Existence and uniqueness of positive traveling fronts in reaction-diffusion equations with spatio-temporal delays

by

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Se informa a la Dirección de Postgrado del Instituto de Matemáticas y Física que la tesis de Doctorado presentada por la candidata:

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I dedicate this thesis to my beloved children Gonzalo, Felipe and Gustavo.

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CHAPTER I

Introduction

1.1 Introduction and definitions

In this thesis a class time-delayed reaction-diffusion equations is investigated:

$$u_t(t,x) = u_{xx}(t,x) - f(u(t,x)) + \int_0^{\infty}\int_{\mathbb{R}} K(s,w)g(u(t-s,x-w))dwds, \tag{1.1}$$

where $x \in \mathbb{R}$ is the spatial variable, t is the time, $f, g \in C(\mathbb{R}_+, \mathbb{R}_+)$ and the non-negative $K \in L^1(\mathbb{R}_+ \times \mathbb{R})$ meet some additional natural conditions. These equations, with appropriate f, g and K, are widely used to model many ecological and biological processes, where wave phenomena are observed and which depend not only on the present state but also on some past occurrences (see, e.g. [8, 20, 21, 27, 28, 29, 36, 42, 47, 49]). The nonlinear g is referred to in ecology literature as the *birth function* and, for example, the biological interpretation of u is the population density of mature species. The main goal of this work is to study the existence, uniqueness (up to translation) and minimal speed of propagation of positive traveling wave solutions for some equations of the form (1.1), in the case where g is non-monotone.

Aiming to develop a more general theory and motivated by recent studies by C. Gómez and S. Trofimchuk (see sections III.2 - III.4 which are included here for the completeness), we propose also renewed version of the Diekmann-Kaper theory (DK theory for short) of a non-linear convolution equation:

$$\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) g(\varphi(t-s),\tau) ds, \quad t \in \mathbb{R}, \tag{1.2}$$

Our version of DK theory allows to consider new types of models which include e.g. the nonlocal KPP-Fisher equations (with either symmetric or anisotropic dispersal kernel), nonlocal lattice equations and delayed reaction- diffusion equations; to include the critical case (which corresponds to the slowest wavefronts) into the consideration; to weaken or to remove various restrictions on kernels and nonlinearities, including the subtangential Lipschitz condition $|g(u) - g(v)| \leq g'(0)|u - v|$.

Definition I.1. A travelling wave solution of equation (1.1) is a solution of the form $u(x,t) = \phi(x+ct)$, where c is the wave speed. In the event that (1.1) has two spatially homogeneous equilibria u_1 and u_2 ($u(x,t) = u_1$ and $u(x,t) = u_2$ are constant solution of (1.1)) with $u_1 < u_2$ and the profile ϕ of the wave satisfies the boundary conditions $\phi(-\infty) = u_1$ and $\phi(+\infty) = u_2$, the travelling wave solution is called a wavefront. If ϕ is bounded and satisfies $\phi(-\infty) = u_1$, is called semi-wavefront.

Example I.2. Consider the logistic equation

$$u_t(t,x) = u(t,x)(1 - u(t,x)), \ u \geq 0, \ x \in \mathbb{R}. \tag{1.3}$$

We recall that the classical solution $u(x,t) = \phi(x + ct)$, is a wavefront for (1.3), if the profile function ϕ is positive and satisfies $\phi(-\infty) = 0, \ \phi(+\infty) = 1$. Note that ϕ is solution of equation $c\phi'(t) = \phi(t)(1 - \phi(t))$ (see figure 1.1).

Definition I.3. Equation (1.1) is the monostable type if has only two non-negative spatially homogeneous equilibria $u_1 = 0 < u_2$.

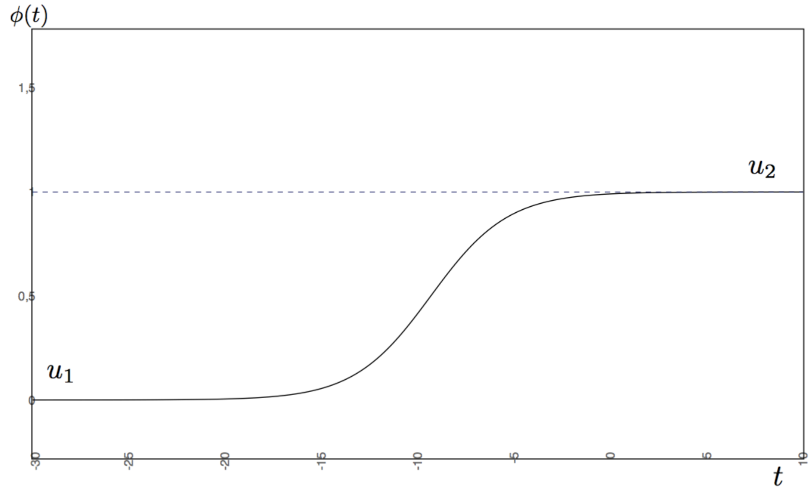

Figure 1.1: An example of a profile ϕ of equation (1.3) with $c = 2$ and $\phi(0) = 106$.

1.2 Results achieved

The results can be grouped into several parts: below we give a brief description of each of them.

1.2.1 On existence and uniqueness of wavefronts solution for local reaction-diffusion equations with delay

First, we begin our study by taking in (1.1) $f(u) = u$ and $K(s, w) = \delta(s-h)\delta(w)$, with $h > 0$. Then equation (1.1) reads as

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h,x)), \ x \in \mathbb{R}. \tag{1.4}$$

In this case, we suppose that (1.4) is monostable and our main result says that for every fixed and sufficiently large velocity c, the positive wavefront (traveling front) $u(t, x) = \phi(x + ct)$ of (1.4) is unique (modulo translations).

If we take $h = 0$ in (1.4), we obtain a monostable reaction-diffusion equations without delay. The problem of existence of travelling fronts for this equation is quite well understood. In particular, for each such equation we can indicate a positive real number c_* such that, for every $c \geq c_*$, it has exactly one travelling front $u(x,t) = \phi(x+ct)$. Furthermore, equation (1.4) does not have any travelling front propagating at the velocity $c < c_*$. The profile ϕ is necessarily a strictly increasing function. See, for example, Theorem 8.3 (ii), Theorem 8.7 and Theorem 2.39 in [24].

However, the situation will change drastically if we take $h > 0$. Even at the present moment, we are far from proving similar results concerning the existence, uniqueness and geometric properties of wavefronts for delayed equation (1.4). This despite the fact that the existence of travelling fronts in (1.4) was intensively studied for some specific subclasses of birth functions. E.g. see [21, 40, 47, 53, 58] and references wherein. Certainly, the so called monotone case (when g is monotone on $[0, \kappa]$) is the one for which the most information is available. But so far, even for equations with monotone birth functions very little is known about the number of positive wavefronts (modulo translations) for an arbitrary fixed $c \geq c_*$. In fact, at the time we started our research there were very few theoretical studies devoted to the uniqueness problem for equation (1.4) and its non-local extensions. To the best of our knowledge, the first uniqueness result for a non-local version of equation (1.4) has been proved by [49], who have extended an integral-equations approach (see [14, 48]) to scalar non-local reaction-diffusion equations with delay. Besides the work by [49], the uniqueness was established for small delays in [6] and for a family of unimodal piece-wise linear birth functions (i.e. tent maps) in [52]. Since *'asymmetric'* tent maps mimic the main features of general unimodal birth functions, we are able to prove the uniqueness of positive wavefront for delayed equations with the unimodal birth function satisfying the following assumptions

Assumption I.4. *The steady state $y_1(t) \equiv \kappa > 0$ (respectively $y_2(t) \equiv 0$) of the* *equation*

$$y'(t) = -y(t) + g(y(t-h)) \tag{1.5}$$

is exponentially stable and globally attractive (respectively hyperbolic) (see Definitions II.4 -II.8).

Assumption I.5. *$g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $p := g'(0) > 1$, and $g''(s)$ exists and is bounded near 0. We suppose that g has exactly two fixed points 0 and $\kappa > 0$. Set $\zeta_2 = \max_{s\in[0,\kappa]} g(s)$, we assume that $g(s) > 0$ for $s \in (0, \zeta_2]$.*

Example I.6. If we consider $g(s) = pse^{-s}, s \geq 0, p > 1$ in (1.5), we obtain the Nicholson's blowflies equation, where y is the size of an adult population. Is easy to see that g satisfy assumption I.5 with $\kappa = \ln(p)$ and $\zeta_2 = 1$ (see figure 1.2). Moreover, we will prove in Section 4.9 that the steady states 0 and κ satisfy assumption I.4.

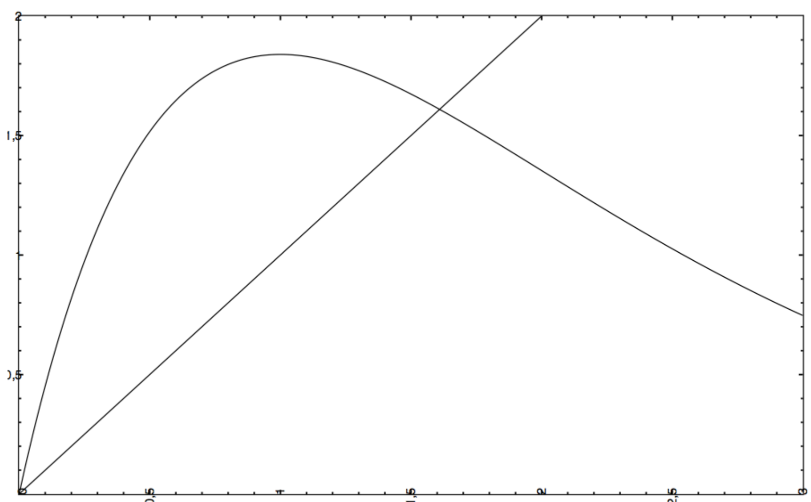

Figure 1.2: $g(s) = pse^{-s}$, $p = 5$.

We established the following result developed in Chapter IV (see also [4]):

Theorem I.7. *Assume I.4, I.5. Then there exists a unique (modulo translations) positive wavefront of equation (1.4) for each sufficiently large speed c.*

Notice that the wavefront, whose existence and uniqueness is established in Theorem I.7, may be non-monotone. For other results concerning the existence, uniqueness and oscillation properties of a non-monotone wavefront for equation (1.4), see [53]. We would like to mention also the asymptotic formulas given in Chapter IV: these formulas explains why the differential equations describing the wave profile is singular at $\varepsilon := 1/c = 0$ and the solutions admits an asymptotic expansion which are uniform in small ε. These formulas were generalized and then used in [22] for the case of multi-dimentional systems of nonlinear reaction-diffusion equations.

1.2.2 Existence of wavefronts solution for non-local reaction-diffusion equations with delay

Our second goal is to extend some previous results concerning the existence and uniqueness of wavefronts to non-local situation. We are interested in the particular case when $f(u) = u$ and $K(s, w) = \delta(s - h)K(w)$ with $h > 0$, equation (1.1) reduces

$$u_t(t, x) = u_{xx}(t, x) - u(t, x) + \int_{\mathbb{R}} K(x - w)g(u(t - h, w))dw, \ x \in \mathbb{R}, \tag{1.6}$$

where K satisfies $\int_{\mathbb{R}} K(w)dw = 1$, $\int_{\mathbb{R}} K(w)e^{\lambda w}dw \in \mathbb{R}$, for every $\lambda \in \mathbb{R}$. Here, we establish the existence of a continuous family of fast positive wavefronts $u(t, x)$ $= \phi(x + ct)$ of equation (1.6). We also prove that the fast wavefronts are non-monotone if $g'(\kappa)he^{h+1} < -1$.

Example I.8. Consider the gaussian kernel $K_\alpha = \frac{1}{\sqrt{4\pi\alpha}}e^{-s^2/4\alpha}$, $\alpha > 0$, then

$\int_{\mathbb{R}} K_\alpha(w)dw = 1$ and

$$\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\alpha}}e^{-w^2/4\alpha}e^{\lambda w}dw = \frac{e^{\alpha\lambda}}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-u^2/4\alpha}du \ \in \mathbb{R}, \quad \text{for every} \quad \lambda \in \mathbb{R}.$$

Before our research, the problem of existence of wavefronts for equation (1.6) was considered in [20, 28, 36, 37, 40, 47, 49, 51, 54, 55] by means of different methods. In the mentioned papers, a typical existence result requires several conditions on g (in particular, monotonicity of $g|_{[0,\kappa]}$ was assumed in [36, 37, 47, 49]) and K (for example, even/gaussian kernel was considered in [28, 36, 37, 40, 47, 54, 55]). Note that, the so called non-monotone case (when the restriction $g|_{[0,\kappa]}$ is not monotone) seems to be considerably more complicated than the monotone one. Its systematic study has started very recently in [20, 40, 52] (see also [51] for some further references).

An interesting approach was proposed in [20], where the Lyapunov-Schmidt reduction was used to study systems of delayed reaction-diffusion equations with non-local response. In the case of equation (1.6), the approach of [20] requires the existence of a positive heteroclinic solution ψ of (1.5). Under certain conditions imposed on g and h, the authors of [20] succeeded to establish the existence of a smooth manifold $\mathcal{M}$ of fast traveling fronts. In some sense, this $\mathcal{M}$ is generated by the mentioned heteroclinic ψ of (1.5). However, the main result of [20] does not answer the question about the presence of *positive* wavefront solutions of (1.6) in $\mathcal{M}$ when $\varepsilon := 1/c \neq 0$. We recall here that only non-negative solutions to (1.6) are biologically meaningful.

For establish the existence of positive traveling front solutions of (1.6) we suppose (I.4) and

Assumption I.9. $g \in C^1(\mathbb{R}_+, \mathbb{R}_+),\ p := g'(0) > 1,$ *and* $g''(s)$ *exists and is bounded near* 0. *We suppose that* g *has exactly two fixed points* 0 *and* $\kappa > 0$.

Below, we present our main result developed in Chapter V (see also [1]):

Theorem I.10. *Assume I.4 and I.9. Then there is* $c_* > 0$ *such that equation (1.6) has a continuous family of positive wavefronts* $u(t,x) = \phi_c(x + ct),\ c > c_*$. *These wavefronts are non-monotone if* $g'(\kappa)he^{h+1} < -1$. *Moreover,* $\phi_c(s)$ *is oscillating about* κ *if* $g'(\kappa)he^{h+1} < -1$ *and* $K(s)$ *has a compact support.*

1.2.3 Uniqueness of semi-wavefronts for non-local reaction-diffusion equations

In order to prove the uniqueness we study in [2] the uniqueness of semi-wavefronts for a class more general of equations. When g is a Lipschitzian function differentiable at 0 and f is strictly increasing, we prove the uniqueness (up to translations) of positive semi-wavefront solutions for equation (1.1). The uniqueness result is proved for all speeds $c > c_\star$, where the determination of $c_\star$ is similar to the determination of the minimal speed of propagation. We present a new result concerning the uniqueness (up to translation) of semi-wavefronts for non-local reaction-diffusion equations (1.1). In the case, when $K(s, w) = \delta(s - h)K(w)$ and $f(u) = du$, equation (1.1) was analyzed in [29, 36, 42, 47, 49]. If $f(u) = \beta u^2$, then equation (1.6) reduces to the model studied in [27].

During the last decade, the existence and uniqueness of the traveling wave solutions $u(t, x) = \phi(x + ct)$ for equation (1.1) have been investigated in a series of papers. Let us mention here [1, 20, 21, 37, 40, 51, 54, 55] where the existence problem was approached by means of different methods and assuming different conditions on f, K and g. Remarkably, the aspect of uniqueness appears to be considerably more complicated than the existence part of the problem. In fact, a very few theoretical studies have considered this important question. As far as we know, the list of references includes only several contributions: [4, 6, 19, 49, 52, 55, 59]. It should be noted that all these publications suggest the following heuristic principle: *"The existence of semi-wavefronts implies their uniqueness"* which was justified for only the important cases (however, always under strong technical restrictions imposed on the nonlinearities f, g). In fact, none example of multiple of traveling waves for equation (1.1) can be found in the literature. In many cases it was possible to prove the existence of waves while their uniqueness (or non-uniqueness) was left as an open problem. For example, it is well known (e.g. see [51]) that equation (1.6) with the unimodal *continuous* birth function g has semi-wavefronts for each $c \geq c_{\#}$, where (no necessary optimal) speed $c_{\#}$ is defined as the minimal value of c for which the characteristic equation

$$z^2 - cz - d + p_{\#}e^{-zch}\int_{\mathbb{R}} K(w)e^{-zw}dw = 0, \text{ with } \quad p_{\#} = \sup_{s>0}\frac{g(s)}{s},$$

has at least one positive root. This value of $c_{\#}$ is optimal (equivallenty, $c_{\#}$ is the minimal speed of propagation) if $p_{\#} = g'(0)$. However, the uniqueness of semi-wavefronts for (1.6) was proved only in several special cases (listed below) and almost always assuming that

Assumption I.11. $g : \mathbb{R}_+ \to \mathbb{R}_+$ *is a Lipschitzian function differentiable at* 0*:*

$$|g(s_1) - g(s_2)| \leq L|s_1 - s_2|, \quad s_1, s_2 \geq 0, \tag{1.7}$$

for some $L > 0$*,* $g(0) = 0$ *and* $g(s) > 0$ *if* $s > 0$*.*

The above mentioned special cases are the following ones:

(i) For *monotone* g satisfying (I.11) with $L = g'(0)$ and for the Gaussian kernel K, Thieme and Zhao [49] proved the uniqueness of fronts to (1.6) for each $c > c_{\#}$.

(ii) For a family of non-monotone unimodal piece-wise linear birth functions satisfying (I.11) with $L = g'(0)$ and when $K = \delta(w)$, Trofimchuk *et al* [52] established the uniqueness of fronts to (1.6) for each $c \geq c_{\#}$.

(iii) Following the approach of [49, 15], Wu and Liu [59] proved the uniqueness of wavefronts to (1.6) when $L = g'(0)$ in (I.11) and $K = \delta(w)$, for each $c > c_{\#}$.

(iv) For g satisfying (I.11) with $L = g'(0)$ and even K, the uniqueness of the traveling wave of (1.6) for each $c > c_{\#}$ was proved by Fang and Zhao in [19].

(v) Also [6] can be applied to (1.6) that proves the uniqueness of fronts in the particular case when $h = 0$, $K(s) = \frac{1}{\tau}e^{-\frac{|s|}{\tau}}$, $g \in C^4(\mathbb{R})$, τ is sufficiently small and c belongs to some specially defined compact interval.

It is worthwhile to mention that the main ideological and technical ingredients in the proofs of uniqueness in (i), (iii), (iv) are due to the seminal paper [15] by Diekmann and Kaper. In fact, the uniqueness statement of [59] is a direct consequence of [15, Theorem 6.4].

In any case, condition $L = g'(0)$ is essential in constructions [15, 19, 49, 59] and can not be omitted or weakened within the framework of [59]. However, as it was shown recently in [4] (as part of this thesis), the global Lipschitz condition (I.11) with $L = g'(0)$ is not necessary to have the uniqueness in (1.6) with $h > 0$. It was proved in [4] that each *sufficiently fast* front solution of (1.6) with $K(w) = \delta(w)$ is unique. In order to establish this, a small parameter $\varepsilon = 1/c$ was introduced and the Lyapunov-Schmidt reduction in a scale of Banach spaces was realized.

Here, the main result extends further the result of [4]. In particular, as a direct application of Theorem I.16 below, we obtain that semi-wavefront solution of equation (1.6) with the *Lipschitzian* birth function g is unique (modulo translation) for each fixed $c > c_*$, where (no necessary optimal) speed c_* is defined as the minimal value of c for which the characteristic equation

$$z^2 - cz - d + p_* e^{-zch} \int_{\mathbb{R}} K(w)e^{-zw}dw = 0,$$

with $p_* = \operatorname{ess\,sup}_{s>0} |g'(s)|$ has at least one positive root. Clearly, $p_\# \leq p_*$ so that $c_* \geq c_\#$. In the particular case when g is differentiable on $\mathbb{R}_+$ and $|g'(s)| \leq g'(0),\ s \geq 0$, we get $c_* = c_\#$ in complete accordance with [19, 49, 52, 59].

Let us list now the additional conditions imposed on f, K and define the critical velocity $c_\star$:

Assumption I.12. *$f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is strictly increasing, $f(0) = 0$ and $f'(0) < g'(0)$.*

Assumption I.13. *For every $\lambda, \mu \in \mathbb{R}$ the non-negative $K \in L^1(\mathbb{R}_+ \times \mathbb{R})$*

$$\int_0^\infty \int_{\mathbb{R}} K(s,w)e^{\mu s + \lambda w} dw ds \ \text{ is finite and } \int_0^\infty \int_{\mathbb{R}} K(s,w) dw ds = 1.$$

Assumption I.14. *There are $Q, \varepsilon > 0$ and $\theta \in (0,1)$ such that $g'(s)$ exists on $[0, \varepsilon]$ and*

$$|g'(s) - g'(0)| + \left| \frac{f(s)}{s} - f'(0) \right| \leq Q s^\theta, \quad s \in (0, \varepsilon].$$

Definition I.15. The speed $c_\star := c_\star(L, \inf_{s \geq 0} f'(s)) > 0$ is defined as the minimal value of c for which the characteristic equation

$$\chi_L(z,c) := z^2 - cz - \inf_{s \geq 0} f'(s) + L \int_0^\infty \int_{\mathbb{R}} K(s,w) e^{-z(cs+w)} dw ds = 0. \tag{1.8}$$

has at least one positive root. The speed c_* is defined similarly from the equation

$$\chi_0(z,c) := z^2 - cz - f'(0) + g'(0) \int_0^\infty \int_{\mathbb{R}} K(s,w) e^{-z(cs+w)} dw ds = 0. \tag{1.9}$$

Now, we are ready to state our main result developed in Chapter VI (see also [2]). Let $\lambda_1(c) < \lambda_\infty(c)$ be the positive roots of associated characteristic equation to (1.6).

Theorem I.16. *Assume I.11 - I.14. Then for each fixed $c > c_\star$, equation (1.6) has at most one (modulo translation) positive semi-wavefront solution $u(t,x) = \phi(x+ct)$. Furthermore, $\phi(t - t_0) = \exp(\lambda_1(c)t) + \exp((\lambda_1(c) + \delta)t)o(1)$ as $t \to -\infty$ for some $\delta > 0$ small and $t_0 \in \mathbb{R}$.*

1.2.4 Upper and lower bounds for the minimal speed of propagation

We give constructive upper and lower bounds for the minimal speed of propagation of traveling waves for equation (1.6) with K even. It is known [55] that for various systems modeled by equation (1.6), the minimal wave speed c_* coincides with the spreading speed. Therefore, it is important to study the effects caused by the delay and other parameters (depending on specific models) on c_*, cf. [36, 44, 47, 53, 55]. Another aspect of the problem concerns easily calculable upper and lower bounds for c_*. In particular, in the recent work [57], Wu *et al.* give several nice estimations for c_* when $K_\alpha(s) = \frac{1}{\sqrt{4\pi\alpha}}e^{-s^2/4\alpha}$ and $\alpha \le h$. However, the approach of [57] depends heavily on the condition $\alpha \le h$ and on the special form of K which is the fundamental solution of the heat equation. In the present work, we use a completely different idea to estimate the minimal speed for general kernels and without imposing any restriction on h. We note that this construction of upper and lower bounds for the minimal speed can be applied to speed $c_\star$.

1.2.5 General theory

Finally, we present an abstract setting for our problem. It allows generalization of the Diekmann-Kaper theory of a nonlinear convolution equation. Using our framework we prove the uniqueness of semi-wavefronts to a broad family of monostable equations.

For better understanding of our exposition, we include here several results obtained by C. Gómez and S. Trofimchuk. These results are presented in sections III.2-III.4 and form a part of our joint paper [3].

1.3 Organization of the thesis

The thesis is organized as follows: Chapter II contains the basic definitions and preliminaries results obtained for a scalar delay differential equations. Chapter III is where we prove the uniqueness of semi-wavefronts to a broad family of monostable equations which can be written as a nonlinear convolution equation. Chapter

IV contains studies of the solution of a reaction-diffusion equation with local delay (1.4). Following [21], we realize the Lyapunov-Schmidt reduction in a scale of Banach spaces and we obtain an alternative proof of the existence of positive wavefronts, see Theorem IV.12. In Chapter IV, we show that there exists exactly one wavefront for each fixed fast speed, this wavefront may be non-monotone. Chapter V is where the existence of positive fast wavefronts of the reaction-diffusion equation with non-local delay (1.6) is proven. In addition, in Chapter V, we obtain that this wavefront may to oscillate about the positive equilibrium. In Chapter VI we analyzed the uniqueness the semi-wavefronts solution of (1.1). Chapter VII is devoted to the estimation the minimal speed of propagation of positive traveling wave solutions of (1.6).

CHAPTER II

Preliminaries

2.1 Introduction

Suppose $h \geq 0$ is a given real number, $\mathbb{R}_+$ is an 1-dimensional linear vector space over the reals with norm $|\cdot|$ and $C := C([-h, 0], \mathbb{R}_+)$ is the Banach space of continuous functions mapping the interval $[-h, 0]$ into $\mathbb{R}_+$ with the topology of uniform convergence. We designate the norm of an element $\phi \in C$ by $|\phi| = \sup_{-h \leq s \leq 0} |\phi(s)|$. If $\sigma \in \mathbb{R}, \tau \geq 0$ and $y \in C([-\sigma - h, \sigma + \tau], \mathbb{R}_+)$, then for any $t \in [\sigma, \sigma + \tau]$, we let $y_t \in C$ be define by $y_t(\theta) = y(t + \theta)$, $-h \leq \theta \leq 0$.

Now, consider the scalar functional equation defined by

$$y'(t) = -y(t) + f(y_t), \quad y \geq 0, \tag{2.1}$$

where $f : C([-h, 0], \mathbb{R}_+) \to \mathbb{R}_+$ is a continuous functional which takes closed bounded sets into bounded subsets of $\mathbb{R}_+$. Below, we give some definitions and results obtaining the existence and properties of heteroclinic solutions of (2.1).

Definition II.1. Suppose that (2.1) is the monostable type, with non-negative equilibrium $u_1 < u_2$. Then a heteroclinic solution of (2.1) is a solution $\phi(t)$ such that converges to u_1 as $t \to -\infty$ and to u_2 as $t \to +\infty$.

Definition II.2. A function y is said to be a solution of equation (2.1) on $[\sigma - h, \sigma + \tau]$ if $y \in C([\sigma - h, \sigma + \tau], \mathbb{R}_+)$ and $y(t)$ satisfies (2.1) for all $t \in [\sigma, \sigma + \tau]$. For given

$\sigma \in \mathbb{R}$, $\phi \in C$, we say that $y(\sigma, \phi)$ is a solution of equation (2.1) with initial value ϕ at σ if there is $\tau > 0$ such that $y(\sigma, \phi)$ is solution of (2.1) on $[\sigma - h, \sigma + \tau]$ and $y_\sigma(t) = \phi(t)$ for all $-h \leq t \leq 0$.

Definition II.3. If y is a solution of equation (2.1) on $[\sigma - h, a), a > \sigma$, we say that $\hat{y}$ is a continuation of y if there is a $b > a$ such that $\hat{y}$ is defined on $[\sigma - h, b)$ and coincides with y on $[\sigma - h, a)$. For more details see [31, Chapter 2].

Definition II.4. Suppose that (2.1) is of the monostable type. Then the positive steady state u_2 of (2.1) is exponentially stable if there are $k, \gamma, \delta > 0$ constant such that if $\phi(t)$ is solution of the equation (2.1) with $|\phi(t_0) - u_2| < \delta$, then $|\phi(t) - u_2| < ke^{-\gamma t}$ for all $t > t_0$.

Definition II.5. Suppose that (2.1) is the monostable type. Then the positive steady state u_2 of (2.1) is globally attractive if all solutions $\phi(t)$ of the equation (2.1) converges to u_2 as $t \to +\infty$.

2.2 Heteroclinic solutions of scalar delay differential equation

In this section we start by showing an existence result of heteroclinic solution of equation (2.1) obtained in [21, Section 2]. We suppose that equation (2.1) has exactly two steady states $y_1(t) \equiv 0$ and $y_2(t) \equiv \kappa$.

Lemma II.6. *Let $f : C([-h, 0], \mathbb{R}_+) \to \mathbb{R}_+$ be a continuous functional which takes closed bounded sets into bounded subsets of $\mathbb{R}_+$. Assume further that every nonnegative solution of (2.1) admits a unique extension on the right semi-axis. If $f(0) = 0$ and $f(\kappa) = \kappa > 0$ and $y_2(t) \equiv \kappa$ attracts every solution of (2.1) with nonnegative and nontrivial initial function, then there exists a positive complete (that is defined over $\mathbb{R}$) solution ψ of (2.1) such that $\psi(-\infty) = 0$ and $\psi(+\infty) = \kappa$.*

Proof: See [21, Theorem 5]. □

Lemma II.7. *Suppose that $p > 1$ and $h > 0$. Then the characteristic equation*

$$z = -1 + p\exp(-zh), \; z \in \mathbb{C} \tag{2.2}$$

has only one real root $0 < \lambda < p - 1$. Moreover, all roots $\lambda \in \mathbb{R}, \lambda_j \in \mathbb{C}, \; j = 2, 3, \ldots$ of (2.2) are simple and we can enumerate them in such a way that $\lambda > \Re\lambda_2 = \Re\lambda_3 \geq \ldots$

Proof: Set $H(z) = z + 1$ and $G(z) = p\exp(-zh)$, $z \in \mathbb{R}$. Observe that $H(0) = 1 < G(0)$ and G is a decreasing function and is strictly convex. Hence, there exists a unique $\lambda > 0$ such that $H(\lambda) = G(\lambda)$. Since $\lambda = -1 + p\exp(-\lambda h)$ we have $\lambda > -1 + p$.

Now let $\psi(z) = z + 1 - p\exp(-zh)$, $z \in \mathbb{C}$. Since ψ is analytic, then the set of zeros of ψ is numerable and since $\psi'(\lambda_j) = 1 + hp\exp(-\lambda_j h) \neq 0$ for all $\lambda_j \in \mathbb{C}$ root of (2.2), all roots $\lambda, \lambda_j, \; j = 2, 3, \ldots$ of (2.2) are simple. Finally, from the inequality $\Re\lambda_j + 1 < p\exp(-\Re\lambda_j h)$ we get that these roots we can enumerate them in such a way that $\lambda > \Re\lambda_2 = \Re\lambda_3 \geq \ldots$. □

Everywhere in the sequel, λ_j stands for a root of (2.2). Notice that we write λ instead of λ_1.

Definition II.8. The trivial equilibrium of (2.1) is called hyperbolic if the roots of the characteristic equation (2.2), have nonzero real parts.

Now, we give a uniqueness result of heteroclinic solution of equation (2.1) for $f(\phi) = g(\phi(-h))$.

Lemma II.9. *Assume I.4, I.9 and let λ be as in Lemma II.7. Then (1.5) has a unique (modulo translations) positive heteroclinic solution ψ. Moreover, $\psi(t - t_0) =$*

$\exp(\lambda t) + O(\exp((2\lambda - \delta)t)),\ t \to -\infty$ *and* $\psi'(t - t_0) = \lambda \exp(\lambda t) + O(\exp((2\lambda - \delta)t)),\ t \to -\infty$, *for each* $\delta > 0$ *and some* $t_0 \in \mathbb{R}$.

Proof: See [21, Lemma 8]. $\square$

2.3 Uniform permanence of wavefront

Definition II.10. Equation (1.5) is said to be uniformly persistent if there exists a positive number m such that $\liminf_{t\to+\infty} y(t) \geq m$ for every solution $y \not\equiv 0$ of (1.5).

In this section we give a result obtained in [38] showing that the equation (1.5) is uniformly persistent under hypothesis I.9.

Notation II.11. Suppose that g satisfies I.9. Set $\zeta_2 = \max_{s\in[0,\kappa]} g(s)$, we assume that $g(s) > 0$ for $s \in (0, \zeta_2]$. Set $\mathrm{A} := \sup\{a \in (0, \kappa/2] : g'(s) > 0,\ s \in [0, a)\}$.

Observation II.12. It should be observed that assumption I.9 implies the existence of a positive $\zeta_1 \leq \min\{g(\zeta_2), \mathrm{A}\}$ such that $g(\zeta_1) = \min_{s\in[\zeta_1,\zeta_2]} g(s)$. Notice that $g([\zeta_1, \zeta_2]) \subseteq [\zeta_1, \zeta_2]$. Without restricting the generality, we may also suppose that $\sup_{s\geq 0} g(s) \leq \zeta_2$ (see figure 2.1).

The following lemma shows the uniform permanence of wavefront of (1.5). The proof is given in [38, Theorem 3.6(a)] and [53, Lemma 4.3].

Lemma II.13. *Assume I.9. If* $y \not\equiv 0$ *is a non negative bounded solution of equation (1.5), then*

$$\zeta_1 \leq \liminf_{t\to+\infty} y(t) \leq \limsup_{t\to+\infty} y(t) \leq \zeta_2.$$

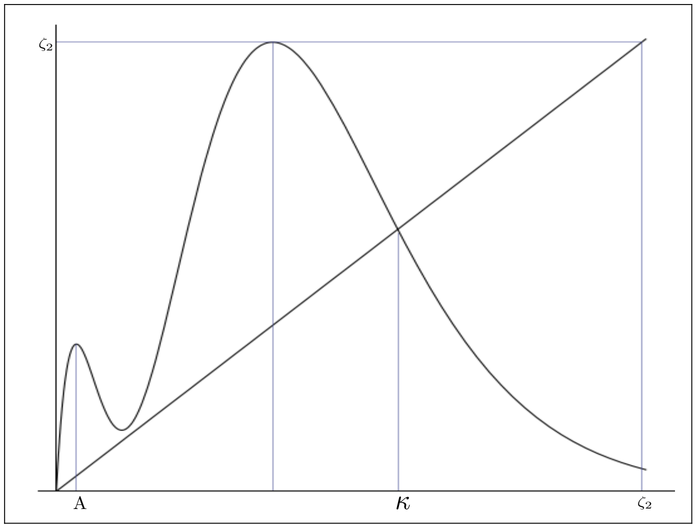

Figure 2.1: An example of birth function g

2.4 Small solutions for equations

Definition II.14. A small solution y at $t = -\infty$ is a solution for some equation, by example of equation (2.1), such that

$$\lim_{t \to -\infty} e^{kt} y(t) = 0 \quad \text{for all } k \in \mathbb{R}.$$

Small solutions that are not identically zero are called nontrivial.

We need the following lemma in Chapter VI:

Lemma II.15. *If y is a non trivial solution of the linear asymptotically autonomous homogeneous equation*

$$y'(t) = -y(t) + p(t)y(t-h),\ p(-\infty) = p > 1,$$

where $P \in C(\mathbb{R}, \mathbb{R})$, then y is not a nontrivial small solution.

Proof: See [21, Lemma 8]. $\square$

2.5 Asymptotic behavior of solutions for linear equations with delay

We consider the homogeneous equation

$$y'(t) + y(t) + py(t-h) = 0, \tag{2.3}$$

and the characteristic equation associated given by $\xi(\lambda) := \lambda + 1 + pe^{-h\lambda} = 0$.

Definition II.16. Let λ be such that $\xi(\lambda) = 0$. We say that the function $z \in C(\mathbb{R}, \mathbb{R})$ is an eigensolutions of (2.3) corresponding to λ, if $z(t) = e^{\lambda t}p(t)$, where $p(t)$ is any polynomial, and z satisfies equation (2.3) (see [41]).

Lemma II.17. *Let $y : \mathbb{R} \to \mathbb{R}$ be a solution of equation*

$$y'(t) + y(t) + py(t-h) = f(t), \tag{2.4}$$

for some $f : \mathbb{R} \to \mathbb{R}$, and $p \in \mathbb{R}$. Assume for some real number $a < b$ that $y(t) = O(e^{-at})$ and $f(t) = O(e^{-bt})$, $t \to +\infty$. Then for every $\delta > 0$, we have that

$$y = z(t) + O(e^{-(b-\delta)t}), \quad t \to +\infty,$$

where z is an eigensolution of (2.4) associated to the roots λ_j of (2.2) such that $\lambda_j \in \{-b < \Re\lambda_j \leq -a\}$

Proof: See [41, Proposition 7.1]. $\square$

2.6 Oscillations of the linear scalar delay equations

Definition II.18. Let ϕ be a continuous function defined on $\mathbb{R}$. The function ϕ is said to *oscillate* about $\rho \in \mathbb{R}$, if for every $t > t_0$ there exists a point $t_1 > t$ such that $\phi(t_1) = \rho$, for some t_0 fix (see figure 2.6).

