:

$$L_r = \sum_{\boldsymbol{i}\in I^n} q^{2i_r}(1 - y_r)e(\boldsymbol{i}) \tag{2.2.16}$$

$$T_r = \sum_{\boldsymbol{i}\in I^n} (\psi_r Q_r(\boldsymbol{i}) - P_r(\boldsymbol{i}))e(\boldsymbol{i}). \tag{2.2.17}$$

CHAPTER 3

Diagrams algebras and combinatorics of tableaux

In this chapter, we recall the realization of the Temperley-Lieb algebra and the blob algebra by planar diagrams. We index the diagrammatical basis of the Temperley-Lieb algebra by pairs of two-columns standard tableaux of the same shape. Similarly, we index the diagrammatical basis of the blob algebra by pairs of one-line standard bitableaux of same shape. Next, we remark the bijection between one-line standard bitableaux and walks on the Bratelli diagram. The set of all of two-columns standard tableaux shall be ordered by dominance, whereas the set of all one-line standard bitableaux shall be ordered in a non-conventional way. We end this chapter by relating these two orders.

3.1 Diagram basis for $Tl_n(q)$

We first recall the diagrammatic realization of the Temperley-Lieb algebra $Tl_n(q)$, first given by L. Kauffman, in which the basis elements are drawn as "(n, n)-bridges" or simply "Temperley-Lieb diagrams". An (n, n)-bridge consists of n points, also called points or nodes, on each of two parallel edges, the "top" resp. "bottom" lines, that are joined pairwise by n non-intersecting lines between the two lines. Figure 3.1 shows two examples.

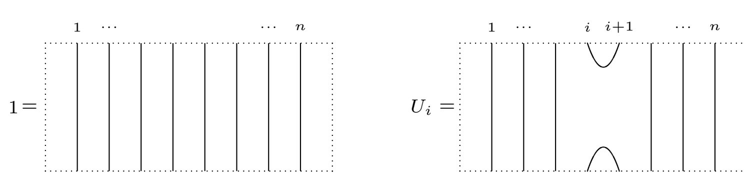

Figure 3.1: Diagrammatic generators of $Tl_n(q)$.

The set of all (n, n)-bridges is denoted by $\mathbb{T}(n)$. We define a multiplication on

$\mathbb{CT}(n)$ by identifying the bottom of the first diagram with the top of the second, and replacing every closed loop that may arise by a factor $-[2]$ (see Figure 3.2).

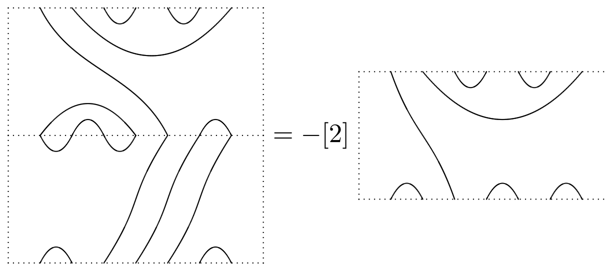

Figure 3.2: Composition in $Tl_7(q)$

With this definition $\mathbb{CT}(n)$ becomes a $\mathbb{C}$-algebra where the one-element is the diagram denoted by 1 in Figure 3.1 . The diagrammatic realization of the Temperley-Lieb algebra refers to the isomorphism of $\mathbb{C}$-algebras $f : Tl_n \to \mathbb{CT}(n)$, given by $f(U_i) = U_i$ where the second U_i is the diagram of Figure 3.1. Using the above isomorphism, we can consider the set $\mathbb{T}(n)$ as a $\mathbb{C}$-basis of $Tl_n(q)$. We refer to this basis as the diagrammatical basis for $Tl_n(q)$.

Our next goal is to index the diagrammatical basis for $Tl_n(q)$ by pairs of two-columns standard tableaux of the same shape. First, we recall some basic combinatorial notions related to partitions and tableaux. Let n be a positive integer. A(n integer) partition of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $|\lambda| := \sum_i \lambda_i = n$ and $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$. The Young diagram of λ is the set

$$[\lambda] = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq j \leq \lambda_i \text{ and } i \geq 1\}.$$

The elements of it are called nodes or entries. It is useful to think of $[\lambda]$ as an array of boxes in the plane, with the indices following matrix conventions. Thus the box with label (i, j) belongs to the i'th row and j'th column. For example if $\lambda = (4, 3, 2, 1)$ then the associated Young diagram is shown in Figure 3.3. If λ is a partition of n we denote by λ' the partition of n obtained from λ by interchanging its rows and columns. A two-column partition of n is a partition λ of n such that $\lambda_i \leq 2$ for all $i \geq 1$. The set of all partitions of n is denoted $\mathrm{Par}(n)$ and the set of two-column partitions of n is denoted by $\mathrm{Par}_2(n)$. A λ-tableau is a bijection $\tau : [\lambda] \to \{1, \dots, n\}$. We say that τ has shape λ and write $\mathrm{Shape}(\tau) = \lambda$. We think of it as a labeling of the diagram of λ, and in this way we can talk of the rows and columns of a tableau. We say that τ is row (resp. column) standard if the entries of

τ increase from left (resp. top) to right (resp. bottom) in each row (resp. column). τ is standard if it is row standard and column standard. The set of all standard λ-tableau is denoted by $\mathrm{Std}(\lambda)$ and the union of all $\mathrm{Std}(\lambda)$ is denoted $\mathrm{Std}(n)$.

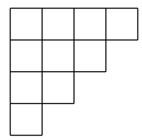

Figure 3.3: Young diagram for $\lambda = (4, 3, 2, 1)$.

Given a node $A = (i, j)$, the (l-)residue of A is defined to be

$$\mathrm{res}(A) := j - i \mod l \tag{3.1.1}$$

For $\tau \in \mathrm{Std}(n)$ and $1 \leq k \leq n$, define the residue of τ at k as $r_\tau(k) := \mathrm{res}(A)$ where A is the node occupied by k in the standard tableau τ.

Example 3.1.1. Figure 3.4 shows the residue diagram for $\lambda = (4, 3, 2, 1)$ when the quantum characteristic is $l = 3$.

0	1	2	0
2	0	1	
1	2		
0			

Figure 3.4: Residue diagram for $\lambda = (4, 3, 2, 1)$ and $l = 3$.

Assume that $\lambda, \mu \in \mathrm{Par}(n)$. We say that λ dominates μ and write $\lambda \trianglerighteq \mu$ if

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$$

for all $j \geq 1$. Then $\mathrm{Par}(n)$ becomes a partially ordered set via $\trianglerighteq$. It can be extended to $\mathrm{Std}(n)$ as follows. For $\sigma, \tau \in \mathrm{Std}(n)$, we say that σ dominates τ and write $\sigma \trianglerighteq \tau$ if $\mathit{Shape}(\sigma_{|_k}) \trianglerighteq \mathrm{Shape}(\tau_{|_k})$, for $k = 1, \ldots, n$, where $\sigma_{|_k}$ and $\tau_{|_k}$ are the tableaux obtained from σ and τ by removing the entries greater than k.

Let τ^λ be the unique standard λ-tableau such that $\tau^\lambda \trianglerighteq \tau$ for all $\tau \in \mathrm{Std}(\lambda)$. In τ^λ the numbers $1, 2, \ldots, n$ are filled in increasingly along the rows from top to bottom. The symmetric group $\mathfrak{S}_n$ acts on the left on the set of λ-tableaux permuting the entries.

For $\tau \in \mathrm{Std}(\lambda)$, we denote by $d(\tau)$ the permutation of $\mathfrak{S}_n$ that satisfies $\tau = d(\tau)\tau^\lambda$.

Example 3.1.2. If $\lambda = (4,3,2,1)$, then Figure 3.5 shows the maximal tableau τ^λ in $\mathrm{Std}(\lambda)$.

1	2	3	4
5	6	7	
8	9		
10			

Figure 3.5: The tableau τ^λ for $\lambda = (4,3,2,1)$.

Let us now recall the bijection between (n,n)-bridges and pairs of two-column standard tableaux of the same shape. Let β be an element of $\mathbb{T}(n)$. We say that a line of β is vertical if it travels from top to bottom, otherwise we say that it is horizontal. Suppose now that β has exactly v vertical lines and set $h = \frac{n-v}{2}$. The associated pair of standard $(h+v,h)'$-tableaux $(\tau_{top}(\beta), \tau_{bot}(\beta))$ is then given by the following rules:

1. k is in the second column of $\tau_{top}(\beta)$ $(\tau_{bot}(\beta))$ if and only if the k-th point is a right endpoint of a horizontal line in the top (bottom) edge.

2. the numbers increase along the columns of $\tau_{top}(\beta)$ and $\tau_{bot}(\beta)$.

For $\lambda \in \mathrm{Par}_2(n)$ and $\sigma, \tau \in \mathrm{Std}(\lambda)$, we denote by $\beta_{\sigma\tau}$ the unique (n,n)-bridge such that $\tau_{top}(\beta_{\sigma\tau}) = \sigma$ and $\tau_{bot}(\beta_{\sigma\tau}) = \tau$.

3.2 Diagram basis for b_n

We aim at generalizing the above results to the case of the blob algebra. For this we first recall the concepts of bipartitions and bitableaux. We provide them with structures of partially ordered sets, in a non-conventional way.

A bipartition of n is a pair $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ of usual (integer) partitions such that $n = |\lambda^{(1)}| + |\lambda^{(2)}|$. By the Young diagram of a bipartition $\boldsymbol{\lambda}$ we mean the set

$$[\boldsymbol{\lambda}] = \{(r,c,d) \in \mathbb{N} \times \mathbb{N} \times \{1,2\} \mid 1 \leq c \leq \lambda_r^{(k)}\}.$$

Its elements are called entries or nodes. We can visualize $[\boldsymbol{\lambda}]$ as a pair of usual Young diagrams called the components of $[\boldsymbol{\lambda}]$. Thus for $j = 1,2$, the j'th component of $[\boldsymbol{\lambda}]$ is $\{(r,c,d) \in [\boldsymbol{\lambda}] \,|\, d = j\}$. A one-line bipartition of n is a bipartition $\boldsymbol{\lambda}$ of n such that $\lambda_r^{(d)} = 0$ for all $r \geq 2$ and $d = 1,2$. Thus, a node $A = (r,c,d) \in [\boldsymbol{\lambda}]$ for some one-line bipartition only if $r = 1$. The set of all one-line bipartitions of n is denoted $\mathrm{Bip}_1(n)$. For $\boldsymbol{\lambda}$ a bipartition, a $\boldsymbol{\lambda}$-bitableau is a bijection $\mathfrak{t} : [\boldsymbol{\lambda}] \to \{1, \dots, n\}$. We say that $\mathfrak{t}$ has shape $\boldsymbol{\lambda}$ and write $\mathrm{Shape}(\mathfrak{t}) = \boldsymbol{\lambda}$. A $\boldsymbol{\lambda}$-bitableau $\mathfrak{t}$ is called standard if the entries of $\mathfrak{t}$ increase from left to right in each component. The set of all standard $\boldsymbol{\lambda}$-bitableaux is denoted by $\mathrm{Std}(\boldsymbol{\lambda})$ and the union $\bigcup_{\boldsymbol{\lambda}} \mathrm{Std}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda}$ running over all bipartitions of n is denoted by $\mathrm{Std}(\boldsymbol{n})$. Given a node $A = (1, c, d)$ the residue of A is defined to be

$$\mathrm{res}(A) = \begin{cases} c - 1 + k, & \text{if } d = 1 \\ c - 1 - k, & \text{if } d = 2 \end{cases} \tag{3.2.1}$$

where $k \in \mathbb{Z}$ such that $2k \equiv m \mod l$. For $\mathfrak{t} \in \mathrm{Std}(n)$ and $1 \leq j \leq n$, define the residue of $\mathfrak{t}$ at j, as $r_{\mathfrak{t}}(j) := \mathrm{res}(A)$ where A is the node occupied by j in $\mathfrak{t}$.

Example 3.2.1. Figure 3.6 shows the residue diagram for $\boldsymbol{\lambda} = ((5), (6))$ when the quantum characteristic is $l = 5$ and $m = 2$.

$$\left(\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 0 \\ \hline \end{array} \; , \; \begin{array}{|c|c|c|c|c|c|} \hline 4 & 0 & 1 & 2 & 3 & 4 \\ \hline \end{array} \right)$$

Figure 3.6: Residue diagram for $\boldsymbol{\lambda} = ((5), (6))$, $m = 2$ and $l = 5$.

There are several ways of endowing $\mathrm{Bip}_1(n)$ with an order structure, the most well known being dominance order, but we shall need a different order on $\mathrm{Bip}_1(n)$ that we now explain. Let Λ_n be the set $\{-n, -n+2, \dots, n-2, n\}$. Then the following definition makes Λ_n into a totally ordered set with order relation $\succ$.

Definition 3.2.2. Suppose $\lambda, \mu \in \Lambda_n$. We then define $\mu \succeq \lambda$ if either $|\mu| < |\lambda|$, or if $|\mu| = |\lambda|$ and $\mu \leq \lambda$.

On the other hand, the map f given by

$$f : \mathrm{Bip}_1(n) \to \Lambda_n, \; ((a), (b)) \to a - b$$

is a bijection and so we can define a total order $\succeq$ on $\mathrm{Bip}_1(n)$ as follows.

Definition 3.2.3. Suppose $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{Bip}_1(n)$. Then we define $\boldsymbol{\lambda} \succeq \boldsymbol{\mu}$ iff $f(\boldsymbol{\lambda}) \succeq f(\boldsymbol{\mu})$.

For $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ let $\mathfrak{t}_{|_k}$ be the bitableau obtained from $\mathfrak{t}$ by removing the entries greater than k. We extend the order $\succeq$ to the set of all $\boldsymbol{\lambda}$-standard bitableaux as follows.

Definition 3.2.4. Suppose that $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. We define $\mathfrak{s} \succeq \mathfrak{t}$ if $\mathrm{Shape}(\mathfrak{s}_{|_k}) \succeq \mathrm{Shape}(\mathfrak{t}_{|_k})$ for all $k = 1, \ldots, n$.

Note that this is only a partial order on $\mathrm{Std}(\boldsymbol{\lambda})$. Let $\mathfrak{t}^{\boldsymbol{\lambda}}$ be the unique standard $\boldsymbol{\lambda}$-bitableau such that $\mathfrak{t}^{\boldsymbol{\lambda}} \succeq \mathfrak{t}$ for all $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. For $\boldsymbol{\lambda} = ((a),(b))$, set $c = \min\{a, b\}$. Then in $t^{\boldsymbol{\lambda}}$ the numbers $1, 2, \ldots, n$ are located increasingly along the rows according to the following rules:

1. even numbers less than or equal to $2c$ are placed in the first component.

2. odd numbers less than $2c$ are placed in the second component.

3. numbers greater than $2c$ are placed in the remaining boxes.

Example 3.2.5. Figure 3.7 shows the maximal bitableaux with respect to $\succeq$ for $\boldsymbol{\lambda} = ((8),(3))$ and $\boldsymbol{\mu} = ((3),(8))$.

$$\mathfrak{t}^{\lambda} = \left(\boxed{2}\boxed{4}\boxed{6}\boxed{7}\boxed{8}\boxed{9}\boxed{10}\boxed{11} \, , \, \boxed{1}\boxed{3}\boxed{5} \right)$$

$$\mathfrak{t}^{\mu} = \left(\boxed{2}\boxed{4}\boxed{6} \, , \, \boxed{1}\boxed{3}\boxed{5}\boxed{7}\boxed{8}\boxed{9}\boxed{10}\boxed{11} \right)$$

Figure 3.7: Maximal bitableaux with respect to the order $\succeq$.

Definition 3.2.6. Suppose that $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and let $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Define a sequence of integers inductively by the rules $\mathfrak{t}(0) = 0$ and for $1 \leq j \leq n$

$$\mathfrak{t}(j) = \mathfrak{t}(j-1) \pm 1$$

where the $+$ $(-)$ sign is used if j is in the first (second) component of $\mathfrak{t}$.

Using this sequence we can now describe the order $\succeq$.

Lemma 3.2.7. *If $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, then $\mathfrak{s} \succeq \mathfrak{t}$ if and only if $|\mathfrak{s}(j)| \leq |\mathfrak{t}(j)|$, for all $1 \leq j \leq n$, and if $|\mathfrak{s}(j)| = |\mathfrak{t}(j)|$ then $\mathfrak{s}(j) \leq \mathfrak{t}(j)$.*

Proof: Note that for all $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $1 \leq j \leq n$, we have $\mathfrak{t}(j) = f(\mathrm{Shape}(\mathfrak{t}|_j))$. Therefore, the result is a direct consequence of Definition 3.2.4. $\square$

As is the case for the Temperley-Lieb algebra, the blob algebra has a diagrammatic realization that we now explain. A "blob diagram on n points", or just a blob diagram when no confusion arises, is an (n,n)-bridge with possible decorations of "blobs" on certain of its lines. The blobs appear subject to the following conditions. Each line is decorated with at most one blob; no line to the right of the leftmost vertical line may be decorated; and to the left of it, only the outermost line in any nested formation of loop lines can be decorated. The set of blob diagrams on n points is denoted $\mathbb{B}(n)$. Figure 3.8 shows an example of a blob diagram on 11 points with a blob on all lines that accept decoration.

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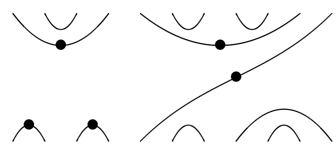


Figure 3.8: A blob diagram in $\mathbb{B}(11)$

Similar to the Temperley-Lieb case, there is now a multiplication on $\mathbb{C}\mathbb{B}(n)$, defined using a concatenation procedure. This may give rise to internal loops and multiple blobs on certain lines. We then impose the rules on the multiplication that any diagram with multiple blobs on one or several lines is considered equal to the same diagram with a single blob on those lines, and any internal loop is removed from the diagram multiplying by y_e, if the loop is decorated, otherwise by $-(q+q^{-1})$. The realization of $b_n(m)$ is now the isomorphism $f : b_n(m) \to \mathbb{C}\mathbb{B}(n)$, mapping U_i and e to the diagrams U_i and e, given in Figure 3.1 and 3.9.

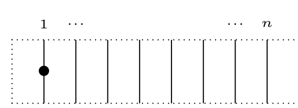
$e =$ 

Figure 3.9: Blob generator e.

Our next goal is to establish a bijection between the set of blob diagrams and the set of pairs of one-line standard bitableaux of same shape. Let $\mathfrak{m}$ be a blob diagram. Given a horizontal line l, in either edge, we put $l = (a,b)$ where a is the left endpoint and b is the right endpoint. Let $l_1 = (a_1, b_1)$ and $l_2 = (a_2, b_2)$ be horizontal lines on the same edge. We say that l_1 covers l_2 if $a_1 < a_2 < b_2 < b_1$. We also say that the leftmost vertical line (if any) covers all lines to the right of it. Now, we say that a node is covered if the line to which it belongs is decorated or the line to which it belongs is covered by a decorated line. If a node is not covered, we call it uncovered.

Definition 3.2.8. Let $\mathfrak{m}$ be a blob diagram. Suppose that $\mathfrak{m}$ has exactly v vertical lines and $h = \frac{n-v}{2}$ horizontal lines on each edge.

- If $v \geq 0$ and the leftmost vertical line is not decorated or there is no vertical lines then we associate to $\mathfrak{m}$ a pair of $\boldsymbol{\lambda}$-bitableaux, $\mathfrak{t}_{top}(\mathfrak{m})$ and $\mathfrak{t}_{bot}(\mathfrak{m})$, with $\boldsymbol{\lambda} = ((h+v),(h))$ by the following rules

  1. k is in the second component of $\mathfrak{t}_{top}(\mathfrak{m})$ ($\mathfrak{t}_{bot}(\mathfrak{m})$) if and only if: either k is uncovered and it is the right endpoint of a horizontal line on the top (bottom) edge, or it is covered and it is the left endpoint of a horizontal line on the top (bottom) edge
  2. the numbers increase along rows.

- If $v > 0$ and the leftmost vertical line is decorated then we associate to $\mathfrak{m}$ a pair of $\boldsymbol{\lambda}$-bitableaux, $\mathfrak{t}_{top}(\mathfrak{m})$ and $\mathfrak{t}_{bot}(\mathfrak{m})$, with $\boldsymbol{\lambda} = ((h),(h+v))$ by the following rules

  1. k is in the first component of $\mathfrak{t}_{top}(\mathfrak{m})$ ($\mathfrak{t}_{bot}(\mathfrak{m})$) if and only if: either it is uncovered and it is the left endpoint of a horizontal line on the top (bottom) edge or it is covered and it is the right endpoint of a horizontal line on the top (bottom) edge
  2. the numbers increase along rows.

We view these rules as a generalization of the bijection between $\mathbb{T}(n)$ and $\mathrm{Par}_2(n)$, with the two components of the bitableau replacing the two columns of the element of $\mathrm{Par}_2(n)$ and with the presence of a cover reversing the roles of left and right.

For $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, we let $\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}$ denote the unique blob diagram such that $\mathfrak{t}_{top}(\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}) = \mathfrak{s}$ and $\mathfrak{t}_{bot}(\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}) = \mathfrak{t}$.

Remark 3.2.9. For all $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $1 \leq j \leq n$, we have

(i) If $\mathfrak{t}(j) < 0$ then the node k is covered in the top edge of $\mathfrak{m}_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\lambda}}}$.

(ii) If the node k is covered in the top edge of $\mathfrak{m}_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\lambda}}}$ then $\mathfrak{t}(j) \leq 0$.

Example 3.2.10. Let $\mathfrak{m}$ be the blob diagram in Figure 3.8 then

$$\mathfrak{t}_{top}(\mathfrak{m}) = \left(\boxed{3 \mid 4 \mid 7 \mid 9 \mid 10}\ ,\ \boxed{1 \mid 2 \mid 5 \mid 6 \mid 8 \mid 11} \right)$$

$$\mathfrak{t}_{bot}(\mathfrak{m}) = \left(\boxed{2 \mid 4 \mid 7 \mid 10 \mid 11}\ ,\ \boxed{1 \mid 3 \mid 5 \mid 6 \mid 8 \mid 9} \right)$$

Figure 3.10: The pair of bitableaux associated with $\mathfrak{m}$.

3.3 Walks on the Bratteli diagram

We now explain how the Bratteli diagram for $b_n(m)$ provides a useful interpretation of the order $\succ$ on $\mathrm{Std}(\boldsymbol{\lambda})$. Let $\mathbb{B}^{top}(n)$ (resp. $\mathbb{B}^{bot}(n)$) denote the set of upper (lower) halves of blob diagrams. To be more precise, $\mathbb{B}^{top}(n)$ (resp. $\mathbb{B}^{bot}(n)$) consists of all blob diagrams on n points with the information on the bottom (top) points of the vertical lines omitted. Thus $\mathbb{B}^{top}(n)$ (resp. $\mathbb{B}^{bot}(n)$) is in bijection with $\mathrm{Std}(\boldsymbol{n})$ via $\mathfrak{m} \mapsto \mathfrak{t}_{top}(\mathfrak{m})$ (resp. $\mathfrak{m} \mapsto \mathfrak{t}_{bot}(\mathfrak{m})$) and so $\mathbb{B}^{top}(n)$ and $\mathbb{B}^{bot}(n)$ are in bijection with each other. On the diagrammatic level, the bijection can be visualized as a reflection through a horizontal axis.

Recall that the Bratteli diagram for $b_n(m)$ gives an enumeration of $\mathbb{B}^{top}(n)$ through a Pascal triangle pattern, see [23]. To be precise, for $\lambda \in \Lambda_n$ the Bratteli diagram associates with the point (λ, n) of the plane the set $\mathbb{B}^{top}(n, \lambda)$, defined as those diagrams from $\mathbb{B}^{top}(n)$ that have exactly $|\lambda|$ vertical lines, where the leftmost vertical line is decorated iff λ is negative. Set $b_{n,\lambda} := |\mathbb{B}^{top}(n, \lambda)|$ with the convention that $\mathbb{B}^{top}(n, \lambda) := \emptyset$ if $\lambda \notin \Lambda_n$. Then there is a bijection between $\mathbb{B}^{top}(n, \lambda)$ and $\mathbb{B}^{top}(n-1, \lambda+1) \cup \mathbb{B}^{top}(n-1, \lambda-1)$, as we explain shortly. The Pascal triangle formula $b_{n,\lambda} = b_{n-1,\lambda+1} + b_{n-1,\lambda-1}$ is a consequence of this bijection.

For $\lambda \in \Lambda_n \setminus \{0\}$ define $\lambda^+ \in \Lambda_{n+1}$ by $\lambda^+ := \lambda \pm 1$ where the sign is positive iff $\lambda > 0$. Similarly, for $\lambda \in \Lambda_n \setminus \{0\}$ define $\lambda^- := \lambda \pm 1$ where the sign is positive iff $\lambda < 0$. Finally, if $\lambda = 0 \in \Lambda_n$ define $\lambda^+ := 1$ and $\lambda^- := -1$. With these definitions we have for any $\lambda \in \Lambda_n$ that $\lambda^- \succ \lambda^+$ in Λ_{n+1}. In other words, the map $\lambda \mapsto \lambda^-$ moves λ closer to the central axis of the Bratteli diagram consisting of the points $\{(0, k), k = 0, 1 \ldots\}$, whereas $\lambda \mapsto \lambda^+$ takes λ away from the central axis.