The first aim in this section is to show a fundamental result for the oscillation of all solutions of equation

$$y'(t) + y(t) + qy(t-h) = 0, \ q \in \mathbb{R}. \tag{2.5}$$

Lemma II.19. *Assume that $h > 0$, $q \in \mathbb{R}$ and let $y : \mathbb{R} \to \mathbb{R}$ be a solution of equation (2.6). Then the following statements are equivalent*

1. *Every solution of equation (2.6) oscillates.*

2. *The characteristic equation $z + 1 + qe^{-zh} = 0,\ z \in \mathbb{C}$, has not real roots.*

Proof: See [30, Theorem 2.1.1.]. □

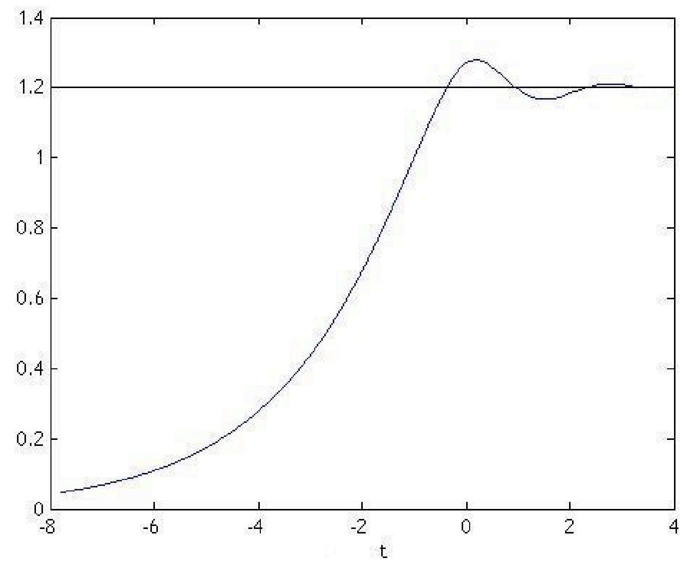

Figure 2.2: Oscillating wave solutions, see [52].

Now we show a sufficient condition for the oscillation of all solution of the linear differential equation with asymptotically constant coefficients:

$$y'(t) + y(t) + Q(t)y(t-h) = 0, \tag{2.6}$$

where $Q \in C(\mathbb{R}, \mathbb{R}_+)$ such that $\lim_{t\to+\infty} Q(t) = q$.

Lemma II.20. *Assume that $h > 0$ and $q \in \mathbb{R}$. If every solution of the limiting equation*

$$y'(t) + y(t) + qy(t-h) = 0.$$

oscillates, then every solution of equation (2.6) also oscillates.

Proof: See [30, Theorem 2.4.1.]. □

CHAPTER III

General theory

3.1 Introduction

The main goal of this chapter is to develop a variant of the fundamental Diekmann-Kaper theory (DK theory for short) of a nonlinear convolution equation [15] for the scalar integral equation

$$\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) g(\varphi(t-s),\tau) ds, \quad t \in \mathbb{R},$$

in the case when the nonlinearity g is of the monostable type and the averaging kernel K can be asymmetric in the first variable. Here (X, μ) will denote a measure space with finite measure μ, $K(s,\tau) \geq 0$ will be integrable on $\mathbb{R} \times X$ with $\int_{\mathbb{R}} K(s,\tau)ds > 0$, $\tau \in X$, while measurable $g : \mathbb{R}_+ \times X \to \mathbb{R}_+$, $g(0,\tau) \equiv 0$, will be continuous in ϕ for every fixed $\tau \in X$. In the case when X is just a single point (i.e. $\#X = 1$) and $\mu(X) = 1$, equation (1.2) coincides with the nonlinear convolution equation from [15].

There are various motivations to study the above equation, mainly from the theory of traveling waves for nonlinear models (e.g. reaction-diffusion equations with delayed response [4, 49, 51, 55], equations with non-local dispersal [12, 9, 13, 34], lattice systems [18] etc). It should be noted that only a few of these models take the simplest form with $\#X = 1$ of (1.2). But even when sometimes they can be written down as equation (1.2) with $\#X > 1$, the lack of a general theory obligates to repeat, at least partially, some ideas and constructions from the seminal work [15]. Hence, the first goal of this paper is to show that the ideology of [15] can be extended to include much broader class of equations than it was initially expected. We are making here the first step to create such a general extension. The further generalizations of (1.2) can be undertaken to include some other interesting applications (for example, equations with distributed delays as in [18, 19]). We would like to mention here [25] where a criterion of the existence and uniqueness of monotone fronts in the KPP-Fisher delayed reaction-diffusion equation was established within the framework of another extension of DK theory. Nevertheless, we do not pursue this direction in our current work.

In a biological context, φ is the size of an adult population, so we consider only non-negative solutions of (1.2). Due to the possible applications, it is convenient to call bounded non-negative continuous solution $\varphi : \mathbb{R} \to \mathbb{R}_+$ a semi-wavefront to equation (1.2) if $\varphi(-\infty) = 0$ or $\varphi(+\infty) = 0$ [24]. We observe however that it suffices to consider only the situation when $\varphi(-\infty) = 0$, since the case $\varphi(+\infty) = 0$ can be easily transformed to the first one via the change of variables $\zeta(t) = \varphi(-t)$, which sends (1.2) into

$$\zeta(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K_1(s,\tau) g(\zeta(t-s),\tau) ds, \quad t \in \mathbb{R},$$

with $K_1(s,\tau) = K(-s,\tau)$. We would like to emphasize that semi-wavefronts are generally non-monotone, see [21]. On the other hand, it is well known that the monotonicity of waves is very helpful for analyzing their properties. For instance, wave uniqueness sometimes is established *only* within a subclass of monotone waves, e.g. cf. [13, 25, 55]. In a similar line, we mention here that some wave properties (e.g. uniqueness) in monostable dynamics in general does not hold without assumption of their non-negativity, see [4, 20].

Actually the uniqueness aspect will be central in our research, where we fully agree with Chen and Guo [10, p. 126] in that "it seems that uniqueness of traveling waves for discrete monostable dynamics is largely open". So as in [15], after assuming the existence of a non-trivial semi-wavefront to (1.2), we will study its asymptotic behavior at infinity that sometimes will allow us to conclude about its uniqueness (up to a translation, observe that our equation is translation invariant). Here, similarly to other works using asymptotic expansions at infinity, we will work with the first positive eigenvalue of the linearization of (1.2) at zero, thus our analysis also excludes from the consideration so called "pushed" fronts [16, 24]). Similarly to [15], the existence of semi-wavefronts to (1.2) is not studied here.

As a by product of this strategy elaborated by Diekmann and Kaper, we are able to establish a non-existence result as well as asymptotic properties of the kernel K which proved to satisfy exponential convergence estimates (Mollison's condition [13]). Here the fulfillment of the Mollison's condition means that the characteristic function

$$\chi(z) = 1 - \int_X g'(0,\tau)d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-zs}ds, \tag{3.1}$$

is well defined for all z from some maximal non-degenerate interval $\mathcal{J}$ (which can be open, closed, half-closed, finite or infinite). One of the crucial results of the paper says that, under rather mild assumptions on g, K the presence of a semi-wavefront $\varphi, \varphi(-\infty) = 0$, guarantees the existence of a minimal positive root λ_l to (3.1).

Next, as it is well known the DK uniqueness theorem does not apply to the critical (minimal) fronts when $\chi(\lambda_l) = \chi'(\lambda_l) = 0$. To overcome this difficulty in the case of a nonlocal analogue of the KPP-Fisher equation (and in the case of its discrete counterpart as well)

$$u_t = J * u - u + g(u),\ x \in \mathbb{R},\ g(0) = g(1) = 0,\ f > 0 \text{ on } (0,1), \tag{3.2}$$

Carr and Chmaj in their influential paper [9] achieved an important extension of the DK theory. Assuming that J is even, compactly supported and that $g'(s) \leq g'(0)$, $s \in (0,1)$, $g(s) = g'(0)s + O(s^2),\ s \to 0+$, they showed that the minimal wavefront $\varphi(x + c_0 t)$ to (3.2) with profile $0 \leq \varphi(s) \leq 1,\ s \in \mathbb{R}$, is unique up to translation. Carr and Chmaj's work has motivated the second goal of our research: to get an improvement of the DK theory that includes the critical case. Theorem III.23 below gives such an extension for general model (1.2). In special case of equation (3.2) our result requires less assumptions on J and f than [15] does. In particular, J can be asymmetric and non compactly supported, see Section 6.1 for more details. This agrees with the initial idea of Kolmogorov, Petrovsky and Piskunov [34] who interpreted $J(x)dx$ as the probability that an individual passes a distance lying between x and $x + dx$. By Theorem III.23, the continuous birth function f is supposed to be differentiable at 0, with $g(s) = g'(0)s + O(s^{1+\alpha}),\ s \to 0+$, for some $\alpha > 0$, and to meet the *obligatory* [9, 15, 18, 49] subtangetial Lipcshitz condition of the DK uniqueness theorem:

$$|g(s) - g(t)| \leq g'(0)|t - s|, \quad s, t \geq 0. \tag{3.3}$$

The necessity of condition (3.3) could be considered as a weak point of the DK theory, cf. [4, 10, 13, 24] For instance, as it was established recently by Coville, Dávila and Martínez [13], neither (3.3) nor $g'(s) \leq g'(0)$, $s \in (0,1)$, is necessary to prove the uniqueness of non-stationary monotone traveling fronts to (3.2). Instead of that, it was supposed in [13] that generally asymmetric $J \in C^1(\mathbb{R})$ is compactly supported with $J(a) > 0,\ J(b) > 0$ for some $a < 0 < b$, while $g \in C^1(\mathbb{R})$ has to satisfy $g'(0)g'(1) < 0,\ g(s) \leq g'(0)s,\ s \geq 0$, and $g \in C^{1,\alpha}$ near 0. The proof in

[13] follows ideas of [12] and is mainly based on the sliding methods proposed by Berestycki and Nirenberg (see [12, 13] for a nice state-of-art overview about (3.2) as well as for the further references). The above discussion explains our third goal in this thesis: to weaken various convergence and smoothness conditions of DK theory, and especially condition (3.3). The related improvements can be found in Theorems III.23 and III.28. In the latter theorem, we remove condition (3.3) by assuming a little more smoothness for g and exploiting the absence of zeros for $\chi(z)$ in the vertical strip $\lambda_l < \Re z < \lambda_r$ (see Lemma III.12). Incidentally, Theorems III.28 justifies the following principle for monostable equations which seems to be rather general: "fast positive semi-wavefronts are unique (modulo translation)". In the last section of this chapter, we apply this principle to reaction-diffusion equations with delayed Mackey-Glass type nonlinearities to improve the uniqueness result of [4].

Finally, we observe that our approach to equation (1.2) differs from the methods used by Diekmann-Kaper and Carr-Chmaj in many key points. Even if the logical sequence of results here basically is the same as in [15], our proofs, starting from the deduction of Mollison's condition, are essentially different. In particular, we use neither the Titchmarsh theory of Fourier integrals [15, 18] nor the powerful Ikehara Tauberian theorem [9, 13, 55] in order to obtain necessary asymptotic expansions of solutions. We have found more convenient for our purpose the use of a suitable L^2-variant of the bootstrap argument (as this one described by Mallet-Paret in [41, p. 9-10]).

The main uniqueness results of this chapter are stated as Theorems III.23, III.28. We apply them to nonlocal integro-differential equations (Section 6.1), nonlocal lattice systems (Section 6.2), nonlocal (Section 6.3) and local (Section 6.4) reaction-diffusion equations with discrete delays. We would like also mention here a short proof of Theorem III.2 (concerning the Mollison's condition) and Theorem III.9 (providing a non-existence result).

3.2 Mollison's condition and the exponential rate of convergence

In this section, we consider general equation convolution type

$$\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) g(\varphi(t-s), t-s, \tau) ds, \tag{3.4}$$

where measurable $g : \mathbb{R} \times \mathbb{R} \times X \to \mathbb{R}_+$ is continuous in the first two variables for every fixed $\tau \in X$. We will suppose additionally that, for some measurable $p(\tau) \geq 0$ and $\delta > 0,\ \bar{s} \leq 0$, it holds

$$g(v, s, \tau) \geq p(\tau) v,\ v \in (0,\delta),\ s \leq \bar{s},\ \tau \in X. \tag{3.5}$$

Example III.1. If we consider $g(v,s,\tau) = pve^{-v}, v \geq 0, p > 1$, is easy to see that g satisfy assumption (3.5) with $p(\tau) = 1$ and $\delta > 0$ small.

First, we present a simple proof of the necessity of the following Mollison's condition (cf. [13]) for the existence of the semi-wavefronts:

$$\int_{\mathbb{R}} \int_X K(s,\tau) p(\tau) d\mu(\tau) e^{-sz} ds \text{ is finite for some } z \in \mathbb{R} \setminus \{0\}. \tag{3.6}$$

Theorem III.2. *Let continuous $\varphi : \mathbb{R} \to [0, +\infty)$ satisfy (3.4) and suppose that $\varphi(-\infty) = 0$ and $\varphi(t) \not\equiv 0,\ t \leq t'$ for each fixed t'. If (3.5) holds and*

$$\int_X \int_{\mathbb{R}} K(s,\tau) p(\tau) ds d\mu(\tau) \in (1, \infty), \tag{3.7}$$

then $\int_{-\infty}^0 \varphi(s) e^{-s\bar{x}} ds$ and $\int_{\mathbb{R}} \int_X K(s,\tau) p(\tau) d\mu(\tau) e^{-s\bar{x}} ds$ are convergent for an appropriate $\bar{x} > 0$. Furthermore, $\operatorname{supp} K \cap (\mathbb{R}_+ \times X) \neq \emptyset$.

Remark III.3. Looking for heteroclinic solutions of the simple logistic equation $x' = -\beta x + x(1+\beta-x)$ with $\beta > 0$, we obtain an example of (1.2) where $\operatorname{supp} K \cap (\mathbb{R}_- \times X) = \emptyset$ under conditions of the above theorem.

Proof: Since the support of K generally is unbounded, we will truncate K by choosing integer N such that

$$\kappa := \int_X \int_{-N}^{N} K(s,\tau)p(\tau)dsd\mu(\tau) > 1, \text{ and } 0 \le \varphi(t) < \delta,\ t < \bar{s} - N.$$

Integrating equation (3.4) between t' and $t < \bar{s} - N$, we find that

$$\begin{aligned}
\int_{t'}^{t} \varphi(v)dv \geq & \quad \int_X d\mu(\tau) \int_{-N}^{N} K(s,\tau) \int_{t'}^{t} g(\varphi(v-s), v-s, \tau)dvds \\
\geq & \quad \int_X p(\tau)d\mu(\tau) \int_{-N}^{N} K(s,\tau) \int_{t'}^{t} \varphi(v-s)dvds \\
= & \quad \int_X p(\tau)d\mu(\tau) \int_{-N}^{N} K(s,\tau)\Big(\int_{t'-s}^{t'} + \int_{t'}^{t} + \int_{t}^{t-s}\Big)\varphi(v)dvds,
\end{aligned}$$

from which

$$\int_{t'}^{t} \varphi(v)dv \le \frac{2\delta \int_X \int_{-N}^{N} |s|K(s,\tau)p(\tau)dsd\mu(\tau)}{\int_X \int_{-N}^{N} K(s,\tau)p(\tau)dsd\mu(\tau) - 1}, \quad t' < t < \bar{s} - N.$$

Hence, the increasing function

$$\psi(t) = \int_{-\infty}^{t} \varphi(s)ds \tag{3.8}$$

is well defined for all $t \in \mathbb{R}$ and

$$\psi(t) \ge \int_X p(\tau)d\mu(\tau) \int_{-N}^{N} K(s,\tau)\psi(t-s)ds \ge \kappa\psi(t-N), \quad t < \bar{s} - N.$$

Consider $h(t) = \psi(t)e^{-\gamma t}$ where $\kappa = e^{\gamma N}$, cf. [9]. For all $t < \bar{s} - N$ we have

$$h(t-N) = \psi(t-N)e^{-\gamma(t-N)} \le \frac{1}{\kappa}\psi(t)e^{-\gamma t}e^{\gamma N} = h(t)$$

and $\gamma = N \ln \kappa > 0$. Hence $\sup_{t \le 0} h(t) < \infty$ and $\psi(t) = O(e^{\gamma t})$, $t \to -\infty$. After taking $\bar{x} \in (0, \gamma)$ and integrating by parts, we obtain

$$\int_{-\infty}^{t} \varphi(s)e^{-\bar{x}s}ds = \psi(t)e^{-\bar{x}t} + \bar{x}\int_{-\infty}^{t} \psi(s)e^{-\bar{x}s}ds$$

that proves the first statement of the theorem. Finally,

$$e^{-\bar{x}t}\psi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} e^{-\bar{x}s} K(s,\tau) e^{-\bar{x}(t-s)} \psi_1(t-s,\tau) ds,$$

where $\psi_1(u,\tau) := \int_{-\infty}^{u} g(\varphi(s),s,\tau)ds \geq p(\tau) \int_{-\infty}^{u} \varphi(s)ds,\ u \leq \bar{s} - N$. The latter yields

$$\int_{-\infty}^{\bar{s}-N} e^{-\bar{x}v}\psi(v)dv = \int_X d\mu(\tau) \int_{\mathbb{R}} e^{-\bar{x}s} K(s,\tau) \int_{-\infty}^{\bar{s}-N} e^{-\bar{x}(v-s)} \psi_1(v-s,\tau) dv ds \geq$$

$$\int_X p(\tau)d\mu(\tau) \int_{-\infty}^{0} e^{-\bar{x}s} K(s,\tau)ds \int_{-\infty}^{\bar{s}-N} e^{-\bar{x}v}\psi(v)dv,$$

where

$$(3.9)\ \ \mathcal{K}_-(\bar{x}) := \int_X p(\tau)d\mu(\tau) \int_{-\infty}^{0} e^{-\bar{x}s} K(s,\tau)ds \leq 1,\ (\text{note that } \psi(s) > 0,\ s \in \mathbb{R}),$$

so that

$$\mathcal{K}_-(0) = \int_X p(\tau)d\mu(\tau) \int_{-\infty}^{0} K(s,\tau)ds \leq 1 < \int_X p(\tau)d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)ds,$$

which completes the proof of the theorem. □

Remark III.4. Suppose that $|g(\varphi(s),s,\tau)| \leq C$ where C does not depend on s,τ. Then

$$|\varphi(t+h) - \varphi(t)| \leq C \int_{\mathbb{R}} |K_a(s+h) - K_a(s)|ds$$

since $K_a(s) := \int_X K(s,\tau)d\mu(\tau) \in L_1(\mathbb{R})$ and the translation is continuous in $L_1(\mathbb{R})$ [17, Example 5.4]. Thus $\varphi(t)$ is uniformly continuous on $\mathbb{R}$. It is easy to see that the convergence of the integral $\int_{-\infty}^{0} \varphi(s)ds < \infty$ combined with the uniform continuity of φ gives $\varphi(-\infty) = 0$. In this way, $\int_{-\infty}^{0} \varphi(s)ds < \infty$ implies that $\int_{-\infty}^{0} e^{-xs}\varphi(s)ds < \infty$ for small positive x.

Remark III.5. The non-negativity of g is an important restriction of the Theorem III.2. This assumption, for instance, does not hold for non-monotone solutions of the KPP-Fisher equation. However, it can be easily checked that this condition can be omitted in the case of K having bounded support (uniformly in $\tau \in X$).

Now, let $\varphi, K, g, \bar{x}$ be as in Theorem III.2. Set

$$\Phi(z) = \int_{\mathbb{R}} e^{-zs}\varphi(s)ds,\ \mathcal{K}(z) = \int_{\mathbb{R}}\int_X K(s,\tau)p(\tau)d\mu(\tau)e^{-sz}ds,$$

and denote the maximal open vertical strips of convergence for these two integrals as $\sigma_\phi < \Re z < \gamma_\phi$ and $\sigma_K < \Re z < \gamma_K$, respectively. Evidently, $\sigma_\phi, \sigma_K \le 0$ and $\gamma_\phi, \gamma_K \ge \bar{x} > 0$. Since φ, K are both non-negative, by [56, Theorem 5b, p. 58], $\gamma_\phi,\ \gamma_K, \sigma_\phi,\ \sigma_K$ are singular points of $\Phi(z), \mathcal{K}(z)$ (whenever they are finite). A simple inspection of the proof of Theorem III.2 suggests the following

Lemma III.6. *Assume φ, g, K are as in Theorem III.2. Then $\sigma_K \le \sigma_\phi < \gamma_\phi \le \gamma_K$. Furthermore, $\mathcal{K}(\gamma_\phi)$ is always a finite number.*

Proof: For all $z \in (0, \gamma_\phi),\ t \le 0$, we have

$$\psi(t) = \int_{-\infty}^{t} (\varphi(s)e^{-zs})e^{zs}ds \le e^{zt}\int_{-\infty}^{0} \varphi(s)e^{-zs}ds,$$

so that $\int_{-\infty}^0 \psi(s)e^{-z's}ds < \infty$ for each $z' \in (0, \gamma_\phi)$ and, due to (3.9), we get

$$\mathcal{K}_-(z) := \int_X p(\tau)d\mu(\tau)\int_{-\infty}^{0} e^{-zs}K(s,\tau)ds \le 1$$

for all $z \in (0, \gamma_\phi)$. Hence, using the Beppo Levi monotone convergence theorem, we obtain that $\mathcal{K}_-(\gamma_\phi) \le 1$. As a consequence, $\mathcal{K}(\gamma_\phi)$ is finite and $\gamma_K \ge \gamma_\phi$. $\square$

Corollary III.7. *Assume that*

$$\lim_{z\to\gamma_K-}\int_{\mathbb{R}}\int_X K(s,\tau)p(\tau)d\mu(\tau)e^{-sz}ds = +\infty.$$

Then γ_ϕ is a finite number and $\gamma_\phi < \gamma_K$.

3.3 Abscissas of convergence

In this section, we consider the abscissas of convergence for the bilateral Laplace transforms of K and bounded non-negative φ satisfying $\varphi(-\infty) = 0$, $\varphi(t) \not\equiv 0,\ t \le t'$

for each fixed t', and solving our main equation

$$\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) g(\varphi(t-s),\tau) ds. \tag{3.10}$$

Now we are supposing that the continuous $g(\cdot,\tau) : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable at 0 with $g'(0+,\tau) > 0$ for each fixed τ. Then the non-negative functions

$$\lambda_\delta^+(\tau) := \sup_{u\in(0,\delta)} \frac{g(u,\tau)}{u},\ \lambda_\delta^-(\tau) := \inf_{u\in(0,\delta)} \frac{g(u,\tau)}{u}, \quad \delta > 0,\ \tau \in X,$$

are well defined, measurable, monotone in δ and pointwise converging:

$$\lim_{\delta\to 0+} \lambda_\delta^\pm(\tau) = g'(0+,\tau).$$

The *characteristic* function χ associated with the variational equation along the trivial steady state of (3.10) is defined by

$$\chi(z) := 1 - \int_{\mathbb{R}} \int_X K(s,\tau) g'(0+,\tau) d\mu(\tau) e^{-sz} ds.$$

It is supposed to take a negative value at $z = 0$: $\chi(0) < 0$.

Example III.8. Consider $X = \{\tau_1\}$, $\mu(X) = 1$ and $g(v,\tau_1) = pve^{-v}, v \geq 0, p > 1$. Since $g'(0+,\tau_1) = p$ is easy to see that $\chi(0) = 1 - p < 0$

Since condition (3.5) is obviously satisfied with $p(\tau) = \lambda_\delta^-(\tau)$ and

$$\lim_{\delta\to 0+} \int_{\mathbb{R}} \int_X K(s,\tau)\lambda_\delta^-(\tau) d\mu(\tau) ds = \int_{\mathbb{R}} \int_X K(s,\tau) g'(0+,\tau) d\mu(\tau) ds > 1$$

by the monotone convergence theorem, all results of Section 2 hold true for equation (3.10). Furthermore, we have the following

Theorem III.9. *Assume* $\chi(0) < 0$. *Let* $\varphi : \mathbb{R} \to [0,+\infty)$ *be a semi-wavefront to equation (3.10). If* $\varphi(-\infty) = 0$ *and* $\varphi(t) \not\equiv 0,\ t \leq t'$ *for each fixed* t', *then* $\chi(z)$ *has a zero on* $(0, \gamma_K] \subset \mathbb{R} \cup \{+\infty\}$.

Remark III.10. 1) If $\varphi(+\infty)=0$ then a similar statement can be proved. Namely, in such a case $\chi(z)$ has a zero on $[\sigma_K, 0)$. 2) It should be noted that Theorem III.9 also provides a non-existence result: if $\chi(x)<0$ for all $x\in(0,\gamma_K]$ then equation (3.10) does not have any semi-wavefront vanishing at $-\infty$.

Proof: For real positive $z\in(0,\gamma_\phi)$ we consider the integrals

$$\Phi(z)=\int\limits_{\mathbb{R}} e^{-zs}\varphi(s)ds, \mathcal{G}(z,\tau):=\int\limits_{\mathbb{R}} e^{-zs}g(\varphi(s),\tau)ds, \mathcal{K}(z,\tau):=\int\limits_{\mathbb{R}} e^{-zs}K(s,\tau)ds.$$

Since φ is non-negative and bounded, and since $g'(0+,\tau)>0$ exists, the convergence of $\mathcal{G}(z,\tau)$ (for positive z) is equivalent to the convergence of $\Phi(z)$. Applying the bilateral Laplace transform to equation (3.10), we obtain that

$$\Phi(z)=\int_X \mathcal{K}(z,\tau)\mathcal{G}(z,\tau)d\mu(\tau). \tag{3.11}$$

Obviously, $\mathcal{K},\mathcal{G},\Phi$ are positive at each real point of the convergence.

Let us prove that $\chi(z)$ has a zero on $(0,\gamma_K]$. First, we suppose that $\Phi(\gamma_{\phi}-)=\lim_{z\to\gamma_\phi-}\Phi(z)=\infty$. In such a case, we claim that

$$\lim_{z\to\gamma_\phi-}\frac{\mathcal{G}(z,\tau)}{\Phi(z)}=g'(0,\tau).$$

Indeed, let T_δ be the rightmost non-positive number such that $\varphi(s)\le\delta$ for $s\le T_\delta$. Then

$$\lambda_\delta^-\int_{-\infty}^{T_\delta} e^{-zs}\varphi(s)ds\le\int_{-\infty}^{T_\delta} e^{-zs}g(\varphi(s),\tau)ds\le\lambda_\delta^+\int_{-\infty}^{T_\delta} e^{-zs}\varphi(s)ds,$$

$$\int_{T_\delta}^{+\infty} e^{-zs}(g(\varphi(s),\tau)+\varphi(s))ds\le\frac{\sup_{s\in\mathbb{R}}(g(\varphi(s),\tau)+\varphi(s))}{z}e^{-\gamma_\phi T_\delta}.$$

As a consequence, for each positive $\delta>0$,

$$\lambda_\delta^-\le\liminf_{z\to\gamma_\phi-}\frac{\mathcal{G}(z,\tau)}{\Phi(z)}\le\limsup_{z\to\gamma_\phi-}\frac{\mathcal{G}(z,\tau)}{\Phi(z)}\le\lambda_\delta^+,$$

that proves our claim.

Now, by using the Fatou lemma as $z \to \gamma_\phi-$ in

$$\int_X \mathcal{K}(z,\tau)\frac{\mathcal{G}(z,\tau)}{\Phi(z)}d\mu(\tau) = 1,$$

we obtain

$$1 - \chi(\gamma_\phi) = \int_X \mathcal{K}(\gamma_\phi,\tau)g'(0,\tau)d\mu(\tau) \le 1.$$

Therefore $\chi(\gamma_\phi) \ge 0$, and since $\chi(0) < 0$ we obtained the required assertion.

Hence, we have to consider only the situation when $\Phi(\gamma_\phi) = \lim_{z\to\gamma_\phi-}\Phi(z) > 0$ is finite. Since $\varphi(t) \not\equiv 0,\ t \le t'$ for each fixed t', in such a case $\gamma_\phi < \infty$. Due to Lemma III.6, the value $\mathcal{K}(\gamma_\phi)$ is also finite. Set

$$\zeta(t) := \varphi(t)e^{-\gamma t},\ \ K_1(s,\tau) := e^{-\gamma s}K(s,\tau),\ \text{ where }\ \gamma := \gamma_\phi.$$

Then, for $t < T_\delta - N$, we have from (3.10) that $\int_{-\infty}^t \zeta(v)dv =$

$$\int_{-\infty}^t \varphi(v)e^{-\gamma v}dv \ge \int_X d\mu(\tau)\int_{-N}^N K_1(s,\tau)\int_{-\infty}^t g(\varphi(v-s),\tau)e^{-\gamma(v-s)}dvds \ge$$
$$\int_X d\mu(\tau)\int_{-N}^N \lambda_\delta^-(\tau)K_1(s,\tau)\int_{-\infty}^t \zeta(v-s)dvds \ge$$
$$(\int_X d\mu(\tau)\int_{-N}^N \lambda_\delta^-(\tau)K_1(s,\tau)ds)\int_{-\infty}^{t-N}\zeta(v)dv.$$

Suppose now on the contrary that the characteristic equation

$$\chi(z) := 1 - \int_{\mathbb{R}}\int_X K(s,\tau)g'(0+,\tau)d\mu(\tau)e^{-sz}ds = 0$$

has not real roots on $[0,\gamma_\phi]$. Then $\chi(0) < 0$ implies $\chi(\gamma) < 0$. As a consequence, in virtue of the monotone convergence theorem,

$$\lim_{\delta\to 0+, N\to+\infty}\int_X d\mu(\tau)\int_{-N}^N \lambda_\delta^-(\tau)K_1(s,\tau)ds = 1 - \chi(\gamma) > 1.$$

Hence, for some appropriate $\delta, N > 0$, increasing function $\xi(t) = \int_{-\infty}^t \zeta(s)ds$ satisfies $\xi(t) \ge \kappa_\delta\xi(t-N)$, $t < T_\delta - N$ with $\kappa_\delta > 1$. Arguing now as in the proof of Theorem

III.2 below (3.8) we conclude that the integral $\int_{-\infty}^{t}\zeta(s)e^{-zs}$ converges for all small positive z, contradicting to the definition of γ_ϕ. $\square$

Remark III.11. It is clear that $\chi(z)$ is concave on (σ_K, γ_K), where $\chi''(z) < 0$. Since $\chi(0)$ is negative, χ can have at most two real zeros, and they must be of the same sign. We will denote them (if they exist) by $\lambda_l \leq \lambda_r$. Under assumption of the existence of a semi-wavefront φ vanishing at $-\infty$, χ has at least one positive root λ_l. Finally, it is clear that χ is analytical in the vertical strip $\Re z \in (0, \gamma_K)$.

Notation At this stage, it is convenient to introduce the following notation:

$$\lambda_{rK} = \begin{cases} \lambda_r, & \text{if } \lambda_r \text{ exists,} \\ \gamma_K, & \text{otherwise.} \end{cases}$$

Lemma III.12. *Equation $\chi(z) = 0$ does not have roots in the open strip $\Sigma := \Re z \in (\lambda_l, \lambda_{rK})$. Furthermore, the only possible zeros on the boundary Σ are λ_l, λ_r.*

Proof: Observe that if $\chi(z_0) = 0$ for some $z_0 \in \Sigma$, then $\chi(\Re z_0) > 0$ since χ is concave, $\chi(\lambda_l) = 0$ and $\Re z_0 \in (\lambda_l, \min\{\lambda_r, \gamma_K\})$. On the other hand,

$$1 = |\int_{\mathbb{R}}\int_X K(s,\tau)g'(0+,\tau)d\mu(\tau)e^{-sz_0}ds| \leq \int_{\mathbb{R}}\int_X K(s,\tau)g'(0+,\tau)d\mu(\tau)e^{-s\Re z_0}ds$$

and therefore $\chi(\Re z_0) \leq 0$, a contradiction. Now, if $\chi(\lambda_l + i\omega) = 0$ for some $\omega \neq 0$ then similarly

$$1 = \chi(\lambda_l + i\omega) = |\chi(\lambda_l + i\omega)| \leq \chi(\lambda_l) = 1,$$

so that

$$\int_{\mathbb{R}}\int_X K(s,\tau)g'(0+,\tau)d\mu(\tau)e^{-s\lambda_l}(1 - \cos\omega s)ds = 0.$$

Thus $K(s,\tau)(1-\cos\omega s) = 0$ for almost all $\tau \in X$, so that $K(s,\tau) = 0$ a.e. on $X \times \mathbb{R}$, a contradiction. $\square$

3.4 A bootstrap argument

The main purpose of this section is to prove several auxiliary statements needed in the studies of the asymptotic behavior of solutions $\varphi(t)$ at $t = -\infty$. Usually proofs of the uniqueness are based on the derivation of appropriate asymptotic formulas with one or two leading terms (at $t = -\infty$ as in [9, 15, 18, 55] or at $t = +\infty$ as in [25]). As we have mention in the introductory section, our approach is based on an asymptotic integration technique often used in the theory of functional differential equations, e.g. see [32], [41, Proposition 7.1] or [22]. Thus, we use neither the Titchmarsh theory of Fourier integrals [50] nor the powerful Ikehara Tauberian theorem [9, 15]. First we will apply our methods to get an asymptotic formula for the integral $\psi(t) := \int_{-\infty}^{t} \varphi(s)ds$. Since $\psi \in C^1(\mathbb{R})$ is strictly increasing and positive, this function is somewhat easier to treat than the solution $\varphi(t)$.

Everywhere in the sequel, we continue assuming all conditions of Section 3 on φ, K, g, χ. We also will use the following hypotheses:

Assumption III.13. $\gamma_\phi < \gamma_K$ *and, for some measurable* $C(\tau) > 0$ *and* $\alpha, \sigma \in (0, 1]$,

$$|g'(0,\tau) - \frac{g(u,\tau)}{u}| \le C(\tau)u^\alpha, \; u \in (0, \sigma),$$

$$\zeta(x) := \int_{X\times\mathbb{R}} C(\tau)K(s,\tau)e^{-sx}dsd\mu < +\infty, \; x \in (0, \gamma_K). \tag{3.12}$$

Example III.14. Consider $X = \{\tau_1\}$, $\mu(X) = 1$ and $g(v, \tau_1) = pve^{-v}, v \ge 0, p > 1$. Since $g''(0+, \tau_1)$ exists and is bounded, we have that $|g'(0,\tau)u - g(u,\tau)| \le Cu^2$ for all small $u \ge 0$ and $C > 0$. Moreover, if we choose the heat kernel $K_\alpha(s, \tau) = (4\pi\alpha)^{-1/2} \exp(-s^2/(4\alpha))$, then we obtain that $\zeta(x) = Ce^{4\alpha^2x^2}$.