The above mentioned bijection is now induced by injective maps

$$f^+_{n,\lambda} : \mathbb{B}^{top}(n-1, \lambda) \to \mathbb{B}^{top}(n, \lambda^+), \ \ f^-_{n,\lambda} : \mathbb{B}^{top}(n-1, \lambda) \to \mathbb{B}^{top}(n, \lambda^-) \qquad (3.3.1)$$

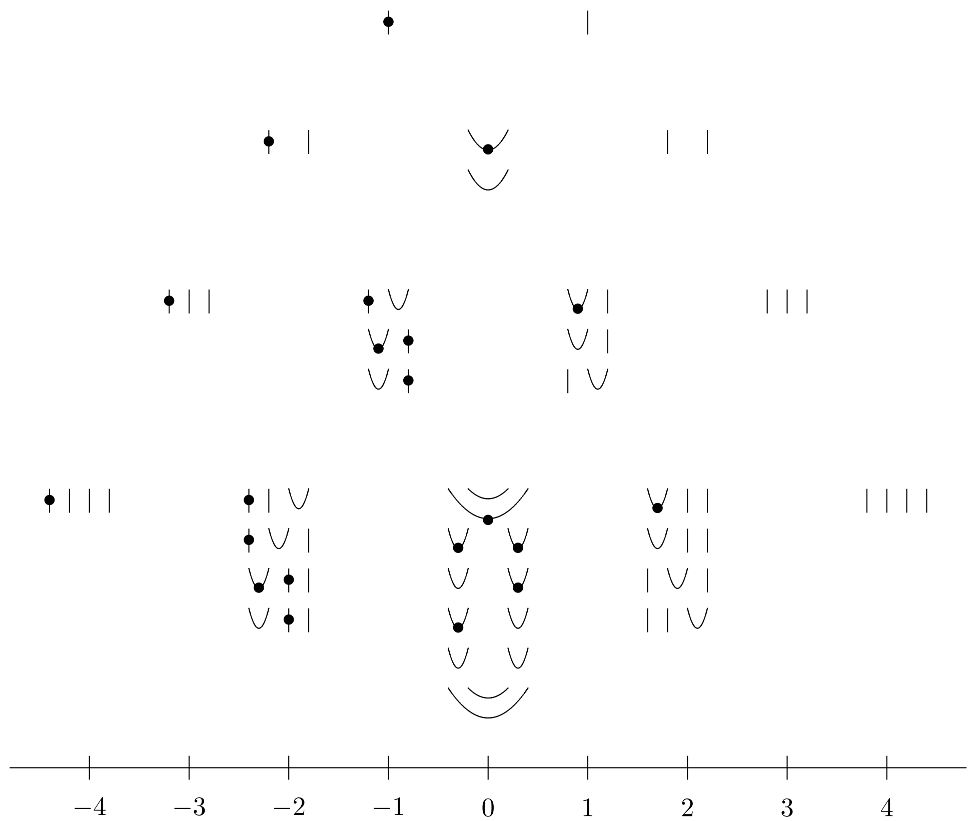


Figure 3.11: Bratteli diagram

that can be described concretely as follows. If $\mathfrak{m} \in \mathbb{B}^{top}(n-1,\lambda)$ then $f^{+}_{n,\lambda}$ adds an undecorated vertical line on the right hand side of $\mathfrak{m}$. If $\lambda \neq 0$ then $f^{-}_{n,\lambda}$ joins the rightmost vertical line of $\mathfrak{m}$ with the new n'st point of the (top) edge whereas $f^{+}_{n,0}$ adds a decorated vertical line on the right hand side of $\mathfrak{m}$. Finally, by convention $f^{+}_{1,0}$ (resp. $f^{-}_{1,0}$) maps the empty diagram to the unique diagram of $\mathbb{B}^{top}(1,1)$ (resp. $\mathbb{B}^{top}(1,-1)$).

For us the main point of this construction is that any element of $\mathfrak{m} \in \mathbb{B}^{top}(n)$ can be written uniquely as

$$\mathfrak{m} = f^{\sigma_n}_{n,\lambda_n} \cdots f^{\sigma_1}_{1,0}\, \emptyset \text{ where } \sigma_k \in \{+,-\} \text{ for } k = 1,\ldots,n. \tag{3.3.2}$$

In other words, the sequence of signs $\{\sigma_k\}_{k=1,\ldots,n}$ uniquely determines $\mathfrak{m}$ and hence $\mathbb{B}^{top}(n)$ is in bijection with walks on the Bratteli diagram, starting with the empty partition in position $(0,0)$ and at the k'th step, where the walk is situated in (k,λ_k), going inwards or outwards according to the value of σ_k. We denote by $W(\mathfrak{m})$ the walk associated with $\mathfrak{m} \in \mathbb{B}^{top}(n)$.

Let us now return to the order $\succeq$ on $\mathrm{Std}(\boldsymbol{\lambda})$ introduced above. Suppose that $\mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$. Then $\mathfrak{s}$ also gives rise to a walk, denoted $w(\mathfrak{s})$, on the points of the Bratteli diagram. It starts in $(0,0)$ and for $k = 0, 1, \ldots, n-1$ goes from (k, j) to $(k+1, j-1)$ if $k+1$ is located in the second component of $\mathfrak{s}$ and to $(k+1, j+1)$ if $k+1$ is located in the first component of $\mathfrak{s}$. In other words, at the k'th step the walk $w(\mathfrak{s})$ is situated in $(k, \mathfrak{s}(k))$ where $\{\mathfrak{s}(k) \,|\, k = 0, 1, \ldots, n\}$ is the sequence of integers associated with $\mathfrak{s}$ as in Definition 3.2.6. With this walk realization of the bitableaux, we can visualize the order $\succeq$. Indeed, let $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then $\mathfrak{s} \succeq \mathfrak{t}$ iff at each step of the two walks $w(\mathfrak{s})$ is either strictly closer than $w(\mathfrak{t})$ to the central vertical axis of the Bratteli diagram or they are at the same distance from the central axis and $w(\mathfrak{s})$ is located (weakly) to the left of $w(\mathfrak{t})$.

Let us now explain the relationship between the two walks. We denote by s the bijection $\mathbb{B}^{top}(n) \to \mathrm{Std}(\boldsymbol{n})$, $\mathfrak{m} \mapsto \mathfrak{t}_{top}(\mathfrak{m})$, mentioned above.

Lemma 3.3.1. *Let $\mathfrak{m} \in \mathbb{B}^{top}(n)$. Then we have $W(\mathfrak{m}) = w(s(\mathfrak{m}))$.*

Proof: This is a consequence of Remark 3.2.9 and the definitions. □

There is a natural surjective map $\pi : \mathbb{B}(n) \to \mathbb{T}(n)$, which sends a blob diagram $\mathfrak{m}$ to the (n,n)-bridge obtained by deleting all decorations in $\mathfrak{m}$. On the other hand, $\mathbb{T}(n)$ is in bijection with pairs of two-column standard tableaux of the same shape and $\mathbb{B}(n)$ is in bijection with pairs of one-line standard bitableaux of the same shape by Definition 3.2.8, and so our next goal is to describe the above map π in terms of one-line bitableaux and two-column tableaux. For this we make a couple of definitions.

Definition 3.3.2. Suppose that $\boldsymbol{\lambda} = ((a),(b)) \in \mathrm{Bip}_1(n)$ and let $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Set $\mu_1 = \max\{a, b\}$ and $\mu_2 = \min\{a, b\}$. Let μ be the two-column partition of n given by $\mu = (\mu_1, \mu_2)'$. Then we define $\tau_{\mathfrak{t}}$ as the unique μ-standard tableau that satisfies

$$k \textit{ is in the second column of } \tau_{\mathfrak{t}} \textit{ if and only if } |\mathfrak{t}(k)| < |\mathfrak{t}(k-1)|.$$

We claim that $\tau_{\mathfrak{t}}$ defined in this way is a standard tableau. For this we use that a node k of the blob diagram given by $\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}$ is a right endpoint in the top (resp. bottom) edge if and only if $|\mathfrak{s}(k)| < |\mathfrak{s}(k-1)|$ (resp. $|\mathfrak{t}(k)| < |\mathfrak{t}(k-1)|$), as can easily be seen by analyzing Definition 3.2.8. In other words, $\tau_{\mathfrak{s}}$ and $\tau_{\mathfrak{t}}$ can be described as the unique two-column tableaux that satisfy $\pi(\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}) = \beta_{\tau_{\mathfrak{s}}\tau_{\mathfrak{t}}}$, where π is the map defined above, and our claim follows.

For $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ we let $|w(\mathfrak{s})|$ denote the walk on the Bratteli diagram that at the k'th step is located in the point $(k, |\mathfrak{s}(k)|)$. The two components of its associated bitableau are then the conjugates of the columns of $\tau_{\mathfrak{s}}$, as follows from the above.

Definition 3.3.3. For $\boldsymbol{\lambda} \in \operatorname{Bip}_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, we write $\mathfrak{s} \sim \mathfrak{t}$ if $\tau_{\mathfrak{s}} = \tau_{\mathfrak{t}}$. Thus $\mathfrak{s} \sim \mathfrak{t}$ if and only if $|\mathfrak{s}(k)| = |\mathfrak{t}(k)|$ for all $1 \leq k \leq n$.

We give a couple of Lemmas related to these definitions.

Lemma 3.3.4. *Suppose that $\boldsymbol{\lambda} \in Bip_1(n)$ and let $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Then, $\tau_{\mathfrak{s}} \trianglerighteq \tau_{\mathfrak{t}}$ if and only if $|\mathfrak{s}(k)| \leq |\mathfrak{t}(k)|$ for all $1 \leq k \leq n$. In particular, if $\mathfrak{s} \succeq \mathfrak{t}$ then $\tau_{\mathfrak{s}} \trianglerighteq \tau_{\mathfrak{t}}$.*

Proof: Notice that

$$\operatorname{Shape}(\tau_{\mathfrak{s}}|_k) = \left(\frac{k + |\mathfrak{s}(k)|}{2}, \frac{k - |\mathfrak{s}(k)|}{2}\right)'$$

$$\operatorname{Shape}(\tau_{\mathfrak{t}}|_k) = \left(\frac{k + |\mathfrak{t}(k)|}{2}, \frac{k - |\mathfrak{t}(k)|}{2}\right)'$$

for all $1 \leq k \leq n$. Using the property of the usual dominance order that $\mu \trianglerighteq \nu \iff \nu' \trianglerighteq \mu'$ we deduce that $\tau_{\mathfrak{s}} \trianglerighteq \tau_{\mathfrak{t}}$ if and only if $|\mathfrak{s}(k)| \leq |\mathfrak{t}(k)|$ for all $1 \leq k \leq n$, which is the first claim of the Lemma. The second claim follows now from Lemma 3.2.7. □

Using the natural embedding $\iota : \mathbb{T}(n) \to \mathbb{B}(n)$ we obtain a walk description of the elements of $\mathbb{T}(n)$ as well. Under this description, $\mathbb{T}(n)$ corresponds to the walks on the Bratteli diagram for $b_n(m)$ that always stay in the positive half of the Bratteli diagram, including the central vertical axis.

The left action of $\mathfrak{S}_n$ on tableaux generalizes to an action of $\mathfrak{S}_n$ on bitableaux. Using it we have the following Lemma.

Lemma 3.3.5. *Suppose that $\boldsymbol{\lambda} \in Bip_1(n)$ and let $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Suppose moreover that $\mathfrak{s} \succ \mathfrak{t}$, that $s_k\mathfrak{s} = \mathfrak{t}$ for some $s_k \in \mathfrak{S}_n$ and that $\mathfrak{s} \nsim \mathfrak{t}$. Then $s_k\tau_{\mathfrak{s}} = \tau_{\mathfrak{t}}$ and $\tau_{\mathfrak{s}} \triangleright \tau_{\mathfrak{t}}$.*

Proof: Note first that by the assumptions we have $\mathfrak{s}(j) = \mathfrak{t}(j)$ for all $j \neq k$. Let us first assume that $\mathfrak{s}(k+1) \geq 1$. Then $\mathfrak{s}(k) \geq 0$ since $\mathfrak{s}(j)$ changes by ± 1 when j is increased by 1. But similarly $\mathfrak{t}(k) \geq 0$ and then we must have $\mathfrak{t}(k) = \mathfrak{s}(k) + 2$ since $\mathfrak{s} \succ \mathfrak{t}$. Since k and $k+1$ are located in different components in $\mathfrak{s}$ and in $\mathfrak{t}$, this gives us the equalities

$$\mathfrak{s}(k-1) = \mathfrak{t}(k-1) = \mathfrak{s}(k) + 1 = \mathfrak{t}(k) - 1 = \mathfrak{s}(k+1) = \mathfrak{t}(k+1)$$

from which we get by Definition 3.3.2 that k (resp. $k+1$) is located in second (resp. first) column of $\tau_{\mathfrak{s}}$ whereas k (resp. $k+1$) is placed in first (resp. second) column of $\tau_{\mathfrak{t}}$. Since j is located in the same column of $\tau_{\mathfrak{s}}$ and $\tau_{\mathfrak{t}}$ for $j \neq k, k+1$ we now conclude that $s_k \tau_{\mathfrak{s}} = \tau_{\mathfrak{t}}$ and $\tau_{\mathfrak{s}} \rhd \tau_{\mathfrak{t}}$, as needed.

The case $\mathfrak{s}(k+1) \leq -1$ is treated similarly and so the only remaining case is $\mathfrak{s}(k+1) = 0$. Then $\mathfrak{t}(k+1) = \mathfrak{s}(k-1) = \mathfrak{t}(k-1) = 0$. Moreover since $\mathfrak{s} \succ \mathfrak{t}$ we have $\mathfrak{s}(k) = -1$ and $\mathfrak{t}(k) = 1$. But this implies that $\mathfrak{s} \sim \mathfrak{t}$, finishing the proof. □

Definition 3.3.6. Suppose that $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and that $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then we say that "$\mathfrak{s}$ has a hook at position k" if $\mathfrak{s}(k-1) = \mathfrak{s}(k+1) = \mathfrak{s}(k) \pm 1$ where $1 \leq k \leq n-1$. Moreover we say that "$\mathfrak{t}$ is obtained from $\mathfrak{s}$ by making a hook at position k smaller" if $\mathfrak{s}(j) = \mathfrak{t}(j)$ for $j \neq k$, $\mathfrak{s}(k) = \mathfrak{t}(k) \pm 2$ and $\mathfrak{s} \succ \mathfrak{t}$.

The last condition can also be written as $s_k \mathfrak{s} = \mathfrak{t}$ and $\mathfrak{s} \succ \mathfrak{t}$. Geometrically, if $\mathfrak{t}$ is obtained from $\mathfrak{s}$ by making a hook at position k smaller then $\mathfrak{t}$ is obtained from $\mathfrak{s}$ by either replacing a configuration of three consecutive points in $w(\mathfrak{t})$ forming a "$\langle$" by a configuration "$\rangle$" at these three points, or reversely, depending on which side of the Bratteli diagram the configuration is located.

Lemma 3.3.7. *For $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ we define $\mathfrak{d}(\mathfrak{t})$ as the element of $\mathfrak{S}_n$ that satisfies $\mathfrak{t} = \mathfrak{d}(\mathfrak{t})\mathfrak{t}^{\boldsymbol{\lambda}}$. Then $\mathfrak{d}(\mathfrak{t})$ can be written as product of simple transpositions $\mathfrak{d}(\mathfrak{t}) = s_{i_k} s_{i_{k-1}} \dots s_{i_1}$ such that $s_{i_j} \dots s_{i_1} \mathfrak{t}^{\boldsymbol{\lambda}}$ is standard and such that $s_{i_j} s_{i_{j-1}} \dots s_{i_1} \mathfrak{t}^{\boldsymbol{\lambda}} \prec s_{i_{j-1}} \dots s_{i_1} \mathfrak{t}^{\boldsymbol{\lambda}}$ for all $1 \leq j \leq k$.*

Proof: This can be seen via the walk realization of $\mathrm{Std}(\boldsymbol{\lambda})$. Indeed the walk $w(\mathfrak{t}^{\boldsymbol{\lambda}})$ first zigzags on and off the central vertical line of the Bratteli diagram, using the sign $-$ an even number of times, and then finishes using the sign $+$ repeatedly, if $\boldsymbol{\lambda}$ is located in the positive half, or using once the sign $-$ followed by the sign $+$ repeatedly, if $\boldsymbol{\lambda}$ is located in the negative half.

This walk can be converted into $w(\mathfrak{t})$ through a series of k walks, say, where at each step the new walk is obtained from the previous one by making a hook at position j smaller, for some j. At tableau level, each of these transformations is given by the action of a simple transposition s_j. The Lemma follows from this. □

Example 3.3.8. We illustrate the above Lemma. Let $\boldsymbol{\lambda} = ((4),(2)) \in \mathrm{Bip}_1(6)$ and

$$\mathfrak{t} = \left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array} \right)$$

Then, we have $\mathfrak{d}(\mathfrak{t}) = s_3 s_4 s_2 s_3 s_1$. Now, define the bitableaux $\mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4$ and $\mathfrak{t}_5$ as follows:

$$\mathfrak{t}_0 = \mathfrak{t}^{\boldsymbol{\lambda}} = \left(\begin{array}{|c|c|c|c|}\hline 2&4&5&6\\\hline\end{array} \, , \begin{array}{|c|c|}\hline 1&3\\\hline\end{array} \right)$$

$$\mathfrak{t}_1 = s_1\mathfrak{t}_0 = \left(\begin{array}{|c|c|c|c|}\hline 1&4&5&6\\\hline\end{array} \, , \begin{array}{|c|c|}\hline 2&3\\\hline\end{array} \right)$$

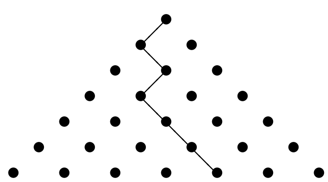
$$\mathfrak{t}_2 = s_3\mathfrak{t}_1 = \left(\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline\end{array} \, , \begin{array}{|c|c|}\hline 2&4\\\hline\end{array} \right)$$

$$\mathfrak{t}_3 = s_2\mathfrak{t}_2 = \left(\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline\end{array} \, , \begin{array}{|c|c|}\hline 3&4\\\hline\end{array} \right)$$

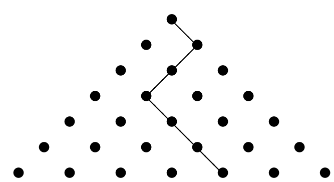
$$\mathfrak{t}_4 = s_4\mathfrak{t}_3 = \left(\begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline\end{array} \, , \begin{array}{|c|c|}\hline 3&5\\\hline\end{array} \right)$$

$$\mathfrak{t}_5 = s_3\mathfrak{t}_4 = \left(\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline\end{array} \, , \begin{array}{|c|c|}\hline 4&5\\\hline\end{array} \right)$$

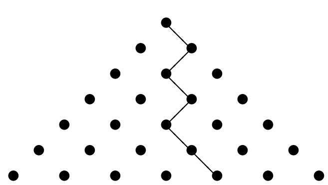
It is straightforward to check that $\mathfrak{t}^{\boldsymbol{\lambda}} = \mathfrak{t}_0 \succ \mathfrak{t}_1 \succ \mathfrak{t}_2 \succ \mathfrak{t}_3 \succ \mathfrak{t}_4 \succ \mathfrak{t}_5 = \mathfrak{t}$. The figures above show how the walk $w(\mathfrak{t}^{\boldsymbol{\lambda}})$ is converted into $w(\mathfrak{t})$.



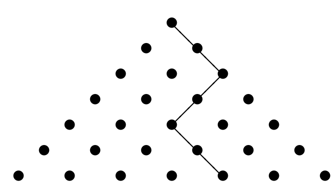
$w(\mathfrak{t}_0)$



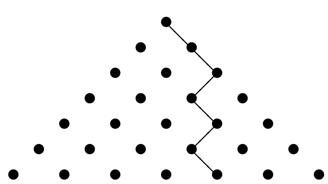
$w(\mathfrak{t}_1)$



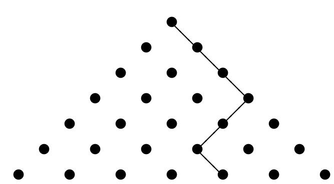
$w(\mathfrak{t}_2)$



$w(\mathfrak{t}_3)$



$w(\mathfrak{t}_4)$



$w(\mathfrak{t}_5)$

Remark 3.3.9. Although we do not need it directly, we note that $l(\mathfrak{d}(\mathfrak{t})) = k$ and that the expression $\mathfrak{d}(\mathfrak{t}) = s_{i_k} s_{i_{k-1}} \dots s_{i_1}$ is reduced.

CHAPTER 4

Graded representation theory

In this chapter we introduce the basic notions of graded representation theory. Our main emphasis will be the concept of graded cellular algebra, this concept plays a central role in the remainder of this thesis. At the end of this chapter we show that the $Tl_n(q)$ and $b_n(m)$ are $\mathbb{Z}$-graded algebras.

4.1 Basic definitions

From now on we adopt the convention that all modules considered are left modules, unless otherwise specified. Fix an integral domain R. A graded R-module is a R-module M which has a direct sum decomposition $M = \bigoplus_{k\in\mathbb{Z}} M_k$. For $k \in \mathbb{Z}$, if $m \in M_k$ we say that m is an homogeneous element of degree k and we set $\deg(m) = k$. If M is a graded R-module denote by $\underline{M}$ to the ungraded R-module obtained by forgetting the grading on M. For any $z \in \mathbb{Z}$ and graded R-module $M = \bigoplus_{k\in\mathbb{Z}} M_k$, let $M\langle z\rangle$ be the graded R-module obtained by shifting the grading on M up by z, thus $M\langle z\rangle_k = M_{k-z}$ for all $k \in \mathbb{Z}$. A graded R-algebra is an unital associative R-algebra A which is graded as R-module such that $A_kA_l \subset A_{k+l}$, for all $(k,l) \in \mathbb{Z}^2$, where $A = \bigoplus_{k\in\mathbb{Z}} A_k$ is the direct sum decomposition in homogeneous components given by the grading on A. A graded A-module is a graded R-module $M = \bigoplus_{k\in\mathbb{Z}} M_k$ such that $\underline{M}$ is an $\underline{A}$-module and $A_kM_l \subset M_{k+l}$, for all $(k,l) \in \mathbb{Z}^2$.

A graded R-module $M = \bigoplus_k M_k$ is positively graded if $M_k = 0$ whenever $k < 0$. That is, all of the homogeneous elements of M have non-negative degree. Suppose that A is a R-algebra positively graded and that $M = \bigoplus_k M_k$ is a finite dimensional graded A-module. For each $j \in \mathbb{Z}$ let $\mathcal{G}_j M = \bigoplus_{k \geq j} M_k$. Since A is positively graded $\mathcal{G}_j M$ is a graded A-submodule of M. Let a be maximal and z be minimal such that $\mathcal{G}_a M = M$ and $\mathcal{G}_z M = 0$, respectively. Then the grading filtration of M is the filtration

$$0 = \mathcal{G}_z M \leq \mathcal{G}_{z-1} M \leq \ldots \mathcal{G}_a M = M \tag{4.1.1}$$

Let t be an indeterminate over $\mathbb{Z}$. Let M be a finite dimensional graded A-module and let $M = \bigoplus_{k \in \mathbb{Z}} M_k$ its direct sum decomposition in homogeneous components, define its graded dimension by an explicit Laurent polynomial

$$\dim_t(M) := \sum_{k \in \mathbb{Z}} (\dim M_k) t^k \in \mathbb{Z}[t, t^{-1}] \tag{4.1.2}$$

For a simple graded A-module L let $[M : L\langle k \rangle]$ be the multiplicity of the simple module $L\langle k \rangle$ as a graded composition factor of M for $k \in \mathbb{Z}$. Then, we can define the graded decomposition number as

$$[M : L]_t := \sum_{k \in \mathbb{Z}} [M : L\langle k \rangle] t^k \in \mathbb{Z}[t, t^{-1}] \tag{4.1.3}$$

We finish this section by relating the graded decomposition numbers of A and some graded subalgebras of A. Let $\mathfrak{e} \in A$ be a homogeneous idempotent and let $A_{\mathfrak{e}}$ denote the subalgebra $\mathfrak{e}A\mathfrak{e}$ of A. Then $A_{\mathfrak{e}}$ is a graded subalgebra of A and the inclusion $i : A_{\mathfrak{e}} \hookrightarrow A$ is a homogeneous map of degree zero. We write $\mathrm{mod} - A$ (resp. $\mathrm{mod} - A_{\mathfrak{e}}$) for the category of finite dimensional left A-modules (resp. $A_{\mathfrak{e}}$-modules). We define the functor $f : \mathrm{mod} - A \to \mathrm{mod} - A_e$, where for $V \in \mathrm{mod} - A$, fV is the subspace $\mathfrak{e}V$ of V regarded as $A_{\mathfrak{e}}$-module.

Theorem 4.1.1. *Let L and V graded A-modules. Assume that L is simple and that $\mathfrak{e}L \neq 0$. Then,*

$$[V : L]_t = [\mathfrak{e}V : \mathfrak{e}L]_t \tag{4.1.4}$$

where the left (resp. right) side of (4.1.4) correspond to the graded decomposition number for A-modules (resp. $A_{\mathfrak{e}}$-modules).

Proof: First, we note that a non-zero homogeneous idempotent must have degree zero. Hence, $\mathfrak{e}V$ is a graded module, where for a homogeneous element $v \in V$ with $\mathfrak{e}v \neq 0$ we have $\deg(v) = \deg(\mathfrak{e}v)$. The same is true for $\mathfrak{e}L$. Therefore, (4.1.4) follows exactly as in the ungraded case [10, Appendix A1]. $\square$

4.2 Graded cellular algebras

All algebras studied in this thesis are graded cellular algebras so we briefly recall the definition and some properties of these algebras. Actually, the claim that $b_n(m)$ is a graded cellular algebra is the main result in this thesis. Graded cellular algebras was introduced by J. Hu and A. Mathas in [14], following and extending ideas of J. Graham and G. Lehrer [11].

Definition 4.2.1. Let A be graded R-algebra which is free of finite rank over R. A graded cell datum for A is a quadruple $(\Lambda, T, C, \deg)$, where $(\Lambda, \succ)$ is the weight poset, $T(\lambda)$ is a finite set for $\lambda \in \Lambda$, and

$$\begin{array}{rcl} C : \coprod_{\lambda\in\Lambda} T(\lambda)\times T(\lambda) & \to & A \\ (\mathfrak{s},\mathfrak{t}) & \to & c^{\lambda}_{\mathfrak{s}\mathfrak{t}} \end{array} \qquad \deg : \coprod_{\lambda\in\Lambda} T(\lambda) \to \mathbb{Z}$$

are two functions such that C is injective and

(a) For $\lambda \in \Lambda$ and $\mathfrak{s},\mathfrak{t} \in T(\lambda)$, $c^{\lambda}_{\mathfrak{s}\mathfrak{t}}$ is a homogeneous element of A of degree $\deg c^{\lambda}_{\mathfrak{s}\mathfrak{t}} = \deg(\mathfrak{s}) + \deg(\mathfrak{t})$.

(b) The set $\{c^{\lambda}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in T(\lambda) \text{ for } \lambda \in \Lambda\}$ is a R-basis of A.

(c) The R-linear map $* : A \to A$ determined by $(c^{\lambda}_{\mathfrak{s}\mathfrak{t}})^* = c^{\lambda}_{\mathfrak{s}\mathfrak{t}}$, for all $\lambda \in \Lambda$ and all $\mathfrak{s},\mathfrak{t} \in T(\lambda)$, is an algebra anti-automorphism of A

(d) If $\mathfrak{s},\mathfrak{t} \in T(\lambda)$, for some $\lambda \in \Lambda$, and $a \in A$ then there exist scalars $r_{\mathfrak{u}\mathfrak{s}}(a) \in R$ such that

$$ac^{\lambda}_{\mathfrak{s}\mathfrak{t}} \equiv \sum_{\mathfrak{u}\in T(\lambda)} r_{\mathfrak{u}\mathfrak{s}}(a) c^{\lambda}_{\mathfrak{u}\mathfrak{t}} \mod A^{\lambda}$$

where A^{λ} is the R-submodule of A spanned by $\{c^{\mu}_{\mathfrak{a}\mathfrak{b}} \mid \mu \succ \lambda; \mathfrak{a},\mathfrak{b} \in T(\mu)\}$

The set $\{c^{\lambda}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in T(\lambda) \text{ for } \lambda \in \Lambda\}$ is a graded cellular basis for A. If A has a graded cellular basis we say that A is a graded cellular algebra.

If we omit the axiom (a) in the definition of graded cellular algebra, then we recover the original definition of cellular algebras given by J. Graham and G. Lehrer. A R-algebra satisfying the axioms (b),(c) and (d) is called cellular algebra.

Fix a graded cell datum $(\Lambda, T, C, \deg)$ for A. For $\lambda \in \Lambda$ the graded cell module $C(\lambda)$ is the R-module with basis $\{c^{\lambda}_{\mathfrak{s}} \mid \mathfrak{s} \in T(\lambda)\}$ and A-action given by

$$ac_{\mathfrak{s}}^{\lambda} = \sum_{\mathfrak{u} \in T(\lambda)} r_{\mathfrak{u}\mathfrak{s}}(a) c_{\mathfrak{u}}^{\lambda} \tag{4.2.1}$$

where the scalars $r_{\mathfrak{u}\mathfrak{s}}(a) \in R$ are the same scalars appearing in Definition 4.2.1 (d). Note that the scalars $r_{\mathfrak{u}\mathfrak{s}}(a)$ not depend on $\mathfrak{t}$, consequently the cell modules are well defined. The cell modules $C(\lambda)$ are equipped with a homogeneous bilinear form $\langle \cdot, \cdot \rangle_{\lambda}$ of degree zero determined by

$$c_{\mathfrak{a}\mathfrak{s}}^{\lambda} c_{\mathfrak{t}\mathfrak{b}}^{\lambda} \equiv \langle c_{\mathfrak{s}}^{\lambda}, c_{\mathfrak{t}}^{\lambda} \rangle_{\lambda} c_{\mathfrak{a}\mathfrak{b}}^{\lambda} \mod A^{\lambda} \tag{4.2.2}$$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{s}, \mathfrak{t} \in T(\lambda)$. The radical of this form

$$\text{rad } C(\lambda) = \{x \in C(\lambda) \mid \langle x, y \rangle_{\lambda} = 0 \text{ for all } y \in C(\lambda)\}$$

is a graded A-submodule of $C(\lambda)$ so that $D(\lambda) = C(\lambda)/\text{rad } C(\lambda)$ is a graded A-module. (See [25, Proposition 2.9] and [14, Lemma 2.7]).
Let $\Lambda_0 = \{\lambda \in \Lambda \mid D(\lambda) \neq 0\}$. The next theorems classifies the simple graded A-modules over a field and describe the respective graded decomposition numbers.

Theorem 4.2.2. *([14, Theorem 2.10]) Suppose that R is a field. Then*

$$\{D(\lambda)\langle k \rangle \mid \lambda \in \Lambda_0 \text{ and } k \in \mathbb{Z}\}$$

is a complete set of pairwise non-isomorphic simple graded A-modules.

Theorem 4.2.3. *Suppose that $\lambda \in \Lambda$ and $\mu \in \Lambda_0$. Then*

(a) $[C(\lambda) : D(\mu)]_t \in \mathbb{N}[t, t^{-1}]$;

(b) $[C(\lambda) : D(\mu)]_{t=1} = [\underline{C}(\lambda) : \underline{D}(\mu)]$;

(c) $[C(\mu) : D(\mu)]_t = 1$ *and* $[C(\lambda) : D(\mu)]_t \neq 0$ *only if* $\lambda \succeq \mu$.

(d) $\dim_t C(\lambda) = \sum_{\lambda \geq \mu} [C(\lambda) : D(\mu)]_t \dim_t D(\mu)$.

Proof: Claims (a), (b) and (c) are [14, Lemma 2.13]. Part (d) is a direct consequence of the definitions. □

It is straightforward to check that a graded cellular algebra is positively graded if and only if $\deg(\mathfrak{s}) \geq 0$ for all $\mathfrak{s} \in T(\lambda)$ and $\lambda \in \Lambda$. Consequently, if A is positively graded then so is each cell module of A.

4.3 GRADING $Tl_n(q)$ AND $b_n(m)$

In this section we show that the Temperley-Lieb algebra $Tl_n(q)$ and the blob algebra $b_n(m)$ are $\mathbb{Z}$-graded algebras. We do this by proving that the kernels of the surjections given in Theorem 2.1.6 and Theorem 2.1.5 are graded ideals. In the $Tl_n(q)$-case we rely on certain properties of Murphy's standard basis that are proved in [13]. These properties are missing in the $b_n(m)$-case and so our argument is somewhat different in that case. We need the following theorem.

Theorem 4.3.1. *Let A be a $\mathbb{Z}$-graded algebra. Assume that I is an ideal of A generated by homogeneous elements, then it is graded. Consequently, the quotient algebra A/I is a $\mathbb{Z}$-graded algebra with the grading induced from the one on A.*

Proof: See [6, Theorem 1.3]. □

4.3.1 GRADING $Tl_n(q)$

Let us briefly recall Murphy's standard basis for the Hecke algebra $\mathcal{H}_n(q^2)$. For $w = s_{i_1} \dots s_{i_k}$ a reduced expression of $w \in \mathfrak{S}_n$ we define $T_w := T_{i_1} \dots T_{i_k}$. Then $\{T_w | w \in \mathfrak{S}_n\}$ is a basis for $\mathcal{H}_n(q)$. For $\lambda \in \mathrm{Par}(n)$ we let $\mathfrak{S}_\lambda \leq \mathfrak{S}_n$ denote the row stabilizer of τ^λ under the left action of $\mathfrak{S}_n$ on tableaux and define

$$x_\lambda := \sum_{w \in \mathfrak{S}_\lambda} T_w$$

We let $*$ denote the anti-automorphism of $\mathcal{H}_n$ determined by $T_i^* = T_i$ for all $1 \leq i < n$ and define for $\sigma, \tau \in \mathrm{Std}(\lambda)$

$$x_{\tau\sigma} = T^*_{d(\tau)} x_\lambda T_{d(\sigma)}.$$

Then $\{x_{\tau\sigma}\}$, with τ and σ running over standard tableaux of the same shape, is Murphy's standard basis for $\mathcal{H}_n(q)$, see [29, Theorem 4.17].

We set $\mathcal{I}_n := \ker \Phi_2$ where $\Phi_2 : \mathcal{H}_n(q^2) \longrightarrow Tl_n(q)$ is the second surjection given in Theorem 2.1.5. Then $\mathcal{I}_n$ is an ideal of $\mathcal{H}_n(q^2)$ and we have $\mathcal{H}_n(q^2)/\mathcal{I}_n = Tl_n(q)$. We can now state our first Theorem.

Theorem 4.3.2. *$\mathcal{I}_n$ is a graded ideal of $\mathcal{H}_n(q^2)$. Hence $Tl_n(q)$ is a $\mathbb{Z}$-graded algebra, with the grading induced from the one on $\mathcal{H}_n(q^2)$, via Theorem 2.2.1.*

Proof: We first note that by the results of Härterich, [13, Theorem 4], we know that $\mathcal{I}_n$ is spanned (over $\mathbb{C}$!) by those $\{x_{\tau\sigma}\}$ for which the underlying shape has strictly more than two columns, that is $\mathrm{Shape}(\tau), \mathrm{Shape}(\sigma) \notin \mathrm{Par}_2(n)$. In other words, $\{x_{\tau\sigma} \,|\, \sigma, \tau \in \mathrm{Std}(\lambda), \lambda \in \mathrm{Par}(n) \setminus \mathrm{Par}_2(n)\}$ is a basis for $\mathcal{I}_n$.

On the other hand, in [14] J. Hu and A. Mathas construct a basis $\{\psi_{\tau\sigma}\}$ for $\mathcal{H}_n(q^2)$, such that each $\psi_{\tau\sigma}$ is a homogeneous element of $\mathcal{H}_n(q^2)$; here (τ, σ) is running over the same set as for the standard basis. They furthermore show in [14, Lemma 5.4] that for each pair (τ, σ) like this, there is a non-zero scalar $c \in \mathbb{C}$ such that

$$\psi_{\tau\sigma} = c x_{\tau\sigma} + \sum_{(\upsilon,\varsigma) \rhd (\tau,\sigma)} r_{\upsilon\varsigma} x_{\upsilon\varsigma} \tag{4.3.1}$$

where $r_{\upsilon\varsigma} \in \mathbb{C}$ and where $(\upsilon, \varsigma) \rhd (\tau, \sigma)$ means that $\upsilon \trianglerighteq \tau$, $\varsigma \trianglerighteq \sigma$, and $(\upsilon, \varsigma) \neq (\tau, \sigma)$. But this shows that also the $\{\psi_{\tau\sigma}\}$ such that $\mathrm{Shape}(\tau)$, $\mathrm{Shape}(\sigma) \notin \mathrm{Par}_2(n)$, are a basis for $\mathcal{I}_n$. From this we get, via Theorem 4.3.1 that $\mathcal{I}_n$ is a graded ideal as claimed. □

Remark 4.3.3. There is a version of the Theorem involving the homomorphism Φ_1. For this, in the proof one should replace $\{\psi_{\tau\sigma}\}$ by the dual basis $\{\psi'_{\tau\sigma}\}$ of [14].

Remark 4.3.4. In spite of the important role of the Temperley-Lieb algebra in recent categorification theory, see eg. [32], the above graded structure has not been mentioned before in the literature, to the best of our knowledge. Our grading is also not immediately comparable with the supergrading used in [34].

4.3.2 Grading $b_n(m)$

Let us now turn to the blob algebra. In order to treat that case we need the following Theorem. Note that the congruence $2k \equiv m \mod l$ can always be solved because we have assumed that l is odd

Theorem 4.3.5. *Let $k \in \mathbb{Z}$ such that $2k \equiv m \mod l$. Then, the elements $E_1, E_2 \in \mathcal{H}_n(m)$ are homogeneous of degree zero. More precisely, they can be written as a sum of homogeneous elements of degree zero as follows*

$$E_1 = \sum_{\boldsymbol{i}} e(\boldsymbol{i}) \qquad\qquad E_2 = \sum_{\boldsymbol{j}} e(\boldsymbol{j})$$

where the left sum runs over all $\boldsymbol{i} \in I^n$ such that $i_1 = k$ and $i_2 = k-1$, and the right sum runs over all $\boldsymbol{j} \in I^n$ such that $j_1 = -k$ and $j_2 = -k-1$.

Proof: We only prove the result for E_1, the result for E_2 is proved similarly.

In [5, Section 4.4], Brundan, Kleshchev and Wang note that under the embedding $\mathcal{H}_n(m) \hookrightarrow \mathcal{H}_{n+1}(m)$ one has $e(\boldsymbol{i}) \mapsto \sum_{i \in I} e(\boldsymbol{i}, i)$, and so it is enough to prove the case $n = 2$, that is that $E_1 = e(k, k-1)$ holds. Using the uniqueness statement for E_1, in order to prove this, it is enough to show that $e(k, k-1)$ verifies the equations (2.1.5) and (2.1.6), since it is clearly an idempotent.

Note first that $y_1 = 0$ as it follows by combining the relations (2.2.3), (2.2.4) and (2.2.6). Put now $\boldsymbol{j} = (k, k-1)$. Multiplying (2.2.16) by $e(\boldsymbol{j})$ for $n = 2$ and $r = 1$, we get $L_1 e(\boldsymbol{j}) = q^{2k} e(\boldsymbol{j})$, or equivalently $L_1 e(\boldsymbol{j}) = q^m e(\boldsymbol{j})$. Hence (2.1.6) holds.

To show (2.1.5) we first recall from [14, Lemma 4.1(c)] that in general $e(\boldsymbol{i}) \neq 0$ iff $\boldsymbol{i} \in I^n$ is a residue sequence coming from a standard bi-tableau of a bipartition of n. Combining this fact with the standing conditions on q given in (2.1.4), we deduce $e(s_1 \boldsymbol{j}) = 0$ and hence $\psi_1 e(\boldsymbol{j}) = 0$ by (2.2.8). Multiplying this equation on the left by ψ_1 and using (2.2.14) we obtain $y_2 e(\boldsymbol{j}) = 0$.

Now, recall that by definition $P_1(\boldsymbol{j})$ and $Q_1(\boldsymbol{j})$ are power series in y_1 and y_2. Furthermore, in this particular case we have that the constant coefficient of $P_1(\boldsymbol{j})$ is 1 and so (2.2.17) gives $(T_1 + 1)e(\boldsymbol{j}) = 0$ as needed. □

We are now in position to establish the main objective of this section, namely to provide a graded structure on $b_n(m)$. In the forthcoming Section 5.2, we refine this graded structure on $b_n(m)$ to a graded cellular basis structure.

Corollary 4.3.6. *The kernel of the surjection $\Phi : \mathcal{H}_n(m) \longrightarrow b_n(m)$ from Theorem 2.1.6 is a graded ideal. Hence, the algebra $b_n(m)$ has a presentation with generators*

$$\{\psi_1, \ldots, \psi_{n-1}\} \cup \{y_1, \ldots, y_n\} \cup \{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I^n\}$$

subject to the same relations as in Theorem 2.2.1, with the additional relation

$$e(\boldsymbol{i}) = 0$$

whenever $i_1 = k$ and $i_2 = k-1$, or $i_1 = -k$ and $i_2 = -k-1$. These relations are homogeneous with respect to the degree function defined in Definition 2.2.1. Therefore, $b_n(m)$ can be provided with the structure of a $\mathbb{Z}$-graded algebra such that Φ is a homogeneous homomorphism.

Proof: The result follows by direct application of Theorems 2.1.6, 4.3.5 and 4.3.1. □

Remark 4.3.7. We can also give an homogeneous presentation for $Tl_n(q)$ as follows. First, note that for $\lambda = (3) \in \mathrm{Par}(3)$ we have

$$x_\lambda = T_1T_2T_1 + T_1T_2 + T_2T_1 + T_1 + T_2 + 1$$

On the other hand, by [14, Corollary 4.16] if q is not a cubic root of unity then we have in $\mathcal{H}_3(q^2)$ that $x_\lambda = ce(0,1,2)$, where $c \in \mathbb{C}^\times$. Therefore, in order to obtain an homogeneous presentation of $Tl_n(q)$ we impose in the homogeneous presentation of $\mathcal{H}_n(q^2)$ the additional relation

$$e(\boldsymbol{i}) = 0 \qquad \text{if } i_1 = 0, i_2 = 1 \text{ and } i_3 = 2.$$

If q is a cubic root of unity, again using [14, Corollary 4.16], we impose the additional relation

$$e(\boldsymbol{i})y_3 = 0 \qquad \text{if } i_1 = 0, i_2 = 1 \text{ and } i_3 = 2$$

in the homogeneous presentation of $\mathcal{H}_n(q^2)$ to obtain an homogeneous presentation of $Tl_n(q)$.

We end this chapter by expressing the generator $e \in b_n(m)$ in terms of homogeneous generators. We remark that in general the elements $U_i \in b_n(m)$, $i \geq 1$ are not homogeneous in $b_n(m)$.

Lemma 4.3.8. *Let $k \in \mathbb{Z}$ such that $2k \equiv m \mod l$. Then, the element $e \in b_n(m)$ is homogeneous of degree zero. More precisely, it can be written as a sum of homogeneous elements of degree zero as follows*

$$e = \sum_{\substack{\boldsymbol{i} \in I^n \\ i_1 = -k}} e(\boldsymbol{i}) \tag{4.3.2}$$

Proof: The claim follows by combining Theorem 2.1.6, (2.2.16) and $U_0 = -[m]e$.
□

CHAPTER 5

Graded cellular basis for $Tl_n(q)$ and $b_n(m)$

The main goal in this thesis is to study graded representation theory of the Temperley-Lieb algebra and the blob algebra, that is, to understand the structure of the graded simple modules of these algebras. It is known that all of the simple modules over finite dimensional $\mathbb{Z}$-graded algebras can be graded in a unique way up to degree shift. Thus, in studying the graded simple modules we do not lose information about the simple modules, but actually gain additional insight into the structure of the ungraded irreducible modules. In this chapter we construct $\mathbb{Z}$-gradings on the cell and simple modules of $Tl_n(q)$ and $b_n(m)$.

5.1 Jucys-Murphy elements on $b_n(m)$

In Corollary 4.3.6, we gave a new (homogeneous) presentation for $b_n(m)$, while in Chapter 3 we described the diagrammatical basis for the blob algebra. Unfortunately, it seems nontrivial to express the homogeneous generators in terms of the diagram basis of $b_n(m)$. However, it turns out that a graded cellular basis for $b_n(m)$ can be constructed from a precise description of the KLR idempotents in $b_n(m)$. Inspired by the work of J. Hu and A. Mathas [14], we shall obtain in this chapter an expression for them building on the results from [26]. A key point for this is to make $b_n(m)$ fit into the general setting of an algebra with Jucys-Murphy (JM) elements.

The first example of a family of JM elements was given by Jucys [16] and, independently, by Murphy [27] for the group algebra of the symmetric group. A cellular algebra with Jucys-Murphy elements, is essentially, a cellular algebra equipped with a family of commuting elements which acts on the cellular basis (when it is suitable ordered) via upper triangular matrices.

It provides an abstract setting for carrying out much of Murphy's theory for Young's seminormal form. The axiomatization of this concept was given by A. Mathas in [26].

Let A be a cellular algebra with cellular basis

$$\mathcal{C} = \{c_{\mathfrak{st}}^{\lambda} \mid \lambda \in \Lambda; \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$$

as in Definition 4.2.1. Assume furthermore that each $T(\lambda)$ is a poset with respect to an order $<_\lambda$, or just $<$ for simplicity. The following definition is taken from [26].

Definition 5.1.1. A family of Jucys-Murphy elements for A is a set $\{L_1, \ldots, L_k\}$ of commuting elements of A together with a set of scalars,

$$\{c_{\mathfrak{s}}(i) \in R \mid \mathfrak{s} \in T(\lambda),\, \lambda \in \Lambda \text{ and } 1 \leq i \leq k\}$$

such that for $i = 1, \ldots, k$ we have $L_i^* = L_i$ and, for all $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in \Lambda$,

$$L_i c_{\mathfrak{st}}^{\lambda} \equiv c_{\mathfrak{s}}(i) c_{\mathfrak{st}}^{\lambda} + \sum_{\mathfrak{v} > \mathfrak{s}} r_{\mathfrak{sv}} c_{\mathfrak{vt}}^{\lambda} \mod A^{\lambda}$$

for some $r_{\mathfrak{sv}} \in R$ (which depends on i). We call $c_{\mathfrak{s}}(i)$ the content of $\mathfrak{s}$ at i.

The purpose of this section is now to apply this definition to $b_n(m)$. By the above definition, in order to apply the results from [26] we must to first find a cellular basis for $b_n(m)$ and then choose an appropriate set of Jucys-Murphy elements. Actually, the diagrammatical basis of $b_n(m)$ is a cellular basis. We now recall the various elements of this cellular structure.

According to the notation introduced in Definition 4.2.1 we take $\Lambda = \mathrm{Bip}_1(n)$, ordered by $\succeq$. Set $T(\boldsymbol{\lambda}) = \mathrm{Std}(\boldsymbol{\lambda})$, for all $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$. Given $\mathfrak{s}, \mathfrak{t} \in T(\boldsymbol{\lambda})$ define $c_{\mathfrak{st}}^{\boldsymbol{\lambda}} = \mathfrak{m}_{\mathfrak{st}}$. We remark that the diagrammatical basis $\mathbb{B}(n)$ is also cellular for $b_n^{\mathcal{O}}(m)$ and $b_n^K(m)$, since $\mathbb{B}(n)$ is a free basis for both algebras. For $\boldsymbol{\lambda} \in \Lambda$, let $b_n^{\boldsymbol{\lambda}}(m)$ be the ideal of $b_n(m)$ spanned by the set

$$\{\mathfrak{m}_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\mu}) \ ; \ \boldsymbol{\mu} \succ \boldsymbol{\lambda}\}.$$

In the cases of $b_n^{\mathcal{O}}(m)$ and $b_n^K(m)$ we write $b_n^{\mathcal{O},\boldsymbol{\lambda}}(m)$ and $b_n^{K,\boldsymbol{\lambda}}(m)$ for the ideals.

Similarly, the diagrammatical basis for the Temperley-Lieb algebra is cellular [11, Example 1.4]. In this case, the cellular structure is given by $\Lambda = \mathrm{Par}_2(n)$, ordered by dominance. $T(\lambda) = \mathrm{Std}(\lambda)$ for all $\lambda \in \Lambda$, and for σ, $\tau \in T(\lambda)$ we set $c^{\lambda}_{\sigma\tau} = \beta_{\sigma\tau}$.

Since we are assuming that $q + q^{-1} \neq 0$ we get that the bilinear forms $\langle \cdot, \cdot \rangle_\lambda$ are all nonzero, in the Temperley-Lieb case as well as the blob algebra case. From this we get from remark (3.10) of [11] that both algebras are quasi-hereditary and that the cell modules are standard modules in the sense of quasi-hereditary algebras.

We return to the aim of finding a family of Jucys-Murphy elements for $b_n(m)$. By Theorem 2.1.6 we have a homomorphism from $\mathcal{H}_n(m)$ onto $b_n(m)$, it maps the elements $L_k \in \mathcal{H}_n(m)$ to

$$(U_{k-1} + q) \dots (U_1 + q)((q - q^{-1})U_0 + q^m)(U_1 + q) \dots (U_{k-1} + q) \in b_n(m).$$

We shall use the same notation L_k for this element of $b_n(m)$. It satisfies the following commutation rules with the U_i

$$L_k U_i = U_i L_k \qquad \text{if } k \neq i, i+1 \tag{5.1.1}$$

$$(U_k + q^{-1})L_{k+1} = L_k(U_k + q) \qquad \text{for } 1 \leq k < n. \tag{5.1.2}$$

$$L_{k+1}(U_k + q^{-1}) = (U_k + q)L_k \qquad \text{for } 1 \leq k < n. \tag{5.1.3}$$

It is known that the L_k are a family of JM-elements for $\mathcal{H}_n(m)$ with respect to the cellular basis used for example in [14], in which $<_\lambda$ is the dominance order on bitableaux. One might now hope that the set $\{L_1, \dots L_n\}$ is also a family of JM-elements for $b_n(m)$. That this should be the case is not at all obvious. Indeed, the concept of a family of JM-elements depends heavily on the underlying cellular basis and a cellular algebra may in general be endowed with several, completely different, cellular bases with different orders. For example the conjectures of Bonnafé, Geck, Iancu and Lam in [2], indicate that Lusztig's theory of cells for unequal parameters should give rise to a cellular basis on $\mathcal{H}_n(m)$ for each choice of a weight function on the Coxeter group of type B, in dependence of a parameter r. In this setting only the asymptotic case $r > n$ corresponds to the dominance order on $T(\lambda)$. On the contrary, in [31] it is shown that the cell structure on $b_n(m)$ corresponds to the other extreme case $r = 0$ under restriction to $\mathrm{Bip}_1(n)$.