Assumption III.15 ($\mathbf{EC}_\rho$). *For every* $x \in (0, \rho)$, $\rho \le \gamma_\phi$, *there exists some positive*

C_x such that

$$0 \le \varphi(t) \le C_x e^{xt}, \;\; t \le 0. \tag{3.13}$$

There are several situations when the fulfillment of Assumption III.15($\mathbf{EC}_\rho$) can be easily checked:

Lemma III.16. *Condition III.15 is satisfied in either of the following two cases:*

(i) $\varphi \in C^1(\mathbb{R})$ *and the integral* $\int_{\mathbb{R}} e^{-xs}\varphi'(s)ds$ *converges absolutely for all* $x \in (0,\rho)$*;*

(ii) (cf. [15]) $\rho < \gamma_\phi$ *and there exist measurable* $d_1, d_2,\; d_1 d_2 \in L^1(X)$*, such that*

$$0 \le K(s,\tau) \le d_1(\tau)e^{\rho s}, \;\; s \in \mathbb{R}, \;\; \tau \in X,$$

$$|g(u,\tau)| \le d_2(\tau)u, \;\; u \ge 0. \tag{3.14}$$

Proof: (i) For each $x \in (0,\rho)$ we have that

$$\varphi(t) = \int_{-\infty}^t \varphi'(s)ds = \int_{-\infty}^t e^{xs}\varphi'(s)e^{-xs}ds \le e^{xt}\int_{-\infty}^t e^{-xs}|\varphi'(s)|ds =: C_x e^{xt}.$$

(ii) Since $\rho < \gamma_\phi$, the integral $\int_{\mathbb{R}} e^{-xs}\varphi(s)ds$ converges for all $x \in (0,\rho]$. If $x \in (0,\rho],\; t \le 0$, then

$$\varphi(t)e^{-xt} \le \varphi(t)e^{-\rho t} = \int_X d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)e^{-\rho s}e^{-\rho(t-s)}g(\varphi(t-s),\tau)ds \le$$

$$C := \int_X d_1(\tau)d_2(\tau)d\mu(\tau)\int_{\mathbb{R}} e^{-\rho s}\varphi(s)ds, \;\; t \in \mathbb{R}. \qquad \square$$

The following simple propositions will be used several times in the sequel:

Lemma III.17. *Assume that* $h(s)e^{-sx} \in L^1(\mathbb{R})$ *for all* $x \in [a,b]$*. Then*

$$H(x,y) := \int_{\mathbb{R}} h(s)e^{-sx-isy}ds, \;\; y \in \mathbb{R}$$

is uniformly (with respect to $y \in \mathbb{R}$*) continuous on* $[a,b]$*.*

Proof: Take an arbitrary $\varepsilon > 0$ and let $N > 0$ be such that

$$\int_{\mathbb{R}\setminus[-N,N]} |h(s)|e^{-sx}ds < 0.5\varepsilon,\ x \in [a,b].$$

Since e^t is uniformly continuous on compact sets, there exists $\delta > 0$ such that $|x_1 - x_2| \le \delta,\ s \in [-N,N]$ implies $|e^{-x_1 s} - e^{-x_2 s}| < 0.5\varepsilon/|h|_1$. But then

$$|H(x_1,y) - H(x_2,y)| \le 0.5\varepsilon + \int_{-N}^{N} |h(s)||e^{-x_1 s} - e^{-x_2 s}|ds < \varepsilon,\ y \in \mathbb{R}. \qquad \square$$

Corollary III.18. *With h as in Lemma III.17, we have that $\lim_{y\to\infty} H(x,y) = 0$ uniformly on $x \in [a,b]$.*

Proof: Due to Lemma III.17, for each $\varepsilon > 0$ there exists a finite sequence $a := x_0 < x_1 < x_2 < \cdots < x_m =: b$ possessing the following property: for each x there is x_j such that $|H(x_j,y) - H(x,y)| < 0.5\epsilon$ uniformly on y. Now, due to Riemann-Lebesgue lemma, $\lim_{y\to\infty} H(x_j,y) = 0$ for every j. Therefore, for all j and some $M > 0$, we have that $|H(x_j,y)| < 0.5\epsilon$ if $|y| \ge M$. This implies that

$$|H(x,y)| \le |H(x_j,y) - H(x,y)| + |H(x_j,y)| < \epsilon, \quad |y| \ge M, x \in [a,b],$$

and the corollary is proved. $\square$

As we know, the property $\varphi(-\infty) = 0$ implies the exponential decay $\psi(t) = O(e^{zt})$ at $-\infty$ for each $z \in (0,\gamma_\phi)$. It is clear also that $\psi(t) = O(t)$ as $t \to +\infty$. Hence, for each fixed $z \in (0,\gamma_\phi)$, we can integrate equation (3.10) twice, to find that $\Psi(z) := \int_{\mathbb{R}} e^{-zv}\psi(v)dv$ satisfies

$$\Psi(z) = \int_X d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)e^{-zs}\int_{\mathbb{R}} e^{-z(v-s)}\int_{-\infty}^{v-s} g(\varphi(u),\tau)dudvds =$$

$$\int_X d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)e^{-zs}\int_{\mathbb{R}} e^{-zv}\int_{-\infty}^{v} g(\varphi(u),\tau)dudvds =$$

$$\left(\int_X d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)g'(0,\tau)e^{-zs}ds\right)\int_{\mathbb{R}} e^{-zv}\psi(v)dv + \mathcal{R}(z), \quad \text{where}$$

$$\mathcal{R}(z) := \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-zs}ds \int_{\mathbb{R}} e^{-zv} \int_{-\infty}^{v} (g(\varphi(u),\tau) - g'(0,\tau)\varphi(u))dudv.$$

Therefore $\chi(z)\Psi(z) = \mathcal{R}(z)$. Set now

$$\mathfrak{G}(z,\tau) := \int_{\mathbb{R}} e^{-zv}G(v,\tau)dv, \quad G(v,\tau) := \int_{-\infty}^{v} (g(\varphi(u),\tau) - g'(0,\tau)\varphi(u))du.$$

Lemma III.19. *Assume (3.14), III.13 and III.15(*$\mathbf{EC}_{2\epsilon}$*) for some small* $2\epsilon \in (0, \gamma_K - \gamma_\phi)$. *Then given* $a, b \in (0, \gamma_\phi + \alpha\epsilon)$ *there exists* $\rho > 0$ *depending on* φ, a, b *such that*

$$|\mathfrak{G}(z,\tau)| \le \rho(\tau)/|z| := \rho(C(\tau) + d_2(\tau) + g'(0,\tau))/|z|, \quad \Re z \in [a,b] \subset (0, \gamma_\phi + \alpha\epsilon).$$

Proof: For $x := \Re z \in (0, \gamma_\phi + \alpha\epsilon),\ v \le 0$, we have

$$e^{-xv}|G(v,\tau)| \le e^{-xv}C(\tau)\int_{-\infty}^{v} (\varphi(u))^{1+\alpha}du \le e^{-xv}C_\epsilon^\alpha C(\tau)\psi(v)e^{\alpha\epsilon v},$$

so that $e^{-x\cdot}|G(\cdot,\tau)| \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. After integrating by parts, we obtain

$$\int_{-N}^{N} e^{-zv}G(v,\tau)dv = \frac{G(-N,\tau)e^{zN} - G(N,\tau)e^{-zN}}{z} +$$

$$+\frac{1}{z}\int_{-N}^{N} e^{-zu}(g(\varphi(u),\tau) - g'(0,\tau)\varphi(u))du.$$

This yields

$$|\int_{\mathbb{R}} e^{-zv}G(v,\tau)dv| = \frac{1}{|z|}|\int_{\mathbb{R}} e^{-zu}(g(\varphi(u),\tau) - g'(0,\tau)\varphi(u))du| \le$$

$$\frac{1}{|z|}\left(C_\epsilon^\alpha C(\tau)\int\limits_{-\infty}^{0} e^{-(\Re z - \alpha\epsilon)u}\varphi(u)du + |\varphi|_\infty(g'(0,\tau) + d_2(\tau))\int\limits_{0}^{+\infty} e^{-\Re z u}du\right). \square$$

Corollary III.20. *In addition, assume that* $\int_{\mathbb{R}\times X} K(s,\tau)\rho(\tau)e^{-sx}d\mu ds$ *converges for all* $x \in (0, \gamma_K)$. *Then* $\chi(\gamma_\phi) = 0$ *and, for appropriate* $\varepsilon_1 > 0$, $m \in \mathbb{R}$ *and* $k \in \{0, 1\}$, *and continuous* $r \in L^2(\mathbb{R})$,

$$\psi(t+m) = (a-t)^k e^{\gamma_\phi t} + e^{(\gamma_\phi+\varepsilon)t}r(t),\ t \in \mathbb{R}.$$

Proof: Set $z := x + iy$. For a fixed $0 < x < \gamma_\phi + \alpha\epsilon$ we have

$$|\mathcal{R}(z)| = |\int_X \mathfrak{G}(z,\tau) \int_{\mathbb{R}} K(s,\tau) e^{-zs} ds d\mu| \leq \frac{1}{|z|} \int_X \rho(\tau) \int_{\mathbb{R}} K(s,\tau) e^{-xs} ds d\mu,$$

so that $\mathcal{R}(z)$ is regular in the strip $0 < \Re z < \gamma_\phi + \alpha\epsilon$. Thus we can deduce from $\Psi(z) = \mathcal{R}(z)/\chi(z)$ that $\gamma_\phi = \gamma_\psi$ (e.g. see [15, Lemma 4.4], the definition of γ_ψ is similar to that of γ_ϕ) must be a positive zero of $\chi(z)$ and $\Psi(\gamma_\phi) = \infty$. It is clear that $\mathcal{R}(x + i\cdot)$ is also bounded and square integrable on $\mathbb{R}$ (for each fixed x). Take now γ', γ'' such that $0 < \gamma' < \gamma_\phi < \gamma'' < \gamma_\phi + \alpha\epsilon$. Then we may shift the path of integration in the inversion formula for the Laplace transform (e.g. see [41, p. 10]) to obtain

$$\psi(t) = \frac{1}{2\pi i} \int_{\gamma' - i\infty}^{\gamma' + i\infty} e^{zt} \Psi(z) dz = -\mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathcal{R}(z)}{\chi(z)} + \frac{e^{\gamma'' t}}{2\pi i} \left\{ \int_{-\infty}^{+\infty} e^{ist} a_1(s) ds \right\},$$

where the first term is different from 0 and $a_1(s) = \mathcal{R}(\gamma'' + is)/\chi(\gamma'' + is)$ is square integrable on $\mathbb{R}$. Here we recall that, by Corollary III.18, $\lim_{y \to \infty} \chi(x + iy) = 1$ uniformly on $x \in [\gamma', \gamma'']$. Since $\chi''(x) > 0, \ x \in (0, \gamma_K)$, for some $m \in \mathbb{R}$ we get $\psi(t + m) = (a - t)^k e^{\gamma_\phi t} + e^{\gamma'' t} r(t)$. $\square$

It should be noted here that depending on the geometric properties of g, the value of γ_ϕ can be minimal (the case of a pulled semi-wavefront [16, 24]) or maximal (the case of a pushed semi-wavefront [16, 24]) positive root of $\chi(z) = 0$. Observe that, due to the monotonicity of ψ, we can also use here the Ikehara Tauberian theorem [9]. However it gives a slightly different result.

Lemma III.21. *Assume all conditions of Lemma III.19 excepting $\gamma_\phi < \gamma_K$. If*

$$1 - \chi_1(x_0) := \int_{\mathbb{R}} \int_X K(s,\tau) d_2(\tau) d\mu(\tau) e^{-sx_0} ds \leq 1,$$

for some $x_0 \in (0, \gamma_K)$, then γ_ϕ coincides with the minimal positive zero λ_l of $\chi(z)$.

*Proof:*Since $d_2(\tau) \geq g'(0,\tau)$, we obtain that $x_0 \in [\lambda_l, \lambda_{rK}]$ and $\lambda_l < \gamma_K$. Case I: $\gamma_\phi < \gamma_K$. Then, by Corollary III.20, we have $\chi(\gamma_\phi) = 0$ so that $\gamma_\phi \in \{\lambda_l, \lambda_r\}$. Suppose that $\gamma_\phi > \lambda_l$, this implies $x_0 \leq \gamma_\phi = \lambda_r$. We have

$$\Psi(z) = \left(\int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) d_2(\tau) e^{-zs} ds\right) \int_{\mathbb{R}} e^{-zv}\psi(v)dv + \mathcal{R}_1(z), \quad \text{where}$$

$$\mathcal{R}_1(z) := \int_X d\mu(\tau) \int_{\mathbb{R}} K(s,\tau) e^{-zs} ds \int_{\mathbb{R}} e^{-zv} \int_{-\infty}^{v} (g(\varphi(u),\tau) - d_2(\tau)\varphi(u)) du dv,$$

or, in a shorter form,

$$\chi_1(z)\Psi(z) = \mathcal{R}_1(z). \tag{3.15}$$

It is clear that $x_0 = \gamma_\phi = \lambda_r > \lambda_l$ implies immediately that $g'(0,\tau) = d_2(\tau)$ a.e. on X and that $\chi_1(z) = \chi(z),\ \mathcal{R}(z) = \mathcal{R}_1(z)$. As we have seen in the proof of Corollary III.20, this guarantees that $\mathcal{R}_1(x_0)$ is a finite number. Of course, $\mathcal{R}_1(x_0)$ is also well defined if $x_0 < \gamma_\phi$. Now, it is clear that $\mathcal{R}_1(x_0) \leq 0$ because of $g(u,\tau) \leq d_2(\tau)u,\ u \geq 0$. We claim that, in fact, $\mathcal{R}_1(x_0) < 0$. Indeed, otherwise $g(u,\tau) = d_2(\tau)u,\ u \geq 0$, for almost all $\tau \in X$ that yields $d_2(\tau) = g'(0,\tau)$ and $\mathcal{R}_1(z) \equiv 0$ leading to a contradiction: $\Psi(z) \equiv 0$ and $\psi(t) \equiv 0$.

Now, from $\mathcal{R}_1(x_0) < 0, \Psi(x_0) > 0, \chi_1(x_0) \geq 0$, we deduce that Ψ must have a pole at $x_0 = \gamma_\phi < \gamma_K$. But then $\chi_1(\gamma_\phi) = \chi(\gamma_\phi)$ implies $\chi_1(z) \equiv \chi(z)$, $\mathcal{R}(z) = \mathcal{R}_1(z)$. Hence, $\lambda_l < \lambda_r = x_0 < \gamma_K$ and $\gamma_\phi = x_0$ is a simple pole of Ψ. Therefore we can proceed as in the proof of Corollary III.20 taking $0 < \gamma' < \gamma_\phi = \lambda_r < \gamma'' < \gamma_\phi + \alpha\epsilon$ to obtain

$$\psi(t) = \frac{1}{2\pi i}\int_{\gamma'-i\infty}^{\gamma'+i\infty} e^{zt}\Psi(z)dz = -\mathrm{Res}_{z=\lambda_r}\frac{e^{zt}\mathcal{R}(z)}{\chi(z)} + e^{\gamma'' t} r_1(t) =$$

$$= Ae^{\gamma_\phi t} + e^{\gamma'' t} r_1(t), \quad \text{where } A := -\frac{\mathcal{R}(\lambda_r)}{\chi'(\lambda_r)} < 0,\ r_1 \in L^2(\mathbb{R}),$$

contradicting to the positivity of ψ.

<u>Case II:</u> $\gamma_\phi = \gamma_K$. Since $x_0 < \gamma_K = \gamma_\phi$ and $\mathcal{R}_1(x_0) < 0$, we similarly deduce from (3.15) that x_0 is a singular point of $\Psi(z)$, a contradiction. $\square$

3.5 The uniqueness theorems

To prove our uniqueness results we will need more strong property of φ than the merely convergence of $\int_{\mathbb{R}} e^{-zs}\varphi(s)ds$ for all $\Re z \in (0, \gamma_\phi)$ (even combined, as in Section 4, with Assumption III.15($\mathbf{EC}_\epsilon$) for some small $\epsilon > 0$). This property, assumed everywhere in the sequel, is Assumption III.15($\mathbf{EC}_{\gamma_\phi}$). The nonlinearity g is supposed to satisfy the Assumption III.13

The following assertion is crucial for extension of DK theory on the critical case $\chi(\lambda_l) = \chi'(\lambda_l) = 0$.

Lemma III.22. *Suppose that, for some $a, b > \delta > 0$, continuous $v : \mathbb{R} \to [0, 1)$ satisfies $v(t) = 1 + O(e^{at}),\ t \to -\infty$, $v(t) = O(e^{-bt}),\ t \to +\infty$, and*

$$v(t) \le \int_{\mathbb{R}} N(s)v(t-s)ds,$$

where measurable $N(s) \ge 0,\ s \in \mathbb{R}$, is such that

$$\int_{\mathbb{R}} N(s)ds = 1, \quad \int_{\mathbb{R}} sN(s)ds = 0, \quad \int_{\mathbb{R}} N(s)e^{xs}ds < \infty, \quad \text{for all } |x| \le \delta.$$

Then $v(t) \equiv 0$.

Proof: First we observe that, without restricting the generality, we may assume that $v \in C^2(\mathbb{R})$ with the finite norm $|v|_{C^2} := \sup_{s\in\mathbb{R}, j=0,1} |v^{(j)}(s)|$. Indeed, if we set

$$w(t) := \int_t^{t+1} v(s)ds,\ t \in \mathbb{R},$$

then $w \in C^1(\mathbb{R})$ has the same properties as v, $|w'(t)| < 1,\ t \in \mathbb{R}$, while $v(t) \equiv 0$ if and only if $w(t) \equiv 0$. For instance, if $v(t) \le ce^{-bt}$ for $t \ge t_0,\ b > 0$, then

$w(t) \leq ce^{-bt} \int_0^1 e^{-bs} ds \leq ce^{-bt},\ t \geq t_0$. Furthermore, $w'(t) = v(t+1) - v(t)$ behaves as $O(e^{at})$ at $-\infty$ and as $O(e^{-bt})$ at $+\infty$.

Applying the same procedure to w once more, we obtain the desired smoothness property of v with $v'(t), v''(t)$ satisfying

$$v'(t), v''(t) = O(e^{at}),\ t \to -\infty, \quad v'(t), v''(t) = O(e^{-bt}),\ t \to +\infty. \tag{3.16}$$

In any case, the bilateral Laplace transform $V(z)$ of $v(t)$ is well defined in the vertical strip $-b < \Re z < 0$.

Set now

$$f(t) := \int_{\mathbb{R}} N(s)v(t-s)ds - v(t) \geq 0,$$

It follows from this definition that $0 \leq f(t) \leq 1 - v(t)$ and therefore $f(t) = O(e^{at}),\ t \to -\infty$. Additionally, using (3.16), we obtain, for $j = 0, 1, 2$ and some positive $C, C' > 0$,

$$\int_{\mathbb{R}} N(s)|v^{(j)}(t-s)|ds \leq C \int_{\mathbb{R}} N(s)e^{\pm\delta(t-s)}ds = Ce^{\pm\delta t} \int_{\mathbb{R}} N(s)e^{\mp\delta s}ds =: C'e^{\pm\delta t}.$$

Thus we can conclude that the Laplace transform $F(z)$ of C^2-smooth function $f(t)$, $|f|_{C^2} < \infty$, is well defined in the strip $-\delta < \Re z < \delta$, where we have

$$|F(z)| \leq \frac{C_{pq}}{|z|^2}, \quad p \leq \Re z \leq q, \quad p, q \in (-\delta, \delta).$$

Hence, we can apply the Laplace transform to the equation

$$v(t) + f(t) = \int_{\mathbb{R}} N(s)v(t-s)ds,$$

to obtain that

$$V(z) = \frac{F(z)}{\mathcal{N}(z) - 1}, \quad -\delta < \Re z < 0,$$

where the Laplace transform $\mathcal{N}(z) := \int_{\mathbb{R}} e^{-zs}N(s)ds$ of N is an analytical function in the strip $|\Re z| < \delta$. Observe also that

$$\mathcal{N}(0) = 1, \quad \mathcal{N}'(0) = 0, \quad \mathcal{N}''(0) = \int_{\mathbb{R}} s^2 N(s)ds > 0.$$

Now, since $V(z)$ is analytical in the strip $\Pi := \{-\delta < \Re z < 0\}$, the function $F(z)/(\mathcal{N}(z)-1)$ has the same property in Π. On the other hand, for an appropriate $\delta' \in (0,\delta)$ the quotient $F(z)/(\mathcal{N}(z)-1)$ defines a meromorphic function in $\Pi' := \{-\delta < \Re z < \delta'\}$, with a unique singularity (double pole) at $z = 0$. Note that Lemma III.12 is used at this stage. Since the Laplace transform V of $v \in C^2(\mathbb{R})$ is integrable along each vertical line inside of Π, we may apply the inversion formula to get, for arbitrarily fixed $c \in (-\delta, 0),\ r \in (0, \delta')$,

$$v(t) = \frac{1}{2\pi i}\int_{c-i\cdot\infty}^{c+i\cdot\infty} \frac{e^{zt}F(z)}{\mathcal{N}(z)-1}dz = \mathrm{Res}_{z=0}\frac{e^{zt}F(z)}{\mathcal{N}(z)-1} + \frac{1}{2\pi i}\int_{r-i\cdot\infty}^{r+i\cdot\infty} \frac{e^{zt}F(z)}{\mathcal{N}(z)-1}dz.$$

Next, observe that if $f(t) \equiv 0$ then also $F(z) \equiv 0$ so that $v(t) \equiv 0$. Therefore the only case of the interest is when $f(s') > 0$ at some $s' \in \mathbb{R}$ that implies $F(0) > 0$. Now, in such a case, we have that

$$\left|\int_{r-i\cdot\infty}^{r+i\cdot\infty} \frac{e^{zt}F(z)}{\mathcal{N}(z)-1}dz\right| \le c_0 e^{rt}\int_{\mathbb{R}} \frac{ds}{r^2+s^2} \le c_1 e^{rt},\ t \in \mathbb{R},$$

while a direct calculation shows that

$$\mathrm{Res}_{z=0}\frac{e^{zt}F(z)}{\mathcal{N}(z)-1} = \frac{2F(0)}{\mathcal{N}''(0)}t + \frac{F'(0)}{\mathcal{N}''(0)} - \frac{2F(0)\mathcal{N}'''(0)}{3(\mathcal{N}''(0))^2} =: At + B, \quad A > 0.$$

In consequence, as $t \to -\infty$,

$$v(t) = At + B + O(e^{rt}), \quad \text{with } A, r > 0,$$

which contradicts to the boundary condition $v(-\infty) = 1$. $\square$

Now we are ready to prove our first uniqueness result:

Theorem III.23. *Assume III.15 ($\mathbf{EC}_{\gamma_\phi}$) excepting $\gamma_\phi < \gamma_K$ as well as Assumption III.13 and suppose further that $\chi(\gamma_K-) \neq 0$,*

$$|g(u,\tau) - g(v,\tau)| \le g'(0,\tau)|u-v|, \;\; u,v \ge 0. \tag{3.17}$$

$$\int_X d\mu \int_{\mathbb{R}} K(s,\tau) g'(0,\tau) ds \in (1,+\infty) \quad (\textit{equivalently}, \;\; \chi(0) < 0). \tag{3.18}$$

Then equation (3.10) has at most one bounded positive solution φ, $\varphi(-\infty) = 0$. Furthermore, γ_ϕ coincides with the minimal positive zero λ_l of $\chi(z)$ and such a solution (if exists) has the following representation:

$$\varphi(t+m) = (a-t)^k e^{\lambda_l t} + e^{(\lambda_l+\delta)t} r(t), \quad \textit{with continuous } r \in L^2(\mathbb{R}),$$

for some appropriate $m \in \mathbb{R}$, $\delta > 0$. Here $k = 0$ [respectively, $k = 1$] if λ_l is a simple [respectively, double] root of $\chi(z) = 0$.

Remark III.24. By Lemma III.21, the above assumptions exclude the existence of pushed semi-wavefronts, the same lemma also guarantees that $\gamma_\phi = \lambda_l$ and consequently $\gamma_\phi < \gamma_K$. Theorem III.23 holds also when $\chi(\gamma_K-) = 0$ but $\chi'(\gamma_K-) < 0$.

Proof: Step I: Asymptotic behavior at $-\infty$. It is clear that equation (3.10) can be written as the linear inhomogeneous equation

$$\varphi(t) = \int_X d\mu \int_{\mathbb{R}} K(s,\tau) g'(0,\tau) \varphi(t-s) ds + \mathcal{D}(t), \;\; t \in \mathbb{R}, \tag{3.19}$$

where all integrals are converging and

$$\mathcal{D}(t) := \int_X d\mu \int_{\mathbb{R}} K(s,\tau)(g(\varphi(t-s),\tau) - g'(0,\tau)\varphi(t-s)) ds \le 0, \;\; t \in \mathbb{R}.$$

Take $C(\tau), \sigma, \zeta(x)$ as in Assumption III.13 . Observe that without restricting the generality, we can assume in Assumption III.13 that $(1+\alpha)\gamma_\phi < \gamma_K$. Since equation

(3.10) is translation invariant, we can suppose that $\varphi(t) < \sigma$ for $t \le 0$. Applying the bilateral Laplace transform to (3.19), we obtain that

$$\chi(z)\Phi(z) = \mathfrak{D}(z).$$

We claim that, due to conditions III.13 and III.15($\mathbf{EC}_{\gamma_\phi}$), function $\mathfrak{D}$ is regular in the strip $\mathfrak{P} = \{z : \Re z \in (0, (1+\alpha)\gamma_\phi)\}$. Indeed, we have

$$\mathfrak{D}(x+iy) = \int_R e^{-iyt}[e^{-xt}\mathcal{D}(t)]dt.$$

Given $x := \Re z \in (0, (1+\alpha)\gamma_\phi)$, we will choose x' sufficiently close from the left to γ_ϕ to satisfy $-x + (1+\alpha)x' > 0$. Then

$$|e^{-xt}\mathcal{D}(t)| \le e^{-xt}\left[\int_X C(\tau)d\mu \int_t^{+\infty} K(s,\tau)C_{x'}^{1+\alpha}e^{(1+\alpha)x'(t-s)}ds + \right.$$
$$\left. +2|\varphi|_\infty \int_X g'(0,\tau)d\mu \int_{-\infty}^t K(s,\tau)ds\right] \le$$
$$e^{-xt}\left[e^{(1+\alpha)x't}C_{x'}^{1+\alpha}\zeta((1+\alpha)x') + 2|\varphi|_\infty \int_X g'(0,\tau)d\mu \int_{-\infty}^t K(s,\tau)ds\right] =:$$
$$e^{-xt}\left[e^{(1+\alpha)x't}A_1 + 2|\varphi|_\infty \int_X g'(0,\tau)d\mu \int_{-\infty}^t K(s,\tau)e^{-(1+\alpha)x's}e^{(1+\alpha)x's}ds\right] \le$$
$$e^{(-x+(1+\alpha)x')t}\left[A_1 + 2|\varphi|_\infty(1-\chi((1+\alpha)x'))\right] =: A_2 e^{(-x+(1+\alpha)x')t}, \ t \in \mathbb{R}.$$

Since clearly $\mathcal{D}(t)$ is bounded on $\mathbb{R}$, the above calculation shows that $e^{-xt}\mathcal{D}(t)$ belongs to $L^k(\mathbb{R})$, for each $k \in [1,\infty]$ once $x \in (0, (1+\alpha)\gamma_\phi)$. As a consequence, for each such fixed x the function $\mathfrak{d}_x(y) = \mathfrak{D}(x + i \cdot y)$ is bounded and square integrable on $\mathbb{R}$.

By our assumptions, $\chi(z)$ is also regular in the domain $\mathfrak{P}$, while

$$\Phi(z) = \frac{\mathfrak{D}(z)}{\chi(z)},$$

is regular in $\Re z \in (0, \gamma_\phi)$ and meromorphic in $\mathfrak{P}$. In virtue of Lemma III.12, we can suppose that $\Phi(z)$ has a unique singular point γ_ϕ in $\mathfrak{P}$ which is either simple or double pole.

Now, for some $x'' \in (0, \gamma_\phi)$, using the inversion theorem for the Fourier transform, we obtain that for an appropriate sequence of integers $N_j \to +\infty$

$$\varphi(t) = \frac{1}{2\pi i} \lim_{j\to+\infty} \int_{x''-iN_j}^{x''+iN_j} \frac{e^{zt}\mathfrak{D}(z)}{\chi(z)} dz$$

almost everywhere on $\mathbb{R}$, e.g. see [41, p. 9-10]. Next, if $x \in (\gamma_\phi, (1+\alpha\gamma_\phi))$ then

$$\int_{x''-iN}^{x''+iN} \frac{e^{zt}\mathfrak{D}(z)dz}{\chi(z)} = \left(\int_{x-iN}^{x+iN} + \int_{x''-iN}^{x-iN} - \int_{x''+iN}^{x+iN} \right) \frac{e^{zt}\mathfrak{D}(z)dz}{\chi(z)} - 2\pi i \mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathfrak{D}(z)}{\chi(z)}.$$

Since, by Corollary III.18,

$$\lim_{j\to+\infty} \max_{z\in[x''\pm iN_j, x\pm iN_j]} (|\mathfrak{D}(z)| + |1-\chi(z)|) = 0,$$

we conclude that, for each fixed $t \in \mathbb{R}$

$$\lim_{j\to+\infty} \int_{x''\pm iN_j}^{x\pm iN_j} \frac{e^{zt}\mathfrak{D}(z)}{\chi(z)} dz = 0.$$

Therefore

$$\varphi(t) = -\mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathfrak{D}(z)}{\chi(z)} + \frac{e^{xt}}{2\pi} \int_{\mathbb{R}} \frac{e^{iyt}\mathfrak{d}_x(y)}{\chi(x+iy)} dy.$$

It should be noted here that $\mathfrak{D}(\gamma_\phi) < 0$ since otherwise $\mathcal{D}(t) \equiv 0$ implying $\chi(z)\Phi(z) = \mathfrak{D}(z) \equiv 0$ so that $\Phi(z) \equiv 0$, a contradiction. Since

$$\mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathfrak{D}(z)}{\chi(z)} = \frac{e^{\gamma_\phi t}\mathfrak{D}(\gamma_\phi)}{\chi'(\gamma_\phi)}, \quad \text{if } \lambda_l < \lambda_r,$$

$$\mathrm{Res}_{z=\gamma_\phi} \frac{e^{zt}\mathfrak{D}(z)}{\chi(z)} = \frac{2e^{\gamma_\phi t}}{\chi''(\gamma_\phi)} \left(t\mathfrak{D}(\gamma_\phi) + \mathfrak{D}'(\gamma_\phi) - \mathfrak{D}(\gamma_\phi)\frac{\chi'''(\gamma_\phi)}{3\chi''(\gamma_\phi)} \right), \quad \text{if } \lambda_l = \lambda_r,$$

we get the desired representation.

Step II: Uniqueness. By the contrary, suppose that φ_1, φ_2 are two essentially different solutions of (3.10) in the sense that $\varphi_1(t) \notin \{\varphi_2(t+s),\ s \in \mathbb{R}\}$. Due to Step I we can suppose that φ_1, φ_2 have the same main parts of their asymptotic representations:

$$\varphi_j(t) = (a_j - t)^k e^{\gamma_\phi t} + e^{(\gamma_\phi+\delta)t} r_j(t),\ r_j \in L^2(\mathbb{R}).$$

Therefore $\omega(t) := \varphi_2(t) - \varphi_1(t) = e^{(\gamma_\phi+\delta)t}r(t),\ t \in \mathbb{R},\ r \in L^2(\mathbb{R})$, in the case of $\lambda_l < \lambda_r$ and $\omega(t) = (a_2 - a_1)e^{\gamma_\phi t} + e^{(\gamma_\phi+\delta)t}r(t),\ t \in \mathbb{R},\ r \in L^2(\mathbb{R})$, in the case of $\lambda_l = \lambda_r$. Set

$$w(t) := \int_{t-1}^{t} |\omega(s)|ds,$$

it is clear that $w \in C^1(\mathbb{R})$ is bounded and has bounded derivative on $\mathbb{R}$, in fact, $0 < |w'|_\infty, |w|_\infty \le \max\{|\varphi_1|_\infty, |\varphi_2|_\infty\}$. Furthermore, if $\lambda_l < \lambda_r$ then

$$w(t) = |\int_{t-1}^{t} e^{(\gamma_\phi+\delta)s}r(s)ds| \le e^{(\gamma_\phi+\delta)t}\int_{t-1}^{t}|r(s)|ds \le e^{(\gamma_\phi+\delta)t}\sqrt{\int_{t-1}^{t} r^2(s)ds},$$

so that $w(t) = e^{(\gamma_\phi+\delta)t}o(1)$ at $t = -\infty$. Now, if $\lambda_l = \lambda_r$, we know that

$$\omega(t) = ae^{\gamma_\phi t} + e^{(\gamma_\phi+\delta)t}r(t),$$

where we can suppose that $a \ge 0$. Therefore

$$-e^{(\gamma_\phi+\delta)t}|r(t)| \le |\omega(t)| - ae^{\gamma_\phi t} \le e^{(\gamma_\phi+\delta)t}|r(t)|,$$

so that, in view of the above estimation of $w(t)$, we get

$$|\omega(t)| = ae^{\gamma_\phi t} + e^{(\gamma_\phi+\delta)t}r_1(t),\ \text{with } |r_1(t)| \le |r(t)|,$$

$$w(t) = \int_{t-1}^{t}|\omega(s)|ds = \frac{a(1-e^{-\gamma_\phi})}{\gamma_\phi}e^{\gamma_\phi t} + e^{(\gamma_\phi+\delta)t}o(1),\ t \to -\infty.$$

We have the following:

$$\omega(t) = \int_X d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)(g(\varphi_2(t-s),\tau) - g(\varphi_1(t-s),\tau))ds,$$

$$|\omega(t)| \le \int_X g'(0,\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)|\omega(t-s)|ds,$$

$$\int_{t-1}^{t}|\omega(u)|du \le \int_X g'(0,\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)\int_{t-1}^{t}|\omega(u-s)|duds,$$

and, finally,

$$w(t) \le \int_X g'(0,\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)w(t-s)ds. \tag{3.20}$$

<u>Case I (noncritical)</u>. If $\chi'(\lambda_l) \neq 0$, then $\chi(\gamma') > 0$ for some $\gamma' \in (\gamma_\phi, \gamma_\phi + \delta)$. After multiplying the both sides of (3.20) by $e^{-\gamma' t}$ and setting $v(t) := w(t)e^{-\gamma' t}$, we find that

$$v(t) \leq \int_{\mathbb{R}} \left(\int_X g'(0,\tau)K(s,\tau)e^{-\gamma' s} d\mu(\tau) \right) v(t-s)ds.$$

Since $v(t) \geq 0$ and $v(\pm\infty) = 0$, there exists a finite t_m such that

$$v(t_m) = |v|_\infty = \max_{s \in \mathbb{R}} v(s).$$

But then $v(t_m) \leq \left(\int_X g'(0,\tau)d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-\gamma' s}ds \right) v(t_m)$, forcing $0 = v(t_m) \equiv v(t) \equiv w(t)$ in view of $\chi(\gamma') > 0$.

<u>Case II (critical)</u>. Now, if $\lambda_l = \lambda_r$, we set $v(t) := w(t)e^{-\gamma_\phi t}$, to conclude analogously that $v(-\infty) = a(1 - e^{-\gamma_\phi})/\gamma_\phi,\ v(+\infty) = 0$,

$$v(t) \leq \int_{\mathbb{R}} \left(\int_X g'(0,\tau)K(s,\tau)e^{-\gamma_\phi s} d\mu(\tau) \right) v(t-s)ds.$$

Since in the sequel we will work only with the last linear inequality, we can assume that $0 \leq v(t) \leq 1 = \sup_{s \in \mathbb{R}} v(s)$ for all $t \in \mathbb{R}$. If $v(\hat{t}) = 1$ for some finite rightmost $\hat{t}$, then

$$1 = v(\hat{t}) \leq \int_{\mathbb{R}} \left(\int_X g'(0,\tau)K(s,\tau)e^{-\gamma_\phi s} d\mu(\tau) \right) v(\hat{t}-s)ds =:$$
$$\int_{\mathbb{R}} N(s)v(\hat{t}-s)ds \leq \int_X g'(0,\tau)d\mu(\tau) \int_{\mathbb{R}} K(s,\tau)e^{-\gamma_\phi s}ds = 1,$$

which implies that $N(s)v(\hat{t}-s) = N(s)$ a.e. and $v(\hat{t}-s) = 1$ for all s such that $N(s) > 0$. Now, since $\int_{\mathbb{R}} N(s)ds = 1,\ \int_{\mathbb{R}} sN(s)ds = 0$, there is a subset of $\mathbb{R}_-$ of positive measure where $N(s) > 0$. This means that $\hat{t}$ does not possesses the property to be the rightmost point where $v(\hat{t}) = 1$, a contradiction. Thus we have to analyze only the case when $a > 0$ and $0 \leq v(t) < 1 = v(-\infty)$. It is easy to check that in such a case, $v(t)$ and N meet all the conditions of Lemma III.22. In particular, since

$\gamma_\phi < \gamma_K$, there exists $\delta > 0$ such that

$$\int_{\mathbb{R}} N(s)e^{xs}ds = 1 - \chi(\gamma_\phi - x) < \infty \quad \text{for all } |x| < \delta.$$

Hence, $v(t) \equiv 0$, a contradiction. $\Box$

Next, we will consider the situation when the subtangential Lipschitz condition of Theorem III.23 is not satisfied. In such a case, we still are able to prove the uniqueness under somewhat stronger hypotheses:

Assumption III.25. *Either one of the following conditions holds*

$$|g(u,\tau) - g(v,\tau) - g'(0,\tau)(u-v)| \le C(\tau)|u-v|^{1+\alpha}, \;\; u,v \in (0,\sigma),$$

$$|g'(u,\tau) - g'(0,\tau)| \le C(\tau)u^\alpha, \;\; u \in (0,\sigma), \quad (i.e. \quad g \in C^{1,\alpha}[0,\sigma]),$$

for some $\alpha, \sigma \in (0,1]$ *and measurable* $C(\tau) > 0$ *satisfying (3.12). Furthermore, there exist some positive* $\hat{\epsilon} \in (0, \gamma_\phi)$ *and measurable* $d_1(\tau)$ *such that*

$$0 \le K(s,\tau) \le d_1(\tau)e^{\hat{\epsilon}s}, \;\; s \in \mathbb{R}.$$

Example III.26. If $g(v,\tau) = pve^{-v}, v \ge 0, p > 1$ and $K_\alpha(s,\tau) = (4\pi\alpha)^{-1/2}e^{(-s^2/(4\alpha))}$, is easy to see that g and K_α satisfy assumption III.25.