We shall show that in fact $\{L_1, \dots L_n\}$ do form a family of JM-elements for $b_n(m)$ where the poset structure $T(\boldsymbol{\lambda}) = \mathrm{Std}(\boldsymbol{\lambda})$ is the one defined above. Even more, using the surjection $\mathcal{H}^{\mathcal{O}}_n(m) \longrightarrow b^{\mathcal{O}}_n(m)$ given in Theorem 2.1.7, we define elements $\{L_1, \dots, L_n\}$ of $b^{\mathcal{O}}_n(m)$ using the same formula as before and we show that these form a family of JM-elements for $b_n^{\mathcal{O}}(m)$ with respect to $\{\mathfrak{m}_{\mathfrak{st}}\}$, considered as elements of $b_n^{\mathcal{O}}(m)$.

Definition 5.1.2. *Suppose that $\boldsymbol{\lambda} \in Bip_1(n)$ and let $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Let j be an integer with $1 \leq j \leq n$. Define the content of $\mathfrak{t}$ at k to be the scalar given by*

$$c_{\mathfrak{t}}(j) = \begin{cases} q^{2(c-1)}Q & \text{if} \quad d = 1 \\ q^{2(c-1)}Q^{-1} & \text{if} \quad d = 2 \end{cases}$$

where $(1, c, d)$ is the unique node in $[\boldsymbol{\lambda}]$ such that $\mathfrak{t}(1, c, d) = j$. In other words, $c_{\mathfrak{t}}(k)$ is an element of either $\mathcal{O}, \mathbb{C}(q, Q)$ or $\mathbb{C}$, depending on the context. In the $\mathbb{C}$ case, note that $c_{\mathfrak{t}}(j) = q^{2r_{\mathfrak{t}}(j)}$.

Lemma 5.1.3. *Suppose that $\boldsymbol{\lambda} \in Bip_1(n)$ and let k be an integer with $1 \leq k \leq n$. Then we have the identity*

$$L_k \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \equiv c_{\mathfrak{t}^{\boldsymbol{\lambda}}}(k) \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{\boldsymbol{\lambda}}(m)$$

Similar statements hold over $b_n^{\mathcal{O}}(m)$ and $b_n^{\mathcal{K}}(m)$.

Proof: Using the description of $t^{\boldsymbol{\lambda}}$ given after Definition 3.2.4 together with Definition 3.2.8 we find that the diagram corresponding to $\mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ is one of the diagrams that appear in Figure 5.1. But then the statement of the Lemma is equation (25) of [7, Lemma 7.1]. (Note that there is an error in equation (25) as presented in *loc. cit.* As a matter of fact, to get the correct expressions one should subtract 2 from all appearing exponents of x since the relation between L_i and L'_i introduced two pages earlier should be corrected the same way).

□

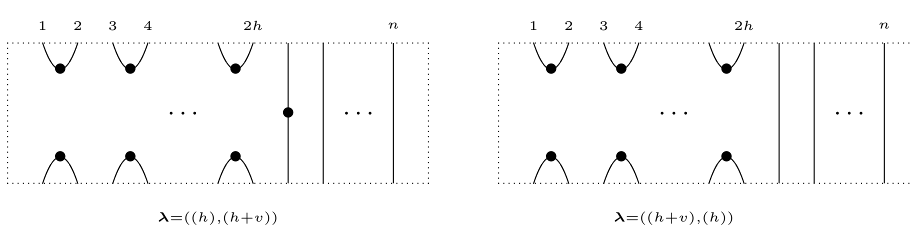


Figure 5.1: The blob diagram $m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$.

Our proof that the $\{L_k\}$ form a family of JM-elements shall be a downwards induction over the partial order $\succeq$ with the preceding Lemma providing the induction basis. To obtain the inductive step we need to understand the relationship between the action of U_i and $\succeq$ and hence we would like to have a formula for the action of U_k in terms of walks on the Bratteli diagram. In general there is no such simple formula. On the other hand, there is one situation where the action of U_k is particularly easy to visualize.

Lemma 5.1.4. *Suppose that $\boldsymbol{\lambda} \in Bip_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Assume moreover that $s_k\mathfrak{s} = \mathfrak{t}$ for the simple transposition s_k and that $\mathfrak{s} \succ \mathfrak{t}$ or equivalently, that $w(\mathfrak{t})$ is obtained from $w(\mathfrak{s})$ by making a hook at position k smaller. Then the following relation holds in $b_n(m)$*

$$U_k \mathfrak{m}_{\mathfrak{s}\mathfrak{t}^{\boldsymbol{\lambda}}} = \begin{cases} \mathfrak{m}_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\lambda}}} & \text{if} \quad \mathfrak{s} \nsim \mathfrak{t} \\ y_e \mathfrak{m}_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\lambda}}} & \text{if} \quad \mathfrak{s} \sim \mathfrak{t}. \end{cases}$$

Similar formulas hold over $\mathcal{O}, K$ and in the Temperley-Lieb algebra (corresponding to $\mathfrak{s} \nsim \mathfrak{t}$).

Proof: This is an immediate consequence of the definition of the maps $f^{\sigma}_{n,k}$. $\square$

The next three Lemmas are preparations for Lemma 5.1.8.

Lemma 5.1.5. *Suppose that $\lambda \in Par_2(n)$ and $\sigma, \tau, u \in \mathrm{Std}(\lambda)$. Suppose moreover that $u \rhd \sigma \rhd \tau$ and that $s_k\sigma = \tau$ for some k. Let $v \in \mathrm{Std}(\lambda)$ be chosen such that $U_k\beta_{ut^\lambda} = r\beta_{vt^\lambda} \mod Tl^\lambda$ for some scalar $r \in \mathbb{C}$ (such v always exists by the diagrammatical realization of the Temperley-Lieb algebra and its cell modules). Then, if r is nonzero we have that $v \rhd \tau$.*

Proof: We identify σ, τ and u with their walks $w(\sigma), w(\tau)$ and $w(u)$ on the Bratteli diagram for Tl_n, and also with their corresponding sign sequences. Then the sign sequences for σ and τ are the same except at the k'th and $k+1$'st positions where the sequence for σ has $-, +$ whereas the sequence for τ has $+, -$. On the other hand, for u all four possibilities of signs may occur at these positions, apriori, and so we proceed by a case by case analysis.

The first case to analyze is the case where the signs for u are $+, -$ at these positions. In this case we get $v = u$ (and $r := -[2]$), and so the claim of the Lemma follows from the assumptions. The next case is the one where the signs are $+, +$ at positions k and $k+1$. On the diagrammatic level we have three options for the top edge of $\beta_{u\tau^\lambda}$, illustrated in Figure 5.2.

In the subcase (a), the signs for u at positions k, $k+1$, a and b are $+, +, -$ and $-$, respectively. For v the signs in these positions are $+, -, +$ and $-$, whereas the

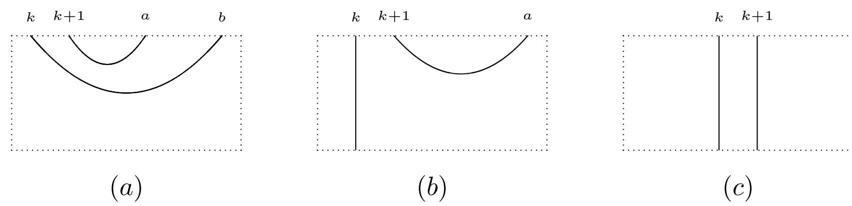


Figure 5.2: Top edge of $\beta_{u\tau^\lambda}$

signs for v and u agree at all other positions. The claim follows from this. The subcase (b) is treated similarly. Finally, in the subcase (c) we have $r = 0$, contrary to the assumptions.

The third case is the one where the signs for u are $-, +$ at the positions $k, k+1$. In that case, at the diagrammatic level, k is connected to a point $a < k$ whereas $k+1$ is either connected to $b > k+1$ or it is the upper endpoint of a vertical line. In both cases, we find that the sign sequence for v is the same as the one for u, except at positions $k, k+1$ where it becomes $+, -$. But by the assumptions, u differs from σ in at least one position and the result follows in this case as well. Note that this is the only case in which $u \rhd v$.

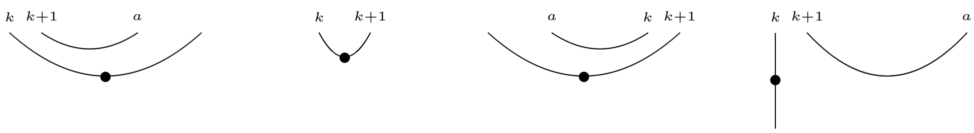
The last case is the one where the signs for u at the positions $k, k+1$ are $-, -$. In this case, k is connected to a and $k+1$ to b and $b < a < k < k+1$. Moreover the signs for u at these positions are $+, +, -, -$. But then the signs for v at these positions are $+, -, +, -$ whereas the signs for v and u agree at all other positions. The claim follows from this. $\square$

Lemma 5.1.6. *Suppose that $\mu \in Par_2(n)$. Let $\sigma, \tau \in \mathrm{Std}(\mu)$. Assume that $U_k\beta_{\sigma\tau^\mu} \equiv \alpha\beta_{\tau\tau^\mu} \mod TL_n^\mu$, with $\alpha \neq 0$ and $1 \leq k < n$. Then, $\tau \unrhd \sigma$, or $\sigma \rhd \tau$ and $s_k\sigma = \tau$.*

Proof: The result follows by a case by case analysis, similar to that given in the proof of the previous Lemma. $\square$

Lemma 5.1.7. *Let $\boldsymbol{\lambda} \in Bip_1(n)$ and $\mathfrak{u} \in \mathrm{Std}(\boldsymbol{\lambda})$. Assume that $U_k\mathfrak{m}_{\mathfrak{u}\mathfrak{t}^{\boldsymbol{\lambda}}} \equiv \alpha\mathfrak{m}_{\mathfrak{v}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{\boldsymbol{\lambda}}(m)$, for $\alpha \in \mathbb{C}$ and $\mathfrak{v} \in \mathrm{Std}(\boldsymbol{\lambda})$ (such $\mathfrak{v}$ always exists by the diagrammatic realization of $b_n(m)$). Suppose moreover that α is nonzero and that the node j is covered in the top edge of $\mathfrak{m}_{\mathfrak{u}\mathfrak{t}^{\boldsymbol{\lambda}}}$, but uncovered in the top edge of $\mathfrak{m}_{\mathfrak{v}\mathfrak{t}^{\boldsymbol{\lambda}}}$. Then $|\mathfrak{u}(j)| = |\mathfrak{v}(j)| \neq 0$ if and only if $j = k$. Similar statements hold over $\mathcal{O}$ and K.*

Proof: In order for the action of U_k to transform a covered node j in the top edge of $\mathfrak{m}_{\mathfrak{u}\mathfrak{t}^{\boldsymbol{\lambda}}}$ to an uncovered node in the top edge of $\mathfrak{m}_{\mathfrak{v}\mathfrak{t}^{\boldsymbol{\lambda}}}$, the diagram of $\mathfrak{m}_{\mathfrak{u}\mathfrak{t}^{\boldsymbol{\lambda}}}$ must be one of those shown in the below Figure 5.3 with the position of j shown in each case. Using this classification, the Lemma follows from Definition 3.2.8. $\square$



$k \leq j < a$ $\quad$ $k \leq j \leq k+1$ $\quad$ $a < j \leq k+1$ $\quad$ $k \leq j < a$

Figure 5.3: Possibilities for $\mathfrak{m}_{\mathfrak{u}\mathfrak{t}^{\boldsymbol{\lambda}}}$

We can now finally prove the property of the order $\succ$ that makes our induction work. It is a generalization to the blob algebra case of Lemma 5.1.5, and in fact we shall deduce it from that Lemma.

Lemma 5.1.8. *Suppose that $\boldsymbol{\lambda} = ((a),(b)) \in Bip_1(n)$ and $\mathfrak{s}, \mathfrak{t}, \mathfrak{u} \in \mathrm{Std}(\boldsymbol{\lambda})$. Suppose furthermore that $s_k\mathfrak{s} = \mathfrak{t}$ and that $\mathfrak{u} \succ \mathfrak{s} \succ \mathfrak{t}$. Let $\mathfrak{v} \in \mathrm{Std}(\boldsymbol{\lambda})$ be chosen such that $U_k\mathfrak{m}_{\mathfrak{u}\mathfrak{t}^{\boldsymbol{\lambda}}} = r\mathfrak{m}_{\mathfrak{v}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{\boldsymbol{\lambda}}(m)$ for some scalar $r \in \mathbb{C}$. Then, if r is nonzero we have that $\mathfrak{v} \succ \mathfrak{t}$. Similar statements are valid for $b_n^{\mathcal{O}}(m)$ and $b_n^{K}(m)$.*

Proof: Set $\mu_1 = \max\{a,b\}$, $\mu_2 = \min\{a,b\}$ and let $\mu = (\mu_1,\mu_2)'$. Then $\mu \in \mathrm{Par}_2(n)$ and in the Temperley-Lieb algebra we have that

$$U_k\beta_{\tau_{\mathfrak{u}}\tau^{\mu}} = \alpha_1\beta_{\tau_{\mathfrak{v}}\tau^{\mu}} \mod Tl^{\mu}$$

where $\tau_{\mathfrak{u}}, \tau_{\mathfrak{v}}$ are as in Definition 3.3.2 and $\alpha_1 \neq 0$; indeed $\alpha_1 = -[2]$ if $\alpha = -[2]$ or if $\alpha = y_e$, and $\alpha_1 = 1$ if $\alpha = 1$. Moreover, by Lemma 3.3.4 we have that $\tau_{\mathfrak{u}} \trianglerighteq \tau_{\mathfrak{s}} \trianglerighteq \tau_{\mathfrak{t}}$.

Case 1 ($\tau_{\mathfrak{u}} \triangleright \tau_{\mathfrak{s}} \triangleright \tau_{\mathfrak{t}}$). In this case we have by Lemma 3.3.5 that $s_k\tau_{\mathfrak{s}} = \tau_{\mathfrak{t}}$ and then Lemma 5.1.5 gives that $\tau_{\mathfrak{v}} \triangleright \tau_{\mathfrak{t}}$. Now by Lemma 3.2.7, in order to prove that $\mathfrak{v} \succ \mathfrak{t}$ it is enough to show that

$$|\mathfrak{v}(j)| = |\mathfrak{t}(j)| \text{ implies } \mathfrak{v}(j) \leq \mathfrak{t}(j). \tag{5.1.4}$$

Hence, assume that $|\mathfrak{v}(j)| = |\mathfrak{t}(j)|$, but $\mathfrak{t}(j) < 0$ and $\mathfrak{v}(j) > 0$ for some $1 \leq j \leq n$. We now split this case into two subcases according to Lemma 5.1.6, that is, $\tau_{\mathfrak{v}} \trianglerighteq \tau_{\mathfrak{u}}$ or, $\tau_{\mathfrak{u}} \triangleright \tau_{\mathfrak{v}}$ and $s_k\tau_{\mathfrak{u}} = \tau_{\mathfrak{v}}$. First, we assume that $\tau_{\mathfrak{v}} \trianglerighteq \tau_{\mathfrak{u}}$. Then we get from $\mathfrak{u} \succ \mathfrak{s} \succ \mathfrak{t}$ and Lemma 3.2.7 that $\mathfrak{u}(j) = \mathfrak{s}(j) = \mathfrak{t}(j)$ and so we get $\mathfrak{u}(j) < 0$, $\mathfrak{v}(j) > 0$ and $|\mathfrak{u}(j)| = |\mathfrak{v}(j)|$. From this we conclude via Lemma 5.1.7 that $j = k$, hence that $\mathfrak{s}(k) = \mathfrak{t}(k)$, which is impossible because $s_k\mathfrak{s} = \mathfrak{t}$.

So we can assume that $\tau_{\mathfrak{u}} \triangleright \tau_{\mathfrak{v}}$ and $s_k\tau_{\mathfrak{u}} = \tau_{\mathfrak{v}}$. Note that in this setting we have $|\mathfrak{u}(j)| = |\mathfrak{v}(j)|$ if $j \neq k$ and $|\mathfrak{u}(k)| + 2 = |\mathfrak{v}(k)|$. Since $s_k\mathfrak{s} = \mathfrak{t}$, $\tau_{\mathfrak{s}} \neq \tau_{\mathfrak{t}}$ and

$\mathfrak{t} < 0$ we also have that $\mathfrak{s}(j) = \mathfrak{t}(j)$ if $j \neq k$ and $\mathfrak{s}(k) + 2 = \mathfrak{t}(k) \leq -2$. Then we get from Lemma 5.1.7 that $j = k$. By $\mathfrak{t}(k) \leq -2$ we obtain that $\mathfrak{v}(k) \geq 2$ since $|\mathfrak{t}(k)| = |\mathfrak{v}(k)|$. Then $\mathfrak{v}(k+1) \geq 1$ since the sequence of integers changes by ± 1 when k is increased by 1. But $|\mathfrak{v}(k+1)| = |\mathfrak{u}(k+1)|$ and $\mathfrak{u}(k+1) < 0$, so we can use the Lemma 5.1.7 again to obtain a contradiction. This completes the proof in Case 1.

Case 2 ($\tau_{\mathfrak{u}} \trianglerighteq \tau_{\mathfrak{s}} = \tau_{\mathfrak{t}}$). By the assumptions $\mathfrak{t}$ is obtained from $\mathfrak{s}$ by making a hook at position k smaller. Moreover, since $\tau_{\mathfrak{s}} = \tau_{\mathfrak{t}}$ this hook is located on the central vertical axis of the Bratteli diagram, that is $\mathfrak{t}(k-1) = \mathfrak{s}(k-1) = \mathfrak{t}(k+1) = \mathfrak{s}(k+1) = 0$. But then, since $\mathfrak{u} \succ \mathfrak{s}$, we have necessarily that $\mathfrak{u}(k-1) = \mathfrak{u}(k+1) = 0$, $\mathfrak{u}(k) = -1$ which implies via Lemma 5.1.4 that $\mathfrak{v}$ is obtained from $\mathfrak{u}$ by making a hook at position k smaller. Hence we get $\mathfrak{v} \succ \mathfrak{t}$ as claimed.

Case 3 ($\tau_{\mathfrak{u}} = \tau_{\mathfrak{s}} \trianglerighteq \tau_{\mathfrak{t}}$). By the hypothesis in this case we have $\tau_{\mathfrak{v}} = \tau_{\mathfrak{t}}$. Recall that at the Bratteli diagram level this implies that at each step the walks $w(\mathfrak{t})$ and $w(\mathfrak{v})$ are either equal or mirror images under the reflection through the central vertical axis of the Bratteli diagram. So, in order to prove the Lemma in this case we must prove that whenever the path $w(\mathfrak{t})$ is on the negative side of the Bratteli diagram, the path $w(\mathfrak{v})$ is also on the negative part. In terms of the sequence of integers the last condition is equivalent to

$$\mathfrak{t}(j) < 0 \text{ implies } \mathfrak{v}(j) < 0 \tag{5.1.5}$$

for all $1 \leq j \leq n$. Suppose by contradiction that (5.1.5) is not true for some $1 \leq j \leq n$. Therefore, $\mathfrak{t}(j) < 0 < \mathfrak{v}(j)$ for some $1 \leq j \leq n$. Using the fact that $\mathfrak{s}(j) = \mathfrak{t}(j)$, for all $j \neq k$, $\tau_{\mathfrak{u}} = \tau_{\mathfrak{s}}$ and $\tau_{\mathfrak{v}} = \tau_{\mathfrak{t}}$, we can conclude via Remark 3.2.9 and Lemma 5.1.7 that $j = k$. Hence, at step k the walk $w(\mathfrak{t})$ (resp. $w(\mathfrak{v})$) is on the negative side of Bratteli diagram and (5.1.5) is true for all $j \neq k$. This implies that $\mathfrak{t}(k-1) = \mathfrak{t}(k+1) = 0$ and $\mathfrak{t}(k) = -1$. But this is impossible because $\mathfrak{s} \succ \mathfrak{t}$ and $s_k\mathfrak{s} = \mathfrak{t}$. This completes the proof of the Lemma. $\square$

We are now in position to prove the triangularity property for $\{L_1, \ldots, L_n\}$. It follows from it that the set $\{L_1, \ldots, L_n\}$ is a family of JM-elements for the blob algebra with respect to the order $\succ$.

Theorem 5.1.9. *Suppose that $\boldsymbol{\lambda} \in Bip_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then*

$$L_k \mathfrak{m}_{\mathfrak{s}\mathfrak{t}} = c_{\mathfrak{s}}(k)\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{u} \in \mathrm{Std}(\boldsymbol{\lambda}) \\ \mathfrak{u} \succ \mathfrak{s}}} a_{\mathfrak{u}} \mathfrak{m}_{\mathfrak{u}\mathfrak{t}} \mod b_n^{\boldsymbol{\lambda}}(m)$$

for some scalars $a_{\mathfrak{u}}$. A similar statements holds for $b_n^{\mathcal{O}}(m)$ and $b_n^K(m)$.

Proof: By the cellularity of the diagram basis, the statement of the Lemma is independent of $\mathfrak{t}$. We proceed by induction on the order $\succeq$. The induction basis $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}}$ is provided by Lemma 5.1.3. Assume now that $\mathfrak{s} \neq \mathfrak{t}^{\boldsymbol{\lambda}}$. Then we can find i and $\mathfrak{s}'$ such that $\mathfrak{s}' \succ \mathfrak{s}$ and $s_i\mathfrak{s}' = \mathfrak{s}$. By the inductive hypothesis the Theorem is valid for $\mathfrak{s}'$. We first assume that $\mathfrak{s} \nsim \mathfrak{s}'$ and $k \neq i, i+1$. Using Lemma 5.1.4 and the commutation rule (5.1.1) we then get

$$L_k\mathfrak{m}_{\mathfrak{st}} = L_kU_i\mathfrak{m}_{\mathfrak{s}'\mathfrak{t}} = U_iL_k\mathfrak{m}_{\mathfrak{s}'\mathfrak{t}} = c_{\mathfrak{s}'}(k)\mathfrak{m}_{\mathfrak{st}} + \sum_{\substack{\mathfrak{u}\in\mathrm{Std}(\boldsymbol{\lambda})\\ \mathfrak{u}\succ\mathfrak{s}'}} a_{\mathfrak{u}}U_i\mathfrak{m}_{\mathfrak{ut}} \mod b_n^{\boldsymbol{\lambda}}(m).$$

On the other hand, by the previous Lemma the sum is a linear combination of elements of the form $\mathfrak{m}_{\mathfrak{ut}}$ where $\mathfrak{u} \succ \mathfrak{s}$ and since $c_{\mathfrak{s}}(k) = c_{\mathfrak{s}'}(k)$ we are done in this case.

If $\mathfrak{s} \sim \mathfrak{s}'$ and $k \neq i, i+1$ we find similarly

$$L_k\mathfrak{m}_{\mathfrak{st}} = y_e^{-1}L_kU_i\mathfrak{m}_{\mathfrak{s}'\mathfrak{t}} = y_e^{-1}U_iL_k\mathfrak{m}_{\mathfrak{s}'\mathfrak{t}} = c_{\mathfrak{s}'}(k)\mathfrak{m}_{\mathfrak{st}} + \sum_{\substack{\mathfrak{u}\in\mathrm{Std}(\boldsymbol{\lambda})\\ \mathfrak{u}\succ\mathfrak{s}'}} y_e^{-1}a_{\mathfrak{u}}U_i\mathfrak{m}_{\mathfrak{ut}} \mod b_n^{\boldsymbol{\lambda}}(m)$$

and may conclude the same way as before. We next treat the case $\mathfrak{s} \nsim \mathfrak{s}'$ and $i = k$ where we find, using the commutation rule (5.1.2) that

$$L_k\mathfrak{m}_{\mathfrak{st}} = L_kU_k\mathfrak{m}_{\mathfrak{s}'\mathfrak{t}} = L_k(U_k + q - q)\mathfrak{m}_{\mathfrak{s}'\mathfrak{t}} = (U_k + q^{-1})L_{k+1}\mathfrak{m}_{\mathfrak{s}'\mathfrak{t}} - qL_k\mathfrak{m}_{\mathfrak{s}'\mathfrak{t}}.$$

By the inductive hypothesis, $L_k\mathfrak{m}_{\mathfrak{s}'\mathfrak{t}}$ and $L_{k+1}\mathfrak{m}_{\mathfrak{s}'\mathfrak{t}}$ are linear combination of elements of the form $\mathfrak{m}_{\mathfrak{ut}}$ where $\mathfrak{u} \succ \mathfrak{s}$ and hence we find, using the inductive hypothesis and Lemma 5.1.4 once more, that $L_k\mathfrak{m}_{\mathfrak{st}}$ is equal to

$$U_kL_{k+1}\mathfrak{m}_{\mathfrak{s}'\mathfrak{t}} = c_{\mathfrak{s}'}(k+1)\mathfrak{m}_{\mathfrak{st}} + \sum_{\substack{\mathfrak{u}\in\mathrm{Std}(\boldsymbol{\lambda})\\ \mathfrak{u}\succ\mathfrak{s}'}} a_{\mathfrak{u}}U_k\mathfrak{m}_{\mathfrak{ut}} \mod b_n^{\boldsymbol{\lambda}}(m).$$

But $c_{\mathfrak{s}}(k) = c_{\mathfrak{s}'}(k+1)$ and we may conclude this case using the previous Lemma as before. The remaining cases are treated similarly. □

5.2 GRADED CELLULAR BASIS FOR $Tl_n(q)$ AND $b_n(m)$

In this section we obtain our main results showing that $b_n(m)$ is a graded cellular algebra. Our methods are inspired by the ones used by Hu and Mathas in [14, Section 4 and 5], who construct a graded cellular basis $\{\psi_{st}\}$ for the cyclotomic

Hecke algebra, in terms of the Khovanov-Lauda-Rouquier generators. But unfortunately is not possible to use their results directly. In fact, the homomorphism $\Phi : \mathcal{H}_n(m) \longrightarrow b_n(m)$ may easily map linearly independent elements to linearly dependent elements. Moreover, due to the incompatibility between the dominance order used for $\{\psi_{st}\}$ and the order $\succ$ for $b_n(m)$, we do not know how to find a basis for $\ker \Phi$ consisting of elements from $\{\psi_{st}\}$, and so in general it seems intractable to determine which are the subsets of $\{\psi_{st}\}$ that stay independent under Φ.