Assumption III.27. *Either one of the following two assumptions is satisfied:*

(i) Each solution of (3.10) is C^1*-smooth and if* $\varphi_1, \varphi_2 \in C^1(\mathbb{R})$ *satisfy (3.10) and the integral* $\int_{\mathbb{R}} e^{-zs}(\varphi_2(s) - \varphi_1(s))ds$ *converges absolutely then the integral* $\int_{\mathbb{R}} e^{-zs}(\varphi_2'(s) - \varphi_1'(s))ds$ *also converges absolutely.*

(ii) There exists $\delta_0 > 0$ *such that, for each* $x \in (\lambda_{rK} - \delta_0, \lambda_{rK})$*, it holds*

$$0 \le K(s,\tau) \le d_{2,x}(\tau)e^{xs}, \;\; s \in \mathbb{R},$$

for some $\mu-$*measurable* $d_{2,x}(\tau)$.

Theorem III.28. *Suppose that*

$$|g(u,\tau) - g(v,\tau)| \le \lambda(\tau)|u-v|, \;\; u,v \ge 0, \tau \in X,$$

for some μ-measurable λ such that $\mu\{\tau : \lambda(\tau) > g'(0,\tau)\} > 0$ and function

$$\chi_1(z) = 1 - \int_{\mathbb{R}} \int_X K(s,\tau)\lambda(\tau) d\mu(\tau) e^{-sz} ds$$

is well defined on $[0, \lambda_{rK})$. If, in addition, Assumptions III.25, III.27, (3.12) hold with $\lambda d_j \in L^1(X)$, $j = 1, 2$, $\chi(0) < 0$ and $\chi_1(m) \ge 0$ for some $m \in (0, \lambda_{rK})$, then equation (3.10) has at most one bounded positive solution φ, $\varphi(-\infty) = 0$. Finally, γ_ϕ coincides with the minimal simple positive zero λ_l of $\chi(z)$ and such a solution (if exists) has the following asymptotic representation:

$$\varphi(t+m) = e^{\lambda_l t} + e^{(\lambda_l+\delta)t} r(t), \quad \textit{with continuous } r \in L^2(\mathbb{R}),$$

for some appropriate $m \in \mathbb{R}$, $\delta > 0$.

Proof: It should be noted first that, due to Lemma III.16, the assumptions of the theorem guarantee the fulfillment of the hypotheses III.13 and III.15($\mathbf{EC}_{\gamma_\phi}$). Furthermore, all arguments of Step I in the proof of Theorem III.23 can be repeated (with a unique change in the estimation of $e^{-xt}\mathcal{D}(t)$ where $g'(0,\tau)$ should be replaced with $\lambda(\tau)$). Evidently, $\lambda_l < \lambda_r$ so that by Lemma III.21 each pair φ_1, φ_2 of solutions of (3.10) can be supposed to have the same main parts of their asymptotic representations: $\varphi_j(t) = e^{\lambda_l t} + e^{(\lambda_l+\delta)t} r_j(t)$, $r_j \in L^2(\mathbb{R})$. The further proof is divided in several steps.

Step I. Again, we consider bounded function $\omega(t) := \varphi_2(t) - \varphi_1(t) = e^{(\lambda_l+\delta)t} r(t)$, $t \in \mathbb{R}$, $r \in L^2(\mathbb{R})$. If $\Re z \in (0, \lambda_l + \delta)$, then $\int_{\mathbb{R}} e^{-zs}\omega(s)ds$ converges absolutely and from condition III.27(i) we have

$$|\omega(t)| = |\int_{-\infty}^t \omega'(s)ds| = |\int_{-\infty}^t e^{xs}\omega'(s)e^{-xs}ds| \le e^{xt}\int_{\mathbb{R}} e^{-xs}|\omega'(s)|ds =: C_x e^{xt},$$

for all $x \in (0, \lambda_l + \delta)$ and $t \in \mathbb{R}$. Similarly, we obtain from III.25, III.27(ii) that

$$|\omega(t)| = |\int_X d\mu \int_{\mathbb{R}} K(s,\tau)\Big(g(\varphi_1(t-s),\tau) - g(\varphi_2(t-s),\tau)\Big)ds \le$$

$$e^{xt}\int_X \lambda(\tau)d\mu \int_{\mathbb{R}} K(s,\tau)e^{-xs}e^{-x(t-s)}|\omega(t-s)|ds \le$$

$$e^{xt}\int_X \lambda(\tau)(d_1(\tau) + d_{2,\lambda_l+\delta}(\tau))d\mu \int_{\mathbb{R}} e^{-xs}|\omega(s)|ds,\ x \in (\hat{\epsilon}, \lambda_l + \delta),\ t \in \mathbb{R}.$$

In any of these cases, for any $x \in (\hat{\epsilon}, \lambda_l + \delta)$ there exists an appropriate $C_x > 0$ such that $|\omega(t)| \le C_x e^{xt},\ t \in \mathbb{R}$. Set

$$\Gamma = \sup\{x \ge \lambda_l |\ \exists C_x : |\omega(t)| \le C_x e^{xt},\ t \in \mathbb{R}\},$$

we claim that $\Gamma \ge \lambda_{rK}$. Indeed, arguing on the contrary, suppose that $\Gamma < \lambda_{rK}$ and let $x_0 \in (\hat{\epsilon}, \Gamma), \alpha > 0, \gamma_0 \in (\hat{\epsilon}, \lambda_l)$ be such that $\{x_0(1+\alpha),\ x_0 + \alpha\gamma_0\} \subset (\Gamma, \lambda_{rK})$. We will denote as x_* the minimal of these two numbers. We have that

$$\omega(t) = \int_X d\mu \int_{\mathbb{R}} K(s,\tau)g'(0,\tau)\omega(t-s)ds + \mathcal{E}(t),\ t \in \mathbb{R}, \tag{3.21}$$

with bounded

$$\mathcal{E}(t) := \int_X d\mu \int_{\mathbb{R}} K(s,\tau)\Big(g(\varphi_1(t-s),\tau) - g(\varphi_2(t-s),\tau) - g'(0,\tau)\omega(t-s)\Big)ds.$$

Now, depending of assumptions chosen in Assumption III.25 , we have either

$$|g(\varphi_1(s),\tau) - g(\varphi_2(s),\tau) - g'(0,\tau)\omega(s)| \le C(\tau)|\omega(s)|^{1+\alpha} \le$$

$$C(\tau)\min\{C_{x_0}e^{x_0(1+\alpha)s}, (|\varphi_1|_\infty + |\varphi_2|_\infty)^{1+\alpha}\} \le k_1 C(\tau)e^{x_* s},\ s \in \mathbb{R},$$

or

$$|g(\varphi_1(s),\tau) - g(\varphi_2(s),\tau) - g'(0,\tau)\omega(s)| \le C(\tau)|\omega(s)|(|\varphi_1(s)| + |\varphi_2(s)|)^{\alpha} \le$$

$$k_2 C(\tau)\min\{C_{x_0}e^{(x_0+\alpha\gamma_0)s}, (|\varphi_1|_\infty + |\varphi_2|_\infty)^{1+\alpha}\} \le k_3 C(\tau)e^{x_* s},\ s \in \mathbb{R},$$

where k_i depend on x_0 and $|\varphi_j|_\infty$ only. Hence,

$$|\mathcal{E}(t)| \le 4e^{x_* t}\max\{|\varphi_1|_\infty, |\varphi_2|_\infty\}\int_X \lambda(\tau)d\mu \int_{-\infty}^t K(s,\tau)e^{-x_* s}ds$$
$$+ ke^{x_* t}\int_X C(\tau)d\mu \int_t^{+\infty} K(s,\tau)e^{-x_* s}ds \le$$
$$e^{x_* t}\Big(4\max\{|\varphi_1|_\infty, |\varphi_2|_\infty\}(1-\chi_1(x_*)) + k\zeta(x_*)\Big) =: Ae^{x_* t},\ t \in \mathbb{R}.$$

Therefore $e^{-xt}\mathcal{E}(t)$ belongs to $L^k(\mathbb{R})$, for each $k \in [1,\infty]$ once $x \in (\hat{\epsilon}, x_*)$. Using Lemma III.12, we can repeat now the arguments of Step I of Theorem III.23 (below the estimation of $|e^{-xt}\mathcal{D}(t)|$) to conclude that $\omega(t) = e^{xt}r_x(t)\ \ t \in \mathbb{R},\ r_x \in L^2(\mathbb{R})$, for each $x \in (\lambda_l, x_*)$. This implies the absolute convergence of $\int_{\mathbb{R}} e^{-xs}\omega(s)ds$ for every $x \in (\lambda_l, x_*)$. But as we have seen at the beginning of Step I, this yields $|\omega(s)| \le B_x e^{xs},\ s \in \mathbb{R},\ x \in (\lambda_l, x^*)$ for appropriate B_x. Therefore $\Gamma \ge x_* > \Gamma$, a contradiction. In this way, we have proved that

$$|\omega(s)| \le B_x e^{xs},\ s \in \mathbb{R},\ x \in (\hat{\varepsilon}, \min\{\lambda_r, \gamma_K\}). \tag{3.22}$$

Step II. Suppose that $\chi_1(m) > 0$ for some $m \in (0, \lambda_{rK})$, it is clear that $m > \lambda_l$ and

$$\int_{\mathbb{R}}\int_X K(s,\tau)\lambda(\tau)d\mu(\tau)e^{-sm}ds < 1.$$

We now define $\bar{\omega}(t) := |\omega(t)|e^{-mt} \ge 0$, $t \in \mathbb{R}$. By (3.22), we obtain that $\bar{\omega}(\pm\infty) = 0$ and $\bar{\omega}(t_m) = \max_{s\in\mathbb{R}} \bar{\omega}(s) > 0$ for some $t_m \in \mathbb{R}$. Since

$$\omega(t) = \int_X d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)(g(\varphi_2(t-s),\tau) - g(\varphi_1(t-s),\tau))ds,$$

we have

$$\bar{\omega}(t_m) = |\omega(t_m)|e^{-mt_m} \le \int_X \lambda(\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)e^{-ms}|\omega(t_m - s)|e^{-m(t_m-s)}ds$$
$$\le \bar{\omega}(t_m)\int_X \lambda(\tau)d\mu(\tau)\int_{\mathbb{R}} K(s,\tau)e^{-ms}ds < \bar{\omega}(t_m),$$

a contradiction. Hence, $\bar{\omega}(\tau) = 0$ and the uniqueness follows.

Step III. Suppose now that $\chi_1(m) = \max_{s\in(0,\lambda_{rK})} \chi(s) = 0$. Then additionally $\chi_1'(m) = 0$. Furthermore, $\bar{\omega}(t) := |\omega(t)|e^{-mt} \geq 0$, $t \in \mathbb{R}$ has the same properties as in Step II: $\bar{\omega}(\pm\infty) = 0$, $\omega(t_m) = \max_{s\in\mathbb{R}} \bar{\omega}(s) > 0$ for some $t_m \in \mathbb{R}$ and

$$\bar{\omega}(t) \leq \int_{\mathbb{R}} \left(\int_X K(s,\tau)\lambda(\tau)e^{-ms}d\mu(\tau) \right) \bar{\omega}(t-s)ds.$$

Here, we can assume that $0 \leq \bar{\omega}(t) \leq 1$ for all $t \in \mathbb{R}$. If $\bar{\omega}(\hat{t}) = 1$ for some finite rightmost $\hat{t}$, then

$$1 \leq \int_{\mathbb{R}} N_\lambda(s)\bar{\omega}(\hat{t}-s)ds \leq \int_{\mathbb{R}} N_\lambda(s)ds = 1,$$

where $N_\lambda(s) := \int_X K(s,\tau)\lambda(\tau)e^{-ms}d\mu(\tau)$. This implies that $N_\lambda(s)\bar{\omega}(\hat{t}-s) = N_\lambda(s)$ a.e. and $\bar{\omega}(\hat{t}-s) = 1$ for all s such that $N_\lambda(s) > 0$. Now, since $\int_{\mathbb{R}} N_\lambda(s)ds = 1$, $\int_{\mathbb{R}} sN_\lambda(s)ds = 0$, there is a subset of $\mathbb{R}_-$ of positive measure where $N_\lambda(s) > 0$. This means that $\hat{t}$ does not possesses the property to be the rightmost point where $\bar{\omega}(\hat{t}) = 1$, a contradiction. In consequence, $\bar{\omega}(t) \equiv 0$ that proves the uniqueness. □

3.6 Applications

In this section, Theorems III.23 and III.28 are applied to various models which can be written as (3.4). This allows to improve or complement the uniqueness results in [4, 9, 13, 15, 18, 49]. Everywhere in this section we assume that continuos $g : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable at 0 with $g'(0) > 0$.

3.6.1 Nonlocal integro-differential equations

Consider the equation

$$u_t = J * u - u + g(u), \tag{3.23}$$

where $J \geq 0$, $\int_{\mathbb{R}} Jds > 0$. Let $\gamma^{\#}$ denote an extended positive real number such that $\int_{\mathbb{R}} J(s)e^{-zs}ds$ is convergent when $z \in [0, \gamma^{\#})$ and is divergent when $z > \gamma^{\#}$. As it can be easily deduced from Theorem III.2, the existence of such $\gamma^{\#}$ is automatically assured by the existence of positive semi-wavefronts $u(t,x) = \phi(x+ct),\ \phi(-\infty) = 0$ to (3.4). Traveling wave profile ϕ must solve

$$c\phi' = J * \phi - \phi + g(\phi). \tag{3.24}$$

We assume that $g,\ g(0) = 0$, is a non-negative locally Lipschitzian function, in order to replace usual condition (3.3) with more weak requirement

$$g'(s) \leq g'(0) \text{ a.e. on } \mathbb{R}_+, \tag{3.25}$$

we will realize the following trick. Set $g_\beta(s) = g(s) + \beta s$ for some positive β. We claim that β can be chosen in such a way that g_β satisfies the Lipschitz condition with a constant $\beta + g'(0)$. First observe that our proof of uniqueness compares two different solutions ϕ_1, ϕ_2. Since they are uniformly bounded by some positive $M > 0$, we can restrict our attention to a finite interval $[0, M]$ where g is obviously globally Lipschitzian. This means there exists $\beta > 0$ such that $g'(0) \geq g'(s) \geq -2\beta - g'(0)$ almost everywhere on $[0, M]$ and, in consequence, we get the necessary estimation

$$-g'(0) - \beta \leq g'_\beta(s) = \beta + g'(s) \leq \beta + g'(0) \text{ a.e. on } \mathbb{R}_+.$$

Hence, instead of (3.24) we will consider

$$c\phi' = J * \phi - (1+\beta)\phi + g_\beta(\phi). \tag{3.26}$$

Let us suppose that $c > 0$ (the case $c < 0$ is similar). Since ϕ is non-negative bounded solution, it should satisfy

$$\begin{aligned}
\phi(t) &= \frac{1}{c}\int_{-\infty}^{t} e^{-(t-s)(1+\beta)/c}\Big(J * \phi(s) + g_\beta(\phi(s))\Big)ds \\
&= \frac{1}{c}\int_{0}^{+\infty} e^{-s(1+\beta)/c}\Big(J * \phi(t-s) + g_\beta(\phi(t-s))\Big)ds \\
&= k * (J * \phi)(t) + k * g_\beta(\phi)(t) = (k * J) * \phi(t) + k * g_\beta(\phi)(t), \qquad (3.27)
\end{aligned}$$

where $k(s) = c^{-1}e^{-s(1+\beta)/c}$, $s \geq 0$ and $k = 0$ if $s < 0$. Thus, equation (3.27) can be written as (3.4), with $X = \{\tau_1, \tau_2\}$ and

$$K(s,\tau) = \begin{cases} k * J(s), & \tau = \tau_1 \\ k(s), & \tau = \tau_2 \end{cases}, \quad g(s,\tau) = \begin{cases} s, & \tau = \tau_1 \\ g_\beta(s), & \tau = \tau_2 \end{cases}.$$

Finally, independently on the sign of c, we find that

$$\chi(z,c) = 1 - \int_{\mathbb{R}} K(s,\tau_1)e^{-zs}ds - (g'(0)+\beta)\int_{\mathbb{R}} K(s,\tau_2)e^{-zs}ds =$$
$$1 - \frac{1}{1+\beta+cz}\int_{\mathbb{R}} J(s)e^{-zs}ds - \frac{g'(0)+\beta}{1+\beta+cz} =: \frac{\tilde{\chi}(z,c)}{1+\beta+cz}.$$

Let c_* be the minimal value of c for which

$$\tilde{\chi}(z,c) := 1 - g'(0) + cz - \int_{\mathbb{R}} J(s)e^{-sz}ds$$

has at least one positive zero. It is easy to see that

$$c_* = \inf_{z>0} \frac{1}{z}\left\{-1 + g'(0) + \int_{\mathbb{R}} J(s)e^{-sz}ds\right\}$$

can be positive, negative (in these cases inf can be replaced with min) or zero. By Theorem III.9, $c \geq c_*$ for each admissible wave speed c. The next result is a direct consequence of Theorem III.23.

Theorem III.29. *Suppose (3.25) together with* $1 - \int_{\mathbb{R}} J(s)ds < g'(0)$ *and*

$$|g(u) - g'(0)u| \leq Cu^{1+\alpha}, \ u, v \in (0,\sigma) \ \text{ for some } \ \alpha, \sigma \in (0,1], \tag{3.28}$$

Then equation (3.24) has at most one bounded positive solution φ, $\varphi(-\infty) = 0$, *for each* $c \neq 0$ *(if* $\tilde{\chi}(\gamma^{\#}-, c_*) \neq 0$*) or for each* $c \neq 0, c_*$ *(if* $\tilde{\chi}(\gamma^{\#}-, c_*) = 0$*).*

Proof: Suppose that $c > 0$ (the case $c < 0$ is similar). We only have to check the assumptions III.15($\mathbf{EC}_{\gamma_\phi}$), III.13 except $\gamma_\phi(c) < \gamma_K(c)$, $\chi(0,c) < 0$ and $\chi(\gamma_K-, c) \neq 0$ of Theorem III.23.

Step I. It is clear that $g(\cdot,\tau)$ satisfies (3.17), where $g'(0,\tau_1) = 1$, $g'(0,\tau_2) = g'(0)+\beta$. Moreover, we have $|g(u,\tau) - g'(0,\tau)u| \leq C(\tau)u^{1+\alpha},\ u,v \in (0,\sigma)$, where $C(\tau) = 0$ if $\tau = \tau_1$ and $C(\tau) = C$ if $\tau = \tau_2$.

Step II. For each $z > -\frac{1+\beta}{c}$ we have $\int_{\mathbb{R}} k(s)e^{-zs}ds = \frac{1}{1+\beta+cz} < +\infty$ so that $\gamma_K(c) = \gamma^{\#}$ because of $\int_{\mathbb{R}} k * J(s)e^{-zs}ds = \int_{\mathbb{R}} J(s)e^{-zs}ds/(1+\beta+cz)$. (Observe here that $\gamma_K(c) = \min\{\gamma^{\#}, -(1+\beta)/c\}$ if $c < 0$. However, if $\gamma_K(c) = -(1+\beta)/c$ then $\chi(\gamma_K(c), c) = \infty$ so that $\gamma_\phi(c) < \gamma_K(c)$ due to Corollary III.7).

Step III. If φ solves (3.24), then $\varphi \in C^1(\mathbb{R})$ and for each $0 < z < \gamma_\phi$ we obtain

$$c\int_{\mathbb{R}} e^{-zs}|\varphi'(s)|ds \leq \int_{\mathbb{R}} e^{-zs} J*\varphi(s)ds + \int_{\mathbb{R}} e^{-zs}\varphi(s)ds + \int_{\mathbb{R}} e^{-zs} g(\varphi(s))ds \leq$$

$$(\int_{\mathbb{R}} e^{-zs}J(s)ds + 1 + g'(0))\int_{\mathbb{R}} e^{-zs}\varphi(s)ds < +\infty.$$

Thus, by Lemma III.16, condition III.15($\mathbf{EC}_{\gamma_\phi}$) is satisfied.

Step IV. We have $\chi(0,c) = (1 - \int_{\mathbb{R}} J(s)ds - g'(0))/(1+\beta) < 0$. Now, if $\gamma^{\#} < +\infty$, then $\tilde{\chi}(\gamma^{\#}-, c_*) \neq 0$ implies that $\chi(\gamma^{\#}-, c_*) \neq 0$ and $\gamma_\phi(c_*) = \lambda_l(c_*) < \gamma^{\#}$. Since $\chi(z,c)$ is strictly increasing in c for each fixed $z > 0$, function $\lambda_l(c)$ is strictly decreasing. Hence $\gamma_\phi(c) = \lambda_l(c) < \gamma^{\#}$ for each $c \geq c_*$. Similar considerations shows that $\gamma_\phi(c) < \gamma^{\#}$ for each $c > c_*$ if $\chi(\gamma^{\#}-, c_*) = 0$. Finally, in the case $\gamma^{\#} = +\infty$ we have that $\chi(+\infty, c) \in \{1, -\infty\} \not\ni 0$, so that $\chi(\gamma_K-, c) \neq 0$ holds automatically. $\square$

Remark III.30. Our approach allows to remove several restrictions on J and g assumed in Carr and Chmaj uniqueness result [9, Theorem 2.1]. In the cited work g is supposed to satisfy (3.3) and J to be an even compactly supported function with $\int_{\mathbb{R}} Jds = 1$. These properties were essential in the proof of Theorem 2.1 in [9] even if (3.3) was not mentioned explicitly there. Similarly, conditions $J \in C^1(\mathbb{R})$, $J(a) > 0, J(b) > 0$ for some $a < 0 < b$, and of J compactly supported were used in [13]. Nevertheless, Coville *et al.* have used $g(u)/u \leq g'(0),\ u > 0$, instead of more restrictive $g'(u) \leq g'(0),\ u > 0$. They also established non-uniqueness of stationary traveling fronts ($c = 0$). Next, Schumacher [45], using completely different approach, established uniqueness of regular and non-critical semi-wavefronts to equation (3.23) for general J and g satisfying (3.25). In fact, it seems that the latter conditions was proposed in [45]. The trick allowing to weaken the Lipschitz restriction (3.3) is due to Thieme and Zhao [49] (up to our knowledge at least). However, usually it was applied under reversed inequality $f'(s) \geq f'(0)$ to the second (damping) term of equation, e.g. see also [19] and Section 6.3 for further generalizations. Here we show that this trick shows to be useful also in the case of birth functions.

3.6.2 Nonlocal lattice equations

Now we consider semi-wavefronts $w_j(t) = u(j + ct), u(-\infty) = 0$, of the nonlocal lattice equation

$$\frac{dw_j(t)}{dt} = D[w_{j+k}(t) - w_j(t)] - dw_j(t) + \sum_{k \in \mathbb{Z}} \beta(j-k) g(w_k(t-r)),\ j \in \mathbb{Z},$$

where $\beta(k) \geq 0$ with $\sum_{k\in\mathbb{Z}} \beta(k) = 1$. Let $\gamma^{\#}$ denote an extended positive real number such that $\sum_{k\in\mathbb{Z}} \beta(k) e^{-zk}$ is convergent when $z \in [0, \gamma^{\#})$ and is divergent when $z > \gamma^{\#}$. As it can be easily deduced from Theorem III.2, the existence of such $\gamma^{\#}$ is automatically assured by the existence of positive semi-wavefronts $w_j(t) = u(j + ct),\ u(-\infty) = 0$ to the above lattice equation. The wave profile u satisfies

$$(3.29)\quad cu'(x) = D[u(x+1) + u(x-1) - 2u(x)] - du(x) + \sum_{k\in\mathbb{Z}} \beta(k) g(u(x - k - cr)).$$

Again we take $c > 0$ for simplicity. Since u is bounded, from (3.29) we get

$$u(t) = \frac{1}{c}\int_{-\infty}^{t} e^{-\frac{2D+d}{c}(t-s)}\left[Du(s+1) + Du(s-1) + \sum_{k\in\mathbb{Z}}\beta(k)g(u(s-k-cr))\right] ds$$
$$= \frac{D}{c}\int_{-\infty}^{t+1} e^{-\frac{2D+d}{c}(t-s+1)}u(s)ds + \frac{D}{c}\int_{-\infty}^{t-1} e^{-\frac{2D+d}{c}(t-s-1)}u(s)ds +$$
$$+\sum_{k\in\mathbb{Z}}\frac{\beta(k)}{c}\int_{-\infty}^{t-k-cr} e^{-\frac{2D+d}{c}(t-s-k-cr)}g(u(s))ds.$$
$$= \frac{D}{c}e^{-\frac{2D+d}{c}}\int_{-1}^{+\infty} e^{-\frac{2D+d}{c}s}u(t-s)ds + \frac{D}{c}e^{\frac{2D+d}{c}}\int_{1}^{+\infty} e^{-\frac{2D+d}{c}s}u(t-s)ds$$
$$+\sum_{k\in\mathbb{Z}}\frac{\beta(k)}{c}e^{\frac{2D+d}{c}(k+cr)}\int_{k+cr}^{+\infty} e^{-\frac{2D+d}{c}s}g(u(t-s))ds$$

(3.30)

$$= (H_1 + H_2) * u(t) + \sum_{k\in\mathbb{Z}}\beta(k)H_3^k * g(u)(t),$$

where

$$H_1(t) = \begin{cases} \frac{D}{c}e^{-\frac{2D+d}{c}(t+1)}, & t \geq -1 \\ 0, & t < -1 \end{cases}, \quad H_2(t) = \begin{cases} \frac{D}{c}e^{-\frac{2D+d}{c}(t-1)}, & t \geq 1 \\ 0, & t < 1 \end{cases},$$

$$H_3^k(t) = \begin{cases} \frac{1}{c}e^{-\frac{2D+d}{c}(t-k-cr)}, & t \geq k+cr \\ 0, & t < k+cr \end{cases}.$$

Thus (3.30) can be written as (3.4), with $X = \{\tau_1, \tau_2\}$ and

$$K(s,\tau) = \begin{cases} H_1(s) + H_2(s), & \tau = \tau_1 \\ \sum_{k\in\mathbb{Z}}\beta(k)H_3^k, & \tau = \tau_2 \end{cases}, \quad g(s,\tau) = \begin{cases} s, & \tau = \tau_1 \\ g(s), & \tau = \tau_2 \end{cases}.$$

Next, $\chi(z,c) = 1 - \int_{\mathbb{R}} K(s,\tau_1)e^{-sz}ds - g'(0)\int_{\mathbb{R}} K(s,\tau_2)e^{-sz}ds =$

$$1 - \frac{2D\cosh(z)}{2D+d+cz} - \frac{g'(0)e^{-crz}}{2D+d+cz}\sum_{k\in\mathbb{Z}}\beta(k)e^{-kz} =: \frac{\tilde{\chi}(z,c)}{2D+d+cz}.$$

Let c_* be the minimal value of c for which

$$\tilde{\chi}(z,c) := d + 2D + cz - D(e^z + e^{-z}) - g'(0)e^{-crz}\sum_{k\in\mathbb{Z}}\beta(k)e^{-kz}$$

has at least one positive zero. It is easily seen that c_* is well defined and is finite. By Theorem III.9, $c \geq c_*$ for each admissible wave speed c.

After these transformations, we can apply our uniqueness results to (3.29).

Theorem III.31. *Suppose that g satisfies (3.3), (3.28) and $g'(0) > d$. Then equation (3.29) has at most one bounded positive solution u, $u(-\infty) = 0$, for each $c \neq 0$ (if $\tilde{\chi}(\gamma^{\#}-, c_*) \neq 0$) or for each $c \neq 0, c_*$ (if $\tilde{\chi}(\gamma^{\#}-, c_*) = 0$).*

Proof: Step I. Obviously, $g(\cdot, \tau)$ meets (3.3) with $g'(0, \tau_1) = 1$ and $g'(0, \tau_2) = g'(0)$. Moreover, we have $|g(u, \tau) - g'(0, \tau)u| \leq C(\tau)u^{1+\alpha}$, $u, v \in (0, \sigma)$, where $C(\tau_1) = 0$ and $C(\tau_2) = C$.

Step II. If $0 < z < \gamma_{\#}$, we get

$$\begin{aligned}
&\int_X \int_{\mathbb{R}} K(s,\tau)e^{-zs}ds = \int_{\mathbb{R}} (H_1(s) + H_2(s))e^{-zs}ds + \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \beta(k) H_3^k e^{-zs}ds \\
&= \frac{D}{c}\left(\int_{-1}^{+\infty} e^{-\frac{2D+d}{c}(s+1)-zs}ds + \int_1^{+\infty} e^{-\frac{2D+d}{c}(s-1)-zs}ds\right) \\
&\quad + \frac{1}{c}\sum_{k\in\mathbb{Z}} \beta(k) \int_{k+cr}^{+\infty} e^{-\frac{2D+d}{c}(s-k-cr)-zs}ds \\
&= \frac{D}{2D+d+cz}(e^z + e^{-z}) + \frac{e^{-cr}}{2D+d+cz}\sum_{k\in\mathbb{Z}} \beta(k)e^{-kz}.
\end{aligned}$$

Therefore $\gamma_K = \gamma_{\#}$ (if $c > 0$) and $\gamma_K = \min\{\gamma_{\#}, -(2D+d)/c\}$ (if $c < 0$).

Step III. If u solves (3.29) with $c > 0$, then for each $0 < z < \gamma_\phi$ we obtain

$$\begin{aligned}
&c\int_{\mathbb{R}} |u'(s)|e^{-zs}ds \leq D\int_{\mathbb{R}} |u(s+1) + u(s-1) - 2u(s)|e^{-zs}ds + d\int_{\mathbb{R}} u(s)e^{-zs}ds \\
&+ \sum_{k\in\mathbb{Z}} \beta(k) \int_{\mathbb{R}} g(u(s-k-cr))e^{-zs}ds \leq D\int_{\mathbb{R}} |u(s+1) + u(s-1) - 2u(s)|e^{-zs}ds + \\
&d\int_{\mathbb{R}} u(s)e^{-zs}ds + g'(0)\sum_{k\in\mathbb{Z}} \beta(k) \int_{\mathbb{R}} u(s-k-cr)e^{-zs}ds \\
&= D\int_{\mathbb{R}} |u(s+1) + u(s-1) - 2u(s)|e^{-zs}ds + d\int_{\mathbb{R}} u(s)e^{-zs}ds \\
&+ g'(0)e^{-zr}\sum_{k\in\mathbb{Z}} \beta(k)e^{-zk}\int_{\mathbb{R}} u(w)e^{-zw}dw < +\infty.
\end{aligned}$$

Thus, by Lemma III.16, condition III.15($\mathbf{EC}_{\gamma_\phi}$) is satisfied.

Step IV . We have $\chi(0,c) = (d - g'(0))/(2D + d) < 0$. The proof of $\gamma_\phi(c) < \gamma^{\#}$ is the same as in Step IV of the previous section and is omitted. □

Remark III.32. Our approach allows to improve the uniqueness results of [18, Theorem 3.1], where additional conditions $\beta(k) = \beta(-k)$ and $\chi(\gamma_K -) = -\infty$ are assumed. Moreover, [18, Theorem 3.1] does not establish the uniqueness of the minimal waves.

3.6.3 Nonlocal reaction-diffusion equation

Here, we consider positive semi-wavefronts solutions $u(t,x) = \phi(x+ct)$ satisfying $\phi(-\infty) = 0$, for non-local delayed reaction-diffusion equations

$$u_t(t,x) = u_{xx}(t,x) - f(u(t,x)) + \int_{\mathbb{R}} k(w)g(u(t-h, x-w))dw,\ h > 0 \tag{3.31}$$

where $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and non-negative and generally asymmetric $k \in L^1(\mathbb{R})$. The reader is referred to [51] for further details concerning wave solutions in the presence of asymmetric non-local interaction. Let $\gamma^{\#}$ denote an extended positive real number such that $\int_{\mathbb{R}} k(s)e^{-zs}ds$ is convergent when $z \in [0, \gamma^{\#})$ and is divergent when $z > \gamma^{\#}$. As it can be easily deduced from Theorem III.2, the existence of such $\gamma^{\#}$ is automatically assured by the existence of positive semi-wavefronts $u(t,x) = \phi(x + ct)$, $\phi(-\infty) = 0$ to (3.31). Is clear that the profile ϕ must satisfy

$$y''(t) - cy'(t) - f(y(t)) + \int_{\mathbb{R}} k(w)g\left(y(t - ch - w)\right)dw = 0,\ \ s \in \mathbb{R}. \tag{3.32}$$

Equation (3.32) can be written as

$$y''(t) - cy'(t) - \beta y(t) + f_\beta(y(t)) + \int_{\mathbb{R}} k(w)g(y(t - ch - w))dw = 0,\ t \in \mathbb{R},$$

where $f_\beta(s) = \beta s - f(s)$ for some $\beta > 0$.

Being ϕ a positive bounded function, it should satisfy the integral equation

$$\phi(t) = \frac{1}{\sigma(c)}\left(\int_{-\infty}^{t} e^{\nu(t-s)}\mathcal{G}(\phi(s-ch))ds + \int_{t}^{+\infty} e^{\mu(t-s)}\mathcal{G}(\phi(s-ch))ds\right), \tag{3.33}$$

where $\sigma(c) = \sqrt{c^2 + 4\beta}$, $\nu < 0 < \mu$ are the roots of $z^2 - cz - \beta = 0$ and

$$\mathcal{G}(\phi(t)) := \int_{\mathbb{R}} k(w)g(\phi(t-w))dw + f_\beta(\phi(t)) = k * g(\phi)(t) + f_\beta(\phi(t)).$$

Thus, we can to write (3.33) as

$$\phi(t) = (\mathcal{K} * k) * g(\phi)(t) + \mathcal{K} * f_\beta(\phi)(t), \tag{3.34}$$

where

$$\mathcal{K}(s) = \begin{cases} \frac{1}{\sigma(c)} e^{\nu(s-ch)}, & s \geq ch \\ \frac{1}{\sigma(c)} e^{\mu(s-ch)}, & s < ch \end{cases}.$$

Hence, we see that (3.31) can be written as (3.4), with $X = \{\tau_1, \tau_2\}$ and

$$K(s,\tau) = \begin{cases} (\mathcal{K} * k)(s), & \tau = \tau_1 \\ \mathcal{K}, & \tau = \tau_2 \end{cases}, \quad g(s,\tau) = \begin{cases} g(s), & \tau = \tau_1 \\ f_\beta(s), & \tau = \tau_2 \end{cases}.$$

Now, we have to check the assumptions of Theorem III.23. Here we are assuming that g satisfies (3.3).

Step I. Suppose that $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is strictly increasing and $f(0) = 0$. We claim that, without restricting the generality, we may assume that β is such that f_β satisfies the Lipschitz condition with a constant $\beta - \inf_{s\geq 0} f'(s)$. First observe that our proof of uniqueness compares two different solutions ϕ_1, ϕ_2. Since they are uniformly bounded by some positive $M > 0$, we can restrict our attention to a finite interval $[0, M]$ where f is obviously Lipschitzian. Now, since f is continuously differentiable on $[0, M]$ and $f(0) = 0$, we can choose $\beta > \inf_{s\geq 0} f'(s)$ such that $f_\beta(s) = \beta s - f(s) \geq 0$ for all $s \in [0, M]$ and

$$\max_{s\in[0,M]} f'(s) \leq 2\beta - \inf_{s\geq 0} f'(s).$$

Take $s_1 < s_2$ in $[0, M]$, then $f(s_2) - f(s_1) = f'(s_0)(s_2 - s_1)$ for some $s_0 \in [s_1, s_2]$.

Thus

$$\frac{f_\beta(s_2)-f_\beta(s_1)}{s_2-s_1} = \beta - \frac{f(s_2)-f(s_1)}{s_2-s_1} = \beta - f'(s_0) \le \beta - \inf_{s\ge 0} f'(s),$$
$$\frac{f_\beta(s_2)-f_\beta(s_1)}{s_2-s_1} \ge \beta - \Big(2\beta - \inf_{s\ge 0} f'(s)\Big) = -\beta + \inf_{s\ge 0} f'(s).$$

Therefore $\left|\frac{f_\beta(s_2)-f_\beta(s_1)}{s_2-s_1}\right| \le \Big(\beta - \inf_{s\ge 0} f'(s)\Big), \quad s_1, s_2 \in [0, M],$

so that we can assume that $g(\cdot,\tau)$ meets (3.3) with $g'(0,\tau_1) = g'(0)$ and $g'(0,\tau_2) = \beta - \inf_{s\ge 0} f'(s)$. Note here that if $f'(0) \le f'(v)$ for all $v \ge 0$, as in [49], then $\beta - \inf_{s\ge 0} f'(s) = \beta - f'(0)$.