Our solution to this problem is indirect. It is based on an alternative realization of the KLR-idempotents $e(\boldsymbol{i})$ which is possible in the setting of an algebra with JM-elements, see Lemma 4.2 of [26]. It also plays a key role in [14] in the setting of cyclotomic Hecke algebras. To explain it we first setup the relevant notation.

We fix $\mathcal{O}$ and $\mathfrak{m}$ as above. Recall that $K = \mathbb{C}(q, Q)$ and $b_n^K(m) = b_n^{\mathcal{O}}(m) \otimes_{\mathcal{O}} K$. Over K the contents from Definition 5.1.2 trivially verify the separation criterion of [26] and so $b_n^K(m)$ is semisimple. Hence we can apply [26] to the algebras $b_n(m)$, $b_n^{\mathcal{O}}(m)$ and $b_n^K(m)$. We repeat the necessary definitions in our setting.

Definition 5.2.1. Suppose that $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and $\mathfrak{s}$, $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then we define

$$F_{\mathfrak{t}} := \prod_{k=1}^{n} \prod_{\substack{\mathfrak{s} \in \mathrm{Std}(n) \\ c_{\mathfrak{s}}(k) \neq c_{\mathfrak{t}}(k)}} \frac{L_k - c_{\mathfrak{s}}(k)}{c_{\mathfrak{t}}(k) - c_{\mathfrak{s}}(k)} \in b_n^K(m)$$

and set $f_{\mathfrak{st}} = F_{\mathfrak{s}} \mathfrak{m}_{\mathfrak{st}} F_{\mathfrak{t}}$.

We extend the order $\succeq$ to pairs of bitableaux of the same shape by declaring $(\mathfrak{u}, \mathfrak{v}) \succeq (\mathfrak{s}, \mathfrak{t})$ if $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\mu})$, and if either $\boldsymbol{\mu} \succeq \boldsymbol{\lambda}$ or $\boldsymbol{\mu} = \boldsymbol{\lambda}$ and $\mathfrak{u} \succeq \mathfrak{s}$ and $\mathfrak{v} \succeq \mathfrak{t}$. Then we get that

$$f_{\mathfrak{st}} = \mathfrak{m}_{\mathfrak{st}} + \sum_{(\mathfrak{u},\mathfrak{v}) \succ (\mathfrak{s},\mathfrak{t})} r_{\mathfrak{uv}} \mathfrak{m}_{\mathfrak{uv}} \tag{5.2.1}$$

for some $r_{\mathfrak{uv}} \in K$ and hence

$$\{f_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathrm{Bip}_1(n)\}$$

is a basis for $b_n^K(m)$, the seminormal basis. Moreover, by [26, Theorem 3.7], for $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ there exists a non-zero scalar $\gamma_{\mathfrak{t}} \in K$ such that

$$f_{\mathfrak{tt}} f_{\mathfrak{tt}} = \gamma_{\mathfrak{t}} f_{\mathfrak{tt}} \tag{5.2.2}$$

Let $\approx$ be the equivalence relation on $\mathrm{Std}(\boldsymbol{n})$ given by $\mathfrak{s} \approx \mathfrak{t}$ if $r_{\mathfrak{s}}(k) = r_{\mathfrak{s}}(k)$ for $k = 1, 2, \ldots, n$. The equivalence classes for $\approx$ are parametrized by residue sequences I^n of length n; for $\boldsymbol{i} \in I^n$ we denote by $\mathrm{Std}(\boldsymbol{i})$ the corresponding class. Any tableau $\mathfrak{s}$ gives rise to a residue sequence that is denoted $\boldsymbol{i}^{\mathfrak{s}}$. Then we have $\mathfrak{s} \in \mathrm{Std}(\boldsymbol{i}^{\mathfrak{s}})$ but in general $\mathrm{Std}(\boldsymbol{i})$ may be empty, of course. For each $\boldsymbol{i} \in I^n$ we define idempotents $e^b(\boldsymbol{i}) \in b_n^K(m)$ by

$$e^b(\boldsymbol{i}) := \sum_{\mathfrak{s} \in \mathrm{Std}(\boldsymbol{i})} \frac{1}{\gamma_{\mathfrak{s}}} f_{\mathfrak{s}\mathfrak{s}}.$$

Then it follows from [26] that actually $e^b(\boldsymbol{i}) \in b_n^{\mathcal{O}}$ and so we may reduce $e^b(\boldsymbol{i})$ modulo $\mathfrak{m}$ to obtain idempotents of $b_n(m)$ that we denote the same way $e^b(\boldsymbol{i})$.

The next result plays a key role in [14] in the setting of cyclotomic Hecke algebras.

Lemma 5.2.2. *For $\boldsymbol{i} = (i_1, i_2, \dots, i_n) \in I^n$ let*

$$b_n(m)(\boldsymbol{i}) := \{v \in b_n(m) \mid (L_r - q^{2i_r})^k v = 0 \text{ for } r = 1, \dots, n \text{ and } k \gg 0\}$$

be the generalized weight space for the action of $L_i \in b_n(m)$. Then we have $b_n(m)(\boldsymbol{i}) = e^b(\boldsymbol{i}) b_n(m)$.

Proof: The proof of Proposition 4.8 of [14] carries over. □

Lemma 5.2.3. *Let $\Phi : \mathcal{H}_n(m) \longrightarrow b_n(m)$ be as above and let $\boldsymbol{i} \in I^n$. Then $\Phi(e(\boldsymbol{i})) = e^b(\boldsymbol{i})$. In particular, $e^b(\boldsymbol{i})$ is a homogeneous element of $b_n(m)$ of degree 0.*

Proof: Since Φ is surjective and maps the JM-elements of $\mathcal{H}_n(m)$ to the JM-elements of $b_n(m)$, we have $\Phi(\mathcal{H}_n(m)(\boldsymbol{i})) = b_n(m)(\boldsymbol{i})$. But then

$$e^b(\boldsymbol{i}) b_n(m) = b_n(m)(\boldsymbol{i}) = \Phi(\mathcal{H}_n(m)(\boldsymbol{i})) = \Phi(e(\boldsymbol{i})\mathcal{H}_n(m)) = \Phi(e(\boldsymbol{i})) b_n(m).$$

Moreover, $\Phi(e(\boldsymbol{i}))$ lies in the subalgebra of $b_n(m)$ generated by the JM-elements since $e(\boldsymbol{i})$ has the corresponding property, and so $\Phi(e(\boldsymbol{i})) = e^b(\boldsymbol{i})$ as claimed. On the other hand, by Corollary 4.3.6 we know that Φ is homogeneous and so the second claim holds as well. □

We next define elements ψ_i^b, y_i^b of $b_n(m)$ by $\psi_i^b := \Phi(\psi_i)$ and $y_i^b := \Phi(y_i)$. As is the case for $e^b(\boldsymbol{i})$, these elements are homogeneous of the same degree as their Hecke algebra counterparts. We are now in position to give the key definition of this section.

Definition 5.2.4. *Suppose that $\boldsymbol{\lambda} \in Bip_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Let $\mathfrak{d}(\mathfrak{s}) = s_{i_1} \dots s_{i_k}$ and $\mathfrak{d}(\mathfrak{t}) = s_{j_1} \dots s_{j_l}$ be reduced expressions for $\mathfrak{d}(\mathfrak{s})$ and $\mathfrak{d}(\mathfrak{t})$, chosen as in Lemma 3.3.7. Then we define*

$$\psi^b_{\mathfrak{s}\mathfrak{t}} := \psi^b_{i_1} \dots \psi^b_{i_k} e^b(\boldsymbol{i}^{\boldsymbol{\lambda}}) \psi^b_{j_l} \dots \psi^b_{j_1} \in b_n(m).$$

Note that although our $\psi^b_{\mathfrak{s}\mathfrak{t}}$ look much like the elements $\psi_{\mathfrak{s}\mathfrak{t}}$ introduced in [14], this resemblance is only formal and in general there is no obvious connection between the two families of elements, due to the differences between the tableaux. Note also that in our definition there is no y factor, contrary to the [14] situation. Finally, note that our $\psi^b_{\mathfrak{s}\mathfrak{t}}$ can be shown to be independent of the choices of reduced expressions as above, this is also contrary to the situation in [14].

Our next result is parallel to Theorem 4.14 of [14], but has no y term. This 'missing' y is the reason why there is no y factor in Definition 5.2.4.

Theorem 5.2.5. *Suppose that $\boldsymbol{\lambda} = ((a),(b)) \in Bip_1(n)$. Then there exists a nonzero scalar $r \in \mathbb{C}^\times$ such that*

$$e^b(\boldsymbol{i}^{\boldsymbol{\lambda}}) \equiv r \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{\boldsymbol{\lambda}}(m).$$

Proof: We begin by determining $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}$. For this we use (5.2.1) and (5.2.2) and find

$$\begin{aligned} \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} &= f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \\ &\equiv \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{K,\boldsymbol{\lambda}}(m) \\ &\equiv (y_e)^c \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{K,\boldsymbol{\lambda}}(m) \\ &\equiv (y_e)^c f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{K,\boldsymbol{\lambda}}(m) \end{aligned}$$

where $c = \min\{a,b\}$ and where $\mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ can be conveniently found via the diagrammatic realization of $\mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ in Figure 5.1. From this we deduce that $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} = (y_e)^c$.

On the other hand, for $\mathfrak{s} \in \mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$ with $\mathfrak{s} \neq \mathfrak{t}^{\boldsymbol{\lambda}}$, we get by combining the description of $\mathfrak{t}^{\boldsymbol{\lambda}}$ given just after Definition 3.2.4 with the standing conditions on the parameters (2.1.4) that $\mathrm{Shape}(\mathfrak{s}) \succ \boldsymbol{\lambda}$. But then (5.2.1) and the definition of $e(\boldsymbol{i}^{\boldsymbol{\lambda}})$ imply

$$e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \equiv \frac{1}{(y_e)^c} \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b_n^{K,\boldsymbol{\lambda}}(m). \tag{5.2.3}$$

Since $e(\boldsymbol{i}^{\boldsymbol{\lambda}})$ and $\frac{1}{(y_e)^c} \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ both belong to $b_n^{\mathcal{O},\boldsymbol{\lambda}}(m)$, we can now replace $b_n^{K,\boldsymbol{\lambda}}(m)$ by $b_n^{\mathcal{O},\boldsymbol{\lambda}}(m)$ in (5.2.3). From this the proof is obtained by reducing modulo $\mathfrak{m}$. $\square$

We can now prove that the elements from Definition 5.2.4 form a basis for $b_n(m)$.

Theorem 5.2.6. *Suppose that $\boldsymbol{\lambda} \in Bip_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then there are scalars $r \in \mathbb{C}^{\times}$ and $r_{\mathfrak{u}\mathfrak{v}} \in \mathbb{C}$ such that*

$$\psi^b_{\mathfrak{s}\mathfrak{t}} = r\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v})\succ(\mathfrak{s},\mathfrak{t})} r_{\mathfrak{u}\mathfrak{v}}\mathfrak{m}_{\mathfrak{u}\mathfrak{v}}.$$

Hence $\{\psi^b_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in Bip_1(n)\}$ is a basis for $b_n(m)$.

Proof: For $\mathfrak{d}(\mathfrak{s}) = s_{i_1} \dots s_{i_k}$ a reduced expression for $\mathfrak{d}(\mathfrak{s})$ as above we consider first $\psi^b_{i_1} \dots \psi^b_{i_k} e^b(\boldsymbol{i}^{\boldsymbol{\lambda}})$. Using (2.2.2) and the commutation rules (2.2.10), (2.2.12) and (2.2.13) between the y_i and ψ_j, we get that it can be expressed as a linear combination of elements of the form $\Phi(T_{i_{j_1}} \dots T_{i_{j_r}} f_{j_1,\dots,j_r}(y_1, \dots, y_n)e(\boldsymbol{i}^{\boldsymbol{\lambda}}))$ where $(i_{j_1}, \dots, i_{j_r})$ is a subsequence of $(i_1, \dots, i_k)$ and where $f_{j_1,\dots,j_r}(y_1, \dots, y_n)$ is a polynomial in the y_i. But $\Phi(T_i) = qU_i + q^2$ and hence this can also be written as a linear combination of elements of the form $U_{i_{j_1}} \dots U_{i_{j_r}} g_{j_1,\dots,j_r}(y^b_1, \dots, y^b_n)e^b(\boldsymbol{i}^{\boldsymbol{\lambda}}))$ where $(i_{j_1}, \dots, i_{j_r})$ is a subsequence of $(i_1, \dots, i_k)$ and $g_{j_1,\dots,j_r}(y^b_1, \dots, y^b_n)$ is a polynomial in the y^b_i. But from (2.2.1) and the previous Theorem this is a linear combination of elements of the form $U_{i_{j_1}} \dots U_{i_{j_r}} \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b^{\boldsymbol{\lambda}}_n(m)$.

Going through the above argument once more, we get that the coefficient of $U_{i_1} \dots U_{i_k} \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ in $\psi^b_{i_1} \dots \psi^b_{i_k} e^b(\boldsymbol{i}^{\boldsymbol{\lambda}})$ is nonzero, in fact it is essentially the product of the constant terms of the polynomials Q appearing in (2.2.2). But Lemma 5.1.4 implies, by the choice of reduced expression for $\mathfrak{d}(\mathfrak{s}) = s_{i_1} \dots s_{i_k}$, that $U_{i_1} \dots U_{i_k} \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = y^l_e \mathfrak{m}_{\mathfrak{s}\mathfrak{t}^{\boldsymbol{\lambda}}}$ for some $l \in \mathbb{Z}_{\geq 0}$ and then Lemma 5.1.8 implies that

$$U_{i_{j_1}} \dots U_{i_{j_r}} \mathfrak{m}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = r\mathfrak{m}_{\mathfrak{u}\mathfrak{t}^{\boldsymbol{\lambda}}} \mod b^{\boldsymbol{\lambda}}_n(m)$$

for some scalar $r \in \mathbb{C}$ and some $\mathfrak{u}$ such that $\mathfrak{u} \succ \mathfrak{s}$. Summing up, this proves the Theorem in the case where $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$.

To prove the general case, we first note that the same argument as above, only acting on the right instead of on the left, proves the Theorem in the case where $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}}$. The general case then follows by multiplying the two versions together and using cellularity. □

Remark 5.2.7. It follows from the Theorem that the subalgebra of $b_n(m)$ generated by the $e^b(\boldsymbol{i})$ and the ψ^b_i is equal to $b_n(m)$ itself.

To establish our main theorem we must define a degree function on the set of all one-line standard bitableaux. Let $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ and $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then we define the degree of $\mathfrak{t}$ as

$$\deg \mathfrak{t} := \deg \psi^b_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\lambda}}}. \tag{5.2.4}$$

We can now prove our main result, namely to construct a graded cellular basis for $b_n(m)$. Given our previous work, we can essentially follow the argument of [14, Theorem 5.8], just making the corresponding changes in notation. We sketch the argument because this is the main theorem of the thesis.

Theorem 5.2.8. *The blob algebra $b_n(m)$ is a graded cellular algebra with graded cellular basis $\{\psi^b_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in Bip_1(n)\}$.*

Proof: First of all it follows from the triangularity property of Theorem 5.2.6 that

$$\{\psi^b_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathrm{Bip}_1(n)\}$$

is a cellular basis for $b_n(m)$, since $\{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)\}$ is it. Moreover, by the definitions, $\psi^b_{\mathfrak{s}\mathfrak{t}}$ is a homogeneous elements of $b_n(m)$ of degree

$$\deg \psi^b_{\mathfrak{s}\mathfrak{t}} = \deg \mathfrak{s} + \deg \mathfrak{t}.$$

Using Corollary 4.3.6 one sees that there is a unique anti-automorphism $*$ of $b_n(m)$ that fixes the generators ψ^b_i, y^b_j and $e^b(\mathbf{i})$. Then by the definition it is clear that $(\psi^b_{\mathfrak{s}\mathfrak{t}})^* = \psi^b_{\mathfrak{t}\mathfrak{s}}$ and so the anti-automorphism induced by the basis $\{\psi^b_{\mathfrak{s}\mathfrak{t}}\}$ coincides with $*$. The Theorem is proved. □

Since $b_n(m)$ is a graded cellular algebra with graded cellular basis $\{\psi^b_{\mathfrak{s}\mathfrak{t}}\}$ we can define graded cell and simple $b_n(m)$-modules which we denote by $\Delta(\boldsymbol{\lambda})$ and $L(\boldsymbol{\lambda})$, respectively, with $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$. Therefore, we have obtained the main goal in this chapter for the $b_n(m)$-case. The graded cell module, $\Delta(\boldsymbol{\lambda})$, has a $\mathbb{C}$-basis $\{\psi^b_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})\}$ where the $b_n(m)$-action comes from (4.2.1).

By Theorem 5.2.6 the cell modules induced by the graded cellular bases agree with the cell modules induced by the diagram bases $\{\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}\}$, that is the standard modules for $b_n(m)$. The following result gives a formula for the graded dimension of $\Delta(\boldsymbol{\lambda})$.

Corollary 5.2.9. *Let $\boldsymbol{\lambda} \in Bip_1(n)$. Then*

$$\dim_t \Delta(\boldsymbol{\lambda}) = \sum_{\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})} t^{\deg(\mathfrak{t})} \tag{5.2.5}$$

Proof: The result is a direct consequence of the definitions and Theorem 5.2.8. □

For completeness, we give the analogous Theorem for the Temperley-Lieb algebra. This proof relies here on Theorem 4.3.2 and the compatibility of Murphy's standard basis with the diagram basis, as proved in [13], and could have been given earlier in the thesis. Let $\Phi_2 : \mathcal{H}_n(q^2) \longrightarrow Tl_n(q)$ be as above and define $\psi^{Tl}_{\mathfrak{st}} := \Phi_2(\psi_{st})$ for $s, t \in \mathrm{Std}(n)$ where ψ_{st} is the graded cellular basis for $\mathcal{H}_n(q^2)$ introduced by Hu and Mathas and $\mathrm{Shape}(s) = \mathrm{Shape}(t) \in \mathrm{Par}_2(n)$.

Theorem 5.2.10. *The Temperley-Lieb algebra $Tl_n(q)$ is a graded cellular algebra with graded cellular basis $\{\psi^{Tl}_{\mathfrak{st}}\}$ and degree function defined as above.*

Proof: According to Theorem 9 of [13], the diagram basis for $Tl_n(q)$ is upper triangularily related to the standard basis, with respect to the dominance order. But ψ_{st} is also upper triangularily related to the standard basis with respect to the dominance order, as already mentioned above, and hence the Theorem follows. □

5.3 EXAMPLES

In this section we illustrate our results on two examples.

Example 5.3.1. Our first example is $Tl_3(q)$, with q chosen to be a primitive cubic root of unity, that is $e = 3$. This is a non-semisimple algebra and so we expect the grading to be nontrivial. Using the homogeneous basis for $\psi^{Tl}_{\mathfrak{st}}$ for $Tl_3(q)$, we aim at determining a homogeneous basis for $Tl_3(q)$, in terms of the diagrams. Define first

$$\sigma = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \qquad\qquad \tau = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

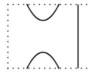
Then σ and τ are the only tableaux of shape $(2, 1)$. The only other possible shape in $\mathrm{Par}_2(3)$ is $\lambda = (1, 1, 1)$ whose only standard tableau is t^λ. Hence we get that $Tl_3(q)$ has dimension five with homogeneous basis consisting of the elements

$$\psi^{Tl}_{\mathfrak{ss}}, \psi^{Tl}_{\mathfrak{st}}, \psi^{Tl}_{\mathfrak{ts}}, \psi^{Tl}_{\mathfrak{tt}}, \psi^{Tl}_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda}.$$

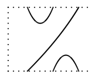
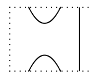
The residue sequences for σ and τ are $\boldsymbol{i}^\sigma = (0, 1, 2)$ and $\boldsymbol{i}^\tau = (0, 2, 1)$ and the degrees are $\deg(\sigma) = 0$ and $\deg(\tau) = 1$. (See [14, (3.8) and Definition 4.7]). Therefore, using the orthogonality of the KLR-idempotents, we have

$$\psi^{Tl}_{\sigma\tau}\psi^{Tl}_{\sigma\sigma} = \psi^{Tl}_{\sigma\sigma}\psi^{Tl}_{\tau\sigma} = 0, \tag{5.3.1}$$

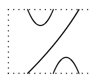
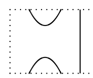
see [14, Lemma 5.2]. Now by the triangular expansion property mentioned in the above Theorem 5.2.8 there exists $c \in \mathbb{C}^{\times}$ such that

$\psi^{Tl}_{\sigma\sigma} = c$  $= cU_1$

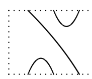
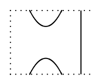
and hence U_1 is homogeneous of degree 0. Using the triangular expansion property of Theorem 5.2.8 once again, there are scalars $c_1, c_2 \in \mathbb{C}$ with $c_1 \neq 0$ such that

$\psi^{Tl}_{\sigma\tau} = c_1$  $+ c_2$ 

Multiplying this equality on the right by $\psi^{Tl}_{\sigma\sigma} = cU_1$, and using equation (5.3.1), we get that $c_1 = [2]c_2$. Hence the element

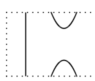
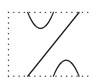
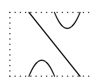
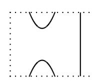
$A := [2]$  $+$ 

is a scalar multiple of $\psi^{Tl}_{\sigma\tau}$ and homogeneous of degree 1. Similarly we obtain that the element

$B := [2]$  $+$ 

is a scalar multiple of $\psi^{Tl}_{\tau\sigma}$ and homogeneous of degree 1.

Now, σ is the maximal tableau of shape $(2, 1)$ and so we have $\psi^{Tl}_{\tau\sigma}\psi^{Tl}_{\sigma\tau} = \psi^{Tl}_{\tau\tau}$. From this we obtain that $\psi^{Tl}_{\tau\tau}$ is a scalar multiple of

$C =$  $+[2]$  $+[2]$  $+$ 

which is a homogeneous element of degree 2.

The last homogeneous basis element can now be determined by expanding $\psi^{Tl}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ in the diagram basis. On the other hand, since the unity 1 is always homogeneous of degree 0 and since it is linearly independent of A, B, C and U_1, we use it. All in all, the set $\{1, U_1, A, B, C\}$ is a homogeneous bases for $Tl_3(q)$. In particular, $Tl_3(q)$ is a positively graded algebra and $\mathcal{F}_1 := \text{span}_{\mathbb{C}}\{A, B, C\}$ and $\mathcal{F}_2 := \text{span}_{\mathbb{C}}\{C\}$ are ideals in $Tl_3(q)$. In general, $Tl_n(q)$ is not positively graded.

Example 5.3.2. We now describe a homogeneous basis for $b_3 = b_3(q, y_e)$ in terms of blob diagrams, with q a primitive quintic root of unity and $y_e = -\frac{[1]}{[2]}$, so in this case $l = 5$ and $m = 2$. First, we list all elements in Std(3), with their respective residues sequences and degrees.

Bi-partitions	Bitableaux	Res. Sequence	Degree
$\boldsymbol{\lambda} = ((1),(2))$	$\mathfrak{t}^{\boldsymbol{\lambda}} = (\boxed{2}\,, \boxed{1}\boxed{3})$	$\boldsymbol{i}^{\boldsymbol{\lambda}} = (4,1,0)$	0
	$\mathfrak{s} = (\boxed{3}\,, \boxed{1}\boxed{2})$	$\boldsymbol{i}^{\mathfrak{s}} = (4,0,1)$	1
	$\mathfrak{t} = (\boxed{1}\,, \boxed{2}\boxed{3})$	$\boldsymbol{i}^{\mathfrak{t}} = (1,4,0)$	0
$\boldsymbol{\mu} = ((2),(1))$	$\mathfrak{t}^{\boldsymbol{\mu}} = (\boxed{2}\boxed{3}\,, \boxed{1})$	$\boldsymbol{i}^{\boldsymbol{\mu}} = (4,1,2)$	0
	$\mathfrak{v} = (\boxed{1}\boxed{3}\,, \boxed{2})$	$\boldsymbol{i}^{\mathfrak{v}} = (1,4,2)$	0
	$\mathfrak{u} = (\boxed{1}\boxed{2}\,, \boxed{3})$	$\boldsymbol{i}^{\mathfrak{u}} = (1,2,4)$	0
$\boldsymbol{\nu} = ((0),(3))$	$\mathfrak{t}^{\boldsymbol{\nu}} = (\emptyset, \boxed{1}\boxed{2}\boxed{3})$	$\boldsymbol{i}^{\boldsymbol{\nu}} = (4,0,1)$	0
$\boldsymbol{\kappa} = ((3),(0))$	$\mathfrak{t}^{\boldsymbol{\kappa}} = (\boxed{1}\boxed{2}\boxed{3}\,, \emptyset)$	$\boldsymbol{i}^{\boldsymbol{\kappa}} = (1,2,3)$	0

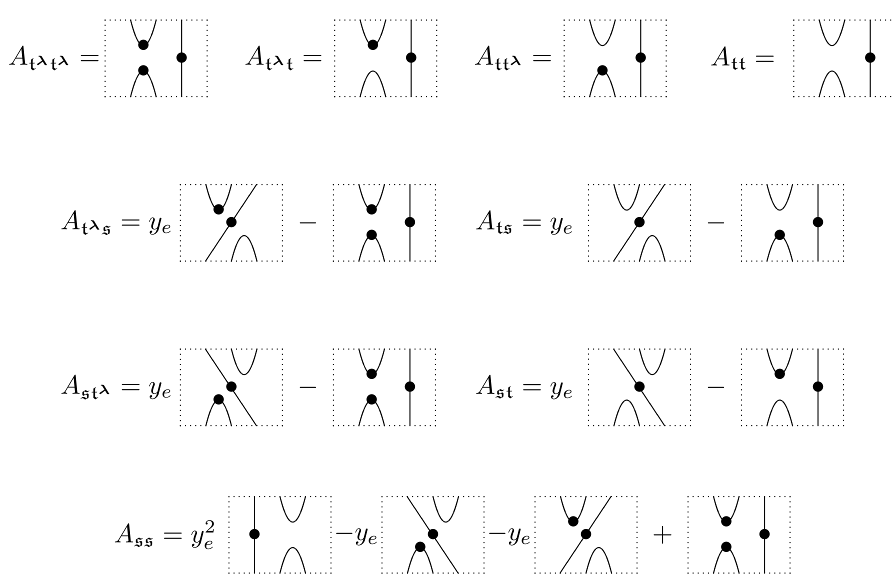
In order to obtain a diagrammatic homogeneous basis for b_3 we need the following lemma.

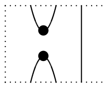
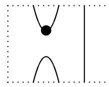
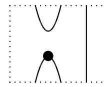
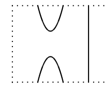
Lemma 5.3.3. *Let $\boldsymbol{\lambda} \in Bip_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then, we have*

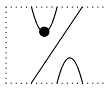
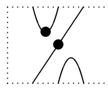
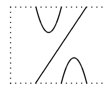
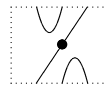
$$e\psi^b_{\mathfrak{s}\mathfrak{t}}e = \begin{cases} \psi^b_{\mathfrak{s}\mathfrak{t}}, & \text{if } 1 \text{ is located in the second component of } \mathfrak{s} \text{ and } \mathfrak{t}; \\ 0, & \text{otherwise.} \end{cases}$$

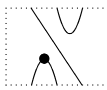
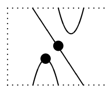
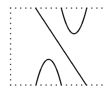
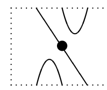
Proof: The claim is a direct consequence of the definition of $\psi^b_{\mathfrak{s}\mathfrak{t}}$, the orthogonality of the KLR-idempotents and Lemma 4.3.8. □

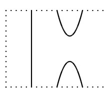
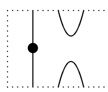
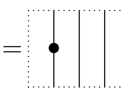
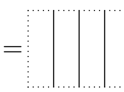
Using the triangularity property given in Lemma 5.2.6, the orthogonality of the KLR-idempotents and the previous Lemma 5.3.3, we can obtain a homogeneous basis for b_3 in terms of diagrams arguing as in the previous example. We omit such arguments for brevity, since $\dim_{\mathbb{C}}(b_3) = 20$.



$A_{\mathfrak{t}^\mu\mathfrak{t}^\mu} =$  $A_{\mathfrak{t}^\mu\mathfrak{v}} =$  $A_{\mathfrak{v}\mathfrak{t}^\mu} =$  $A_{\mathfrak{v}\mathfrak{v}} =$ 

$A_{\mathfrak{t}^\mu\mathfrak{u}} =$  $-$  $A_{\mathfrak{v}\mathfrak{u}} =$  $-$ 

$A_{\mathfrak{u}\mathfrak{t}^\mu} =$  $-$  $A_{\mathfrak{u}\mathfrak{v}} =$  $-$ 

$A_{\mathfrak{u}\mathfrak{u}} =$  $-$  $A_{\mathfrak{t}^\nu\mathfrak{t}^\nu} =$  $A_{\mathfrak{t}^\kappa\mathfrak{t}^\kappa} =$ 

The set $\{A_{\mathfrak{a}\mathfrak{b}} \mid \mathfrak{a}, \mathfrak{b} \in \mathrm{Std}(\boldsymbol{\lambda}) \ , \boldsymbol{\lambda} \in \mathrm{Bip}_1(n)\}$ is a homogeneous basis for b_3 with $\deg(A_{\mathfrak{a}\mathfrak{b}}) = \deg(\mathfrak{a}) + \deg(\mathfrak{b})$. We remark that the elements $A_{\mathfrak{a}\mathfrak{b}}$ in general not coincide with the elements $\psi^b_{\mathfrak{a}\mathfrak{b}}$. Just like in the case of the Temperley-Lieb algebra, in general the blob algebra is not positively graded .

CHAPTER 6

Graded decomposition number for $b_n(m)$

In the previous Chapter we have constructed a graded cellular basis $\{\psi^b_{\mathfrak{st}}\}$ for $b_n(m)$. The existence of this basis allows define graded cell and simple $b_n(m)$-modules which we have denoted by $\Delta(\boldsymbol{\lambda})$ and $L(\boldsymbol{\lambda})$, respectively, for $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$. The graded dimension of the graded cell modules $\Delta(\boldsymbol{\lambda})$ has been computed in Corollary 5.2.9. In this Chapter, we want to calculate the graded dimension of the graded simple modules $L(\boldsymbol{\lambda})$. The graded dimensions of graded cell and simple modules are related to the graded decomposition numbers, $[\Delta(\boldsymbol{\mu}) : L(\boldsymbol{\lambda})]_t$, via equation (iv) in Theorem 4.2.3. Since the graded dimension of the graded cell modules for b_n is known from Corollary 5.2.9, the problem of finding the graded dimensions of the irreducible b_n-modules is equivalent to the problem of finding the graded decomposition numbers for b_n. The main goal in this chapter is then to find $[\Delta(\boldsymbol{\mu}) : L(\boldsymbol{\lambda})]_t$ for all $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$. We shall refer to these polynomials as graded decomposition numbers for $b_n(m)$.

In the ungraded setting, the decomposition numbers for b_n were determined in [24] and [31] by using algebraic methods. Our approach is essentially combinatorial, and therefore different from those used in the ungraded case. A main point of our approach is the existence of a family of positively graded cellular subalgebras $b_n(m, \boldsymbol{\lambda})$ of b_n, with $\boldsymbol{\lambda}$ a one-line bipartition of total degree n. The graded decompositions numbers for b_n and $b_n(m, \boldsymbol{\lambda})$ are closely related and hence we reduce the main problem in this chapter to calculate the graded decomposition numbers for $b_n(m, \boldsymbol{\lambda})$. Now, since $b_n(m, \boldsymbol{\lambda})$ is a positively graded algebra we can define a filtration induced by the grading on $b_n(m, \boldsymbol{\lambda})$ for each cell $b_n(m, \boldsymbol{\lambda})$-module. This filtration together with a counting argument is sufficient to determine the graded decomposition numbers for $b_n(m, \boldsymbol{\lambda})$ (and therefore for b_n).

6.1 Degree function

In this section we study the degree function on $\mathrm{Std}(n)$ defined in (5.2.4). We remark that in general this function does not coincide with the degree function defined [5, (3.5)], the main reason for this is that we do not work with the dominance order on $\mathrm{Std}(n)$ and both degree functions depend heavily on the order considered on $\mathrm{Std}(n)$. First we give an interpretation of the degree function in terms of addable and removable nodes, similar to the one given in [14, Definition 4.7] for standard tableaux using the dominance order. Finally, we give a formula for the degree function depending on walks and *walls* on the Bratteli diagram.

Let $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$. The node $\alpha = (1, c, d)$ is called an addable node of $\boldsymbol{\lambda}$ if $\alpha \notin \lambda$ and $\lambda \cup \{\alpha\}$ is the diagram of a one-line bipartition of $n+1$. Similarly, $\rho \in \lambda$ is called a removable node of $\boldsymbol{\lambda}$ if $\lambda \backslash \{\rho\}$ is the diagram of a one-line bipartition of $n-1$. Note that any one-line bipartition has exactly two addable nodes. Furthermore, a one-line bipartition may have one or two removable nodes.

Given two nodes $\alpha = (1, c_1, d_1)$ and $\beta = (1, c_2, d_2)$ then α is said to be below β if $c_1 > c_2$, or $c_1 = c_2$, $d_1 = 1$ and $d_2 = 2$. The concept of *to be below* could have been defined in terms of $\mathfrak{t}^{\boldsymbol{\lambda}}$. In fact, given two nodes α and β choose a bipartition $\boldsymbol{\lambda}$ such that $\alpha, \beta \in [\boldsymbol{\lambda}]$. Then, the node α is below β if and only if $\mathfrak{t}^{\boldsymbol{\lambda}}(\alpha) > \mathfrak{t}^{\boldsymbol{\lambda}}(\beta)$. Using the dominance order there is a similar interpretation of the concept of *to be below* introduced in [14, Section 4]. But since $\mathfrak{t}^{\boldsymbol{\lambda}}$ does not coincide with the unique maximal bitableau for the dominance order, the two concepts do not coincide in general.

Let $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. For $k = 1, \ldots, n$ let $\mathcal{A}_{\mathfrak{t}}(k)$ be the set of all addable nodes of the bipartition $\mathrm{Shape}(\mathfrak{t}_k)$ which are below of $\mathfrak{t}^{-1}(k)$. Similarly, let $\mathcal{R}_{\mathfrak{t}}(k)$ be the set of all removable nodes of the bipartition $\mathrm{Shape}(\mathfrak{t}_k)$ which are below of $\mathfrak{t}^{-1}(k)$. Now define the sets $\mathcal{A}^m_{\mathfrak{t}}(k)$ and $\mathcal{R}^m_{\mathfrak{t}}(k)$ by

$$\mathcal{A}^m_{\mathfrak{t}}(k) = \{\alpha \in \mathcal{A}_{\mathfrak{t}}(k) \mid \mathrm{res}(\alpha) = r_{\mathfrak{t}}(k)\}$$

$$\mathcal{R}^m_{\mathfrak{t}}(k) = \{\rho \in \mathcal{R}_{\mathfrak{t}}(k) \mid \mathrm{res}(\rho) = r_{\mathfrak{t}}(k)\}$$

It is easy to check that the sets $\mathcal{A}^m_{\mathfrak{t}}(k)$ and $\mathcal{R}^m_{\mathfrak{t}}(k)$ are empty or contain a single node, for all $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ and $1 \leq k \leq n$. Let g be the function defined by

$$\begin{array}{rcl} g : \mathrm{Std}(n) & \to & \mathbb{Z} \\ \mathfrak{t} & \to & \sum_{k=1}^{n} \left(|\mathcal{A}_{\mathfrak{t}}^{m}(k)| - |\mathcal{R}_{\mathfrak{t}}^{m}(k)| \right) \end{array}$$

Using the above notation we can now give a characterization of the degree function. We need the following Lemma.

Lemma 6.1.1. *Let $\boldsymbol{\lambda} \in Bip_1(n)$ and $\mathfrak{s}, \mathfrak{t} \in Std(\boldsymbol{\lambda})$. Assume that $\mathfrak{s} \succ \mathfrak{t}$ and $s_r \mathfrak{s} = \mathfrak{t}$. Then*

$$g(\mathfrak{t}) - g(\mathfrak{s}) = \deg(\psi_r^b e^b(\boldsymbol{i}^{\mathfrak{s}}))$$

Proof: First note that $\mathcal{A}_{\mathfrak{s}}^{m}(k) = \mathcal{A}_{\mathfrak{t}}^{m}(k)$ and $\mathcal{R}_{\mathfrak{s}}^{m}(k) = \mathcal{R}_{\mathfrak{t}}^{m}(k)$, for all $k \neq r, r+1$, since $s_r \mathfrak{s} = \mathfrak{t}$. Hence

$$g(\mathfrak{t}) - g(\mathfrak{s}) = \sum_{k=r}^{r+1} \left(|\mathcal{A}_{\mathfrak{t}}^{m}(k)| - |\mathcal{R}_{\mathfrak{t}}^{m}(k)| \right) - \sum_{k=r}^{r+1} \left(|\mathcal{A}_{\mathfrak{s}}^{m}(k)| - |\mathcal{R}_{\mathfrak{s}}^{m}(k)| \right)$$

Set $e(\boldsymbol{i}^{\mathfrak{s}}) = (i_1, \ldots, i_r, i_{r+1}, \ldots, i_n)$. We remark that $i_j = r_{\mathfrak{s}}(j)$. The numbers involved in the above sums depends on $i_r - i_{r+1}$ modulo l, so we split the proof in four cases according to the followings options:

$$i_r - i_{r+1} \equiv \begin{cases} -1 \mod l \\ 0 \mod l \\ 1 \mod l \\ \text{otherwise.} \end{cases} \tag{6.1.1}$$

We consider the first case, the remaining three cases follow in a similar way. Thus we assume that $i_r - i_{r+1} \equiv -1 \mod l$. Note that the node occupied by r in $\mathfrak{t}$ is below the node occupied by $r+1$ in $\mathfrak{t}$ since $\mathfrak{s} \succ \mathfrak{t}$. In this setting, the values involved in the sums are

$$|\mathcal{A}_{\mathfrak{s}}^{m}(r)| = 0 \quad |\mathcal{A}_{\mathfrak{s}}^{m}(r+1)| = 0 \quad |\mathcal{R}_{\mathfrak{s}}^{m}(r)| = 1 \quad |\mathcal{R}_{\mathfrak{s}}^{m}(r+1)| = 0$$

$$|\mathcal{A}_{\mathfrak{t}}^{m}(r)| = 0 \quad |\mathcal{A}_{\mathfrak{t}}^{m}(r+1)| = 0 \quad |\mathcal{R}_{\mathfrak{t}}^{m}(r)| = 0 \quad |\mathcal{R}_{\mathfrak{t}}^{m}(r+1)| = 0$$

Therefore $g(\mathfrak{t}) - g(\mathfrak{s}) = 1 = \deg(\psi_r^b e^b(\boldsymbol{i}^{\mathfrak{s}}))$, completing the proof in this case. $\square$

Example 6.1.2. *Assume that $l = 5$ and $m = 2$. Let $\boldsymbol{\lambda} = ((5),(1)) \in Bip_1(6)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ given by*

$$\mathfrak{s} = \left(\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{6}\ , \boxed{5} \right) \qquad \mathfrak{t} = \left(\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\ , \boxed{6} \right)$$

Note that $\mathfrak{s} \succ \mathfrak{t}$, $s_5\mathfrak{s} = \mathfrak{t}$ *and* $e(\boldsymbol{i}^{\mathfrak{s}}) = (1,2,3,4,4,0)$. *We also have* $\mathcal{R}^m_{\mathfrak{s}}(5) = \{\mathfrak{s}^{-1}(4)\}$, $\mathcal{R}^m_{\mathfrak{s}}(j) = \emptyset$ *for* $j \neq 5$ *and* $\mathcal{A}^m_{\mathfrak{s}}(j) = \mathcal{R}^m_{\mathfrak{t}}(j) = \mathcal{A}^m_{\mathfrak{t}}(j) = \emptyset$ *for all* $1 \leq j \leq 6$. *Thus* $g(\mathfrak{s}) = -1$ *and* $g(\mathfrak{t}) = 0$. *Therefore,* $g(\mathfrak{t}) - g(\mathfrak{s}) = 1 = \deg \psi^b_5 e^b(\boldsymbol{i}^{\mathfrak{s}})$.

Corollary 6.1.3. *Let* $\boldsymbol{\lambda} \in Bip_1(n)$ *and* $\mathfrak{t} \in Std(\boldsymbol{\lambda})$. *Then* $g(\mathfrak{t}) = \deg(\mathfrak{t})$.

Proof: By Lemma 3.3.7 there is a sequence of one-line standard bitableaux

$$\mathfrak{t} = \mathfrak{t}_0 \prec \mathfrak{t}_1 \prec \ldots \prec \mathfrak{t}_{k-1} \prec \mathfrak{t}_k = \mathfrak{t}^{\boldsymbol{\lambda}}$$

such that $s_{i_j}\mathfrak{t}_{j-1} = \mathfrak{t}_j$ for $1 \leq j \leq k$ and $\mathfrak{d}(\mathfrak{t}) = s_{i_1} \ldots s_{i_k}$ is a reduced expression for $\mathfrak{d}(\mathfrak{t})$. Now using the above Lemma 6.1.1 and the fact that $g(\mathfrak{t}^{\boldsymbol{\lambda}}) = 0$ we have

$$g(\mathfrak{t}) = \sum_{j=1}^{k} (g(\mathfrak{t}_{j-1}) - g(\mathfrak{t}_j)) = \sum_{j=1}^{k} \deg(\psi^b_{i_j} e^b(\boldsymbol{i}^{\mathfrak{t}_j})) = \deg(\psi^b_{i_1} \ldots \psi^b_{i_k} e^b(\boldsymbol{i}^{\boldsymbol{\lambda}})) = \deg(\mathfrak{t})$$

□

Recall from Chapter 3 that for any $\mathfrak{t} \in \mathrm{Std}(n)$ we have associated a walk on the Bratteli diagram $w(\mathfrak{t})$. The walk $w(\mathfrak{t})$ at the j-th step is situated in $(j, \mathfrak{t}(j))$, where $\{\mathfrak{t}(j) | j = 0, 1, ..., n\}$ is the sequence of integers associated to $\mathfrak{t}$ in Definition 3.2.6. In this chapter we describe a walk as a sequence of weights. More precisely, for $\mathfrak{t} \in \mathrm{Std}(n)$ we write $w(\mathfrak{t}) = (w(\mathfrak{t})_0, w(\mathfrak{t})_1, \ldots, w(\mathfrak{t})_n)$, where $w(\mathfrak{t})_j = \mathfrak{t}(j)$ for all $0 \leq j \leq n$. We refer to this notation as the weight sequence of $w(\mathfrak{t})$. With this at hand, we can give a formula for the residues in terms of the weight sequence. In fact, a routine analysis of (3.2.1) reveals that

$$2r_{\mathfrak{t}}(j) \equiv j - 2 + (w(\mathfrak{t})_j - w(\mathfrak{t})_{j-1})(w(\mathfrak{t})_j + m) \mod l \tag{6.1.2}$$

Since l is assumed to be odd the above formula determines uniquely the residue sequence. In the next lemma we will use the above sequence to determine when $\mathcal{A}^m_{\mathfrak{t}}(j)$ and $\mathcal{R}^m_{\mathfrak{t}}(j)$ are non-empty. Recall that these sets have a single element or are empty.

Lemma 6.1.4. *Let* $\mathfrak{t} \in \mathrm{Std}(n)$. *Consider the walk* $w(\mathfrak{t}) = (w(\mathfrak{t})_0, w(\mathfrak{t})_1, \ldots, w(\mathfrak{t})_n)$ *determined in Definition* **??** *written as a sequence of weights. Define the sets*

$$A^1_{\mathfrak{t}} = \{1 \leq j \leq n \mid w(\mathfrak{t})_j < 0, w(\mathfrak{t})_{j-1} \equiv -m \mod l, w(\mathfrak{t})_j \equiv -m+1 \mod l\}$$

$$A^2_{\mathfrak{t}} = \{1 \leq j \leq n \mid w(\mathfrak{t})_j > 0, w(\mathfrak{t})_{j-1} \equiv -m \mod l, w(\mathfrak{t})_j \equiv -m-1 \mod l\}$$

$$R^1_{\mathfrak{t}} = \{1 \leq j \leq n \mid w(\mathfrak{t})_j < 0, w(\mathfrak{t})_{j-1} \equiv -m-1 \mod l, w(\mathfrak{t})_j \equiv -m \mod l\}$$

$$R^2_{\mathfrak{t}} = \{1 \leq j \leq n \mid w(\mathfrak{t})_j > 0, w(\mathfrak{t})_{j-1} \equiv -m+1 \mod l, w(\mathfrak{t})_j \equiv -m \mod l\}$$

Define also the sets $A_{\mathfrak{t}} = A^1_{\mathfrak{t}} \cup A^2_{\mathfrak{t}}$ *and* $R_{\mathfrak{t}} = R^1_{\mathfrak{t}} \cup R^2_{\mathfrak{t}}$. *Then we have*

(i) $j \in A_{\mathfrak{t}}$ *if and only if* $|\mathcal{A}^m_{\mathfrak{t}}(j)| = 1$

(ii) $j \in R_{\mathfrak{t}}$ *if and only if* $|\mathcal{R}^m_{\mathfrak{t}}(j)| = 1$

Proof: We only prove (i), the result (ii) is proved similarly. Recall that $|\mathcal{A}^m_{\mathfrak{t}}(j)| = 0$ or 1, for all $\mathfrak{t} \in \mathrm{Std}(n)$ and $1 \leq j \leq n$. Suppose that $j \in A_{\mathfrak{t}}$, then $j \in A^1_{\mathfrak{t}}$ or $j \in A^2_{\mathfrak{t}}$. Assume that $j \in A^1_{\mathfrak{t}}$, the case $j \in A^2_{\mathfrak{t}}$ is treated similarly. By definition of $A^1_{\mathfrak{t}}$ we have

$$w(\mathfrak{t})_j < 0, \qquad w(\mathfrak{t})_{j-1} \equiv -m \mod l, \qquad w(\mathfrak{t})_j \equiv -m+1 \mod l$$

Recall that for any one-line bipartition there is two addable nodes, one in each component. Let N_1 and N_2 the addable nodes to $\mathrm{Shape}(\mathfrak{t}_{j-1})$ in the first and second component, respectively. By $w(\mathfrak{t})_{j-1} \equiv -m \mod l$ and (6.1.2), the nodes N_1 and N_2 have the same residue. On the other hand, $w(\mathfrak{t})_j \equiv -m+1 \mod l$ implies that $N_1 = \mathfrak{t}^{-1}(j)$. Hence, the node N_2 is addable to $\mathrm{Shape}(\mathfrak{t}_j)$ with the same residue of N_1. Finally, note that N_2 is below N_1 since $w(\mathfrak{t})_j < 0$. Therefore, $N_2 \in \mathcal{A}^m_{\mathfrak{t}}(j)$. Consequently, $|\mathcal{A}^m_{\mathfrak{t}}(j)| = 1$.

Conversely, suppose that $|\mathcal{A}^m_{\mathfrak{t}}(j)| = 1$. Let N be the unique node in $\mathcal{A}^m_{\mathfrak{t}}(j)$. Define M to be the node occupied by j in $\mathfrak{t}$. Then, N and M have the same residue and N is below M. Set $N = (1, c_1, d_1)$ and $M = (1, c_2, d_2)$. Using the conditions on the parameters q and m given in (2.1.4), one can check that two nodes with the same residue can not be located in the same column . Hence, $c_2 \neq c_1$. Since N is below M we actually have $c_2 > c_1$. Note also that N and M can not be located in the same component. Hence, $d_1 \neq d_2$. Assume that $d_2 = 1$, thus $d_1 = 2$. Then, the fact that N and M have the same residue is equivalent to

$$c_2 - c_1 \equiv -m \mod l \tag{6.1.3}$$

On the other hand, recall that for any $1 \leq i \leq n$ the weight $w(\mathfrak{t})_i$ is equal to the number of nodes in the first component of $\mathrm{Shape}(\mathfrak{t}_{|i})$ minus the number of nodes in the second component of $\mathrm{Shape}(\mathfrak{t}_{|i})$. Therefore, it is easy to note that $w(\mathfrak{t})_{j-1} = c_2 - c_1$ and $w(\mathfrak{t})_j = c_2 - c_1 + 1$. Then, by (6.1.3) and $c_2 > c_1$ we obtain

$$w(\mathfrak{t})_{j-1} \equiv -m \mod l; \qquad w(\mathfrak{t})_j \equiv -m+1 \mod l \qquad \text{and} \qquad w(\mathfrak{t})_j > 0.$$

This proves that $j \in A^1_{\mathfrak{t}} \subset A_{\mathfrak{t}}$. If $d_2 = 2$ then arguing similarly we get that $j \in A^2_{\mathfrak{t}} \subset A_{\mathfrak{t}}$, completing the proof. □

Corollary 6.1.5. *Let $\mathfrak{t} \in Std(n)$. Let $A_\mathfrak{t}$ and $R_\mathfrak{t}$ the sets defined in the above Lemma. Then*

$$\deg(\mathfrak{t}) = |A_\mathfrak{t}| - |R_\mathfrak{t}| \tag{6.1.4}$$

Proof: This is a direct consequence of Corollary 6.1.3 and Lemma 6.1.4. □

Bratteli diagram

We are now in position to give a graphical interpretation on the Bratteli diagram of the degree function. For this we need to draw *walls* in the Bratteli diagram. This means drawing vertical lines in each weight, $\boldsymbol{\lambda}$, such that $\boldsymbol{\lambda} \equiv -m \mod l$, as shown in Figure 6.1 for the case $l = 5$ and $m = 2$.