Step II. Now, we suppose that $g, f \in C^{1,\alpha}$ in some neighborhood of 0. Since $|f'_\beta(0) - f'_\beta(u)| = |f'(0) - f'(u)|$, we see that $|g'(u,\tau) - g'(0,\tau)u)| \le C(\tau)u^\alpha,\ u \in (0,\sigma)$, for same $C(\tau) > 0$ and σ small.

Step III. Note that if $\nu < z < \mu$, then

$$\int_{\mathbb{R}} \mathcal{K}(w)e^{-zw}dw = \frac{-1}{z^2 - cz - \beta} < \infty, \int_{\mathbb{R}} \mathcal{K} * k_h(w)e^{-zw}dw =$$
$$\int_{\mathbb{R}} \mathcal{K}(u)e^{-zu}du \int_{\mathbb{R}} k_h(s)e^{-zs}ds = \frac{-e^{-zch}}{z^2 - cz - \beta}\int_{\mathbb{R}} k(s)e^{-zs}ds < +\infty.$$

Thus, $\gamma_K = \min\{\mu, \gamma^{\#}\}$ so that $\gamma_\phi < \mu$.

Observe that

$$\chi_1(z,c) = 1 - g'(0)\int_{\mathbb{R}} K(s,\tau_1)e^{-sz}ds - (\beta - \inf_{s\ge 0} f'(s))\int_{\mathbb{R}} K(s,\tau_2)e^{-sz}ds =$$
$$1 - \frac{\beta - \inf_{s\ge 0} f'(s)}{\beta + cz - z^2} - \frac{g'(0)e^{-zch}}{\beta + cz - z^2}\int_{\mathbb{R}} k(s)e^{-zs}ds =: \frac{\tilde{\chi}_1(z)}{\beta + cz - z^2}.$$

We see that $\gamma_K = \min\{\mu, \gamma^{\#}\}$ so that $\gamma_\phi < \mu$. Let $c_\star$ be the minimal value of c for which

$$\tilde{\chi}_1(z,c) := cz - z^2 + \inf_{s\ge 0} f'(s) - g'(0)e^{-zch}\int_{\mathbb{R}} k(s)e^{-zs}ds$$

has at least one positive zero. This value is finite, well defined and does not depend on β. We will write c_* instead of $c_\star$ in the special case when $f'(0) \le f'(v)$ for all

$v \geq 0$. In such a case, we have $f'(0) = \inf_{s\geq 0} f'(s)$ and therefore $\chi_1 = \chi$. By Theorem III.9, $c \geq c_*$ for each admissible wave speed c.

Step IV. So we will take some $x \in (0, \gamma_\phi)$ and $\phi_1, \phi_2 \in C^1(\mathbb{R})$ satisfy (3.32). Let W be given by $W := |\phi_1 - \phi_2|$ and suppose that the integral $\int_{\mathbb{R}} e^{-xu} W(u)du$ converges. Then, from (3.33) we have

$$|\phi_1'(u) - \phi_2'(u)| \leq \sigma(c)^{-1}\Big(|\nu| \int_{-\infty}^{u} e^{\nu(u-s)}|(\mathcal{G}\phi_1)(s) - (\mathcal{G}\phi_2)(s)|ds$$

$$+ \mu \int_{u}^{+\infty} e^{\mu(u-s)}|(\mathcal{G}\phi_1)(s) - (\mathcal{G}\phi_2)(s)|ds\Big)$$

$$\leq \sigma(c)^{-1}\left(|\nu| \int_{0}^{+\infty} e^{\nu m} F(u-m)dm + \mu \int_{-\infty}^{0} e^{\mu m} F(u-m)dm\right),$$

where $F(s) = \Big(g'(0)\int_{\mathbb{R}} k_h(w)W(s-w)dw + (\beta - \inf_{s\geq 0} f'(s))W(s)\Big)$. In consequence, $\int_{\mathbb{R}} e^{-xu}|\phi_1'(u) - \phi_2'(u)|du \leq$

$$\frac{-2\beta - cx}{\sigma(c)(x^2 - cx - \beta)} \int_{\mathbb{R}} e^{-xu} W(u)du\Big(g'(0)\int_{\mathbb{R}} k_h(w)e^{-xw}dw + \beta - \inf_{s\geq 0} f'(s)\Big)$$

is finite and $\int_{\mathbb{R}} e^{-xs}(\phi_1'(s) - \phi_2'(s))ds$ converges absolutely for each $x \in (0, \gamma_\phi)$.

We are ready to state the main result of this subsection.

Theorem III.33. *Suppose g satisfies (3.3), $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is strictly increasing, and $g, f \in C^{1,\alpha}$ in some neighborhood of 0, and $g(0) = f(0) = 0$, $g'(0) > f'(0)$. Then equation (3.31) has at most one positive semi-wavefront $u(t,x) = \phi(x+ct)$, $\phi(-\infty) = 0$, for each $c \geq c_\star$ (if $\tilde{\chi}(\gamma^\#-, c_\star) \neq 0$) or for each $c > c_\star$ (if $\tilde{\chi}(\gamma^\#-, c_\star) = 0$).*

Proof: Observe that $\beta\chi(0,c) = f'(0) - g'(0) < 0$, and $\chi_1(\gamma^\#-, c_\star) \neq 0$ if $\tilde{\chi}_1(\gamma^\#-, c_\star) \neq 0$. *First let $c \geq c_\star > c_*$,* then $\chi_1(x,c) < \chi(x,c)$ so that $\chi_1(m,c) = 0$ for some $m \in (0, \lambda_{rK}]$. It is clear that $m = \lambda_{rK}$ if and only if $m = \gamma^\#$. Since $\chi_1(z,c)$ is strictly increasing in c for each fixed positive z, this implies that $c = c_\star$

and $\chi_1(\gamma^{\#}-, c_\star) = 0$. Consequently, $m \in (0, \lambda_{rK})$ for each $c \geq c_\star$ (if $\tilde{\chi}(\gamma^{\#}-, c_\star) \neq 0$) or for each $c > c_\star$ (if $\tilde{\chi}(\gamma^{\#}-, c_\star) = 0$).

Next, if $c_\star = c_*$ then $\chi_1 = \chi$ and the inequality $\chi(\gamma^{\#}-, c_*) \neq 0$ guarantees that $\lambda_l(c_*) = \gamma_\phi(c_*) < \gamma^{\#}$ for $c = c_*$. If $c > c_*$ then we have again $\lambda_l(c) = \gamma_\phi(c) < \lambda_l(c_*) < \gamma^{\#}$ because $\lambda_l(c)$ is monotone decreasing in c.

Next, we claim that for each $x \in (0, \gamma_K)$ and some $d_j(x)$ it holds

$$K(s, \tau_j) \leq d_j(x) e^{xs},\ s \in \mathbb{R}.$$

Indeed, since $\gamma_K \leq \mu$ and $\mathcal{K}(s) \leq \dfrac{e^{xs}}{\sigma(c)}$, $s \in \mathbb{R}$, for all $0 < x \leq \mu$, we get that

$$\begin{aligned} K(t, \tau_1) &= \int_{-\infty}^{+\infty} \mathcal{K}(s) k_h(t-s) ds \leq \frac{1}{\sigma(c)} \int_{-\infty}^{+\infty} e^{xs} k(t-s-ch) ds \\ &\leq \frac{e^{-chx}}{\sigma(c)} \Big[\int_{-\infty}^{+\infty} e^{-xu} k(u) du \Big] e^{xt} \end{aligned}$$

Since $\lambda_{rK} \leq \gamma_K = \min\{\gamma^{\#}, \mu\}$, the exponential estimations of K in III.25 and III.27(ii) are verified. This observation completes the proof of the theorem. □

Remark III.34. Our approach allows to improve [49, Theorem 4.3], where the uniqueness was established under assumption that either $f(s) = \beta u$ or $g(s) = \beta u$ and K is the Gaussian kernel. Moreover, [49, Theorem 4.3] does not consider the minimal waves.

3.6.4 Uniqueness of fast traveling fronts in delayed reaction-diffusion equations

Finally, we consider positive semi-wavefronts $u(t,x) = \phi(x+ct), \phi(\infty) = 0$, to

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h,x)),\ x \in \mathbb{R},$$

where $g \in C^{1,\alpha}([0,\sigma])$ is a Lipschitzian function with constant L which is greater than $g'(0)$. Profile ϕ must satisfy the delay differential equation

$$\phi''(t) - c\phi'(t) - \phi(t) + g(\phi(t-hc)) = 0, \quad t \in \mathbb{R}. \tag{3.35}$$

Similarly to Section 3.6.3, we find that ϕ satisfies

$$\phi(t) = \mathcal{K} * g(\phi)(t), \;\; \mathcal{K}(s) = \begin{cases} \frac{1}{\sigma(c)} e^{\nu(s-ch)}, & s \geq ch \\ \frac{1}{\sigma(c)} e^{\mu(s-ch)}, & s < ch \end{cases},$$

which is exactly the form considered in DK theory (formally, we set $X = \{\tau\}$, $K(s,\tau) = \mathcal{K}$ and $g(s,\tau) = g(s)$). Nevertheless, since $L > g'(0)$, Diekmann-Kaper uniqueness theorem does not apply to (3.35).

In order to use Theorem III.28, we realize some elementary computations. First, note that

$$\chi_1(z,c) = 1 - L \int_{\mathbb{R}} \mathcal{K}(s) e^{-sz} ds = 1 - \frac{Le^{-zch}}{1 + cz - z^2}.$$

is defined on (ν, μ). Thus, $\gamma_K = \mu$ and since $\lim_{z \to \mu -} \int_{\mathbb{R}} \mathcal{K}(s) e^{-sz} ds = +\infty$ we obtain $\gamma_\phi < \gamma_K$. Note also that $\chi(0,c) = 1 - g'(0)$ and the exponential estimations of K in III.25, III.27(ii) are also obviously verified. Hence, we only need to verify the Assumptions III.27(i) and $\chi_1(m,c) \geq 0$ for some $m > 0$.

Step I. Assume that $\varphi_1, \varphi_2 \in C^1(\mathbb{R})$ satisfy (3.35) and, in addition, that the integral $\int_{\mathbb{R}} e^{-zs}(\varphi_2(s) - \varphi_1(s))ds$ converges absolutely. Then, for each $\nu < z < \gamma_\phi$,

$$\int_{\mathbb{R}} e^{-zs} |\varphi_2'(s) - \varphi_1'(s)| ds \leq \frac{Le^{-zch}}{\sigma(c)} \left(|\nu| \int_0^{+\infty} e^{(\nu - z)u} du \right.$$
$$\left. + \mu \int_{-\infty}^0 e^{(\mu - z)u} du \right) \int_{\mathbb{R}} e^{-zs} |\varphi_2(s) - \varphi_1(s))| ds < \infty.$$

Step II. Finally, define $c_\star$ as as the minimal value of c for which the equation $z^2 - cz - 1 + Le^{-chz} = 0$ has at least one positive root. This value is well defined and positive. It is easy to see that, for each $c > c_\star$ there exists $m > 0$ close to λ_l from the right and such that $\chi_1(m,c) > 0$.

We are ready present our result concerning the uniqueness in (3.35):

Theorem III.35. *Suppose that* $|g(s) - g(t)| \leq L|t - s|$, $s, t \geq 0$ *and that* $g \in C^{1,\alpha}$

in some neighborhood of 0 *with* $g'(0+) > 1$. *Then, for every* $c > c_\star$ *equation (3.35) has at most one bounded positive solution* ϕ *vanishing at* $-\infty$.

Remark III.36. Theorem III.35 gives another proof of the uniqueness result in [4, Theorem 1.1] where was additionally assumed that $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and that $g''(0+)$ in finite. Moreover, we give here a reasonably good lower bound $c_\star$ for the 'uniqueness' speeds. Observe that if $L = g'(0)$, then $c_\star$ coincides with the minimal speed of propagation c_*.

CHAPTER IV

Existence and uniqueness of fast travelling fronts in reaction-diffusion equation with local delay

4.1 Introduction

In this chapter, we consider the time-delayed reaction-diffusion equation (1.4). We will suppose that $-s + g(s)$ is of the monostable type. Thus equation (1.4) has exactly two non-negative equilibria $u_1 \equiv 0,\ u_2 \equiv \kappa > 0$. If $u(x,t) = \phi(x+ct)$ is a wavefront (or a travelling front) of (1.4), then after scaling such a profile ϕ is a positive heteroclinic solution of the delay differential equation

$$\varepsilon^2\phi''(t) - \phi'(t) - \phi(t) + g(\phi(t-h)) = 0,\ \varepsilon := 1/c > 0, \quad t \in \mathbb{R}. \tag{4.1}$$

In this chapter, we follow the approach of [20] to prove the uniqueness (up to translations) of positive wavefront for a given fast speed c. In the case of (1.4), this approach essentially relies on the fact that, in 'good' spaces and with suitable $g'(0)$, $g'(\kappa)$, the linear operator $(\mathcal{L}y)(t) = y'(t) + y(t) - g'(\psi(t-h))y(t-h)$ is a surjective Fredholm operator. Here ψ is a heteroclinic solution of equation (4.1) considered with $\varepsilon = 0$. In consequence, the Lyapunov-Schmidt reduction can be used to prove the existence of a smooth family of travelling fronts in some neighborhood of ψ. As it was shown in [21] this family contains positive solutions as well. However, an important and natural question about the number of the positive wavefronts has not been answered in the past.

4.2 Spaces and Operators

In this section we assume I.4 and I.5. Let ψ be the positive heteroclinic solution from Lemma II.9.

Notation IV.1. For a fixed $\mu \geq 0$, we set $\#\{\lambda_j : \mu < \Re\lambda_j\} := d(\mu)$, where λ_j is a root of equation (2.2).

Notation IV.2. Let $y \in C(\mathbb{R}, \mathbb{R})$. For a fixed $\mu \geq 0$, we will consider the seminorms $\|y\|^+ = \sup_{\mathbb{R}_+} |y(s)|$ and $\|y\|_\mu^- = \sup_{\mathbb{R}_-} e^{-\mu s}|y(s)|$, and the following Banach space

$$C_\mu(\mathbb{R}) = \{y \in C(\mathbb{R}, \mathbb{R}) : \|y\|_\mu^- < \infty,\ y(-\infty) = 0, \quad \text{and } y(+\infty) \text{ is finite}\},$$

equipped with the norm $|y|_\mu = \max\{\|y\|^+, \|y\|_\mu^-\}$.

Definition IV.3. We define the following operators:

1. The integral operator $\mathcal{N} : C_\mu(\mathbb{R}) \to C_\mu(\mathbb{R})$, such that

   $$(\mathcal{N}y)(t) = \int_{-\infty}^{t} e^{-(t-s)} q(s) y(s-h) ds,$$

   where $q(s) := g'(\psi(s-h))$.

2. The Nemytskii operator $\mathcal{G} : C_\mu(\mathbb{R}) \to C_\mu(\mathbb{R})$, where $(\mathcal{G}y)(t) = g(y(t))$.

3. The integral operators $\mathcal{I}, \mathcal{I}_\varepsilon, \mathcal{I}_\varepsilon^+, \mathcal{I}_\varepsilon^- : C_\mu(\mathbb{R}) \to C_\mu(\mathbb{R})$, where $\mathcal{I} = \mathcal{I}_0^-, \mathcal{I}_0^+ = 0$, $\mathcal{I}_\varepsilon = \sigma^{-1}(\varepsilon)(\mathcal{I}_\varepsilon^+ + \mathcal{I}_\varepsilon^-),\ \sigma(\varepsilon) := \sqrt{1 + 4\varepsilon^2}$, and

   $$(\mathcal{I}_\varepsilon^+ y)(t) = \int_t^{+\infty} e^{\frac{(1+\sigma(\varepsilon))(t-s)}{2\varepsilon^2}} y(s-h) ds,\ (\mathcal{I}_\varepsilon^- y)(t) = \int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} y(s-h) ds.$$

Observation IV.4. Since $g'(t) = p + O(t),\ t \to 0$, and $\psi(t) = O(\exp(\lambda t)),\ t \to -\infty$, we obtain that

$$q(t) = p + \epsilon(t),\ \epsilon(t) = O(\exp(\lambda t)),\ t \to -\infty; \text{ and } q(-\infty) = p > 1,\ q(\infty) = g'(\kappa).$$

Observe that $\mathcal{I}_\varepsilon^{\pm}, \mathcal{N}$ are well defined: e.g. $(\mathcal{N}y)(+\infty) = g'(\kappa)y(+\infty)$ and, for $t \le h$,

$$|(\mathcal{N}y)(t)| \le \int_{-\infty}^{t} e^{-(t-s)}|q(s)|\|y\|_\mu^- e^{\mu(s-h)}ds \le \frac{\|y\|_\mu^- \sup_{t\le h}|q(t)|}{1+\mu}e^{\mu(t-h)}.$$

Lemma IV.5. *Operator families* $\mathcal{I}_\varepsilon^{\pm} : (-1/\sqrt{\mu}, 1/\sqrt{\mu}) \to \mathcal{L}(C_\mu(\mathbb{R})),\ \mu \ge 0,$ *are continuous in the operator norm. In particular,* $\mathcal{I}_\varepsilon \to \mathcal{I}$ *as* $\varepsilon \to 0$.

Proof: The proof of lemma will be divided into three steps. Step I. We first establish that $\|\mathcal{I}_\varepsilon^- - \mathcal{I}_{\varepsilon_0}^-\| \to 0$ as $\varepsilon \to \varepsilon_0 \ne 0$. Fix $y \in C_\mu(\mathbb{R})$, then $|y(t)| \le \|y\|e^{\nu t}$ and $\nu = 0, \mu$, for all $t \in \mathbb{R}$. Hence,

$$\begin{aligned}
|(\mathcal{I}_\varepsilon^- y - \mathcal{I}_{\varepsilon_0}^- y)(t)| &\le \int_{-\infty}^{t} \left| e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} - e^{\frac{-2(t-s)}{1+\sigma(\varepsilon_0)}} \right| |y(s-h)|ds \\
\le \int_{-\infty}^{t} \left| e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} - e^{\frac{-2(t-s)}{1+\sigma(\varepsilon_0)}} \right| \|y\|e^{\nu(s-h)}ds &\le \frac{1}{2}\|y\|e^{\nu(t-h)}|\sigma(\varepsilon) - \sigma(\varepsilon_0)|, \nu = 0, \mu.
\end{aligned}$$

As a consequence, $\|\mathcal{I}_\varepsilon^- - \mathcal{I}_{\varepsilon_0}^-\| \le 0.5|\sigma(\varepsilon) - \sigma(\varepsilon_0)|$.

Step II. Now, we prove that $\mathcal{I}_\varepsilon^+ \to \mathcal{I}_{\varepsilon_0}^+$ uniformly as $\varepsilon \to \varepsilon_0 \ne 0$. An easy computation shows that for each $y \in C_\mu(\mathbb{R})$ and $t \in \mathbb{R}$

$$|(\mathcal{I}_\varepsilon^+ y - \mathcal{I}_{\varepsilon_0}^+ y)(t)| \le \|y\| \int_t^{+\infty} \left| e^{\frac{(1+\sigma(\varepsilon))(t-s)}{2\varepsilon^2}} - e^{\frac{(1+\sigma(\varepsilon_0))(t-s)}{2\varepsilon_0^2}} \right| ds \le 2\|y\||\varepsilon^2 - \varepsilon_0^2|.$$

Next, if $|\varepsilon|, |\varepsilon_0| < \sqrt{\frac{1}{2\mu}}$, then

$$\begin{aligned}
|(\mathcal{I}_\varepsilon^+ y - \mathcal{I}_{\varepsilon_0}^+ y)(t)| &\le \|y\| \int_t^{+\infty} \left| e^{\frac{(1+\sigma(\varepsilon))(t-s)}{2\varepsilon^2}} - e^{\frac{(1+\sigma(\varepsilon_0))(t-s)}{2\varepsilon_0^2}} \right| e^{\mu(s-h)}ds \\
&\le 4\|y\|e^{\mu(t-h)}|\varepsilon^2 - \varepsilon_0^2|.
\end{aligned}$$

Consequently, we get $\|\mathcal{I}_\varepsilon^+ - \mathcal{I}_{\varepsilon_0}^+\| \le 4|\varepsilon^2 - \varepsilon_0^2|$.

Step III. Finally, using the similar arguments, we can see that $\mathcal{I}_\varepsilon^+ \to 0$ and $\mathcal{I}_\varepsilon^- \to \mathcal{I}$ uniformly as $\varepsilon \to 0$. In consequence, $\|\mathcal{I}_\varepsilon - \mathcal{I}\| \to 0$ as $\varepsilon \to 0$. $\square$

Lemma IV.6. *If I.4 holds and* $\mu \notin \{\Re\lambda_j\},\ \mu \ge 0,$ *then* $I - \mathcal{N} : C_\mu(\mathbb{R}) \to C_\mu(\mathbb{R})$ *is a surjective Fredholm operator and* dim Ker $(I - \mathcal{N}) = d(\mu)$.

Proof: First, we establish that $I - \mathcal{N}$ is an epimorphism. Take some $f \in C_\mu(\mathbb{R})$ and consider the following integral equation

$$y(t) - \int_{-\infty}^{t} e^{-(t-s)} q(s) y(s-h) ds = f(t).$$

If we set $z(t) = y(t) - f(t)$, this equation is transformed into

$$z(t) - \int_{-\infty}^{t} e^{-(t-s)} q(s)(z(s-h) + f(s-h)) ds = 0.$$

Hence, in order to establish the surjectivity of $I - \mathcal{N}$, it suffices to prove the existence of $C_\mu(\mathbb{R})$-solution of the equation

$$z'(t) = -z(t) + q(t)z(t-h) + q(t)f(t-h). \tag{4.2}$$

First, notice that all solutions of (4.2) are bounded on the positive semi-axis $\mathbb{R}_+$ due to the boundedness of $q(t)f(t-h)$ and the exponential stability of the homogeneous ω-limit equation $z'(t) = -z(t) + g'(\kappa)z(t-h)$. Here we use the persistence of exponential stability under small bounded perturbations (e.g. see [11, Section 5.2]) and the fact that $q(+\infty) = g'(\kappa)$. Furthermore, since every solution z of (4.2) satisfies $z'(t) = -z(t) + g'(\kappa)z(t-h) + g'(\kappa)f(+\infty) + \epsilon(t)$ with $\epsilon(+\infty) = 0$, we get $z(+\infty) = f(+\infty)g'(\kappa)(1 - g'(\kappa))^{-1}$. Next, by effecting the change of variables $z(t) = \exp(\mu t)v(t)$ to equation (4.2), we get a linear inhomogeneous equation of the form

$$v'(t) = -(1+\mu)v(t) + [p\exp(-\mu h) + \epsilon_1(t)]v(t-h) + \epsilon_{2,\mu}(t), \tag{4.3}$$

where $\epsilon_1(-\infty) = \epsilon_{2,0}(-\infty) = 0$ and $\epsilon_{2,\mu}(t) = O(1)$, $\mu > 0$, at $t = -\infty$. Since the α-limit equation $v'(t) = -(1+\mu)v(t) + p\exp(-\mu h)v(t-h), \mu \notin \{\Re\lambda_j\}$, to the homogeneous part of (4.3) is hyperbolic, due to the above mentioned persistence of the property of exponential dichotomy, we again conclude that equation (4.3) also has an exponential dichotomy on $\mathbb{R}_-$. Thus (4.3) has a solution v_μ^* which is bounded on $\mathbb{R}_-$ (while $v_0^*(-\infty) = 0$) so that $z^*(t) = \exp(\mu t) v_\mu^*(t) = O(\exp(\mu t)),\ t \to -\infty$, is a $C_\mu(\mathbb{R})$-solution of equation (4.2).

Next we prove that dim $\mathrm{Ker}(I - \mathcal{N}) = \#\{\lambda_j : \mu < \Re\lambda_j\}$. It is clear that $\phi_j \in \mathrm{Ker}(I - \mathcal{N})$ if and only if ϕ_j is a $C_\mu(\mathbb{R})-$solution of the equation

$$y'(t) = -y(t) + q(t)y(t-h). \tag{4.4}$$

We already have seen that every solution of (4.4) satisfies $y(+\infty) = 0$, thus we only have to show that there exist solutions ϕ_j with $\|\phi_j\|_\mu^- < \infty$. In fact, we will prove that for each $\Re\lambda_j > \mu$ and $\delta \in (0, \min_{\Re\lambda_j > 0,\ \lambda > \Re\lambda_i > 0}\{\Re\lambda_j, \lambda - \Re\lambda_i\})$ there is $\phi_j(t) = e^{\lambda_j t} + e^{\sigma t} v_j(t) \in \mathrm{Ker}(I - \mathcal{N})$, with $\sigma = \lambda + \delta,\ v_j(t) = O(1),\ t \to -\infty$. Set $q(t) = p + \epsilon(t)$, then $v_j(t)$ can be chosen as a bounded solution of the equation

$$y'(t) + (1+\sigma)y(t) - (p + \epsilon(t))e^{-\sigma h} y(t-h) = e^{-\lambda_j h + (\lambda_j - \sigma)t}\epsilon(t). \tag{4.5}$$

Since $e^{-\lambda_j h + (\lambda_j - \sigma)t}\epsilon(t) = O(e^{(\Re\lambda_j - \delta)t})$ at $-\infty$, we get the following α-limit form of (4.5)

$$y'(t) + (1+\sigma)y(t) - pe^{-\sigma h} y(t-h) = 0.$$

This autonomous equation is exponentially stable since its characteristic equation

$$z + \lambda + \delta = -1 + pe^{-(z+\lambda+\delta)h}$$

has roots $z_j = \lambda_j - \lambda - \delta$ with $\Re z_j = \Re\lambda_j - \lambda - \delta < 0$. Thus (4.5) has a unique solution v_j bounded in $\mathbb{R}_-$. Is clear that $d(\mu)$ solutions $\{\phi_j\}$ are linearly independent, we claim that, in fact, system $\{\phi_j\}$ generates $\mathrm{Ker}(I - \mathcal{N})$. By way of contradiction, suppose that $\varphi \in \mathrm{Ker}(I - \mathcal{N}) - < \phi_j >$.

As φ solves the equation

$$y'(t) = -y(t) + py(t-h) + O(\exp((\lambda + \mu)t)),\ t \to -\infty,$$

we get (e.g. see [41, Proposition 7.1])

$$\varphi(t) = z(t) + O(\exp((\lambda + \mu - \delta)t)),\ t \to -\infty,$$

where $z(t)$ is the eigensolution corresponding to the eigenvalues ζ with $\mu \le \Re\zeta < \lambda + \mu$. In this way,

$$\varphi(t) = C\exp(\lambda t) + \sum_{j=2}^{d(\mu)} C_j \exp(\lambda_j t) + O(\exp((\lambda + \mu - \delta)t)),\ t \to -\infty. \tag{4.6}$$

Now take

$$w(t) = C(\exp(\lambda t) + \exp(\sigma t)v_1(t)) + \sum_{j=2}^{d(\mu)} C_j(\exp(\lambda_j t) + \exp(\sigma t)v_j(t)) \in < \phi_j > .$$

Since $\exp(\sigma t)v_j(t) = O(\exp(\lambda + \delta)t),\ t \to -\infty$, we can write

$$w(t) = C\exp(\lambda t) + \sum_{j=2}^{d(\mu)} C_j \exp(\lambda_j t) + O(\exp((\lambda + \delta)t)),\ t \to -\infty.$$

Thus $r(t) := \varphi(t) - w(t)$ satisfies $r(t) = O(\exp(\lambda - \delta)t),\ t \to -\infty$, and solves

$$y'(t) = -y(t) + py(t-h) + O(\exp((2\lambda - \delta)t)),\ t \to -\infty. \tag{4.7}$$

Applying Proposition 7.1 from [41] we conclude that

$$r(t) = z(t) + O(\exp((2\lambda - \delta - \delta/2)t)),\ t \to -\infty,$$

where $z(t)$ is the eigensolution corresponding to the eigenvalues ζ such that $\lambda - \delta \le \Re\zeta < 2\lambda - \delta$ and in consequence $z(t) = C_1 e^{\lambda t}$, for some C_1. Hence,

$$\varphi(t) = w(t) + r(t) = C'\exp(\lambda t) + \sum_{j=2}^{d(\mu)} C_j \exp(\lambda_j t) + O(\exp((\lambda + \delta)t)),\ t \to -\infty,$$

for small $\delta > 0$. The latter formula improves (4.6), and if we take

$$w_1(t) = C'(\exp(\lambda t) + \exp(\sigma t)v_1(t)) + \sum_{j=2}^{d(\mu)} C_j(\exp(\lambda_j t) + \exp(\sigma t)v_j(t)) \in < \phi_j >,$$

then $r_1(t) = \varphi(t) - w_1(t) = O(\exp{(\lambda+\delta)t}),\ t \to -\infty$. Since $r_1(t)$ satisfies

$$y'(t) = -y(t) + py(t-h) + O(\exp{((2\lambda+\delta)t)}),\ t \to -\infty,$$

we can proceed as before to get $r_1(t) = z_1(t) + O(\exp{(2\lambda+\delta-\delta/2)t}),\ t \to -\infty$, where $z_1(t)$ is the eigensolution corresponding to the eigenvalues ζ such that $\lambda+\delta \le \zeta < 2\lambda+\delta$. Thus $z_1(t) = 0$ and $r_1(t) = O(\exp{(2\lambda+\delta-\delta/2)t}),\ t \to -\infty$. Iterating this procedure (and subtracting $\delta/2^k$ from the exponent $2\lambda+\delta$ on the step k), we can conclude that $r_1(t) = O(\exp{(k\lambda t)}),\ t \to -\infty,\ k \ge 2$. This means that r is a small solution of (4.4). However, equation (4.4) cannot have solutions with superexponential decay at $-\infty$ and thus $r(t) = 0$. This implies that $\varphi \in < \phi_j >$, a contradiction. $\square$

Throughout the rest of the chapter, we will suppose that the C^1-smooth function g is defined and bounded on the whole real axis $\mathbb{R}$. This assumption does not restrict the generality of our framework, since it suffices to take any smooth and bounded extension on $\mathbb{R}_-$ of the nonlinearity g described in (I.5). Notice that, since there exists finite $g'(0)$, we have $g(s) = s\gamma(s)$ for a bounded $\gamma \in C(\mathbb{R})$. Set $\gamma_0 = \sup_{s\in\mathbb{R}} |\gamma(s)|$. As it can be easily checked, $|\mathcal{G}y|_\mu \le \gamma_0|y|_\mu$ so that actually $\mathcal{G}$ is well-defined. Furthermore, we have the following lemma:

Lemma IV.7. *Assume that $g \in C^1(\mathbb{R})$. Then $\mathcal{G}$ is Fréchet continuously differentiable on $C_\mu(\mathbb{R})$ with differential $\mathcal{G}'(y_0) : y(\cdot) \to g'(y_0(\cdot))y(\cdot)$.*

Proof: We have that $|\mathcal{G}'(y)u|_\mu = |g'(y(\cdot))u(\cdot)|_\mu \le \sup_{s\in\mathbb{R}} |g'(y(s))||u|_\mu$. By the Taylor formula, $g(v) - g(v_0) - g'(v_0)(v-v_0) = (g'(\theta) - g'(v_0))(v-v_0),\ \ \theta \in [v, v_0]$. Fix some $y_0 \in C_\mu(\mathbb{R})$. Since functions in $C_\mu(\mathbb{R})$ are bounded and g' is uniformly continuous on bounded sets of $\mathbb{R}$, for any given $\delta > 0$ there is $\sigma > 0$ such that for $|y - y_0|_\mu < \sigma$ we have that $|\mathcal{G}y - \mathcal{G}y_0 - g'(y_0(\cdot))(y-y_0)|_\mu \le \delta|y-y_0|_\mu$ and

$\|\mathcal{G}'(y) - \mathcal{G}'(y_0)\|_{\mathcal{L}(C_\mu(\mathbb{R}))} < \delta.$ $\Box$

4.3 Lyapunov-Schmidt reduction

Being a bounded solution of equation (4.1), each travelling wave should satisfy

$$\phi(t) = \frac{1}{\sigma(\varepsilon)}\left(\int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} g(\phi(s-h))ds + \int_{t}^{+\infty} e^{\frac{(1+\sigma(\varepsilon))(t-s)}{2\varepsilon^2}} g(\phi(s-h))ds\right), \tag{4.8}$$

For $C_\mu(\mathbb{R})$-solutions, this equation takes the form $\phi = (\mathcal{I}_\varepsilon \circ \mathcal{G})\phi$.

Theorem IV.8. *Assume I.5, I.4. Let ψ be the positive heteroclinic from Lemma II.6. Then for every $\mu \neq \Re\lambda_j,\ \mu \in [0, \lambda)$, there are open balls $\mathcal{E}_\mu = (-\varepsilon_\mu, \varepsilon_\mu)$, $\mathcal{V}_\mu \subset \mathbb{R}^{d(\mu)}$, and continuous family of heteroclinics $\psi_{\varepsilon,v} : \mathcal{E}_\mu \times \mathcal{V}_\mu \to C_\mu(\mathbb{R})$ of equation (4.1) such that $\psi_{0,0} = \psi$. For each $\tilde{\varepsilon} \in \mathcal{E}_\mu$, the subset $\{\psi_{\tilde{\varepsilon},v} : v \in \mathcal{V}_\mu\} \subset C_\mu(\mathbb{R})$ is a C^1-manifold of dimension $d(\mu)$. Moreover, there exists a $C_\mu(\mathbb{R})-$neighborhood $\mathcal{U}$ of ψ and $\varepsilon_1 > 0$ such that every solution $\psi_\varepsilon \in \mathcal{U}$, $|\varepsilon| < \varepsilon_1$, of equation (4.1) satisfies $\psi_\varepsilon = \psi_{\varepsilon,v}$ for some $v \in \mathcal{V}_\mu$. Finally, given a closed subinterval $\mathcal{S} \subset [0, \lambda) \setminus \{\Re\lambda_j\}$, we can choose open sets $\mathcal{E}_\mu, \mathcal{V}_\mu$ to be constant on $\mathcal{S}$.*

Proof: Set $R_\mu = (-1/\sqrt{\mu}, 1/\sqrt{\mu})$ and then define $F : R_\mu \times C_\mu(\mathbb{R}) \to C_\mu(\mathbb{R})$ by $F(\varepsilon, \phi) = \psi + \phi - (\mathcal{I}_\varepsilon \circ \mathcal{G})(\psi + \phi)$. We have that $F(0, 0) = 0$. Furthermore, Lemmas IV.5 and IV.7 imply that $F \in C(R_\mu \times C_\mu(\mathbb{R}), C_\mu(\mathbb{R}))$ and $F_\phi(\varepsilon, \phi)$ is continuous in a neighborhood of $(0, 0)$. Set

$$L := F_\phi(0, 0) = I - \mathcal{N},\ V := \mathrm{Ker}L,\ r(\varepsilon, \phi) := F(\varepsilon, \phi) - L\phi.$$

Then $r_\phi(0, 0) = F_\phi(0, 0) - L = 0$. By Lemma IV.6, we have that $\dim V < \infty$ and that L is surjective. Thus V has a topological complement W in $C_\mu(\mathbb{R})$ so that $C_\mu(\mathbb{R}) = V \oplus W$ and any $\phi \in C_\mu(\mathbb{R})$ can be written in the form $\phi = v + w$, $v \in V$ and $w \in W$. Recalling that $Lv = 0$ we get $F(\varepsilon, \phi) = Lw + r(\varepsilon, v + w)$. This suggests the following definition:

$$\Phi(\varepsilon, v, w) := L|_W w + r(\varepsilon, v + w),$$

where $\Phi_w(0, 0, 0) = L|_W$ is the restriction of L to W. Is clear that $\Phi \in C(R_\mu \times V \times W, C_\mu(\mathbb{R}))$ and $\Phi_w(\varepsilon, v, w) = L|_W + r_\phi(\varepsilon, v + w)$ is continuous in a neighborhood of $(0, 0, 0)$. Since $L|_W : W \to C_\mu(\mathbb{R})$ is bijective we have that $(L|_W)^{-1}$ is continuous from $C_\mu(\mathbb{R})$ to W. As a consequence, we can apply the Implicit Function Theorem (e.g. see [7, Theorem 2.3(i)]) to

$$\Phi(\varepsilon, v, w) = L|_W w + r(\varepsilon, v + w) = 0, \quad \Phi(0, 0, 0) = 0.$$

In this way, we find neighborhoods of 0, $\mathcal{E}_\mu \subset R_\mu$, $\mathcal{V}_\mu \subset V$ and $\mathcal{W}_\mu \subset W$ and a continuous map $\gamma \in C^1_v(\mathcal{E}_\mu \times \mathcal{V}_\mu, \mathcal{W}_\mu)$, such that $\Phi(\varepsilon, v, \gamma(\varepsilon, v)) = 0$ for all $(\varepsilon, v) \in \mathcal{E}_\mu \times \mathcal{V}_\mu$. Moreover, without restricting the generality, we can suppose that $\Phi(\varepsilon, v, w) = 0$ with $(\varepsilon, v, w) \in \mathcal{E}_\mu \times \mathcal{V}_\mu \times \mathcal{W}_\mu$ implies $w = \gamma(\varepsilon, v)$ (e.g. see [7, Theorem 2.3(ii)]).