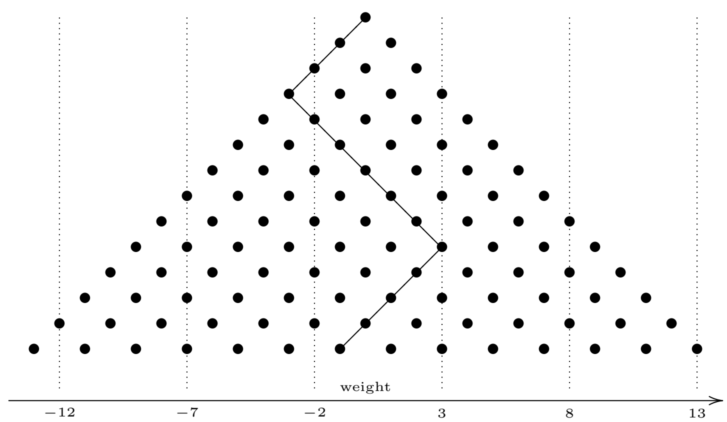


Figure 6.1: Walls in the Bratteli diagram for $l = 5$ and $m = 2$.

Then, for $\mathfrak{t} \in \mathrm{Std}(n)$, $|A_\mathfrak{t}|$ (resp. $|R_\mathfrak{t}|$) is the number of edges in the walk $w(\mathfrak{t})$ such that the initial (resp. final) vertex is on a wall and the final vertex is closer than the initial vertex to the central axis of the Bratteli diagram. For example, let $\mathfrak{t}$ be the bitableau associated with the walk in Figure 6.1, then $A_t = \{5, 10\}$ and $R_t = \{4\}$, consequently $\deg(\mathfrak{t}) = 1$. It is also easy to check that $|A_{\mathfrak{t}^{\boldsymbol{\lambda}}}| = |R_{\mathfrak{t}^{\boldsymbol{\lambda}}}| = 0$ for all $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$, so $\deg(\mathfrak{t}^{\boldsymbol{\lambda}}) = 0$.

The walls drawn on the Bratteli diagram define an alcove structure on $\mathbb{R}$, where the alcoves are the connected components of non-walls elements. We can thus refer to the alcove or wall in which a given weight lies. Note that by the conditions on the parameters (2.1.4), the weight $\boldsymbol{\lambda} = 0$ always belongs to an alcove, that is, it is not on a wall. We refer to the alcove in which $\boldsymbol{\lambda} = 0$ lies as the *fundamental alcove*. Let W be the infinite dihedral group on two generators s_- and s_+, that is $W = \langle s_-, s_+ | s_-^2 = s_+^2 = 1\rangle$. The alcove structure defines an action of W on

$\mathbb{R}$, by mapping s_- (resp. s_+) to the reflection in the left (resp. right) wall of the fundamental alcove. Since the walls were drawn on integral weights, the subset $\mathbb{Z}$ of $\mathbb{R}$ is clearly invariant under this action. Therefore, we can restrict the action of W to $\mathbb{Z}$. Let $\sim$ be the equivalence relation on $\mathbb{Z}$ determined by this action. Figure 6.2 shows the orbit of 0 under this action for $l = 5$ and $m = 2$. In this case, we have $\cdots \sim -10 \sim -4 \sim 0 \sim 6 \sim 10 \sim \cdots$.

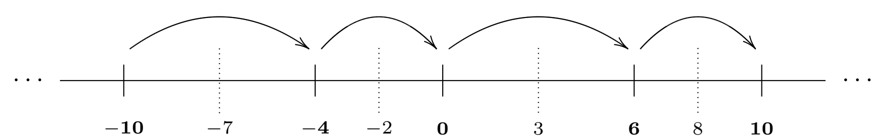


Figure 6.2: The orbit of 0 under the action of W on R, for $l = 5$ and $m = 2$.

For a weight $\boldsymbol{\lambda} \in \Lambda_n$ we denote by $O_n(\boldsymbol{\lambda})$ the set of all $\boldsymbol{\mu} \in \Lambda_n$ such that $\boldsymbol{\mu} \sim \boldsymbol{\lambda}$. Define $M_n(\boldsymbol{\lambda})$ to be the set

$$M_n(\boldsymbol{\lambda}) = \{\boldsymbol{\mu} \in O_n(\boldsymbol{\lambda}) \mid \text{ there exist } \mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}} \text{ with } \operatorname{Shape}(\mathfrak{s}) = \boldsymbol{\mu}\} \tag{6.1.5}$$

Given a walk on the Bratteli diagram we say that a subset of consecutive edges is a *wall to wall step* if these edges form a straight line between two walls of the same alcove. A wall to wall step can be classified into three different types according to whether it crosses the fundamental alcove, it goes away from the central axis or it approaches the central axis. We denote by F, O and I to these types, respectively. For $\mathfrak{s} \in \operatorname{Std}(n)$, we also define integers $n_{\mathfrak{s}}(F)$, $n_{\mathfrak{s}}(I)$ and $n_{\mathfrak{s}}(O)$ as the number of occurrences in $w(\mathfrak{s})$ of wall to wall steps of type F, I and O, respectively. The following lemma is the first step in order to give an easy formula for the degree of $\mathfrak{s} \in \operatorname{Std}(n)$ such that $\mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}}$, for some $\boldsymbol{\lambda} \in \operatorname{Bip}_1(n)$. Recall that $\mathfrak{s} \approx \mathfrak{t}$ if and only if $r_{\mathfrak{s}}(j) = r_{\mathfrak{t}}(j)$, for all $1 \leq j \leq n$.

Lemma 6.1.6. *Let $\boldsymbol{\lambda} \in Bip_1(n)$. A walk $w(\mathfrak{s})$ for $\mathfrak{s} \in \operatorname{Std}(n)$ satisfies $\mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}}$ if and only if the following conditions hold:*

(a) *First, $w(\mathfrak{s})$ and $w(\mathfrak{t}^{\boldsymbol{\lambda}})$ must matches from level 0 to the first contact of $w(\mathfrak{t}^{\boldsymbol{\lambda}})$ with a wall of the fundamental alcove.*

(b) *Next, $w(\mathfrak{s})$ makes wall to wall steps (as many as the number of alcoves between $\boldsymbol{\lambda}$ and 0) of any type.*

(c) *Finally, $w(\mathfrak{s})$ is completed with a straight line to the level n in either direction.*

Proof: First, recall from (6.1.2) that for any $\mathfrak{s} \in \mathrm{Std}(n)$ the residue can be expressed in terms of the weight sequence of $w(\mathfrak{s})$. Assume that $\mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}}$ and suppose that we know the weights $w(\mathfrak{s})_0, w(\mathfrak{s})_1, \ldots, w(\mathfrak{s})_{j-1}$ of the weight sequence of $w(\mathfrak{s})$. Recall that $w(\mathfrak{s})_j = w(\mathfrak{s})_{j-1} \pm 1$, so $w(\mathfrak{s})_j$ has only two options. Therefore, if $w(\mathfrak{t})_{j-1} \not\equiv -m \mod l$ (equivalently, if $w(\mathfrak{t})_{j-1}$ is not on a wall) only one option is acceptable to $w(\mathfrak{s})_j$ because by replacing in (6.1.2) we get two different values for $r_{\mathfrak{s}}(j)$. But $r_{\mathfrak{s}}(j)$ is a known value, actually $r_{\mathfrak{s}}(j) = r_{\mathfrak{t}^{\boldsymbol{\lambda}}}(j)$. Instead, if $w(\mathfrak{t})_{j-1} \equiv -m \mod l$ (equivalently, if $w(\mathfrak{t})_{j-1}$ is on a wall) both options for $w(\mathfrak{t})_j$ are acceptable to $w(\mathfrak{s})_j$ because by replacing in (6.1.2) we get the same value for $r_{\mathfrak{s}}(j)$. The lemma follows then by the description of $w(\mathfrak{t}^{\boldsymbol{\lambda}})$ as the walk that first zigzags on and off the central vertical axis of the Bratteli diagram, and then finishes with a straight line to the weight $\boldsymbol{\lambda}$ at level n. □

Remark 6.1.7. A walk as in the above lemma need not have parts (b) and (c). For example, if $\boldsymbol{\lambda}$ is located in the fundamental alcove, on one of its walls or in one of the two alcoves adjacent to the fundamental alcove a walk $w(\mathfrak{s})$ with $\mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}}$ it does not have part (b). On the other hand, if $\boldsymbol{\lambda}$ is located on a wall a walk $w(\mathfrak{s})$ with $\mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}}$ it does not have part (c). According to the above lemma, we split any walk $w(\mathfrak{s})$ with $\mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}}$ in three parts (a), (b) and (c).

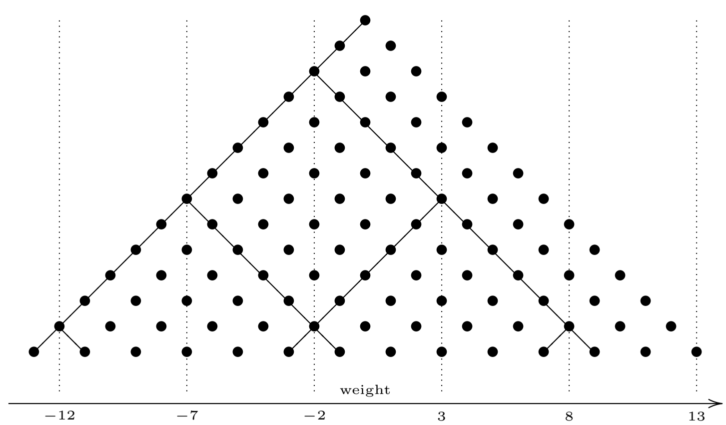


Figure 6.3: Walks $w(\mathfrak{t})$ with $\boldsymbol{i}^{\mathfrak{t}} = \boldsymbol{i}^{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} = ((0), (13))$.

Figure 6.3 shows all walks on the Bratteli diagram, $w(\mathfrak{t})$, with $\mathfrak{t} \approx \mathfrak{t}^{\boldsymbol{\lambda}}$, where $\boldsymbol{\lambda} = ((0), (13)) \in \mathrm{Bip}_1(13)$. In terms of the description given in the above lemma, for all walks in the figure, we have that part (a) goes from level 0 to level 2, part (b) goes from level 2 to level 12, and part (c) goes from level 12 to level 13. Hence, in this case $M_{13}(\boldsymbol{\lambda}) = \{-13, -11, -3, -1, 7, 9\}$.

Theorem 6.1.8. *Let $\boldsymbol{\lambda} \in Bip_1(n)$ and $\mathfrak{s} \in Std(n)$. Suppose that $\mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}}$. Then,*

(a) *If $\mathfrak{s} \in Std(\boldsymbol{\lambda})$ then $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}}$;*

(b) *If part (c) of $w(\mathfrak{s})$ points towards the central axis then* $\deg(\mathfrak{s}) = n_{\mathfrak{s}}(F) + 1$*;*

(c) *If part (c) of $w(\mathfrak{s})$ points away from the central axis then* $\deg(\mathfrak{s}) = n_{\mathfrak{s}}(F)$.

Proof: Statement (a) says that the unique $\mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda})$ with $\mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}}$ is $\mathfrak{t}^{\boldsymbol{\lambda}}$, and this is clear from the Lemma 6.1.6. Now, via Corollary 6.1.5, we can conclude that the part (a) of $w(\mathfrak{s})$ has degree zero, in part (b) the degree is $n_{\mathfrak{s}}(F)$, and part (c) has degree 1 or 0 according to whether the final straight line points towards the central axis of the Bratteli diagram or not. This proves (b) and (c). □

6.2 Graded decomposition numbers

In this section we obtain the main result in this chapter, the graded decomposition numbers for $b_n(m)$. For $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$, denote by $b_n(m, \boldsymbol{\lambda})$ to be the subalgebra $e^b(\boldsymbol{i}^{\boldsymbol{\lambda}})b_n(m)e^b(\boldsymbol{i}^{\boldsymbol{\lambda}})$ of $b_n(m)$. The basic strategy for finding the graded decomposition numbers for $b_n(m)$ is to exploit Theorem 4.1.1 on this subalgebra. We moreover need the known fact that the (ungraded) decomposition numbers of $b_n(m)$ are 0 or 1 (See for instance [31, Theorem 5.5]).

Remark 6.2.1. If we assume that the graded decomposition numbers for b_n, $[\Delta(\boldsymbol{\mu}) : L(\boldsymbol{\lambda})]_t$, are polynomials with constant coefficient equal to zero (for $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$) then we could obtain these numbers using analogous methods to those used by Kleshchev and Nash in [18], without using the prior knowledge about the ungraded decomposition numbers for b_n mentioned in the last paragraph. The additional hypothesis on the graded decomposition numbers can be proved by brute force calculations over the homogeneous presentation for b_n. However, for the sake of readability, we prefer the presentation as it stands.

Theorem 6.2.2. *For $\boldsymbol{\lambda} \in Bip_1(n)$, we have that $b_n(m, \boldsymbol{\lambda})$ is a positively graded cellular algebra with weight poset $(M_n(\boldsymbol{\lambda}), \succeq)$, $T(\boldsymbol{\mu}) = \mathrm{Std}(\boldsymbol{\mu}) \cap \mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$ for $\boldsymbol{\mu} \in M_n(\boldsymbol{\lambda})$, graded cellular basis*

$$\{\psi^b_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\mu}) \cap \mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}}) \textit{ for } \boldsymbol{\mu} \in M_n(\boldsymbol{\lambda})\} \tag{6.2.1}$$

and degree function as (5.2.4).

Proof: By Theorem 5.2.8 and the orthogonality of the KLR-idempotents, $e^b(\boldsymbol{i})$, we have that $b_n(m, \boldsymbol{\lambda})$ has a $\mathbb{C}$-basis consisting of all elements $\psi^b_{\mathfrak{st}}$ such that

$$\boldsymbol{i}^{\mathfrak{s}} = \boldsymbol{i}^{\mathfrak{t}} = \boldsymbol{i}^{\boldsymbol{\lambda}} \tag{6.2.2}$$

Thus, $b_n(m, \boldsymbol{\lambda})$ is a positively $\mathbb{Z}$-graded algebra since all bitableaux satisfying the condition (6.2.2) have non-negative degree by Theorem 6.1.8. All claims about the cellularity of $b_n(m, \boldsymbol{\lambda})$ follow from the cellularity of $b_n(m)$ and the definition of $M_n(\boldsymbol{\lambda})$. □

Now that $\{\psi^b_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\mu}) \cap \operatorname{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}}) \text{ for } \boldsymbol{\mu} \in M_n(\boldsymbol{\lambda})\}$ is known to be a graded cellular basis for $b_n(m, \boldsymbol{\lambda})$ we can define graded cell and simple modules which we denote by $\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ and $L_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ respectively, for $\boldsymbol{\mu} \in M(\boldsymbol{\lambda})$. Note that $e^b(\boldsymbol{i}^{\boldsymbol{\lambda}})\Delta(\boldsymbol{\mu}) = \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ and $e^b(\boldsymbol{i}^{\boldsymbol{\lambda}})L(\boldsymbol{\mu}) = L_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$. We can also define graded decomposition numbers for $b_n(m, \boldsymbol{\lambda})$. Graded decomposition numbers of $b_n(m)$ and $b_n(m, \boldsymbol{\lambda})$ are related via equation (4.1.4).

Lemma 6.2.3. *Let $\boldsymbol{\lambda} \in Bip_1(n)$. If $\boldsymbol{\mu} \notin M_n(\boldsymbol{\lambda})$ then $[\Delta(\boldsymbol{\mu}) : L(\boldsymbol{\lambda})]_t = 0$.*

Proof: First, recall that for any $\boldsymbol{\nu} \in \operatorname{Bip}_1(n)$ the cell b_n-module $\Delta(\boldsymbol{\nu})$ has a $\mathbb{C}$-basis $\{\psi^b_{\mathfrak{s}} \mid \mathfrak{s} \in \operatorname{Std}(\boldsymbol{\nu})\}$ and that by the orthogonality of the KLR-idempotents, $e^b(\boldsymbol{i})$, these act on this basis according to the rule

$$e^b(\boldsymbol{i})\psi^b_{\mathfrak{s}} = \begin{cases} \psi^b_{\mathfrak{s}}, & \text{if } \boldsymbol{i}^{\mathfrak{s}} = \boldsymbol{i} \\ 0, & \text{if } \boldsymbol{i}^{\mathfrak{s}} \neq \boldsymbol{i} \end{cases} \tag{6.2.3}$$

Now, if $\boldsymbol{\mu} \notin M_n(\boldsymbol{\lambda})$ then for all $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\mu})$ we have $\boldsymbol{i}^{\mathfrak{s}} \neq \boldsymbol{i}^{\boldsymbol{\lambda}}$, thus

$$e^b(\boldsymbol{i}^{\boldsymbol{\lambda}})\Delta(\boldsymbol{\mu}) = \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = 0$$

By the description given in Lemma 6.1.6 for all walks $w(\mathfrak{s})$ with $\boldsymbol{i}^{\mathfrak{s}} = \boldsymbol{i}^{\boldsymbol{\lambda}}$ (and therefore for all $\boldsymbol{\mu} \in M_n(\boldsymbol{\lambda})$), it is straightforward to check that $\boldsymbol{\lambda}$ is a minimal element of $M_n(\boldsymbol{\lambda})$. Furthermore, Theorem 6.1.8(a) implies that $\mathfrak{t}^{\boldsymbol{\lambda}}$ is the unique standard bitableau in $\operatorname{Std}(\boldsymbol{\lambda})$ with residue sequence equal to $\boldsymbol{i}^{\boldsymbol{\lambda}}$. Hence

$$e^b(\boldsymbol{i}^{\boldsymbol{\lambda}})\Delta(\boldsymbol{\lambda}) = \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \operatorname{Span}_{\mathbb{C}}\{\psi^b_{\mathfrak{t}^{\boldsymbol{\lambda}}}\}$$

Therefore, by (4.1.4) we have

$$[\Delta(\boldsymbol{\mu}) : L(\boldsymbol{\lambda})]_t = [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t = 0$$

completing the proof of the lemma. □

6.2.1 The non-wall case.

If $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ belongs to the fundamental alcove it is straightforward to check that $M_n(\boldsymbol{\lambda}) = \{\boldsymbol{\lambda}\}$. Hence, by Lemma 6.2.3 the module $L(\boldsymbol{\lambda})$ only appears as a graded composition factor in $\Delta(\boldsymbol{\lambda})$, and in this case by Theorem 4.2.3(c) $[\Delta(\boldsymbol{\lambda}) : L(\boldsymbol{\lambda})]_t = 1$. Therefore, we fix $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ that does not belong to the fundamental alcove. Furthermore, for the rest of this subsection we also assume that $\boldsymbol{\lambda}$ is not on a wall of the Bratteli diagram. The other case will be treated in the forthcoming subsection.

Since $b_n(m, \boldsymbol{\lambda})$ is a positively graded cellular algebra $\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ is also positively graded, for all $\boldsymbol{\mu} \in M_n(\boldsymbol{\lambda})$. Then $\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) \in \mathbb{Z}[t]$, for all $\boldsymbol{\mu} \in M_n(\boldsymbol{\lambda})$. Again by the positive grading on $b_n(m, \boldsymbol{\lambda})$ we can define the grading filtration for each $\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$. In order to know the dimensions of the quotients that appear in this grading filtration it is enough with to know the coefficients of $\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$. This is our next goal. We derive the graded decomposition number for b_n from this. For $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ define the number $\kappa(\boldsymbol{\lambda})$ as the number of alcoves between $\boldsymbol{\lambda}$ and 0.

Lemma 6.2.4. *Let $\boldsymbol{\lambda} \in Bip_1(n)$. Then $|M_n(\boldsymbol{\lambda})| = 2(\kappa(\boldsymbol{\lambda}) + 1)$.*

Proof: Since $\boldsymbol{\lambda}$ is not on a wall, there is in each alcove a unique representative for the orbit of $\boldsymbol{\lambda}$. So, if $\boldsymbol{\lambda}$ is located in the positive (resp. negative) side of the Bratteli diagram then to the right (resp. left) of the (resp. left) right wall of the fundamental alcove there is exactly $\kappa(\boldsymbol{\lambda}) + 1$ elements in $M_n(\boldsymbol{\lambda})$. Reflecting these elements through the right (resp. left) wall of the fundamental alcove we can get all the elements in $M_n(\boldsymbol{\lambda})$ to the left (resp. right) of such wall. Hence, $|M_n(\boldsymbol{\lambda})| = 2(\kappa(\boldsymbol{\lambda}) + 1)$. □

In order to give a precise description of $\dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\mu}))$, for $\boldsymbol{\mu} \in M_n(\boldsymbol{\lambda})$, we need to index the set $M_n(\boldsymbol{\lambda})$. Set $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}$. Assume that $\boldsymbol{\lambda}$ is located in the negative (resp. positive) side of the Bratteli diagram. For $1 \leq i < |M_n(\boldsymbol{\lambda})|$ and i odd, define $\boldsymbol{\lambda}_{i+1}$ as the rightmost (resp. leftmost) weight in $M_n(\boldsymbol{\lambda}) \backslash \{\boldsymbol{\lambda}_j\}_{j=1}^{i}$. On the other hand, if $1 \leq i < |M_n(\boldsymbol{\lambda})|$ and i even then we define $\boldsymbol{\lambda}_{i+1}$ as the leftmost (resp. rightmost) weight in $M_n(\boldsymbol{\lambda}) \backslash \{\boldsymbol{\lambda}_j\}_{j=1}^{i}$.

Example 6.2.5. Figure 6.4 shows an example of the indexation for $M_n(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} = -19$, $l = 5$ and $m = 2$. Note that the maximal element in $M_n(\boldsymbol{\lambda})$, $\boldsymbol{\lambda}_8$, belongs to the fundamental alcove. This fact is true in general, that is, the maximal element in $M_n(\boldsymbol{\lambda})$ is the representative in the orbit of $\boldsymbol{\lambda}$ that belongs to the fundamental alcove. Note also that for the example we have $\boldsymbol{\lambda}_i \succeq \boldsymbol{\lambda}_j$ if and only if $i \leq j$. This fact is not true in general.

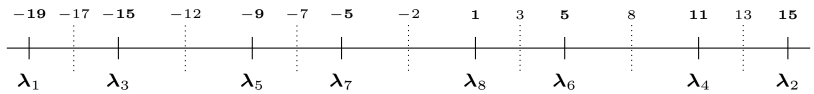


Figure 6.4: The indexation of $M_n(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} = -19$, $l = 5$ and $m = 2$.

Lemma 6.2.6. *Let $\boldsymbol{\lambda} \in Bip_1(n)$ and $\boldsymbol{\lambda}_{4j+1} \in M_n(\boldsymbol{\lambda})$. For $0 \leq i \leq j$, define*

$$D_i^j = \{\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}_{4j+1}) \mid \mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}} \text{ and } \deg(s) = 2i\} \tag{6.2.4}$$

Then, all $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}_{4j+1})$ with $\mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}}$ belong to some D_i^j and $|D_i^j| = |\operatorname{Std}(\mu_i)|$, where μ_i^j is the two-column partition of $\kappa(\boldsymbol{\lambda})$ given by

$$\mu_i^j = (\kappa(\boldsymbol{\lambda}) - j + i, j - i)' \tag{6.2.5}$$

Proof: Let $\boldsymbol{\lambda}_{4j+1} \in M_n(\boldsymbol{\lambda})$. Let $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}_{4j+1})$ with $\mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}}$. According to Lemma 6.1.6 we can split the walk $w(\mathfrak{s})$ in three parts (a), (b) and (c). Now, by a routine analysis of the indexation given for $M_n(\boldsymbol{\lambda})$, it is clear that $\boldsymbol{\lambda}_{4j+1}$ and $\boldsymbol{\lambda}$ are on the same side (positive or negative) of the Bratteli diagram (actually, all weights in $M_n(\boldsymbol{\lambda})$ with odd subscript are on the same side), $\boldsymbol{\lambda}_{4j+1}$ does not belong to the fundamental alcove, and $\boldsymbol{\lambda}_{4j+1}$ is located $2j$ alcoves closer than $\boldsymbol{\lambda}$ to the fundamental alcove.

Furthermore, part (c) of $w(\mathfrak{s})$ always points away from the central axis of Bratteli diagram (actually, this line coincide for all $\mathfrak{s}$ under the above conditions). Thus, Theorem 6.1.8(c) implies that $\deg(\mathfrak{s}) = n_{\mathfrak{s}}(F)$, where we recall that $n_{\mathfrak{s}}(F)$ was defined as the number of occurrences in $w(\mathfrak{s})$ of wall to wall steps of type F (similarly, we have defined the integers $n_{\mathfrak{s}}(I)$ and $n_{\mathfrak{s}}(O)$). Note also that $n_{\mathfrak{s}}(F)$ is even, because $\boldsymbol{\lambda}_{4j+1}$ and $\boldsymbol{\lambda}$ are on the same side of the Bratteli diagram, and for $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}_{4j+1})$ we have $0 \leq n_{\mathfrak{s}}(F) \leq 2j$ because $\boldsymbol{\lambda}_{4j+1}$ is located $2j$ alcoves closer than $\boldsymbol{\lambda}$ of the fundamental alcove. Consequently, $\deg(\mathfrak{s})$ is even and $0 \leq \deg(\mathfrak{s}) \leq 2j$. This proves the first claim of the Lemma.

Let $0 \leq i \leq j$ and assume that $\mathfrak{s} \in D_i^j$. By the above paragraph, we can replace the condition $\deg(\mathfrak{s}) = 2i$ in the definition of D_i^j by $n_{\mathfrak{s}}(F) = 2i$. Parts (a) and (c) of $w(\mathfrak{s})$ are fixed, and part (b) (and therefore, the entire walk $w(\mathfrak{s})$) is determined by a sequence of wall to wall steps. More conveniently, we can describe the walk $w(\mathfrak{s})$ as an ordered word in the alphabet with three letters $\{F, I, O\}$ in the obvious way.

For example, for $\boldsymbol{\lambda} = ((2),(20)) \in \mathrm{Bip}_1(22)$, $l = 5$, and $m = 2$ we have that the bitableau described in Figure 6.5 as a walk on the Bratteli diagram is in $\mathrm{Std}(\boldsymbol{\lambda}_5)$, where $\boldsymbol{\lambda}_5 = ((7),(15))$ and in terms of ordered words on $\{F, I, O\}$ correspond to FFO.

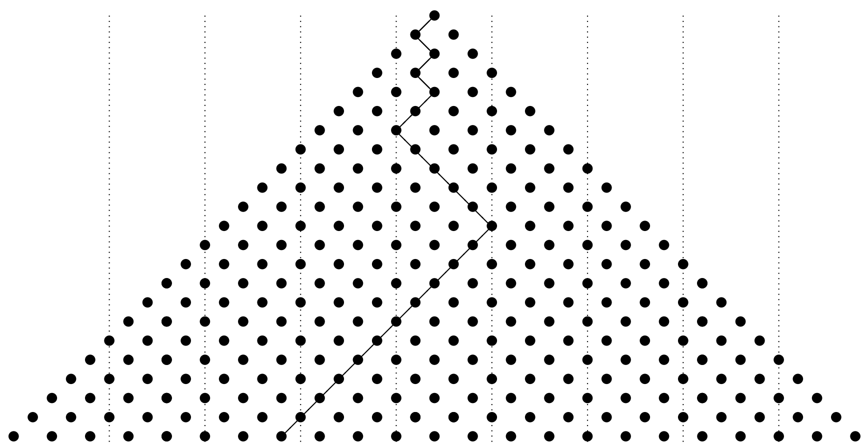


Figure 6.5: A walk corresponding to FFO.

We now associate to $\mathfrak{s}$ a two-column standard tableau of shape μ_i^j as follows: for $w(\mathfrak{s})$ described as a ordered word in $\{F, O, I\}$, the tableau associated to $\mathfrak{s}$ is determined by placing on the second column the entries corresponding to the positions at which the letter I appears in the respective ordered word. Therefore, the shape of the two-column partition associated to $\mathfrak{s}$ is $(n_{\mathfrak{s}}(F) + n_{\mathfrak{s}}(O), n_{\mathfrak{s}}(I))'$.

In order to check that the above assignment is well defined first note that at the first k positions of the ordered word associated to $w(\mathfrak{s})$ the number of occurrences of the letter O is greater or equal than the number of occurrences of the letter I, this shows that the two-column tableau assigned to $\mathfrak{s}$ is standard. Next, recall that $\kappa(\boldsymbol{\lambda})$ is the number the alcoves between $\boldsymbol{\lambda}$ and 0, so by Lemma 6.1.6(b) we have

$$\kappa(\boldsymbol{\lambda}) = n_{\mathfrak{s}}(F) + n_{\mathfrak{s}}(O) + n_{\mathfrak{s}}(I) = 2i + n_{\mathfrak{s}}(O) + n_{\mathfrak{s}}(I) \tag{6.2.6}$$

On the other hand, since $\boldsymbol{\lambda}_{4j+1}$ is located $2j$ alcoves closer than $\boldsymbol{\lambda}$ to the fundamental alcove we have

$$\kappa(\boldsymbol{\lambda}) = 2j + n_{\mathfrak{s}}(O) - n_{\mathfrak{s}}(I) \tag{6.2.7}$$

Combining (6.2.6) and (6.2.7) we obtain $n_{\mathfrak{s}}(I) = j - i$. This proves that the two-column tableau associated to $w(\mathfrak{s})$ has actually shape μ_i^j. Therefore, the assignment is well defined. Finally, for any two-column standard tableau of shape μ_i^j is straightforward to check that one can recover a walk $w(\mathfrak{s})$ with $\mathfrak{s} \in D_i^j$. Hence, the above assignment is a bijection and $|D_i^j| = |\operatorname{Std}(\mu_i^j)|$, completing the proof of the Lemma. □

Theorem 6.2.7. *Let $\boldsymbol{\lambda} \in Bip_1(n)$ and assume that $\kappa(\boldsymbol{\lambda}) \geq 1$. The graded dimension of $\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_i)$, for $\boldsymbol{\lambda}_i \in M_n(\boldsymbol{\lambda})$, is completely determined by the formulas:*

$$(a) \qquad \dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1})) = \textstyle\sum_{i=0}^{j} c_i t^{2i}$$

$$(b) \qquad \dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+2})) = t \dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1}))$$

$$(c) \qquad \dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+3})) = t \dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1}))$$

$$(d) \qquad \dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+4})) = t^2 \dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1}))$$

where $c_i = |\operatorname{Std}(\mu_i^j)|$ and μ_i^j is the two-column partition of $\kappa(\boldsymbol{\lambda})$ defined in (6.2.5).

Proof: By Theorem 6.2.2 we have

$$\dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_i)) = \sum_{\substack{\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}_i) \\ \boldsymbol{i}^{\mathfrak{s}} = \boldsymbol{i}^{\boldsymbol{\lambda}}}} t^{\deg(\mathfrak{s})} \tag{6.2.8}$$

Hence, part (a) in the Theorem follows immediately from Lemma 6.2.6. Now, we prove (b). Assume that $\boldsymbol{\lambda}$ is located on the negative (resp. positive) side of the Bratteli diagram. Then, by the indexing on $M_n(\boldsymbol{\lambda})$, the weight $\boldsymbol{\lambda}_{4j+2}$ is obtained from $\boldsymbol{\lambda}_{4j+1}$ by reflection about the left (resp. right) wall of the fundamental alcove (See Example 6.2.5). Next, define the sets

$$A = \{\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}_{4j+1}) \mid \mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}}\} \qquad B = \{\tilde{\mathfrak{s}} \in \operatorname{Std}(\boldsymbol{\lambda}_{4j+2}) \mid \tilde{\mathfrak{s}} \approx \mathfrak{t}^{\boldsymbol{\lambda}}\}$$

For $\mathfrak{s} \in A$ we associate an element $\tilde{\mathfrak{s}} \in B$ in the following way: Let l be the level at which the walk $w(\mathfrak{s})$ intersects the left (resp. right) wall of the fundamental alcove for the last time. Then the walks $w(\tilde{\mathfrak{s}})$ and $w(\mathfrak{s})$ match from level 0 to level l, and then from level l to level n, $w(\tilde{\mathfrak{s}})$ is obtained from $w(\mathfrak{s})$ by reflection about the left (resp. right) wall of the fundamental alcove. It is clear that this process is reversible, so it defines a bijection between A and B, and via Theorem 6.1.8 we can conclude that $\deg(\tilde{\mathfrak{s}}) = \deg(\mathfrak{s}) + 1$. Consequently, by (6.2.8) we have

$$t \dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1})) = t \sum_{\mathfrak{s} \in A} t^{\deg(\mathfrak{s})} = \sum_{\mathfrak{s} \in A} t^{\deg(\mathfrak{s})+1} = \sum_{\tilde{\mathfrak{s}} \in B} t^{\deg(\tilde{\mathfrak{s}})} = \dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+2}))$$

proving (b). Parts (c) and (d), follow in a similar way. □

Corollary 6.2.8. *Let $\boldsymbol{\lambda} \in Bip_1(n)$ and suppose that $\kappa(\boldsymbol{\lambda}) \geq 1$. For $r = 1, 2, 3, 4$ and $\boldsymbol{\lambda}_{4j+r} \in M_n(\boldsymbol{\lambda})$ we have*

$$\dim_{\mathbb{C}}(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+r})) = \sum_{i=0}^{j} |Std(\mu_i^j)| \tag{6.2.9}$$

where μ_i^j is the two-column partition of $k(\boldsymbol{\lambda})$ defined in (6.2.5).

Proof: This follows immediately by putting $t = 1$ in the above Theorem. □

Corollary 6.2.9. *Let $\boldsymbol{\lambda} \in Bip_1(n)$ with $\kappa(\boldsymbol{\lambda}) \geq 1$. Then, $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k) \neq 0$ if and only if $k = 4j + 1$. Furthermore,*

$$\dim_{\mathbb{C}} L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1}) = \dim_t L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1}) = |\operatorname{Std}(\mu_0^j)| \tag{6.2.10}$$

where $\dim_{\mathbb{C}} L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1})$ is viewed as polynomial over t in the natural way.

Proof: Since $b_n(m, \boldsymbol{\lambda})$ is a positively graded cellular algebra the modules $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k)$ are pure of degree zero. Therefore, $\dim_{\mathbb{C}} L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1}) = \dim_t L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1})$ and using Theorem 6.2.7 we can also conclude that

$$\dim_{\mathbb{C}}(L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k)) \leq \dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k)_{t=0} = \begin{cases} |\operatorname{Std}(\mu_0^j)| & \text{if } k = 4j+1 \\ 0 & \text{otherwise.} \end{cases} \tag{6.2.11}$$

Thus, $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k) \neq 0$ only if $k = 4j + 1$. On the other hand, recall that for $b_n(m)$ it is known that the (ungraded) decomposition numbers are 0 or 1. By putting $t = 1$ in (4.1.4) this is also true for $b_n(m, \boldsymbol{\lambda})$. Now note that $\boldsymbol{\lambda}_{|M_n(\boldsymbol{\lambda})|}$ is in the fundamental alcove, so this is the maximal element in $M_n(\boldsymbol{\lambda})$ with respect to the order $\succeq$. Hence, by (6.2.9) and (6.2.11)

$$\begin{aligned}
\dim_{\mathbb{C}}\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{|M_n(\boldsymbol{\lambda})|}) &\leq \sum_{\boldsymbol{\lambda}_{4j+1} \in M_n(\boldsymbol{\lambda})} \dim_{\mathbb{C}} L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1}) \\
&\leq \sum_{\boldsymbol{\lambda}_{4j+1} \in M_n(\boldsymbol{\lambda})} |\operatorname{Std}(\mu_0^j)| \\
&\leq \sum_{\mu \in \operatorname{Par}_2(\kappa(\boldsymbol{\lambda}))} |\operatorname{Std}(\mu)| \\
&= \dim_{\mathbb{C}}\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{|M_n(\boldsymbol{\lambda})|})
\end{aligned}$$

Therefore, the inequalities become equalities and $\dim_{\mathbb{C}}(L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1})) = |\operatorname{Std}(\mu_0^j)|$. Thus $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k) \neq 0$ if $k = 4j + 1$. □

Remark 6.2.10. Assume that $\kappa(\boldsymbol{\lambda}) \geq 1$ and $\kappa(\boldsymbol{\lambda}) \neq 2$. Then, we have

$$|\operatorname{Std}(\mu_0^j)| \begin{cases} = 1, & \text{if } j = 0 \\ > 1, & \text{if } j \neq 0 \end{cases}$$

Thus, under the above conditions on $\kappa(\boldsymbol{\lambda})$, the algebra $b_n(m, \boldsymbol{\lambda})$ has a unique (up to degree shift) one-dimensional graded simple module. This module is $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{4j+1})$ when $j = 0$, that is, $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_1) = L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})$ since $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}$. If $\kappa(\boldsymbol{\lambda}) = 2$ then by Lemma 6.2.4 we have $|M_n(\boldsymbol{\lambda})| = 6$. Hence, by the above Corollary $b_n(m, \boldsymbol{\lambda})$ has two (up degree shift) non-isomorphic simple modules $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_1)$ and $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5)$, both of dimension one.

We are now able to prove the main theorem in this chapter for the *non-wall case*, to provide the graded decomposition numbers for the blob algebra. Surprisingly, it was more difficult to determine the graded decomposition numbers for the case $\kappa(\boldsymbol{\lambda}) = 2$ than for the general case. This difficulty lies in the fact that for $\kappa(\boldsymbol{\lambda}) = 2$ there is two one-dimensional simple module for $b_n(m, \boldsymbol{\lambda})$, as explained in the previous remark.

Theorem 6.2.11. *Let $\boldsymbol{\lambda} \in Bip_1(n)$. For $\boldsymbol{\lambda}_k \in M_n(\boldsymbol{\lambda})$ we have*

$$[\Delta(\boldsymbol{\lambda}_k) : L(\boldsymbol{\lambda})]_t = \begin{cases} t^{2j}, & \text{if } k = 4j+1; \\ t^{2j+1}, & \text{if } k = 4j+2; \\ t^{2j+1}, & \text{if } k = 4j+3; \\ t^{2j+2}, & \text{if } k = 4j+4. \end{cases} \tag{6.2.12}$$

Proof: By Remark 6.2.10 we know that $\dim_t L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = 1$ so Theorem 4.1.4 implies

$$[\Delta(\boldsymbol{\lambda}_k) : L(\boldsymbol{\lambda})]_t = [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t$$

Therefore, we prove the theorem for the graded decomposition numbers of $b_n(m, \boldsymbol{\lambda})$, $[\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t$. On the other hand, Theorem 4.2.3(d) relates the graded dimension of cell and simple modules with the graded decomposition numbers via the formula

$$\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k) = \sum_{\boldsymbol{\lambda}_j \preceq \boldsymbol{\lambda}_k} [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_j)]_t \dim_t L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_j) \tag{6.2.13}$$

Assume that $\kappa(\boldsymbol{\lambda}) = 0$, then $\boldsymbol{\lambda}$ is located in one of the two alcoves adjacent to the fundamental alcove, and $|M_n(\boldsymbol{\lambda})| = 2$ by Theorem 6.2.4. Write $M_n(\boldsymbol{\lambda}) = \{\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2\}$. Then, $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_2$ is in the fundamental alcove. Combining Lemma 6.1.6 and Theorem 6.1.8 we have

$$\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_2) = t \tag{6.2.14}$$

Since $b_n(m, \boldsymbol{\lambda})$ is a positively graded cellular algebra the modules $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k)$, $k = 1, 2$ are pure of degree zero. Then, $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_2) = 0$ by (6.2.14). By Theorem 4.2.3 we know that $[\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t = 1$ and for $k = 2$ equation (6.2.13) becomes $t = [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_2) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t$, proving the Theorem for the case $\kappa(\boldsymbol{\lambda}) = 0$.

Now we suppose that $\kappa(\boldsymbol{\lambda}) = 2$. By Lemma 6.2.4 we get $|M_n(\boldsymbol{\lambda})| = 6$. In this setting, we have the following three possibilities for the order $\succeq$ on $M_n(\boldsymbol{\lambda})$

$$\boldsymbol{\lambda}_1 \prec \boldsymbol{\lambda}_2 \prec \boldsymbol{\lambda}_3 \prec \boldsymbol{\lambda}_4 \prec \boldsymbol{\lambda}_5 \prec \boldsymbol{\lambda}_6$$
$$\boldsymbol{\lambda}_1 \prec \boldsymbol{\lambda}_3 \prec \boldsymbol{\lambda}_2 \prec \boldsymbol{\lambda}_4 \prec \boldsymbol{\lambda}_5 \prec \boldsymbol{\lambda}_6$$
$$\boldsymbol{\lambda}_1 \prec \boldsymbol{\lambda}_2 \prec \boldsymbol{\lambda}_3 \prec \boldsymbol{\lambda}_5 \prec \boldsymbol{\lambda}_4 \prec \boldsymbol{\lambda}_6$$

In this three cases, the theorem follows by a case to case analysis, we only prove the lemma for the (most interesting) last case. Figure 6.6 shows an example of this case for $\boldsymbol{\lambda} = (0, 16)$, $l = 5$ and $m = 2$. In this figure are also drawn all walks $w(\mathfrak{s})$ with $\mathfrak{s} \approx \mathfrak{t}^{\boldsymbol{\lambda}}$.

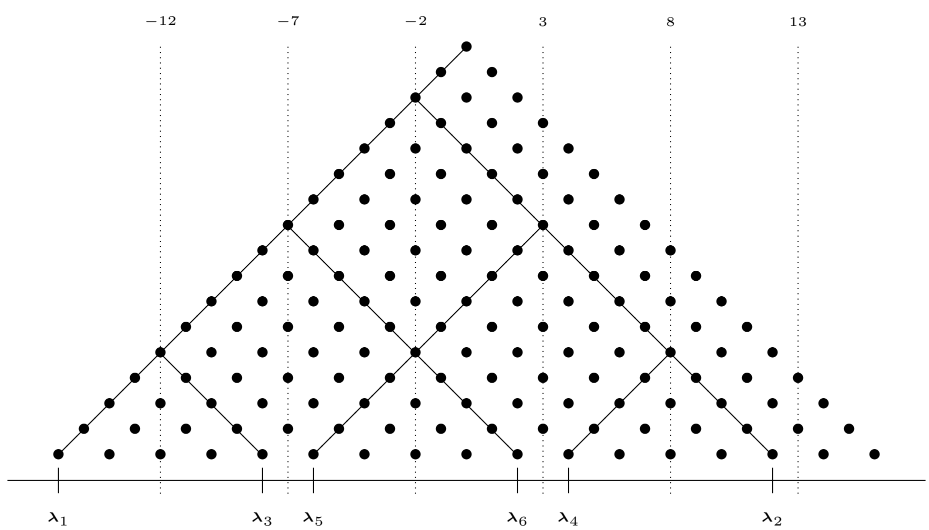


Figure 6.6: An example of the case when $\kappa(\boldsymbol{\lambda}) = 2$.

By Corollary 6.2.9, the modules $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_1)$ and $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5)$ are the unique (up degree shift) graded simple (non-zero) modules for $b_n(m, \boldsymbol{\lambda})$, furthermore by Remark 6.2.10 these modules are one dimensional and pure of degree zero. Now, Theorem 6.2.7 implies

$$\begin{aligned} &\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_1) = 1; \quad &&\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_2) = t; \quad &&\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_3) = t; \\ &\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_4) = t^2; \quad &&\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5) = t^2 + 1; \quad &&\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_6) = t^3 + t. \end{aligned}$$

By Theorem 4.2.3 we have $[\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t = 1$. As in the previous case we analyze equation (6.2.13) for the different values of k. For $k = 2, 3$ equation

(6.2.13) becomes

$$t = [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_2) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t \quad \text{and} \quad t = [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_3) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t$$

Next, if $k = 4$ then equation (6.2.13) becomes

$$t^2 = [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_4) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5)]_t + [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_4) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t$$

but it is straightforward to check that $\boldsymbol{\lambda}_4 \notin M_n(\boldsymbol{\lambda}_5)$, hence by Lemma 6.2.3 we have $[\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_4) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5)]_t = 0$, thus $t^2 = [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_4) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5)]_t$. Now, for $k = 5$ we have

$$t^2 + 1 = [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5)]_t + [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t$$

but by Theorem 4.2.3(c) we have $[\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5)]_t = 1$, so $[\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5)]_t = t^2$. Finally, for $k = 6$ equation (6.2.13) becomes

$$t^3 + t = [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_6) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5)]_t + [\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_6) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t$$

It is not hard to note that $\kappa(\boldsymbol{\lambda}_5) = 0$ and that $M_n(\boldsymbol{\lambda}_5) = \{\boldsymbol{\lambda}_5, \boldsymbol{\lambda}_6\}$, so we know by the first case analyzed in this proof that $[\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_6) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_5)]_t = t$. Consequently, $[\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_6) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t = t^3$. This completes the proof of the Theorem for the case $\kappa(\boldsymbol{\lambda}) = 2$.

Now we can assume that $\kappa(\boldsymbol{\lambda}) \neq 0, 2$. By Remark 6.2.10, $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})$ is the unique (up to degree shift) one-dimensional graded simple module for $b_n(m, \boldsymbol{\lambda})$. Recall that $b_n(m, \boldsymbol{\lambda})$ is a positively graded cellular algebra, so we can consider the grading filtration for $\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k)$, $\boldsymbol{\lambda}_k \in M_n(\boldsymbol{\lambda})$. Now, by Theorem 6.2.7 and Remark 6.2.10 $\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k) \in \mathbb{Z}[t]$ is a monic polynomial with the non-leading coefficients greater than 1. Thus, in the grading filtration of $\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k)$ there is a unique quotient of dimension one. This quotient is pure of degree $\deg(\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k))$ (where here deg denotes the polynomial degree) and must be isomorphic (in the ungraded setting) to $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})$. Since $L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})$ is pure of degree zero, if the grading filtration for $\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k)$ is a graded composition series we have

$$[\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k) : L_{\boldsymbol{\lambda}}(\boldsymbol{\lambda})]_t = t^{\deg(\dim_t \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k))} \tag{6.2.15}$$

If the grading filtration for $\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k)$ is not a graded composition series we can always add graded $b_n(m, \boldsymbol{\lambda})$-submodules of $\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k)$ to the grading filtration in order to obtain a graded composition series. In a graded composition series obtained in this way we can also have only one graded composition factor of dimension one. Otherwise we obtain via Theorem 4.1.4 that the blob algebra $b_n(m)$ has a (ungraded) decomposition number greater than one. Therefore, (6.2.15) is still valid even if the grading filtration is not a graded composition series. Finally, from Theorem 6.2.7 we know that

$$\deg(\dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k))) = \begin{cases} 2j, & \text{if } k = 4j+1; \\ 2j+1, & \text{if } k = 4j+2; \\ 2j+1, & \text{if } k = 4j+3; \\ 2j+2, & \text{if } k = 4j+4. \end{cases} \tag{6.2.16}$$

Now the Theorem follows by combining (6.2.15) and (6.2.16). □

The formula given in the previous theorem for the graded decomposition numbers for b_n is not entirely clear. We want to obtain a formula that reflects the alcove geometry on $\mathbb{R}$. For $\boldsymbol{\lambda} \in \mathrm{Bip}_1(n)$ define $n_{\boldsymbol{\lambda}} \in \mathbb{N}$ as the number of walls between $\boldsymbol{\lambda}$ and 0. Then, it is straightforward to check that we can rewrite (6.2.12) as

$$[\Delta(\boldsymbol{\mu}) : L(\boldsymbol{\lambda})]_t = \begin{cases} t^{n_{\boldsymbol{\lambda}} - n_{\boldsymbol{\mu}}}, & \text{if } \boldsymbol{\mu} \in M_n(\boldsymbol{\lambda}) \\ 0, & \text{otherwise.} \end{cases} \tag{6.2.17}$$

Remark 6.2.12. We stress the importance that has in our method the fact that $b_n(m)$ admits a family of positively graded subalgebras. We expect that the graded decomposition numbers of related graded algebras, that admit positively graded subalgebras, can be calculated by mimicking our approach.

6.2.2 The wall case.

In the previous subsection we determine the graded decomposition numbers for $b_n(m)$, $[\Delta(\boldsymbol{\mu}) : L(\boldsymbol{\lambda})]_t$, when $\boldsymbol{\lambda}$ is not on a wall. In this subsection we consider the wall case, that is we assume that $\boldsymbol{\lambda}$ is on a wall. For brevity, the results shall be presented without proof since the same series of arguments used in the non-wall case work here with minimal changes. If $\boldsymbol{\lambda}$ is on one of the walls of the fundamental alcove we have $M_n(\boldsymbol{\lambda}) = \{\boldsymbol{\lambda}\}$ so by Theorem 4.2.3 and Lemma 6.2.3 we obtain $[\Delta(\boldsymbol{\mu}) : L(\boldsymbol{\lambda})]_t = \delta_{\boldsymbol{\mu}\boldsymbol{\lambda}}$, where $\delta_{\boldsymbol{\mu}\boldsymbol{\lambda}}$ is a Kronecker delta. So we can assume that $\boldsymbol{\lambda}$ is not a wall of the fundamental alcove. As in the non-wall case, define $\kappa(\boldsymbol{\lambda})$ as the number of alcoves between $\boldsymbol{\lambda}$ and 0. Similarly, we index $M_n(\boldsymbol{\lambda})$ and define two-column partitions of $\kappa(\boldsymbol{\lambda})$, μ_i^j, by the same rules used for the non-wall case. The following theorems correspond to Theorem 6.2.7 and Theorem 6.2.11 in the *non-wall* case.

Theorem 6.2.13. *Let* $\boldsymbol{\lambda} \in Bip_1(n)$*. The graded dimension of* $\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_k)$*, for* $\boldsymbol{\lambda}_k \in M_n(\boldsymbol{\lambda})$*, is completely determined by the formulas:*

$$(a) \qquad \dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{2j+1})) = \textstyle\sum_{i=0}^{j} c_i t^{2i}$$

$$(b) \qquad \dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{2j+2})) = t\dim_t(\Delta_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{2j+1}))$$

where $c_i = |\operatorname{Std}(\mu_i^j)|$.

Theorem 6.2.14. *Let* $\boldsymbol{\lambda} \in Bip_1(n)$*. For* $\boldsymbol{\lambda}_k \in M_n(\boldsymbol{\lambda})$ *we have*

$$[\Delta(\boldsymbol{\lambda}_k) : L(\boldsymbol{\lambda})]_t = t^{k-1}$$

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