Hence, the continuous family $\psi_{\varepsilon,v} = \psi + v + \gamma(\varepsilon, v) : \mathcal{E}_\mu \times \mathcal{V}_\mu \to C_\mu(\mathbb{R})$ contains all solutions of equation (4.1) from small neighborhoods of ψ, with $\psi_{0,0} = \psi$. Since $\gamma_v(0, 0) = 0$ and $\gamma_v(\varepsilon, v)$ is continuous for each fixed $\varepsilon \in \mathcal{E}_\mu$, we conclude that $\{\psi_{\varepsilon,v} : v \in \mathcal{V}_\mu\} \subset C_\mu(\mathbb{R})$ is a C^1-smooth manifold of dimension $d(\mu)$. Notice that (4.8) implies that $g(\psi_{\varepsilon,v}(+\infty)) = \psi_{\varepsilon,v}(+\infty)$. Thus $\psi_{\varepsilon,v}(+\infty) = \psi_{0,0}(+\infty) = \kappa$, so that $\{\psi_{\varepsilon,v}\}$ are heteroclinic solutions of (4.1).

Finally, the last conclusion of the theorem follows from the simple observations that (a) the sets $\mathcal{E}_\mu, \mathcal{V}_\mu, \mathcal{W}_\mu$ are non-increasing in μ and (b) the function $d(t)$ is piece-wise constant, with discontinuities at $\{\Re\lambda_j\} \cap [0, \lambda)$. $\square$

4.4 Characteristic equation

Lemma IV.9. *Let $\{\lambda_\alpha(\varepsilon), \alpha \in \mathscr{A}\}$, where $\mathbb{N} \cup \{\infty\} \subset \mathscr{A}$, denote the (countable) set of roots to the equation*

$$\varepsilon^2 z^2 - z - 1 + p\exp(-zh) = 0. \tag{4.9}$$

If $p > 1,\ h > 0,\ \varepsilon \in (0, 1/(2\sqrt{p-1}))$ then (4.9) has exactly two real roots $\lambda_1(\varepsilon), \lambda_\infty(\varepsilon)$ such that

$$0 < \lambda < \lambda_1(\varepsilon) < 2(p-1) < \varepsilon^{-2} - 2(p-1) < \lambda_\infty(\varepsilon) < \varepsilon^{-2} + 1.$$

Moreover:

(i) there exists an interval $\mathcal{O} = \mathcal{O}(p,h) \ni 0$ such that, for every $\varepsilon \in \mathcal{O}$, all roots $\lambda_\alpha(\varepsilon), \alpha \in \mathscr{A}$ of (4.9) are simple and the functions $\lambda_\alpha : \mathcal{O} \to \mathbb{C}$ are continuous;

(ii) we can enumerate $\lambda_j(\varepsilon), j \in \mathbb{N}$, in such a way that there exists $\lim_{\varepsilon\to 0+} \lambda_j(\varepsilon) = \lambda_j$ for each $j \in \mathbb{N}$, where $\lambda_j \in \mathbb{C}$ are the roots of (2.2), with $\lambda_1 = \lambda$;

(iii) for all sufficiently small ε, every vertical strip $\xi \le \Re z \le 2(p-1)$ contains only a finite set of $m(\xi)$ roots (if $\xi \notin \{\Re\lambda_j,\ j \in \mathbb{N}\}$, then $m(\xi)$ does not depend on ε) $\lambda_1(\varepsilon), \dots, \lambda_{m(\xi)}(\varepsilon)$ to (4.9), while the half-plane $\Re z > 2(p-1)$ contains only the root $\lambda_\infty(\varepsilon)$.

Proof: See [21, Lemma 13]. □

4.5 Asymptotic formulae of solutions

Notation IV.10. Throughout this section, we denote by $\beta, \gamma, \eta, b, C, C_j, C_*, \dots$ some positive constants that are independent of the parameters $\varepsilon \in \Lambda_j := (-\varepsilon_j, \varepsilon_j),\ v \in \Omega$, where $1 > \varepsilon_0 > \varepsilon_1 > \dots > \varepsilon_* > 0$, and $\Omega \subset \mathbb{R}^q$. We also assume that $h > 0, p > 1$.

Lemma IV.11. *Let continuous $y_{\varepsilon,v}(\cdot), f_{\varepsilon,v}(\cdot) : \Lambda_0 \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy*

$$\varepsilon^2 y''(t) + y'(t) - y(t) + py(t+h) = f_{\varepsilon,v}(t), \quad t \in \mathbb{R}. \tag{4.10}$$

Suppose further that $\sup_{t\le 0}[|y_{\varepsilon,v}(t)| + |f_{\varepsilon,v}(t)|] \le C, \ |y_{\varepsilon,v}(t)| \le Ce^{-\gamma t}, \ t \ge 0,$ *and that* $|f_{\varepsilon,v}(t)| \le Ce^{-bt}, \ t \ge 0, \ (\varepsilon, v) \in \Lambda_0 \times \Omega$. *Then, given $\sigma \in (0, b)$, it holds*

$$y_{\varepsilon,v}(t) = z_{\varepsilon,v}(t) + w_{\varepsilon,v}(t), \ t \in \mathbb{R},$$

where, with some continuous and bounded $B_j : (-\varepsilon_*, \varepsilon_*) \times \Omega \to \mathbb{C}$,

$$z_{\varepsilon,v}(t) = \sum_{\gamma \le \Re\lambda_j(\varepsilon) < b-\sigma} B_j(\varepsilon, v) e^{-\lambda_j(\varepsilon)t}$$

is a finite sum of eigensolutions of (4.10) associated to the roots $\lambda_j(\varepsilon) \in \{\gamma \le \Re\lambda_j(\varepsilon) < b - \sigma\}$ of (4.9) and $|w_{\varepsilon,v}(t)| \le C_* e^{-(b-\sigma)t}, \ t \ge 0, \ (\varepsilon, v) \in (-\varepsilon_*, \varepsilon_*) \times \Omega$.

Proof: Applying the Laplace transform $\mathcal{L}$ to equation (4.10), we obtain

$$\chi(z, \varepsilon)\tilde{y}_{\varepsilon,v}(z) = \tilde{f}_{\varepsilon,v}(z) + r_{\varepsilon,v}(z),$$

where $\chi(z, \varepsilon) = \varepsilon^2 z^2 + z - 1 + p\exp(zh), \ \tilde{y}_{\varepsilon,v} = \mathcal{L}\{y_{\varepsilon,v}\}, \ \tilde{f}_{\varepsilon,v} = \mathcal{L}\{f_{\varepsilon,v}\}$, and

$$r_{\varepsilon,v}(z) = \varepsilon^2(y'_{\varepsilon,v}(0) + zy_{\varepsilon,v}(0)) + y_{\varepsilon,v}(0) + pe^{zh}\int_0^h e^{-zu} y_{\varepsilon,v}(u)du.$$

Since $y_{\varepsilon,v}e^{\gamma t}$ is bounded, $\tilde{y}_{\varepsilon,v}$ is holomorphic in the open half-plane $\{\Re z > -\gamma\}$. Similarly, $\tilde{f}_{\varepsilon,v}$ is holomorphic in $\{\Re z > -b\}$. Since $r_{\varepsilon,v}$ is an entire function, the function

$$H_{\varepsilon,v}(z) := (\tilde{f}_{\varepsilon,v}(z) + r_{\varepsilon,v}(z))/\chi(z, \varepsilon)$$

is meromorphic in $\Re z > -b$, with only finitely many poles there.

<u>Step I.</u> We claim that there are $\sigma' \in (0, \sigma), \ \varepsilon_1 > 0$, such that $|H_{\varepsilon,v}(z)| \le C_1/|z|$, if $\Re z = -b + \sigma'$, $(\varepsilon, v) \in \Lambda_1 \times \Omega$. Indeed, take $\sigma' \in (0, \sigma)$ such that the line $\Re z = -b + \sigma'$ does not contain any eigenvalue $-\lambda_j(\varepsilon), \ \varepsilon \in \overline{\Lambda}_1$, and $1 - b + \sigma' \ne 0$. We have

$$|\tilde{f}_{\varepsilon,v}(z)| \le \int_0^{+\infty} e^{-\Re zt}|f_{\varepsilon,v}(t)|dt \le C\int_0^{+\infty} e^{-\Re zt}e^{-bt}dt \le \frac{C}{\sigma'}, \ \Re z \ge -b + \sigma';$$

$$|r_{\varepsilon,v}(z)| \le \varepsilon^2(|y'_{\varepsilon,v}(0)| + |z||y_{\varepsilon,v}(0)|) + |y_{\varepsilon,v}(0)| + pe^{\Re zh}\int_0^h e^{-\Re zu}|y_{\varepsilon,v}(u)|du.$$

As a bounded solution of (4.10), $y_{\varepsilon,v}$ should satisfy, for all $t \in \mathbb{R}$,

$$y_{\varepsilon,v}(t) = \frac{1}{\sqrt{1+4\varepsilon^2}}\left(\int_{-\infty}^t e^{\bar{\lambda}(t-s)}G_{\varepsilon,v}(s)ds + \int_t^{+\infty} e^{\bar{\mu}(t-s)}G_{\varepsilon,v}(s)ds\right), \tag{4.11}$$

where $\bar{\lambda} < 0 < \bar{\mu}$ are the roots of $\varepsilon^2z^2 + z - 1 = 0$ and $G_{\varepsilon,v}(t) := py_{\varepsilon,v}(t+h) - f_{\varepsilon,v}(t)$. Differentiating (4.11), we obtain

$$y'_{\varepsilon,v}(t) = \frac{1}{\sqrt{1+4\varepsilon^2}}\left(\bar{\lambda}\int_{-\infty}^t e^{\bar{\lambda}(t-s)}G_{\varepsilon,v}(s)ds + \bar{\mu}\int_t^{+\infty} e^{\bar{\mu}(t-s)}G_{\varepsilon,v}(s)ds\right), \tag{4.12}$$

so that

$$|y'_{\varepsilon,v}(0)| \le \frac{\bar{\mu}}{\sqrt{1+4\varepsilon^2}}\int_0^{+\infty} e^{-\bar{\mu}s}|G_{\varepsilon,v}(s)|ds + \frac{|\bar{\lambda}|}{\sqrt{1+4\varepsilon^2}}\int_{-\infty}^0 e^{-\bar{\lambda}s}|G_{\varepsilon,v}(s)|ds \le$$
$$(p+1)C\left(\int_0^{+\infty}\bar{\mu}e^{-\bar{\mu}s}ds + |\bar{\lambda}|\int_{-\infty}^0 e^{-\bar{\lambda}s}ds\right) = 2C(p+1).$$

Fix $k > -b+\sigma'$ and consider the vertical strip $\Sigma_k := \{-b+\sigma' \le \Re z \le k\}$, then

$$pe^{\Re zh}\int_0^h |e^{-zu}y_{\varepsilon,v}(u)|du \le Cpe^{kh}\int_0^h e^{bu}du := C_3,\ \ z \in \Sigma_k,$$

so that $|r_{\varepsilon,v}(z)| \le C_4(1+\varepsilon^2|z|),\ \ z \in \Sigma_k$.

Set $b(z) = -1 + pe^{zh}$, then $|b(z)| \le 1 + pe^{kh} := \beta, z \in \Sigma_k$, and

$$|z||H_{\varepsilon,v}(z)| \le \frac{C_5(|z| + \varepsilon^2|z|^2)}{|\varepsilon^2z^2 + z + b(z)|},\quad z \in \Sigma_k. \tag{4.13}$$

Now, set $y_0 = \eta\beta$ for some $\eta > 2$ satisfying $\eta^2 \ge 2\beta^{-1}\sqrt{\eta^2\beta^2 + b^2}$ and $\eta\beta > b - \sigma'$. For all z such that $\Re z = -b+\sigma'$, and $|\Im z| \ge y_0$, we have

$$|\varepsilon z^2 + z| = |z||\varepsilon^2z + 1| \ge y_0|\varepsilon^2z+1| \ge \frac{y_0^2}{\sqrt{y_0^2 + (b-\sigma')^2}} \ge 2\beta.$$

Thus $|\varepsilon^2z^2 + z + b(z)| \ge |\varepsilon^2z^2 + z| - |b(z)| \ge |\varepsilon^2z^2+z| - \beta \ge |\varepsilon^2z^2+z|/2$, so that

$$\frac{(|z| + \varepsilon^2|z|^2)}{|\varepsilon^2z^2 + z + b(z)|} \le 2\frac{1+\varepsilon^2|z|}{|\varepsilon^2z+1|} \le \eta + \sup_{\Re z = -b+\sigma'}\frac{2|\varepsilon^2z|}{|\varepsilon^2z+1|} \le 2\eta, \tag{4.14}$$

for all $|\Im z| \geq y_0$, $\Re z = -b + \sigma'$ and $\varepsilon \in \Lambda_1$.

Finally, for all $(z, \varepsilon) \in \{z : \Re z = -b + \sigma', |\Im z| \leq y_0\} \times \overline{\Lambda}_1$, we have that

$$\frac{|z| + \varepsilon |z|^2}{|\varepsilon z^2 + z + b(z)|} \leq C_6.$$

Combining this inequality with (4.13), (4.14), we prove the main assertion of Step I.

<u>Step II.</u> Taking $k > 0$, in virtue of (4.13) we can use the inversion formula

$$(4.15) \qquad y_{\varepsilon,v}(t) = \frac{1}{2\pi i} \int_{k-\infty i}^{k+\infty i} e^{zt} \tilde{y}_{\varepsilon,v}(z) dz = \frac{1}{2\pi i} \int_{k-\infty i}^{k+\infty i} e^{zt} H_{\varepsilon,v}(z) dz, \ t \geq 0.$$

By Lemma IV.9, $H_{\varepsilon,v}(z)$ has only finitely many poles in the strip $-b < \Re z \leq -\gamma$. Also, $H_{\varepsilon,v}(z) \to 0$ uniformly in the strip $-b + \sigma' \leq \Re z \leq k$, as $|\Im z| \to \infty$, and $H_{\varepsilon,v}(-b + \sigma' + i\cdot) \in L_2$. Thus, we may shift the path of integration in (4.15) to the left, to the line $\Re z = -b + \sigma'$, and obtain $y_{\varepsilon,v}(t) = z_{\varepsilon,v}(t) + w_{\varepsilon,v}(t)$, where

$$z_{\varepsilon,v}(t) = \sum_{\gamma \leq \Re\lambda_j(\varepsilon) < b - \sigma'} \mathrm{Res}_{-\lambda_j(\varepsilon)} e^{zt} H_{\varepsilon,v}(z), \ w_{\varepsilon,v}(t) = \frac{1}{2\pi i} \int\limits_{-b+\sigma'-\infty\cdot i}^{-b+\sigma'+\infty\cdot i} e^{zt} H_{\varepsilon,v}(z) dz.$$

By Lemma IV.9, the roots of equation $\chi(z, \varepsilon) = 0$ are simple for all small ε. Hence

$$z_{\varepsilon,v}(t) = \sum_{\gamma \leq \Re\lambda_j(\varepsilon) < b - \sigma'} e^{-\lambda_j(\varepsilon)t} B_j(\varepsilon, v), \ \text{with } B_j(\varepsilon, v) = \frac{\tilde{f}_{\varepsilon,v}(-\lambda_j(\varepsilon)) + r_{\varepsilon,v}(-\lambda_j(\varepsilon))}{\chi'(-\lambda_j(\varepsilon), \varepsilon)}.$$

It is easy to check that $B_j(\varepsilon, v)$ is continuous on its domain of definition (observe here that the continuity of $y'_{\varepsilon,v}(0)$ follows from (4.12)). Take j such that $-b + \sigma' < -\Re\lambda_j(\varepsilon) \leq -\gamma$, then $|r_{\varepsilon,v}(-\lambda_j(\varepsilon))| \leq C_4(\varepsilon^2 |\lambda_j(\varepsilon)| + 1) \leq C_4(\max_{j,\varepsilon} |\lambda_j(\varepsilon)| + 1) := C_7$. In addition, if $\varepsilon \to 0$ then

$$0 < |\chi'(-\lambda_j(\varepsilon), \varepsilon)| \ = \ |-2\varepsilon^2 \lambda_j(\varepsilon) + 1 + phe^{-\lambda_j(\varepsilon)h}| \to |1 + phe^{-\lambda_j h}| \neq 0.$$

$$\text{Hence, } |B_j(\varepsilon, v)| \ \leq \ \frac{|\tilde{f}_{\varepsilon,v}(-\lambda_j(\varepsilon))| + |r_{\varepsilon,v}(-\lambda_j(\varepsilon))|}{|\chi'(-\lambda_j(\varepsilon), \varepsilon)|} \leq \frac{C/\sigma' + C_7}{\min_{j,\varepsilon} |\chi'(-\lambda_j(\varepsilon), \varepsilon)|} \leq C_8$$

if $\varepsilon \in \Lambda_2$, for some small $\varepsilon_2 > 0$ and $v \in \Omega$.

<u>*Step III.*</u> Consider $u_{\varepsilon,v}(t) = e^{(b-\sigma')t}w_{\varepsilon,v}(t)$ and $v_{\varepsilon,v}(t) = e^{(b-\sigma)t}w_{\varepsilon,v}(t)$. We have

$$u_{\varepsilon,v}(t) = \frac{1}{2\pi i}\int_{-b+\sigma'-\infty\cdot i}^{-b+\sigma'+\infty\cdot i} e^{(s+b-\sigma')t}H_{\varepsilon,v}(s)ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi t}H_{\varepsilon,v}(-b+\sigma'+i\xi)d\xi.$$

By Plancherel theorem,

$$\|u_{\varepsilon,v}\|_2 = \frac{1}{2\pi}\|H_{\varepsilon,v}(-b+\sigma'+i\cdot)\|_2 \le \frac{C_1}{2\sqrt{\pi(b-\sigma')}}.$$

Hence, $v_{\varepsilon,v}(t) = e^{-(\sigma-\sigma')t}u_{\varepsilon,v}(t)$ is integrable on $[0,+\infty)$, and by the Cauchy-Schwarz inequality

$$\|v_{\varepsilon,v}\|_1 \le \frac{\|u_{\varepsilon,v}\|_2}{\sqrt{2(\sigma-\sigma')}} \le \frac{C_1}{2\sqrt{2\pi(b-\sigma')(\sigma-\sigma')}}.$$

<u>*Step IV.*</u> We claim that there exist real numbers $C_9 > 0$ and $\varepsilon_3 > 0$ such that $|w_{\varepsilon,v}(t)| \le C_9 e^{-(b-\sigma)t}, t \ge 0$, for all $(\varepsilon, v) \in \Lambda_3 \times \Omega$. In order to prove this, it suffices to show that $v_{\varepsilon,v}$ is uniformly bounded for small $\varepsilon \in \Lambda_3$. Since

$$\varepsilon^2 w''_{\varepsilon,v}(t) + w'_{\varepsilon,v}(t) - w_{\varepsilon,v}(t) + pw_{\varepsilon,v}(t+h) = f_{\varepsilon,v}(t), \quad t \in \mathbb{R},$$

we find that $v_{\varepsilon,v}(t) = e^{(b-\sigma)t}w_{\varepsilon,v}(t)$ satisfies

$$\varepsilon^2 v''_{\varepsilon,v}(t) + (1-2\varepsilon^2(b-\sigma))v'_{\varepsilon,v}(t) = P_{\varepsilon,v}(t),$$

where $\alpha = 1-2\varepsilon^2(b-\sigma) > 0$ and $P_{\varepsilon,v} \in L_1[0,+\infty)$ is defined by

$$P_{\varepsilon,v}(t) = e^{(b-\sigma)t}f_{\varepsilon,v}(t) + (1+(b-\sigma)-\varepsilon^2(b-\sigma)^2)v_{\varepsilon,v}(t) - pe^{-(b-\sigma)h}v_{\varepsilon,v}(t+h).$$

The variation of constants formula yields

$$v'_{\varepsilon,v}(t) = e^{-\frac{\alpha}{\varepsilon^2}t}\left(v'_{\varepsilon,v}(0) + \frac{1}{\varepsilon^2}\int_0^t e^{\frac{\alpha}{\varepsilon^2}s}P_{\varepsilon,v}(s)ds\right),\ \varepsilon \ne 0. \tag{4.16}$$

A direct integration of (4.16) gives

$$v_{\varepsilon,v}(t) = v_{\varepsilon,v}(0) + \frac{\varepsilon^2}{\alpha}v'_{\varepsilon,v}(0)(1-e^{-\frac{\alpha}{\varepsilon^2}t}) + \frac{1}{\varepsilon^2}\int_0^t\int_0^u e^{\frac{\alpha}{\varepsilon^2}(s-u)}P_{\varepsilon,v}(s)dsdu.$$

After changing the order of integration in the iterated integral, we get

$$\frac{1}{\varepsilon^2}\left|\int_0^t\int_s^t e^{\frac{\alpha}{\varepsilon^2}(s-u)}P_{\varepsilon,v}(s)duds\right| = \frac{1}{\alpha}\left|\int_0^t P_{\varepsilon,v}(s)(1-e^{\frac{\alpha}{\varepsilon^2}(s-t)})ds\right| \le \frac{1}{\alpha}\int_0^t |P_{\varepsilon,v}(s)|ds.$$

Additionally, recalling Step II, we find that $|v'_{\varepsilon,v}(0)| \le (b-\sigma)|w_{\varepsilon,v}(0)| + |w'_{\varepsilon,v}(0)| \le$

$$\le (b-\sigma)(|y_{\varepsilon,v}(0)| + |z_{\varepsilon,v}(0)|) + |\phi'_{\varepsilon,v}(0)| + |z'_{\varepsilon,v}(0)| < C_{10}.$$

As a consequence, for all small ε and $v \in \Omega$, we have that

$$|v_{\varepsilon,v}(t)| \le |v_{\varepsilon,v}(0)| + \frac{\varepsilon^2}{\alpha}C_{10}(1+e^{-\frac{\alpha}{\varepsilon^2}t}) + \frac{1}{\alpha}\int_0^{+\infty}|P_{\varepsilon,v}(s)|ds \le C_{11},\ t \ge 0.$$

Finally, since $w_{\varepsilon,v}(t) = v_{\varepsilon,v}(t)e^{-(b-\sigma)t}$, Lemma IV.11 is proved. $\square$

4.6 Existence of fast traveling wave

Theorem IV.12. *In Theorem IV.8, take $\mu = \lambda - \delta$, with small $\delta > 0$. Assume that ψ is the positive heteroclinic of (1.5) normalized by $\psi(t) = \exp(\lambda t) + O(\exp((2\lambda-\delta)t))$, $t \to -\infty$. Then we can choose a neighborhood $\mathcal{U} \subset C_\mu(\mathbb{R})$ of ψ and a neighborhood $\mathcal{E}^*_\mu \times \mathcal{V}^*_\mu$ of $0 \in \mathbb{R}^2$ in such a way that $\psi_{\varepsilon,v} \in \mathcal{U}$, $(\varepsilon, v) \in \mathcal{E}^*_\mu \times \mathcal{V}^*_\mu$, is positive and unique in $\mathcal{U}$ (up to translations in t) for every fixed ε. Moreover, $\psi_{\varepsilon,v}(t-t_0) = \exp(\lambda_1(\varepsilon)t) + O(\exp(1.99\mu t))$ at $t \to -\infty$ for some $t_0 = t_0(\varepsilon, v) \in \mathbb{R}$.*

Proof: First, we take $\mathcal{V}_\mu$, $\mathcal{E}_\mu \subset (-\varepsilon_1, \varepsilon_1)$, $\mathcal{U}$ as in Theorem IV.8. It follows from Lemma IV.9 and Theorem IV.8 that $\mathcal{V}_\mu \subset \mathbb{R}$ and that we can choose positive δ and $\mathcal{E}_\mu$ such that $\Re\lambda_j(\varepsilon) < \mu < \lambda < \lambda_1(\varepsilon) < 1.99\mu < \lambda_\infty(\varepsilon)$ for all $\varepsilon \in \mathcal{E}_\mu$. If we set $y_{\varepsilon,v}(t) = \psi_{\varepsilon,v}(-t)$, then $y_{\varepsilon,v}$ satisfies (4.10) where

$$|f_{\varepsilon,v}(t)| = |g(y_{\varepsilon,v}(t+h)) - g'(0)y_{\varepsilon,v}(t+h)| \le C_1e^{-2\mu t},\ t \ge -h.$$

Lemma IV.11 assures that there are $\mathcal{V}'_\mu \subset \mathcal{V}_\mu$, $\mathcal{E}'_\mu \subset \mathcal{E}_\mu$ such that

$$y_{\varepsilon,v}(t) = B(\varepsilon, v)e^{-\lambda_1(\varepsilon)t} + w_{\varepsilon,v}(t),\ (\varepsilon, v) \in \mathcal{E}'_\mu \times \mathcal{V}'_\mu.$$

Here $B : \mathcal{E}'_\mu \times \mathcal{V}'_\mu \to \mathbb{R}_+$, $B(0,0) = 1$, is continuous and $|w_{\varepsilon,v}(t)| \le C_* e^{-1.99\mu t}$, $t \ge 0$, for some $C_* > 0$.

Hence, there are $\mathcal{E}''_\mu \times \mathcal{V}''_\mu$ and $T > 0$ (independent of ε, v) such that $y_{\varepsilon,v}(t) > 0.5e^{-\lambda_1(\varepsilon)t}$, $t > T$, for all $(\varepsilon, v) \in \mathcal{E}''_\mu \times \mathcal{V}''_\mu$. On the other side, $\lim_{(\varepsilon,v)\to 0} y_{\varepsilon,v}(t) = \psi(-t)$ uniformly on $\mathbb{R}$. In consequence, since ψ is bounded from below by a positive constant on $[-T, \infty)$, we conclude that $y_{\varepsilon,v}$ is positive on $\mathbb{R}$, if (ε, v) belongs to sufficiently small neighborhood $\mathcal{E}^*_\mu \times \mathcal{V}^*_\mu \subset \mathcal{E}''_\mu \times \mathcal{V}''_\mu$ of the origin. Without the loss of the generality, we can assume additionally that $\psi_{\varepsilon,v} \in \mathcal{U}$ for all $(\varepsilon, v) \in \mathcal{E}^*_\mu \times \mathcal{V}^*_\mu$.

Next, for every fixed $\varepsilon \in \mathcal{E}^*_\mu$, the subset $\mathfrak{F} = \{\psi_{\varepsilon,v} : v \in \mathcal{V}_\mu\} \subset C_\mu(\mathbb{R})$ is homeomorphic to $\mathcal{V}_\mu$. On the other hand, for every $n > 0$, the collection $\mathfrak{P}_n = \{\psi_{\varepsilon,0}(t - s),\ s \in (-n, n)\}$ of positive heteroclinics is a continuous 1-manifold in $C_\mu(\mathbb{R})$. Since $\psi_{\varepsilon,0} \in \mathfrak{F} \cap \mathfrak{P}_n$ we obtain that $\{\psi_{\varepsilon,v} : v \in \mathcal{V}^*_\mu\} \subset \mathfrak{P}_\infty$. In consequence, $\psi_{\varepsilon,v}(t)$ is unique in $\mathcal{U}$ (up to shifts in t) for every fixed small ε. $\square$

Theorem IV.13. *Set $\mathcal{P} = \{(\varepsilon, v) \in \mathcal{E}_0 \times \mathcal{V}_0 : \psi_{\varepsilon,v}(t) > 0,\ t \in \mathbb{R}\}$, where $\mathcal{E}_0, \mathcal{V}_0$ are as in Theorem IV.8. Then there exist a neighborhood $\mathcal{E}^* \times \mathcal{V}^* \subset \mathcal{E}_0 \times \mathcal{V}_0$ of 0 and $C > 0$ such that, for all $(\varepsilon, v) \in \mathcal{P}^* := \mathcal{P} \cap (\mathcal{E}^* \times \mathcal{V}^*)$, we have that*

$$\psi_{\varepsilon,v}(t) = B(\varepsilon, v)e^{\lambda_1(\varepsilon)t} + w_{\varepsilon,v}(t), \tag{4.17}$$

where $|w_{\varepsilon,v}(t)| \le Ce^{1.99\lambda t}, t \le 0$, and $B : \mathcal{E}^* \times \mathcal{V}^* \to (0, \infty)$ is continuous.

Proof: Let $\mathcal{E}' \subset \mathcal{E}_0$ be such that $\lambda_\infty(\varepsilon) > 3\lambda$, for all $\varepsilon \in \mathcal{E}'$. The last assertion of Theorem IV.8 implies that, for some $\gamma > 0$, $C_1 > 0$,

$$\sup_{t\ge 0} |\psi_{\varepsilon,v}(t)| \le C_1,\ |\psi_{\varepsilon,v}(t)| \le C_1 e^{\gamma t},\ t \le 0. \tag{4.18}$$

If we set $y_{\varepsilon,v}(t) = \psi_{\varepsilon,v}(-t)$, then $y_{\varepsilon,v}$ satisfies (4.10) where

$$|f_{\varepsilon,v}(t)| = |g(y_{\varepsilon,v}(t + h)) - g'(0)y_{\varepsilon,v}(t + h)| \le C_2 e^{-2\gamma t},\ t \ge -h.$$

Set $\Gamma = \sup\{\gamma > 0$ such that (4.18) holds for all $(\varepsilon, v) \in \mathcal{P} \cap (\mathcal{E}' \times \mathcal{V}_0)\}$. Applying Lemma IV.11, we get

$$y_{\varepsilon,v}(t) = \sum_{0<\lambda_j(\varepsilon)<2\Gamma} B_j(\varepsilon, v)e^{-\lambda_j(\varepsilon)t} + \tilde{w}_{\varepsilon,v}(t),$$

where $B_j : \mathcal{E}'' \times \mathcal{V}_0 \to \mathbb{C}$ are continuous and $|\tilde{w}_{\varepsilon,v}(t)| \le C_3 e^{-1.99\Gamma t}$, $t \ge 0$, $(\varepsilon, v) \in \mathcal{P} \cap (\mathcal{E}'' \times \mathcal{V}_0)$, for some $C_3 > 0$ and open $\mathcal{E}'' \subset \mathcal{E}'$. Since $\Gamma > 0$ is finite and $y_{\varepsilon,v}(t) > 0$, we obtain

$$\sum_{0<\lambda_j(\varepsilon)<2\Gamma} B_j(\varepsilon, v)e^{-\lambda_j(\varepsilon)t} = B(\varepsilon, v)e^{-\lambda_1(\varepsilon)t},$$

so that $\Gamma \ge \lambda$, see Lemma IV.9. Next, due to Lemma II.9, it holds that $B(0, 0) > 0$. Hence, $\Gamma = \lambda$. $\square$

Corollary IV.14. *Given $\delta \in (0, \lambda)$ and $(\varepsilon_j, v_j) \in \mathcal{P}^*, j = 0, 1, \dots,$ the convergence*

$$\psi_{\varepsilon_j,v_j} \xrightarrow{C_0(\mathbb{R})} \psi_{\varepsilon_0,v_0} \qquad \text{implies} \qquad \psi_{\varepsilon_j,v_j} \xrightarrow{C_{\lambda-\delta}(\mathbb{R})} \psi_{\varepsilon_0,v_0}.$$

Proof: By the contrary, suppose that there are a sequence $\{\psi_{\varepsilon_j,v_j}, (\varepsilon_j, v_j) \in \mathcal{P}^*\}_{j\ge 0}$ and $\eta > 0$ such that

$$\lim_j |\psi_{\varepsilon_j,v_j} - \psi_{\varepsilon_0,v_0}|_0 = 0, \;\; |\psi_{\varepsilon_j,v_j} - \psi_{\varepsilon_0,v_0}|_{\lambda-\delta} > \eta, \;\; j = 1, 2, \dots$$

It follows from (4.17) that there exist $C > 0$ and $T < 0$ such that

$$\psi_{\varepsilon_j,v_j}(t)e^{-(\lambda-\delta)t} \le Ce^{\delta t} < \eta/4, \;\; j = 0, 1, 2, \dots, \;\; t \le T.$$

Thus

$$\sup_{s\le T} \left[e^{-(\lambda-\delta)s}|\psi_{\varepsilon_j,v_j}(s) - \psi_{\varepsilon_0,v_0}(s)|\right] \le \eta/2, \;\; j = 1, 2, \dots .$$

Next, since $\psi_{\varepsilon_j,v_j}(t) \to \psi_{\varepsilon_0,v_0}(t)$ uniformly on $\mathbb{R}$, we can find j_* such that

$$\sup_{s\in[T,0]} \left[e^{-(\lambda-\delta)s}|\psi_{\varepsilon_j,v_j}(s) - \psi_{\varepsilon_0,v_0}(s)|\right] \le \frac{\eta}{2}, \; \sup_{s\ge 0} |\psi_{\varepsilon_j,v_j}(s) - \psi_{\varepsilon_0,v_0}(s)| \le \frac{\eta}{2}, \;\; j \ge j_*.$$

But all this means that $|\psi_{\varepsilon_j,v_j} - \psi_{\varepsilon_0,v_0}|_{\lambda-\delta} \le \eta/2$ for all $j \ge j_*$, a contradiction. $\square$

4.7 Uniqueness Theorems

In this section establishes our main result for a reaction-diffusion equation with local delay. Here we show that there exists exactly one wavefront for each fixed sufficiently fast speed. For to prove the Theorem I.7 we need the following auxiliary result.

Lemma IV.15. *Assume I.5. Consider wavefront* $u(x,t) = \phi(x+ct)$*, to equation (1.4). Then there exists a unique* τ *such that* $\phi(\tau) = \mathrm{A}$*,* $\phi'(s) > 0$ *for all* $s \le \tau$*.*

Proof: See [52, Proposition 2.1]. $\Box$

Now everywhere below, all positive wavefronts ϕ will be normalized by the conditions $\phi(0) = \zeta_1/2$ and $\phi'(s) > 0$, $s < 0$, with $\zeta_1 \le \mathrm{A}$ defined in Chapter II. The possibility of such a normalization was established in Lemma IV.15. Let ψ, $\psi(0) = \zeta_1/2$, $\psi(s) < \zeta_1/2$, $s < 0$, be the positive heteroclinic of (1.5) given in Lemma II.9. By Theorem IV.12, there exists a neighborhood $(-\varepsilon_0, \varepsilon_0) \times \mathcal{U} \subset \mathbb{R} \times C_{\lambda-\delta}(\mathbb{R})$ of $(0, \psi)$ such that for every fixed $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ there is a unique normalized positive wavefront $\psi_\varepsilon \in \mathcal{U}$. We claim that, if ε is sufficiently small, then this ψ_ε will be the unique normalized positive wavefront of equation (4.1). By way of contradiction, let us suppose that we can find a sequence $\varepsilon_j \to 0$ and normalized positive wavefronts $\phi_{\varepsilon_j} \ne \psi_{\varepsilon_j}$.

Lemma IV.16. *Assume I.5 and I.4. Then* $\phi_{\varepsilon_j} \to \psi$ *uniformly on* $\mathbb{R}$*.*

Proof: First, we prove the uniform convergence $\phi_{\varepsilon_j} \to \psi$ on compact subsets of $\mathbb{R}$. Since g is a bounded function, we obtain from (4.8) that

$$|\phi'_{\varepsilon_j}(t)| + |\phi_{\varepsilon_j}(t)| \le \varepsilon^{-2}(\mu-\lambda)^{-1}\max_{s\ge 0} g(s) + \max_{s \ge 0} g(s) \le 2\zeta_2, \ j \in \mathbb{N}.$$

Hence, by the Ascoli-Arzelà theorem combined with the diagonal method, $\{\phi_{\varepsilon_j}\}$ is precompact in $C(\mathbb{R},\mathbb{R})$. Thus, every $\{\phi_{\varepsilon_{j_k}}\}$ has a subsequence converging in $C(\mathbb{R},\mathbb{R})$ to some continuous positive bounded function $\varphi(s)$ such that $\varphi'(s) \geq 0, s \leq 0$, and $\varphi(0) = \zeta_1/2$. Making use of the Lebesgue's dominated convergence theorem, we deduce from equation (4.8) that

$$\varphi(t) = \int_{-\infty}^{t} e^{-(t-s)} g(\varphi(s-h))ds.$$

Therefore φ is a positive bounded solution of equation (1.5) and since the equilibrium κ of equation (1.5) is globally attractive, it holds that $\varphi(+\infty) = \kappa$. On the other hand, since $\varphi(-\infty) \leq \varphi(0) = \zeta_1/2$, we have that $\varphi(-\infty) = 0$. Hence, due to Lemma II.9, we obtain that $\varphi(t) = \psi(t),\ t \in \mathbb{R}$. Next, if $\phi_{\varepsilon_n} \not\to \psi$ uniformly on $\mathbb{R}$ then there exist a subsequence $\{\phi_{\varepsilon_{j_n}}\} \subset \{\phi_{\varepsilon_j}\}$ (for short, we will write again $\{\phi_{\varepsilon_j}\}$ instead of $\{\phi_{\varepsilon_{j_n}}\}$), a sequence $\{S_j\}$ and positive numbers $T, \delta < \kappa/6$ such that

$$|\psi(S_j) - \phi_{\varepsilon_j}(S_j)| = 2\delta,\ |\psi(t)| < 0.25\delta, t \leq -T,\ |\psi(t) - \kappa| < 0.25\delta,\ t \geq T.$$

Since ϕ_{ε_n} converges uniformly on $[-2T, 2T]$ to ψ, and $\phi_{\varepsilon_n},\ \psi$ are monotone increasing on $(-\infty, 0]$, we can suppose that $|\psi(t) - \phi_{\varepsilon_n}(t)| < \delta$ for all $t \in (-\infty, 2T]$ and $n \geq n_0$. In this way, $S_j \to +\infty$ and we can suppose that

$$|\psi(t) - \phi_{\varepsilon_j}(t)| < 2\delta,\ t \in (-\infty, S_j).$$

Consider the sequence $y_j(t) = \phi_{\varepsilon_j}(t + S_j)$ of heteroclinics to equation (4.1). We have that $|y_j(0) - \kappa| > 1.5\delta$ and $|y_j(t) - \kappa| < 3\delta$ when $t \in (T - S_j, 0)$. Arguing as above, we find that $\{y_j\}$ contains a subsequence converging, on compact subsets of $\mathbb{R}$, to some solution $y_*(t)$ of (1.5) satisfying $|y_*(0) - \kappa| \geq 1.5\delta$ and $|y_*(t) - \kappa| \leq 3\delta < \frac{\kappa}{2}$ for all $t < 0$. Lemma II.13 implies that $\inf_{\mathbb{R}} y_*(t) > 0$. Since $y_*(0) \neq \kappa$, we have established the existence of a non-constant positive bounded and separated from 0 solution to (1.5). This contradicts to the global attractivity of κ. $\square$

Corollary IV.17. *$\phi_{\varepsilon_j} \to \psi$ in $C_{\lambda-\delta}(\mathbb{R})$.*

Proof: Since $\phi_{\varepsilon_j} \to \psi$ in $C_0(\mathbb{R})$, we have that $\phi_{\varepsilon_j} = \psi_{\varepsilon_j, v_j}$ for some $v_j \in \mathcal{V}_0$. Now we can apply Corollary IV.14 to find that $\phi_{\varepsilon_j} \to \psi$ in $C_{\lambda-\delta}(\mathbb{R})$.

Lastly, Theorem IV.12 and Corollary IV.17 implies that $\phi_{\varepsilon_j} = \psi_{\varepsilon_j}$, a contradiction which completes the proof of Theorem I.7. □

4.8 Nonmonotonicity of travelling wave

In this section we give the results obtained in [21] where the oscillation of the traveling waves about positive equilibrium is obtained.

Lemma IV.18. *Let $g'(\kappa) < 0$ and $|g'(\kappa)|he^{h+1} > 1$. Then the equation*

$$\varepsilon^2 z^2 - z - 1 + g'(\kappa)\exp(-zh) = 0 \tag{4.19}$$

has no negative real roots, for all sufficiently small ε. Moreover, if the equilibrium κ of (1.5) is hyperbolic, then, for all small ε, there are no roots of (4.19) on the imaginary axis.

Proof: See [21, Lemma 15]. □

Lemma IV.19. *Assume I.4 and $g'(\kappa)he^{h+1} < -1$. Then for small $\varepsilon > 0$, every nonconstant and bounded solution ϕ of (4.1) such that $\phi(+\infty) = \kappa$ oscillates about κ.*

Proof: See [21, Lemma 16]. □

4.9 Application

In order to apply Theorems I.7, we need to find sufficient conditions to ensure the global attractivity of the positive equilibrium of (1.5). Some results in this direction were found in [39] for nonlinearities satisfying a generalized Yorke condition.

The reader can be referred to [43] for the case of unimodal nonlinearities, and for further references. On the other hand, in [33] provide various conditions which are sufficient to guarantee the exponential stability of the positive steady state. The above mentioned works yield the following

Corollary IV.20. *Let $g \in C^3(\mathbb{R}_+, \mathbb{R}_+)$ be such that*

(1) the Schwarz derivative $(Sg)(x) = g'''(x)(g'(x))^{-1} - (3/2)\left(g''(x)(g'(x))^{-1}\right)^2$ is negative for all $x > 0$, $x \neq x_M$;

(2) g has only one critical point x_M (global maximum);

(3) g has exactly two fixed points, 0 and $\kappa > 0$. Moreover, $\Gamma_0 := g'(0) > 1$;

(4) $1 + i\sqrt{\Gamma_0^2 - 1} \neq \Gamma_0 \exp(-ih\sqrt{\Gamma_0^2 - 1})$;

(5) either $\Gamma := g'(\kappa) \in [0, 1)$ or

$$\Gamma < 0 \quad \text{and} \quad e^{-h} > -\Gamma \ln \frac{\Gamma^2 - \Gamma}{\Gamma^2 + 1}.$$

Then there exists a unique (modulo translations) positive wavefront of equation (1.4) for each sufficiently large speed c.

Proof: We only need to check assumptions I.5 and I.4 of Theorem IV.15. Since g is C^3-smooth, it is immediate that $(2), (3)$ imply (I.5). Next, condition (4) ensures that the characteristic equation $\lambda + 1 = g'(0)\exp(-\lambda h)$ has no roots on the imaginary axis. Therefore the trivial steady state is hyperbolic.

In the rest of the proof, we assume that $(1) - (3)$ hold. In consequence, if $g'(\kappa) \in [0, 1)$ then the positive equilibrium is exponentially stable (e.g. see [33, Corollary 3.2]) and globally attracting (e.g. see [43, Proposition 3.2]). The second line of condition (5) also ensures the exponential stability of κ (see [33, Theorem 2.9]) and the global attractivity of κ (see [39, Corollary 2.3]). Therefore $(1) - (5)$ imply (I.4).$\square$

Below, we apply Theorem IV.15 and Corollary IV.20 to two time-delayed reaction-diffusion population models. First, we consider the diffusive Nicholson's blowflies equation

$$u_t(t,x) = \Delta u(t,x) - \delta u(t,x) + pu(t-h,x)e^{-bu(t-h,x)}, \ t \in \mathbb{R}, \ x \in \mathbb{R}^m. \tag{4.20}$$

This equation was introduced in [46], it generalizes the famous Nicholson's blowflies equation

$$y'(t) = -\delta y(t) + py(t-h)e^{-by(t-h)},$$

intensively studied for the last decade. Equation (4.20) takes into account spatial distribution of the species, and nowadays there is growing interest in understanding the factors that influence the spatial spread of the growing population modeled by (4.20). Relevant biological discussion can be found in [28], where various modifications of (4.20) were proposed and studied.

After a linear rescaling of both variables u and t, we can assume that $\delta = b = 1$. Therefore equation (4.20) can be written in the following normalized form

$$u_t(t,x) = \Delta u(t,x) - u(t,x) + pu(t-h,x)e^{-u(t-h,x)}. \tag{4.21}$$

The case of interest is $p > 1$ when (4.21) has a unique positive steady state $\kappa = \ln p$. It is immediate to check that the birth function

$$g(s) = pse^{-s}, s \geq 0,$$

satisfies conditions $(1) - (3)$ of the above corollary. In this way, the conclusion of Corollary IV.20 holds if $\Gamma_0 = p$ and $\Gamma = 1 - \ln p$ satisfy conditions $(4), (5)$. It is worth to mention that (5) trivially holds if $\Gamma \in [-1, 1)$ (that is, when $0 < \ln p \leq 2$).

As a second application, let us consider the birth function

$$g(s) = \frac{ps}{1+s^n}, \ n \geq 1, s \geq 0.$$

This function was proposed in 1977 by Mackey and Glass to model hematopoiesis (blood cell production). The Mackey-Glass equation with non-monotone nonlinearity can be written in the following normalized form

$$y'(t) = -y(t) + \frac{py(t-h)}{1+(y(t-h))^n}. \tag{4.22}$$

The corresponding reaction-diffusion equation with delay is

$$u_t(t,x) = \Delta u(t,x) - u(t,x) + \frac{pu(t-h,x)}{1+(u(t-h,x))^n}. \tag{4.23}$$

Taking $p > 1$ in equation (4.23), we find that conditions $(2), (3)$ and (I.5) are satisfied with $\kappa = (p-1)^{1/n}$. Furthermore, if $n \geq 2$, then the Schwarz derivative of $g(s) = ps/(1+s^n)$, $s \geq 0, n \geq 1$ is negative, see [26, Lemma 3]. Consequently, the conclusion of Corollary IV.20 holds if $n \geq 2$ and $\Gamma_0 = p,\ \Gamma = 1 - n + n/p$, satisfy conditions $(4), (5)$. Now, suppose that $n \in (1, 2]$. Then [33, Corollary 3.2] [respectively, [26, Theorem 2]] guarantees that the positive steady state of equation (4.22) is exponentially stable [respectively, globally attractive]. Therefore, if $n \in (1, 2]$ and $1 + i\sqrt{p^2-1} \neq p\exp(-ih\sqrt{p^2-1})$, then Theorem IV.15 assures the existence of a unique (modulo translations) positive wavefront of equation (4.23) for each sufficiently large speed c.

CHAPTER V

Existence of fast positive wavefronts for a non-local delayed reaction-diffusion equation

5.1 Introduction

The main object of study in this chapter is the time-delayed reaction-diffusion equation (1.6). Here $h \geq 0$ denotes the time delay and it is assumed that the non-negative averaging kernel K satisfies $\int_{\mathbb{R}} K(w)dw = 1$, $\int_{\mathbb{R}} K(w)e^{\lambda w}dw \in \mathbb{R}$, for every $\lambda \in \mathbb{R}$. The function $-s+g(s)$ is of the monostable type and sufficiently smooth and we also suppose that g satisfies I.9. Our main concern are the positive wavefront solutions $u(t,x) = \phi(x+ct)$ of (1.6) After scaling, ϕ is a positive heteroclinic solution of the delay differential equation

$$\varepsilon^2 y''(t) - y'(t) - y(t) + \int_{\mathbb{R}} K(w)g(y(t-h-\varepsilon w))dw = 0, \quad t \in \mathbb{R}, \tag{5.1}$$

where $\varepsilon := 1/c > 0$, c is the wavefront velocity.

In this chapter, inspired by [4, 20, 21], we give the affirmative answer to the existence question. Namely, for a broad family of nonlinearities g satisfying the hypothesis I.4 and I.5 we prove that equation (1.6) has a continuous family of positive wavefronts $u(t,x) = \phi_c(ct+x)$ provided that the wave speed c is sufficiently large.

This result can be viewed as a natural continuation and extension of the main theorem in [21], where a similar problem for a *local* delayed reaction-diffusion equation was analysed (the local equation can be obtained formally from (1.6) by taking the delta of Dirac as $K(w)$). It should be stressed here that, due to the presence of non-local terms, a direct application of the method from [21] to equation (1.6) fails due to technical issues arising. To cope with them, we use here a somewhat different approach proposed in [4] and based on the uniform asymptotic integration formulae (see Lemma V.11 below).

5.2 Spaces and operators

This section contains several lemmas which will be needed later. We will assume I.4 and I.9, where the C^1- smooth function g is defined and bounded on the whole real axis $\mathbb{R}$. Let λ be as in Lemma II.7 and let ψ be as in Lemma II.9.

Notation V.1. For a fixed $\mu > 0$ and $\lambda_* \in (0, \lambda)$, we will consider the following Banach spaces:

$$C_\mu(\mathbb{R}) = \left\{ y \in C(\mathbb{R}, \mathbb{R}) : \lim_{s \to -\infty} e^{-\mu s} y(s), \lim_{s \to +\infty} y(s) \text{ exists and are finite} \right\},$$
$$C_{\psi,\lambda_*}(\mathbb{R}) = \left\{ y \in C_{\lambda_*}(\mathbb{R}) : \int_{-\infty}^{0} y(s)\psi'(s)ds = 0 \right\},$$

equipped with the norm $\|y\|_\mu = \max\{\|y\|^+, \|y\|_\mu^-\}$, where $\|y\|^+ = \sup_{\mathbb{R}_+} |y(s)|$ and $\|y\|_\mu^- = \sup_{\mathbb{R}_-} e^{-\mu s}|y(s)|$(in order to simplify the notation, we shall often write $\|y\|$ instead of $\|y\|_\mu$).

Note that $\psi, \psi' \in C_{\lambda_*}(\mathbb{R}) \setminus C_{\psi,\lambda_*}(\mathbb{R})$.

Definition V.2. The same manner like in Chapter IV, we will also need the integral operator

$$\mathcal{N} : C_{\psi,\lambda_*}(\mathbb{R}) \to C_{\lambda_*}(\mathbb{R}); \quad (\mathcal{N}y)(t) = \int_{-\infty}^{t} e^{-(t-s)} q(s) y(s-h) ds,$$

where $q(s) = g'(\psi(s-h))$.

Since $q(-\infty) = g'(0+) = p > 1$ and $q(+\infty) = g'(\kappa)$, it can be checked directly that $\mathcal{N}$ is well defined.

The following lemma was proved in [21, Lemma 9] for spaces $C_{\psi,\lambda_*}(\mathbb{R})$ defined in a different way (the convergence $\lim_{s\to-\infty} e^{-\lambda_* s}y(s)$ was not required). Since our definition of $C_{\psi,\lambda_*}(\mathbb{R})$ involves this condition of convergence we decided to include the proof.

Lemma V.3. $I - \mathcal{N} : C_{\psi,\lambda_*}(\mathbb{R}) \to C_{\lambda_*}(\mathbb{R})$ *is an isomorphism of Banach spaces, where* $I : C_{\psi,\lambda_*}(\mathbb{R}) \to C_{\lambda_*}(\mathbb{R})$, $I(y) = y$, *and* $\lambda_* \notin \{\Re\lambda_j\}$.

Proof: By [21, Lemma 9], the operator $I - \mathcal{N}$ is injective and the equation $(I - \mathcal{N})y = d$, $d \in C_{\lambda_*}(\mathbb{R})$ has a solution y such that there exists $y(+\infty) \in \mathbb{R}$, $\displaystyle\int_{-\infty}^{0} y(s)\psi'(s)ds = 0$ and $\|y\|^{-}_{\lambda_*} < \infty$. Hence, we only need to show that $y(t)e^{-\lambda_* t}$ converges as $t \to -\infty$. First, observe that $z(t) := y(t) - d(t)$ satisfies

$$z'(t) = -z(t) + q(t)z(t-h) + q(t)d(t-h). \tag{5.2}$$

Set $r_0 = \dfrac{p\lim_{t\to-\infty} d(t-h)e^{-\lambda_* t}}{1 + \lambda_* - pe^{-\lambda_* t}}$. Since $q(t) = p + O(\exp(\lambda t))$, $t \to -\infty$, we obtain that $r(t) = e^{-\lambda_* t}z(t) - r_0$, is a bounded solution of

$$r'(t) = -(1+\lambda_*)r(t) + [p\exp(-\lambda_* h) + \epsilon_1(t)]r(t-h) + \epsilon_2(t),$$

where $\epsilon_1, \epsilon_2 : \mathbb{R} \to \mathbb{R}$ are bounded and $\epsilon_1(-\infty), \epsilon_2(-\infty) = 0$. We claim that $\lim_{t\to-\infty} r(t) = 0$. On the contrary, let us suppose that there exists a real number $\epsilon_0 > 0$ and a sequence $t_n \to -\infty$ such that $|r(t_n)| \geq \epsilon_0$ for all $n \geq 1$. We find that $r_n(t) = r(t + t_n), t \in \mathbb{R}$, is a bounded solution of the equation

$$r_n'(t) = -(1+\lambda_*)r_n(t) + a_n(t)r_n(t-h) + b_n(t),$$

where $a_n(t) = p\exp(-\lambda_* h) + \epsilon_1(t + t_n)$ and $b_n(t) = \epsilon_2(t + t_n)$. Consequently,

$$r_n(t) = \int_{-\infty}^{t} e^{-(1+\lambda_*)(t-s)}[a_n(s)r_n(s-h) + b_n(s)]ds, \tag{5.3}$$

and

$$|r_n(t)| + |r_n'(t)| \le \sup_{t\in\mathbb{R}} |r(t)| \Big(2 + \lambda_* + p + \sup_{t\in\mathbb{R}} |\epsilon_1(t)|\Big) + \sup_{t\in\mathbb{R}} |\epsilon_2(t)|,\ t \in \mathbb{R}.$$

Hence, by the Ascoli-Arzelá compactness criterion the sequence $\{r_n(t)\}$ is pre-compact in the compact open topology of $C(\mathbb{R},\mathbb{R})$. Thus there is a subsequence $\{r_{n_k}(t)\}$ converging in $C(\mathbb{R},\mathbb{R})$ to some continuous bounded function $r_*(t)$, such that $r_*(0) \ge \epsilon_0 > 0$. By the Lebesgue's dominated convergence theorem, it holds that, for every fixed $t \in \mathbb{R}$,

$$\int_{-\infty}^{t} e^{(1+\lambda_*)s}[a_{n_k}(s) r_{n_k}(s-h) + b_{n_k}(s)]ds \to pe^{-\lambda_* h} \int_{-\infty}^{t} e^{(1+\lambda_*)s} r_*(s-h)ds.$$

In consequence, we deduce from equation (5.3) that $r_*(t)$ satisfies

$$r_*'(t) = -(1+\lambda_*) r_*(t) + pe^{-\lambda_* h} r_*(t-h). \tag{5.4}$$

Finally, since (5.4) is hyperbolic and r_* is bounded on $\mathbb{R}$, then $r_*(t) \equiv 0$, a contradiction. Hence, $r(-\infty) = 0$ so that

$$\lim_{t\to-\infty} y(t)e^{-\lambda_* t} = r_0 + \lim_{t\to-\infty} d(t)e^{-\lambda_* t}.$$

$\square$

Definition V.4. Set $\sigma(\varepsilon) = \sqrt{1+4\varepsilon^2}$, $|\varepsilon| \ge 0$. For $\lambda_* \in (0,\lambda)$ define the operators $\mathcal{G}, L_\varepsilon, \mathcal{I}_\varepsilon : C_{\lambda_*}(\mathbb{R}) \to C_{\lambda_*}(\mathbb{R})$ as follows

$$(\mathcal{G}y)(t) = g(y(t)), \quad (L_\varepsilon y)(t) = \int_{\mathbb{R}} K(w) y(t - \varepsilon w) dw,$$

$$(\mathcal{I}_\varepsilon y)(t) = \frac{1}{\sigma(\varepsilon)} \Big(\int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} y(s-h)ds + \int_{t}^{+\infty} e^{\frac{(1+\sigma(\varepsilon))(t-s)}{2\varepsilon^2}} y(s-h)ds \Big),\ \varepsilon \ne 0,$$

where $\frac{-2}{1+\sigma(\varepsilon)}$ and $\frac{1+\sigma(\varepsilon)}{2\varepsilon^2}$ are the roots of $\varepsilon^2 z^2 - z - 1 = 0$.

Notice that, since there exists finite $g'(0)$, we have $g(s) = s\gamma(s)$ for a bounded $\gamma \in C(\mathbb{R})$. As a consequence, $\mathcal{G}$ is well-defined and it can be checked directly that $\mathcal{I}_\varepsilon$ and L_ε are well defined too. Moreover, since $|(L_\varepsilon y)(t)| \leq \|y\|$ for all $t \in \mathbb{R}$, and $e^{-\lambda_* t}|(L_\varepsilon y)(t)| \leq \|y\| \int_{\mathbb{R}} K(w)e^{-\lambda_* \varepsilon w}dw$ for all $t \leq 0$, we conclude that L_ε is a linear continuous operator on $C_{\lambda_*}(\mathbb{R})$.

In order to study the existence of positive heteroclinic solutions $\phi(t)$ of (5.1), we will consider the following integral equation:

$$y(t) = \frac{1}{\sigma(\varepsilon)}\Big(\int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}}(\mathcal{G}_\varepsilon y)(s-h)ds + \int_{t}^{+\infty} e^{\frac{(1+\sigma(\varepsilon))(t-s)}{2\varepsilon^2}}(\mathcal{G}_\varepsilon y)(s-h)ds\Big), \tag{5.5}$$

where the operator $\mathcal{G}_\varepsilon := L_\varepsilon \circ \mathcal{G}$.

Each wavefront ϕ being a bounded function should satisfy (5.5). This equation can be written in a shorter form

$$y - (\mathcal{I}_\varepsilon \circ \mathcal{G}_\varepsilon)y = 0. \tag{5.6}$$

Lemma V.5. *$\mathcal{G}_\varepsilon$ is Fréchet continuously differentiable on $C_{\lambda_*}(\mathbb{R})$, with the differential ${\mathcal{G}_\varepsilon}' : C_{\lambda_*}(\mathbb{R}) \to \mathcal{L}(C_{\lambda_*}(\mathbb{R}), C_{\lambda_*}(\mathbb{R}))$, given by ${\mathcal{G}_\varepsilon}'(y_0) : h(\cdot) \to L_\varepsilon \circ \mathcal{G}'(y_0)h(\cdot)$.*

Proof: Since $L_\varepsilon : C_{\lambda_*}(\mathbb{R}) \to C_{\lambda_*}(\mathbb{R})$ is a linear continuous operator, the proof of Lemma V.5 follows directly from [21, Lemma 11]. $\square$

Definition V.6. Now we consider the integral operators $\mathcal{I}_\varepsilon^-, \mathcal{I}_\varepsilon^+ : C_\mu(\mathbb{R}) \to C_\mu(\mathbb{R})$ defined by

$$(\mathcal{I}_\varepsilon^- y)(t) = \int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} y(s-h)ds, \;\; (\mathcal{I}_\varepsilon^+ y)(t) = \int_{t}^{+\infty} e^{\frac{(1+\sigma(\varepsilon))(t-s)}{2\varepsilon^2}} y(s-h)ds.$$

Set $\mathcal{I} := \mathcal{I}_0^-$ and $\mathcal{I}_0^+ := 0$.

In the following lemmas we study the continuity properties of the operator families $\{\mathcal{I}_\varepsilon^\pm, |\varepsilon| < \frac{1}{\sqrt{2\mu}}\}$.

Lemma V.7. *Families* $\mathcal{I}_\varepsilon^\pm : (-\frac{1}{\sqrt{2\mu}}, \frac{1}{\sqrt{2\mu}}) \to \mathcal{L}(C_\mu(\mathbb{R}), C_\mu(\mathbb{R}))$ *are continuous in the operator norm. In particular,* $\mathcal{I}_\varepsilon \to \mathcal{I}$ *uniformly when* $\varepsilon \to 0$.

Proof: This follows by an analysis similar to that realized in the proof of Lemma IV.5.

The proof of the next lemma uses a nice idea from [20].

Lemma V.8. *Families* $\mathcal{I}_\varepsilon \circ L_\varepsilon : (-\frac{1}{\sqrt{2\mu}}, \frac{1}{\sqrt{2\mu}}) \to \mathcal{L}(C_\mu(\mathbb{R}), C_\mu(\mathbb{R}))$ *are continuous in the operator norm. In particular,* $\mathcal{I}_\varepsilon \circ L_\varepsilon \to \mathcal{I}$ *uniformly as* $\varepsilon \to 0$.

Proof: For the convenience of the reader the proof will be divided into several parts.

<u>*Step I.*</u> Set $D_\mu^1(\mathbb{R}) = \left\{ y \in C_\mu(\mathbb{R}) \cap C^1(\mathbb{R}) : \lim\limits_{s\to\pm\infty} y'(s) = 0 \right\}$. First, we establish the existence of a constant $C > 0$, which does not depend on ε, y such that

$$\|\mathcal{I}_\varepsilon^\pm \circ (L_\varepsilon - L_{\varepsilon_0})y\| \le C|\varepsilon - \varepsilon_0|\|y\|, \ y \in D_\mu^1(\mathbb{R}), \ |\varepsilon_0|, |\varepsilon| < \frac{1}{\sqrt{2\mu}}. \tag{5.7}$$

In consequence,

$$\begin{aligned} \|(\mathcal{I}_\varepsilon^\pm \circ L_\varepsilon - \mathcal{I}_{\varepsilon_0}^\pm \circ L_{\varepsilon_0})y\| &\le \|(\mathcal{I}_\varepsilon^\pm \circ (L_\varepsilon - L_{\varepsilon_0})y\| + \|y\|\|\mathcal{I}_\varepsilon^\pm - \mathcal{I}_{\varepsilon_0}^\pm\|\|L_{\varepsilon_0}\| \\ &\le C_1\|y\|(|\varepsilon - \varepsilon_0| + \|\mathcal{I}_\varepsilon^\pm - \mathcal{I}_{\varepsilon_0}^\pm\|), \quad y \in D_\mu^1(\mathbb{R}). \end{aligned} \tag{5.8}$$

Taking into account that $D_\mu^1(\mathbb{R})$ is dense in $C_\mu(\mathbb{R})$, we conclude that inequality (5.8) holds for all $y \in C_\mu(\mathbb{R})$. Thus, we obtain that

$$\|\mathcal{I}_\varepsilon^\pm \circ L_\varepsilon - \mathcal{I}_{\varepsilon_0}^\pm \circ L_{\varepsilon_0}\| \le C(|\varepsilon - \varepsilon_0| + \|\mathcal{I}_\varepsilon^\pm - \mathcal{I}_{\varepsilon_0}^\pm\|), \ |\varepsilon_0|, |\varepsilon| < \frac{1}{\sqrt{2\mu}}. \tag{5.9}$$

The statement of the lemma follows now from (5.9) and Lemma V.7 .

<u>*Step II.*</u> Here, we estimate $\mathcal{I}_\varepsilon^+ \circ (L_\varepsilon - L_{\varepsilon_0})(y)(t)$ for $y \in D_\mu^1(\mathbb{R})$. Since $y(-\infty) = 0$, by exchanging the order of integration and integrating by parts with respect to the

variable s, we get that

$$\begin{aligned}
&\int_t^\infty e^{\frac{1+\sigma(\varepsilon)}{2\varepsilon^2}(t-s)}\int_{\mathbb{R}} K(w)(y(s-\varepsilon w-h)-y(s-\varepsilon_0 w-h))dwds\\
&=(\varepsilon_0-\varepsilon)\int_t^\infty e^{\frac{1+\sigma(\varepsilon)}{2\varepsilon^2}(t-s)}\int_{\mathbb{R}} K(w)\int_0^1 y'(s-(\varepsilon-\varepsilon_0)\gamma w-h-\varepsilon_0 w)wd\gamma\,dw\,ds\\
&=(\varepsilon-\varepsilon_0)\int_0^1\Big(\int_{\mathbb{R}} K(w)wy(t-(\varepsilon-\varepsilon_0)\gamma w-h-\varepsilon_0 w)dw\\
&-\frac{1+\sigma(\varepsilon)}{2\varepsilon^2}\int_t^\infty e^{\frac{1+\sigma(\varepsilon)}{2\varepsilon^2}(t-s)}\int_{\mathbb{R}} K(w)wy(s-(\varepsilon-\varepsilon_0)\gamma w-h-\varepsilon_0 w)dwds\Big)d\gamma.
\end{aligned}$$

In consequence, for all $t\in\mathbb{R}$,

$$\begin{aligned}
|\mathcal{I}_\varepsilon^+\circ(L_\varepsilon-L_{\varepsilon_0})(y)(t)|\le|\varepsilon-\varepsilon_0|\|y\|\int_{\mathbb{R}} K(w)|w|dw\int_0^1\Big(1+\\
\frac{1+\sigma(\varepsilon)}{2\varepsilon^2}\int_t^\infty e^{\frac{1+\sigma(\varepsilon)}{2\varepsilon^2}(t-s)}ds\Big)d\gamma\le 2|\varepsilon-\varepsilon_0|\|y\|\int_{\mathbb{R}} K(w)|w|dw.
\end{aligned}$$

If $t<0$, then

$$\begin{aligned}
e^{-\mu t}|\mathcal{I}_\varepsilon^+\circ(L_\varepsilon-L_{\varepsilon_0})(y)(t)|\le|\varepsilon-\varepsilon_0|\|y\|\int_0^1 C(\gamma,\varepsilon)\Big(1+\\
\frac{1+\sigma(\varepsilon)}{2\varepsilon^2}\int_t^\infty e^{\frac{1+\sigma(\varepsilon)-2\varepsilon^2\mu}{2\varepsilon^2}(t-s)}ds\Big)d\gamma\le 3|\varepsilon-\varepsilon_0|\|y\|\int_0^1 C(\gamma,\varepsilon)d\gamma,
\end{aligned}\tag{5.10}$$

where $C(\gamma,\varepsilon):=\displaystyle\int_{\mathbb{R}} K(w)|w|e^{-\mu((\varepsilon-\varepsilon_0)\gamma w+h+\varepsilon_0 w)}dw$.

Since, $|\varepsilon_0|,|\varepsilon|<\frac{1}{\sqrt{2\mu}}$ and $\gamma\in[0,1]$, we get

$$C(\gamma,\varepsilon)\le\int_{\mathbb{R}} K(w)|w|e^{\mu|(\varepsilon-\varepsilon_0)\gamma w+\varepsilon_0 w|}dw\le\int_{\mathbb{R}} K(w)|w|e^{\mu(h+3\sqrt{2\mu}|w|)}dw.\tag{5.11}$$

Finally, from (5.10) and (5.11) we obtain that

$$e^{-\mu t}|\mathcal{I}_\varepsilon^+\circ(L_\varepsilon-L_{\varepsilon_0})(y)(t)|\le C_1|\varepsilon-\varepsilon_0|\|y\|,\quad t\le 0,$$

where C_1 is a positive constant which does not depend on ε,y.

<u>*Step III.*</u> We can proceed similarly to estimate $\mathcal{I}^-$:

$$\begin{aligned}
&\int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} \int_{\mathbb{R}} K(w)(y(s-\varepsilon w-h)-y(s-\varepsilon_0 w-h))dwds \\
&=(\varepsilon_0-\varepsilon)\int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} \int_{\mathbb{R}} K(w)\int_0^1 y'(s-(\varepsilon-\varepsilon_0)\gamma w-h-\varepsilon_0 w)wd\gamma dwds \\
&=(\varepsilon_0-\varepsilon)\int_0^1\Big(\int_{-\infty}^{+\infty} K(w)wy(t-(\varepsilon-\varepsilon_0)\gamma w-h-\varepsilon_0 w)dw \\
&+\frac{2}{1+\sigma(\varepsilon)}\int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} \int_{\mathbb{R}} K(w)wy(s-(\varepsilon-\varepsilon_0)\gamma w-h-\varepsilon_0 w)dwds\Big)d\gamma.
\end{aligned}$$

In the same manner of step I, we get

$$|\mathcal{I}_\varepsilon^-\circ(L_\varepsilon-L_{\varepsilon_0})(y)(t)|\le 2|\varepsilon-\varepsilon_0|\|y\|\int_{\mathbb{R}} K(w)|w|dw,\ t\in\mathbb{R},$$

and, for all $t<0$,

$$e^{-\mu t}|\mathcal{I}_\varepsilon^-\circ(L_\varepsilon-L_{\varepsilon_0})(y)(t)|\le 2\|y\||\varepsilon-\varepsilon_0|\int_0^1 C(\gamma,\varepsilon)d\gamma\le C_2\|y\||\varepsilon-\varepsilon_0|,$$

where $C_2>0$ is a constant which does not depend on ε, y. Thus (5.7) is proved and the lemma follows. $\square$

5.3 A charecteristic equation

In this section, we analyze the equation

$$\varepsilon^2z^2-z-1+p\exp(-zh)\int_{\mathbb{R}} K(w)\exp(-\varepsilon zw)dw=0,\ \ h>0. \tag{5.12}$$

The next result relates λ_j described above to the roots $\lambda_j(\varepsilon)$ of equation (5.12) with $\varepsilon\neq 0$. Set $\lambda_j(0)=\lambda_j$, $j>1$ and $\lambda_1(0)=\lambda$.

Lemma V.9. *Suppose that $p>1$.*

(1) There exists a real positive number $\varepsilon_0>0$ such that, for every $\varepsilon\in(0,\varepsilon_0)$, the equation (5.12) has exactly two real roots $0<\lambda_1(\varepsilon)<\lambda_\infty(\varepsilon)$ and if $\varepsilon>\varepsilon_0$, then $\chi(z,\varepsilon)>0$ for all $z>0$. Furthermore, the vertical strip $\lambda_1(\varepsilon)\le\Re z\le\lambda_\infty(\varepsilon)$ does not contain complex roots of (5.12) with $\Im z\neq 0$.

(2) Take some $\xi \notin \{\Re\lambda_j, j \in \mathbb{N}\}$. Then there are $\varepsilon_1 = \varepsilon_1(\xi) > 0$, an interval $\mathcal{O}_\xi := (-\varepsilon_1, \varepsilon_1)$ and a positive integer $m = m(\xi)$ such that, for every $0 < |\varepsilon| < \varepsilon_1$, $0 < \lambda_1(\varepsilon) < 2(p-1)$ and $|\varepsilon|^{-1} < \lambda_\infty(\varepsilon) < \varepsilon^{-2}+1$. Moreover, the vertical strip $\xi \leq \Re z \leq 2(p-1)$ contains exactly m roots $\lambda_1(\varepsilon), \ldots, \lambda_m(\varepsilon)$ of (5.12). Next, the functions $\lambda_j : \mathcal{O}_\xi \to \mathbb{C}$ are continuous and the roots $\lambda_j(\varepsilon)$, $j = 1, \ldots, m$, $\varepsilon \in \mathcal{O}_\xi$, are simple.

Proof: By [51, Lemma 20] we obtain there exists $\varepsilon_0 > 0$ such that the first affirmation of item (1) holds. Now we prove that the vertical strip $\lambda_1(\varepsilon) \leq \Re z \leq \lambda_\infty(\varepsilon)$ does not contain complex roots of (5.12) for each $\varepsilon \in (0, \varepsilon_0)$. For this, we first observe that if $\mu(\varepsilon), \nu(\varepsilon)$ are the (real) roots of $\varepsilon^2 z^2 - z - 1 = 0$, then we obtain

$$|\varepsilon^2 z^2 - z - 1| = \varepsilon^2 |z - \mu(\varepsilon)||z - \nu(\varepsilon)| \geq \varepsilon^2 |\Re z - \mu(\varepsilon)||\Re z - \nu(\varepsilon)| = |\varepsilon^2 (\Re z)^2 - \Re z - 1|.$$

Let $H(z, \varepsilon) := 1 + z - \varepsilon^2 z^2$ and $G(z, \varepsilon) := p \exp(-zh) \int_{\mathbb{R}} K(w) \exp(-\varepsilon z w) dw$. We observe that $H(0, \varepsilon) < G(0, \varepsilon)$. Since $\lambda_1(\varepsilon)$ and $\lambda_\infty(\varepsilon)$ are the unique real roots of equation $H(z, \varepsilon) = G(z, \varepsilon)$, then we get that $H(z, \varepsilon) > G(z, \varepsilon)$ for all $z > 0$ such that $\lambda_1(\varepsilon) < z < \lambda_\infty(\varepsilon)$, see figure 5.1.

Thus for all $z \in \mathbb{C}$ such that $\lambda_1(\varepsilon) < \Re z < \lambda_\infty(\varepsilon)$, we obtain that

$$\begin{aligned} &|pe^{-zh} \int_{\mathbb{R}} K(w) e^{-\varepsilon z w} dw| \leq pe^{-\Re z h} \int_{\mathbb{R}} K(w) e^{-\varepsilon \Re z w} dw < 1 + \Re z - \varepsilon^2 \Re z^2 \\ &= |\varepsilon^2 (\Re z)^2 - \Re z - 1| \leq |\varepsilon^2 z^2 - z - 1|, \end{aligned}$$

so that the vertical strip $\lambda_1(\varepsilon) < \Re z < \lambda_\infty(\varepsilon)$ does not contain roots of (5.12) for each $0 < \varepsilon < \varepsilon_0$.

Next, let $z \in \mathbb{C}$ is such that $\Re z = \lambda_1(\varepsilon)$ or $\Re z = \lambda_\infty(\varepsilon)$. If z is root of (5.12), then

$$\Re \left(\varepsilon^2 z^2 - z - 1 + p \exp(-zh) \int_{\mathbb{R}} K(w) \exp(-\varepsilon z w) dw \right) = 0,$$

this implies that

$$\varepsilon^2(\Im z)^2 = -2\int_{\mathbb{R}} K(w)\exp(-\varepsilon z w)\sin^2\left(\frac{\Im z(h+\varepsilon w)}{2}\right)dw,$$

so that $\Im z = 0$.

On the other hand, set $\psi(z,\varepsilon) = \varepsilon^2 z^2 - z - 1 + p\exp(-zh)\int_{\mathbb{R}} K(w)\exp(-\varepsilon z w)dw$. Then $\psi \in C^1(\mathbb{C}\times\mathbb{R},\mathbb{C})$. Arguing as above, there exists a positive number ε_0 such that the equation $\psi(z,\varepsilon) = 0$ has exactly two positive real roots $\lambda_1(\varepsilon) < \lambda_\infty(\varepsilon)$ if and only if $0 < |\varepsilon| < \varepsilon_0$.

Consider $\varepsilon \neq 0$ and let $z = z_0(\varepsilon)$ be a positive root of (5.12). Then $\varepsilon^2 z_0^2(\varepsilon) - z_0(\varepsilon) - 1 < 0$ and hence $0 < z_0(\varepsilon) < \frac{1+\sqrt{1+4\varepsilon^2}}{2\varepsilon^2} < \varepsilon^{-2}+1$. Moreover, analyzing (5.12) we get that $\inf_{0<|\varepsilon|<\varepsilon_0} z_0(\varepsilon) > 0$. In addition, for $|\varepsilon| > 0$ small, we have that

$$\psi(1/|\varepsilon|,\varepsilon) = -\frac{1}{|\varepsilon|} + p\exp(-h/|\varepsilon|)\int_{\mathbb{R}} K(w)\exp(-w\varepsilon/|\varepsilon|)dw < 0,$$

$$\psi(2(p-1),\varepsilon) \leq -p + p\exp(-2(p-1)h)\int_{\mathbb{R}} K(w)\exp(-2\varepsilon(p-1)w)dw < 0.$$

Thus, since $\psi(0,\varepsilon) = p-1 > 0$ and $\psi(\varepsilon^{-2}+1,\varepsilon) > 0$, we can suppose that, for all $0 < |\varepsilon| < \varepsilon_0$,

$$0 < \lambda_1(\varepsilon) < 2(p-1), \quad |\varepsilon|^{-1} < \lambda_\infty(\varepsilon) < \varepsilon^{-2}+1.$$

Next, we claim that the vertical strip $\xi \leq \Re z \leq 2(p-1)$ contains only a fixed number of roots of (5.12) for each small, $|\varepsilon| \neq 0$. First, we fix $0 < \bar\varepsilon < \varepsilon_0$ such that

$$\int_{\mathbb{R}} K(w)e^{\max\{|\xi|,2(p-1)\}\bar\varepsilon|w|}dw < 2.$$

If $z(\varepsilon)$ is a root of (5.12) and $\Re z(\varepsilon) \in [\xi, 2(p-1)]$, then for $|\varepsilon|$ sufficiently small we find that

$$|\varepsilon^2 z^2(\varepsilon) - z(\varepsilon) - 1| \;\; \geq \;\; |\Im(\varepsilon^2 z^2(\varepsilon) - z(\varepsilon) - 1)| \geq \frac{|\Im z(\varepsilon)|}{2},$$

and, for $0 < |\varepsilon| < \bar{\varepsilon}$, we obtain

$$|pe^{-z(\varepsilon)h} \int_{\mathbb{R}} K(w)e^{-\varepsilon z(\varepsilon)w} dw| \leq pe^{-\Re z(\varepsilon)h} \int_{\mathbb{R}} K(w)e^{-\varepsilon \Re z(\varepsilon)w} dw < 2pe^{-\xi h}.$$

Therefore, $|\Im z(\varepsilon)| < 4pe^{-\xi h}$ for all $|\varepsilon|$ sufficiently small.

Next, let $g(z) = -z - 1 + p\exp(-zh)$ and consider the following rectangle $E_2 = [\xi, 2(p-1)] \times [-4pe^{-\xi h}, 4pe^{-\xi h}]$. If $\Re z = 2(p-1)$ or $\Re z = \xi$, then by Lemma II.7 and the definition of ξ we get that $|g(z)| > 0$. In addition, if $|\Im z| = 4pe^{-\xi h}$, we find that

$$|g(z)| \geq |-\Im z + p\Im(e^{-zh})| \geq ||\Im z| - p|\Im(e^{-zh})|| \geq |\Im z| - pe^{-\Re zh} = 3pe^{-\xi h} > 0.$$

On the other hand, the family of analytic functions $\psi(\cdot, \varepsilon)$, $|\varepsilon| < 1$, is uniformly bounded and converges pointwise to $g(\cdot)$ in E_2 as $\varepsilon \to 0$. Then by Montel's Theorem (e.g. see Lemma IV.4.8 in [23]) we obtain that $\lim_{\varepsilon \to 0} \psi(z, \varepsilon) = g(z)$ uniformly on E_2. Hence, applying Rouché's theorem to the analytic functions $\psi(z, \varepsilon)$ and $g(z)$ on E_2, we get that they have the same number of roots (say, m roots) in the vertical strip $\xi \leq \Re z \leq 2(p-1)$, for all small $|\varepsilon|$.

Next, by Lemma II.7 we get that $\psi(\lambda_j, 0) = 0$ and $\psi_z(\lambda_j, 0) \neq 0$. Thus, by the implicit function theorem there exists intervals $(-\varepsilon_j, \varepsilon_j) \subseteq \mathbb{R}$ and C^1-mappings $\lambda_j : (-\varepsilon_j, \varepsilon_j) \to E_2$ such that $\lambda_j(0) = \lambda_j$ and $\psi(\lambda_j(\varepsilon), \varepsilon) = 0$, $j = 1 \ldots m$. If $\varepsilon_* := \min_{i=1,m} \varepsilon_j$, then we can define $\lambda_j : (-\varepsilon_*, \varepsilon_*) \to E_2$, for all $j = 1, \ldots, m$. Finally, since the root λ_j of (2.2) is simple, for each $j \geq 1$, we obtain that $\lambda_j(\varepsilon)$ is simple for each $j \in \{1, \ldots, m\}$ and $\varepsilon \in (-\varepsilon_*, \varepsilon_*)$. $\square$

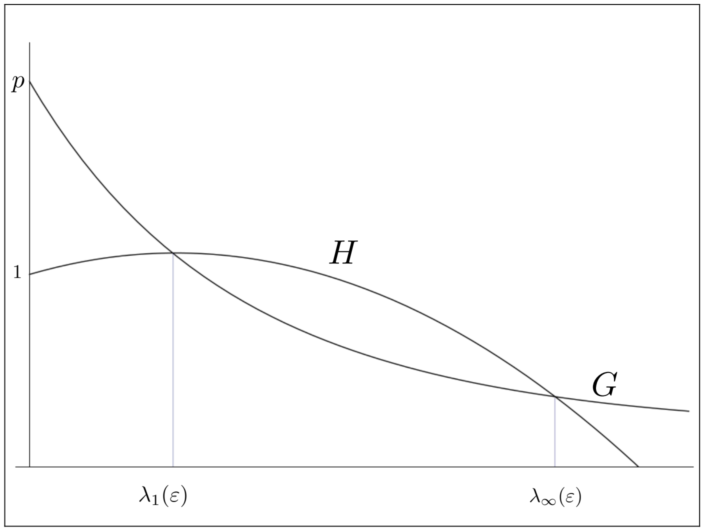

Figure 5.1: $G(z,\varepsilon)$ and $H(z,\varepsilon)$, $z \geq 0$, $|\varepsilon| > 0$ small.

5.4 Asymptotic expansions

In this section, we analyze the asymptotic expansions of solutions to the non-homogenuos equation

$$\varepsilon^2 y''(t) + y'(t) - y(t) + p\int_{\mathbb{R}} K(-s)y(t - \varepsilon s + h)ds = f_\varepsilon(t), \quad t \in \mathbb{R}. \tag{5.13}$$

This equation is singular at $\varepsilon = 0$. Remarkably, under some natural assumptions, each family $\{y_\varepsilon(t)\}$ of bounded solutions of (5.13) admits an asymptotic expansion at $t = +\infty$ which is regular in ε .

In this way, Lemma V.11 extends a result in [4] proved for the local case.

Notation V.10. Throughout the lemma, we denote by $\beta, \gamma, \eta, \rho, b, C, C_1, C_2, C_*, \dots$ some positive constants which do not depend on the parameter $\varepsilon \in \Lambda_j := (-\varepsilon_j, \varepsilon_j)$, where our convention is that $1 > \varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_* > 0$. We also assume that $h > 0, p > 1$.

Lemma V.11. *Let $y_{(\cdot)}(\cdot), f_{(\cdot)}(\cdot) : \Lambda_0 \times \mathbb{R} \to \mathbb{R}$ be continuous functions and y_ε satisfies*

(5.13). Suppose further that $\sup_{t\leq 0}[|y_\varepsilon(t)|+|f_\varepsilon(t)|]\leq C,\ \ |y_\varepsilon(t)|\leq Ce^{-\gamma t},\ t\geq 0,$ *and that* $|f_\varepsilon(t)|\leq Ce^{-bt},\ t\geq 0,\ \varepsilon\in\Lambda_0$*. If* $\gamma<b$*, then given* $\sigma\in(0,b)$*, there exists* $\varepsilon_*>0$ *and continuous bounded functions* $B_j:(-\varepsilon_*,\varepsilon_*)\to\mathbb{C}$ *such that*

$$y_\varepsilon(t)=z_\varepsilon(t)+w_\varepsilon(t),\ t\in\mathbb{R},$$

where $z_\varepsilon(t)=\sum_{\gamma\leq\Re\lambda_j(\varepsilon)<b-\sigma'} B_j(\varepsilon)e^{-\lambda_j(\varepsilon)t}$ *is a finite sum of eigensolutions of (5.13) associated to the roots* $\lambda_j(\varepsilon)\in\{\gamma\leq\Re\lambda_j(\varepsilon)<b-\sigma',\sigma'\in(0,\sigma)\}$ *of (5.12). Furthermore,* $|w_\varepsilon(t)|+|w_\varepsilon'(t)|\leq C_*e^{-(b-\sigma)t},\ t\geq 0,\ \varepsilon\in(-\varepsilon_*,\varepsilon_*)$*.*

Proof: First, observe that the conditions of Lemma V.11 imply that $y_\varepsilon'(t)$ and $y_\varepsilon''(t)$ are bounded on $\mathbb{R}$, for each $\varepsilon\neq 0$. Indeed, y_ε satisfies the equation

$$\varepsilon^2y''(t)+y'(t)-y(t)+G_\varepsilon(t)=0, \tag{5.14}$$

where $G_\varepsilon(t):=p\int_{-\infty}^{+\infty}K(-s)y_\varepsilon(t-\varepsilon s+h)ds-f_\varepsilon(t)$ is uniformly bounded: $|G_\varepsilon(t)|\leq C(p+1)=:C_1,\ t\in\mathbb{R},\ \varepsilon\in\Lambda_0$.

Now as a bounded solution of (5.14), y_ε should satisfy

$$y_\varepsilon(t)=\frac{1}{\sqrt{1+4\varepsilon^2}}\left(\int_{-\infty}^{t}e^{\bar\lambda(t-s)}G_\varepsilon(s)ds+\int_t^{+\infty}e^{\bar\mu(t-s)}G_\varepsilon(s)ds\right),\quad t\in\mathbb{R}, \tag{5.15}$$

where $\bar\lambda=\bar\lambda(\varepsilon)$ and $\bar\mu=\bar\mu(\varepsilon)$ are the roots of $\varepsilon^2z^2+z-1=0$ and $\bar\lambda<0<\bar\mu$.

Differentiating (5.15), we obtain

$$y_\varepsilon'(t)=\frac{1}{\sqrt{1+4\varepsilon^2}}\left(\bar\lambda\int_{-\infty}^{t}e^{\bar\lambda(t-s)}G_\varepsilon(s)ds+\bar\mu\int_t^{+\infty}e^{\bar\mu(t-s)}G_\varepsilon(s)ds\right), \tag{5.16}$$

so that

$$\begin{aligned}|y_\varepsilon'(t)|&\leq\frac{\bar\mu}{\sqrt{1+4\varepsilon^2}}\int_t^{+\infty}e^{\bar\mu(t-s)}|G_\varepsilon(s)|ds+\frac{|\bar\lambda|}{\sqrt{1+4\varepsilon^2}}\int_{-\infty}^{t}e^{\bar\lambda(t-s)}|G_\varepsilon(s)|ds\\ &\leq C_1\Big(\bar\mu\int_t^{+\infty}e^{\bar\mu(t-s)}ds+|\bar\lambda|\int_{-\infty}^{t}e^{\bar\lambda(t-s)}ds\Big)=2C_1,\ \ t\in\mathbb{R}. \qquad (5.17)\end{aligned}$$

In consequence,

$$|y''_\varepsilon(t)| \le \frac{1}{\varepsilon^2}\Big(|y'(t)| + |y(t)| + |G_\varepsilon(t)|\Big) \le \varepsilon^{-2}(3C_1 + C), \quad t \in \mathbb{R}, \varepsilon \neq 0. \tag{5.18}$$

Applying the Laplace transform $\mathcal{L}$ to equation (6.12), we obtain

$$\chi(z,\varepsilon)\tilde{y}_\varepsilon(z) = \tilde{f}_\varepsilon(z) + r_\varepsilon(z), \ \Re z > 0,$$

where $\chi(z,\varepsilon) = \varepsilon^2 z^2 + z - 1 + p\exp(zh)\int_{\mathbb{R}} K(-s)e^{-\varepsilon zs}ds, \ \tilde{y}_\varepsilon = \mathcal{L}\{y_\varepsilon\}, \ \tilde{f}_\varepsilon = \mathcal{L}\{f_\varepsilon\}$ and

$$r_\varepsilon(z) = \varepsilon^2(y'_\varepsilon(0) + zy_\varepsilon(0)) + y_\varepsilon(0) - pe^{zh}\int_{\mathbb{R}} K(-s)e^{-zs\varepsilon}ds\int_{h-s\varepsilon}^0 e^{-zu}y_\varepsilon(u)du.$$

Due to our assumptions, $\tilde{y}_\varepsilon$ is holomorphic in the open half-plane $\{\Re z > -\gamma\}$ and $\tilde{f}_\varepsilon$ is holomorphic in $\{\Re z > -b\}$. Since r_ε is an entire function, we obtain that

$$H_\varepsilon(z) := (\tilde{f}_\varepsilon(z) + r_\varepsilon(z))/\chi(z,\varepsilon)$$

is meromorphic in $\Re z > -b$, with only finitely many poles there.

The rest of the proof is divided into four parts.

Step I. We claim that there are $\sigma' \in (0,\sigma), \ \varepsilon_1 > 0$, such that $|H_\varepsilon(z)| \le C_2/|z|$, if $\Re z = -b + \sigma'$, $\varepsilon \in \Lambda_1$. Indeed, take $\sigma' \in (0,\sigma)$ such that the line $\Re z = -b + \sigma'$ does not contain any eigenvalue $-\lambda_j(\varepsilon), \ \varepsilon \in \Lambda_1$, and $\sigma' - b \neq -1$. We have

$$|\tilde{f}_\varepsilon(z)| \le \int_0^{+\infty} e^{-\Re zt}|f_\varepsilon(t)|dt \le C\int_0^{+\infty} e^{-(\Re z + b)t}dt \le \frac{C}{\sigma'}, \quad \Re z \ge -b + \sigma';$$

$$\begin{aligned} |r_\varepsilon(z)| &\le \varepsilon^2(|y'_\varepsilon(0)| + |z||y_\varepsilon(0)|) + |y_\varepsilon(0)| \\ &\quad + pe^{\Re zh}\int_{\mathbb{R}} K(-s)e^{-\Re zs\varepsilon}\left|\int_{h-s\varepsilon}^0 e^{-zu}y_\varepsilon(u)du\right| ds. \end{aligned}$$

Next, for some fixed $k > -b + \sigma'$, consider the vertical strip $\Sigma_k := \{-b + \sigma' \le \Re z \le k\}$ and set $\rho = \max\{k, b\}$. If we define

$$Q(z) := pe^{\Re zh}\int_{\mathbb{R}} K(-s)e^{-\Re zs\varepsilon}\left|\int_{h-s\varepsilon}^0 e^{-zu}y_\varepsilon(u)du\right| ds,$$

then for all $z \in \Sigma_k$ we have

$$\begin{aligned} Q(z) &\le Cpe^{kh} \int_{\mathbb{R}} K(-s) e^{\rho|s|} e^{\rho(h+|s|)} (h+|s|) ds \\ &\le Cpe^{(k+\rho)h} \int_{\mathbb{R}} K(-s) e^{2\rho|s|} (h+|s|) ds := C_3. \end{aligned}$$

Hence, taking into account (5.17) we find that $|r_\varepsilon(z)| \le C_4(1+\varepsilon^2|z|), z \in \Sigma_k$.

Now set $b_\varepsilon(z) := -1 + pe^{zh} \int_{\mathbb{R}} K(-s) e^{-\varepsilon z s} ds$. We have

$$\begin{aligned} |b_\varepsilon(z)| &\le 1 + pe^{\Re zh} \int_{\mathbb{R}} K(-s) e^{-\varepsilon \Re z s} ds \le 1 + pe^{\Re zh} \int_{\mathbb{R}} K(s) e^{|\varepsilon||\Re z||s|} ds \\ &\le 1 + pe^{kh} \int_{\mathbb{R}} K(s) e^{\rho|s|} ds := \beta, z \in \Sigma_k, \end{aligned} \tag{5.19}$$

so that

$$|z||H_\varepsilon(z)| \le \frac{C_5(|z| + \varepsilon^2|z|^2)}{|\varepsilon^2 z^2 + z + b_\varepsilon(z)|}, \qquad z \in \Sigma_k. \tag{5.20}$$

As it was show in step I in Lemma IV.11, the estimates (5.19) and (5.20) imply the main assertion of Step I.

<u>*Step II.*</u> Taking $k > 0$, we can use the inversion formula

$$y_\varepsilon(t) = \frac{1}{2\pi i} \int_{k-\infty i}^{k+\infty i} e^{zt} \tilde{y}_\varepsilon(z) dz = \frac{1}{2\pi i} \int_{k-\infty i}^{k+\infty i} e^{zt} H_\varepsilon(z) dz, \ t \ge 0. \tag{5.21}$$

By Lemma V.9, $H_\varepsilon(z)$ has only finitely many poles in the strip $-b < \Re z \le 0$. Also, $H_\varepsilon(z) \to 0$ uniformly in the strip $-b + \sigma' \le \Re z \le k$, as $|\Im z| \to \infty$, and $H_\varepsilon(-b + \sigma' + i\cdot) \in L_2(\mathbb{R})$. Thus, we may shift the path of integration in (5.21) to the left (e.g. see [41, p. 8]), to the line $\Re z = -b + \sigma'$, obtaining $y_\varepsilon(t) = z_\varepsilon(t) + w_\varepsilon(t)$, where

$$z_\varepsilon(t) = \sum_{0 \le \Re\lambda_j(\varepsilon) < b - \sigma'} \operatorname{Res}_{-\lambda_j(\varepsilon)} e^{zt} H_\varepsilon(z), \ w_\varepsilon(t) = \frac{1}{2\pi i} \int\limits_{-b+\sigma'-\infty\cdot i}^{-b+\sigma'+\infty\cdot i} e^{zt} H_\varepsilon(z) dz.$$

By Lemma V.9, the roots of equation $\chi(z, \varepsilon) = 0$ are simple for all small $|\varepsilon|$. Hence

$$z_\varepsilon(t) = \sum_{0 \le \Re\lambda_j(\varepsilon) < b - \sigma'} e^{-\lambda_j(\varepsilon)t} B_j(\varepsilon), \text{ with } B_j(\varepsilon) = \frac{\tilde{f}_\varepsilon(-\lambda_j(\varepsilon)) + r_\varepsilon(-\lambda_j(\varepsilon))}{\chi'(-\lambda_j(\varepsilon), \varepsilon)}.$$

It is easy to check that $B_j(\varepsilon)$ is continuous in some open neighborhood of 0 (observe here that the continuity of $y'_\varepsilon(0)$ follows from (5.16)). Take $\lambda_j(\varepsilon)$ such that $-b+\sigma' < -\Re\lambda_j(\varepsilon) \le 0$, then $|r_\varepsilon(-\lambda_j(\varepsilon))| \le C_4(\varepsilon^2|\lambda_j(\varepsilon)|+1) \le C_4(\max_{j,\varepsilon}|\lambda_j(\varepsilon)|+1) := C_7$. In addition, by Lebesgue's theorem on dominated convergence, if $\varepsilon \to 0$ then

$$0 < \Big|\chi'(-\lambda_j(\varepsilon),\varepsilon)\Big| = \Big| -2\varepsilon^2\lambda_j(\varepsilon) + 1 + phe^{-\lambda_j(\varepsilon)h}\int_{\mathbb{R}} K(-s)e^{\varepsilon\lambda_j(\varepsilon)s}ds$$
$$- \varepsilon pe^{-\lambda_j(\varepsilon)h}\int_{\mathbb{R}} K(-s)se^{\varepsilon\lambda_j(\varepsilon)s}ds\Big| \to |1+phe^{-\lambda_j h}| \ne 0.$$

$$\text{Hence, } |B_j(\varepsilon)| \le \frac{|\tilde{f}_\varepsilon(-\lambda_j(\varepsilon))| + |r_\varepsilon(-\lambda_j(\varepsilon))|}{|\chi'(-\lambda_j(\varepsilon),\varepsilon)|} \le \frac{C/\sigma' + C_7}{\min_{j,\varepsilon}|\chi'(-\lambda_j(\varepsilon),\varepsilon)|} \le C_8,$$

if $\varepsilon \in \Lambda_2 = (-\varepsilon_2, \varepsilon_2)$, for some small $\varepsilon_2 > 0$.

Step III. Consider $u_\varepsilon(t) = e^{(b-\sigma')t}w_\varepsilon(t)$ and $v_\varepsilon(t) = e^{(b-\sigma)t}w_\varepsilon(t)$. We have

$$u_\varepsilon(t) = \frac{1}{2\pi i}\int_{-b+\sigma'-\infty\cdot i}^{-b+\sigma'+\infty\cdot i} e^{(s+b-\sigma')t}H_\varepsilon(s)ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi t}H_\varepsilon(-b+\sigma'+i\xi)d\xi.$$

By Plancherel's theorem,

$$\|u_\varepsilon\|_2 = \frac{1}{2\pi}\|H_\varepsilon(-b+\sigma'+i\cdot)\|_2 \le \frac{C_2}{2\sqrt{\pi(b-\sigma')}}.$$

Hence, $v_\varepsilon(t) = e^{-(\sigma-\sigma')t}u_\varepsilon(t)$ is integrable on $[0,+\infty)$, and by the Cauchy-Schwarz inequality

$$\|v_\varepsilon\|_1 \le \frac{\|u_\varepsilon\|_2}{\sqrt{2(\sigma-\sigma')}} \le \frac{C_2}{2\sqrt{2\pi(b-\sigma')(\sigma-\sigma')}}.$$

Step IV. We claim that there exist real numbers $C_9 > 0$ and $\varepsilon_3 > 0$ such that $|w_\varepsilon(t)| \le C_9 e^{-(b-\sigma)t}, t \ge 0$, for all $\varepsilon \in \Lambda_3 = (-\varepsilon_3, \varepsilon_3)$. In order to prove this, it suffices to show that v_ε is uniformly bounded for $\varepsilon \in \Lambda_3$. First for $\varepsilon > 0$, note that

$$\varepsilon^2 w''_\varepsilon(t) + w'_\varepsilon(t) - w_\varepsilon(t) + p\int_{-\infty}^{\frac{t+h}{\varepsilon}} K(-s)w_\varepsilon(t-\varepsilon s+h)ds = F_\varepsilon(t), \quad t \in \mathbb{R},$$

where $F_\varepsilon(t) = f_\varepsilon(t) - p\int_{\frac{t+h}{\varepsilon}}^{+\infty} K(-s)(y_\varepsilon(t-\varepsilon s+h) - z_\varepsilon(t-\varepsilon s+h))ds$. Therefore, $v_\varepsilon(t) = e^{(b-\sigma)t}w_\varepsilon(t)$ satisfies

$$\varepsilon^2 v''_\varepsilon(t) + \alpha v'_\varepsilon(t) = P_\varepsilon(t), \tag{5.22}$$

where $\alpha = 1 - 2\varepsilon^2(b-\sigma) > 0$ and

$$P_\varepsilon(t) = e^{(b-\sigma)t}F_\varepsilon(t) + (1 + (b-\sigma) - \varepsilon^2(b-\sigma)^2)v_\varepsilon(t) \\ - p\int_{-\infty}^{\frac{t+h}{\varepsilon}} K(-s)e^{(b-\sigma)(\varepsilon s - h)}v_\varepsilon(t - \varepsilon s + h)ds.$$

We claim that $P_\varepsilon \in L_1[0,+\infty]$ and $\|P_\varepsilon\|_1 \leq C_{10}$. First, we show that $e^{(b-\sigma)t}F_\varepsilon(t)$ is integrable on $[0,+\infty[$. Indeed, is clear that $e^{(b-\sigma)t}f_\varepsilon(t)$ is integrable on $\mathbb{R}_+$. Fix $\rho > b - \sigma > 0$, we have

$$\begin{aligned} &\int_0^\infty e^{(b-\sigma)t}\int_{\frac{t+h}{\varepsilon}}^\infty K(-s)|y_\varepsilon(t-\varepsilon s+h)|dsdt \\ &\leq C\int_0^\infty e^{(b-\sigma)t}\int_{\frac{t+h}{\varepsilon}}^\infty K(-s)e^{\rho(s-\frac{t+h}{\varepsilon})}dsdt \\ &\leq Ce^{-h\rho}\int_0^\infty e^{(b-\sigma-\rho)t}\int_0^{+\infty} K(-s)e^{\rho s}dsdt := C_{11}. \end{aligned} \tag{5.23}$$

In addition,

$$\begin{aligned} &\int_0^\infty e^{(b-\sigma)t}\int_{\frac{t+h}{\varepsilon}}^{+\infty} K(-s)|z_\varepsilon(t-\varepsilon s+h)|dsdt \\ &\leq \int_0^\infty e^{(b-\sigma)t}\sum_{0\leq\Re\lambda_j(\varepsilon)<b-\sigma'}|B_j(\varepsilon)|\int_{\frac{t+h}{\varepsilon}}^{+\infty} K(-s)e^{(-b+\sigma')(t-\varepsilon s+h)}dsdt \\ &\leq \int_0^\infty e^{(\sigma'-\sigma)t}\sum_{0\leq\Re\lambda_j(\varepsilon)<b-\sigma'}|B_j(\varepsilon)|\int_0^{+\infty} K(-s)e^{-(-b+\sigma')s}dsdt \leq C_{12}. \end{aligned} \tag{5.24}$$

We conclude from (5.23) and (5.24) that $e^{(b-\sigma)t}F_\varepsilon(t) \in L_1[0,+\infty]$. Furthermore, since

$$\begin{aligned} &\int_0^{+\infty}\int_{-\infty}^{\frac{t+h}{\varepsilon}} K(-s)e^{(b-\sigma)(\varepsilon s-h)}|v_\varepsilon(t-\varepsilon s+h)|dsdt \\ &\leq \int_0^{+\infty}\left(\frac{1}{\varepsilon}\int_0^{+\infty} K\Big(\frac{u-t-h}{\varepsilon}\Big)e^{(b-\sigma)|\frac{u-t-h}{\varepsilon}|}dt\right)|v_\varepsilon(u)|du \\ &\leq \int_0^{+\infty}\left(\int_{-\infty}^{+\infty} K(s)e^{(b-\sigma)|s|}ds\right)|v_\varepsilon(u)|du \leq \quad C_{13}\int_0^{+\infty}|v_\varepsilon(u)|du \leq C_{14}, \end{aligned}$$

we find that $\|P_\varepsilon\|_1 \leq C_{10}$ for some $C_{10} > 0$.

Next, by the variation of constants formula, we obtain from (5.22) that

$$v'_\varepsilon(t) = e^{-\frac{\alpha}{\varepsilon^2}t}\left(v'_\varepsilon(0) + \frac{1}{\varepsilon^2}\int_0^t e^{\frac{\alpha}{\varepsilon^2}s}P_\varepsilon(s)ds\right), \; \varepsilon \neq 0. \tag{5.25}$$

A direct integration of (5.25) yields

$$v_\varepsilon(t) = v_\varepsilon(0) + \frac{\varepsilon^2}{\alpha}v'_\varepsilon(0)(1-e^{-\frac{\alpha}{\varepsilon^2}t}) + \frac{1}{\varepsilon^2}\int_0^t\int_0^u e^{\frac{\alpha}{\varepsilon^2}(s-u)}P_\varepsilon(s)dsdu.$$

After changing the order of integration in the iterated integral, we obtain

$$\frac{1}{\varepsilon^2}\left|\int_0^t\int_s^t e^{\frac{\alpha}{\varepsilon^2}(s-u)}P_\varepsilon(s)duds\right| = \frac{1}{\alpha}\left|\int_0^t P_\varepsilon(s)(1-e^{\frac{\alpha}{\varepsilon^2}(s-t)})ds\right| \leq \frac{1}{\alpha}\int_0^t |P_\varepsilon(s)|ds.$$

Additionally, since $v_\varepsilon(t) = e^{(b-\sigma)t}w_\varepsilon(t)$, we get $v'_\varepsilon(0) = (b-\sigma)(y_\varepsilon(0)-z_\varepsilon(0)) + y'_\varepsilon(0) - z'_\varepsilon(0)$. Recalling (5.17), we find that

$$|v'_\varepsilon(0)| \leq (b-\sigma)(|y_\varepsilon(0)| + |z_\varepsilon(0)|) + |y'_\varepsilon(0)| + |z'_\varepsilon(0)| \leq C_{15}.$$

As a consequence, for all small $|\varepsilon|$, we have that

$$|v_\varepsilon(t)| \leq |v_\varepsilon(0)| + \frac{\varepsilon^2}{\alpha}C_{15} + \frac{1}{\alpha}\int_0^{+\infty}|P_\varepsilon(s)|ds \leq C_{16}, \; t \geq 0.$$

Taking into account the estimates

$$|P_\varepsilon(t)| \leq e^{(b-\sigma)t}|F_\varepsilon(t)| + C_{16}\Big(1 + (b-\sigma) + (b-\sigma)^2 + p\int_{\mathbb{R}} K(-s)e^{(b-\sigma)|s|}ds\Big),$$

and

$$e^{(b-\sigma)t}|F_\varepsilon(t)| \leq C + p\int_0^{+\infty} K(-s)\Big(Ce^{\rho s} + \sum_{0\leq \Re\lambda_j(\varepsilon) < b-\sigma'} |B_j(\varepsilon)|e^{-(-b+\sigma')s}\Big)ds,$$

we obtain that $|P_\varepsilon(t)| \leq C_{17}, t \geq 0$ (cf. (5.23), (5.24)). This implies that $|v'_\varepsilon(t)| \leq C_{15} + C_{17}/\alpha$. Since $\alpha = 1 - 2\varepsilon^2(b-\sigma) > 0$, for all ε such that $\varepsilon^2 < \frac{1}{4(b-\sigma)}$, we obtain that $|v'_\varepsilon(t)| \leq C_{18}$. In consequence, $|w'_\varepsilon(t)| \leq C_{19}e^{-(b-\sigma)t}$. Now, a similar reasoning applies to the case $\varepsilon < 0$.

Finally, since $|y_\varepsilon(t)| \le Ce^{-\gamma t}$, $t \ge 0$, $\gamma < b$ and $|w_\varepsilon(t)| \le C_9 e^{-(b-\sigma)t}$, $t \ge 0$, we find that

$$z_\varepsilon(t) = \sum_{\gamma \le \Re\lambda_j(\varepsilon) < b-\sigma'} e^{-\lambda_j(\varepsilon)t} B_j(\varepsilon),\ t \ge 0.$$

□

5.5 Existence of a continuous family of positive wavefronts

In this section, we prove the existence of positive wavefronts of equation (1.6). This amounts to prove the existence of positive heteroclinic solutions of equation (5.1).

Theorem V.12. *Assume I.4 and I.9 . Let* ψ *be some positive heteroclinic solution of equation (1.6):* $\psi(-\infty) = 0, \psi(+\infty) = \kappa$*. Then, for every* $\delta > 0$ *there is a continuous family of positive heteroclinic solutions* $\psi_\varepsilon : (-\varepsilon_*, \varepsilon_*) \to C_{\lambda-\delta}(\mathbb{R})$, $\psi_0 = \psi$, *of equation (5.1). Furthermore, for some continuous* $t_0 = t_0(\varepsilon)$ *we have* $\psi_\varepsilon(t-t_0) = \exp(\lambda_1(\varepsilon)t) + \theta^1_\varepsilon(t)$, $\psi'_\varepsilon(t-t_0) = \lambda_1(\varepsilon)\exp(\lambda_1(\varepsilon)t) + \theta^2_\varepsilon(t) > 0$, *where* $|\theta^i_\varepsilon(t)| \le C\exp((2\lambda-\delta)t)$, $t_0 \le t_0(\varepsilon)$.

Proof: We will suppose here that the C^1-smooth function g is defined and bounded on the whole real axis $\mathbb{R}$.

For $\delta > 0$ small, consider $\lambda_* = \lambda - \delta$. As we have seen, there is a neighborhood Λ of 0 such that the operator $F : C_{\psi,\lambda_*}(\mathbb{R}) \times \Lambda \to C_{\lambda_*}(\mathbb{R})$, $F(\phi,\varepsilon) = \alpha\psi' + \phi - (\mathcal{I}_\varepsilon \circ \mathcal{G}_\varepsilon)(\alpha\psi' + \phi)$ is well defined. Set $\phi_0 := (\psi - \alpha\psi') \in C_{\psi,\lambda_*}(\mathbb{R})$ where $\alpha = \psi^2(0)(2\int_{-\infty}^0 (\psi'(s))^2 ds)^{-1}$. From Lemma V.5 and Lemma V.8 it follows that $F \in C(C_{\psi,\lambda_*}(\mathbb{R}) \times \Lambda, C_{\lambda_*}(\mathbb{R}))$ and $F_\phi(\phi,\varepsilon)$ is continuous in a neighborhood of $(\phi_0, 0)$. Note that

$$F_\phi(\phi,\varepsilon)y = y - \mathcal{I}_\varepsilon \circ \mathcal{G}'_\varepsilon(\alpha\psi' + \phi)y, \quad y \in C_{\psi,\lambda_*}(\mathbb{R}).$$

On the other hand, since

$$\psi(t) - (\mathcal{I}_0^- \circ \mathcal{G})(\psi)(t) = \psi(t) - \int_{-\infty}^{t} e^{-(t-s)} g(\psi(s-h)ds = 0,$$

and

$$(\mathcal{I}_0^- \circ \mathcal{G}'(\psi))(y)(t) = \int_{-\infty}^{t} e^{-(t-s)} g'(\psi(s-h))y(s-h)ds,$$

for all $t \in \mathbb{R}$ and $y \in C_{\psi,\lambda_*}(\mathbb{R})$, we conclude that

$$F(\phi_0, 0) = \alpha\psi' + \phi_0 - (\mathcal{I}_0 \circ \mathcal{G}_0)(\alpha\psi' + \phi_0) = \psi - (\mathcal{I}_0^- \circ \mathcal{G})(\psi) = 0,$$

$$F_\phi(\phi_0, 0) = I - \mathcal{I}_0 \circ \mathcal{G}_0'(\alpha\psi' + \phi_0) = I - \mathcal{I}_0^- \circ \mathcal{G}'(\psi) = I - \mathcal{N}.$$

Hence, applying the Implicit Function Theorem (e.g. see Lemma 2.1 and Remark 2.2(i) in [7, pp. 36-37]), we establish the existence of a continuous family $\phi_\varepsilon : (-\varepsilon_0, \varepsilon_0) \to C_{\psi,\lambda_*}(\mathbb{R})$ of solutions of $F(\phi, \varepsilon) = 0$. Since $\psi_0 = \psi$, $\psi_\varepsilon = \alpha\psi' + \phi_\varepsilon \in C_{\lambda_*}(\mathbb{R})$ satisfy equation (5.6), we obtain that

$$\psi_\varepsilon(+\infty) = \frac{1}{\sigma(\varepsilon)}\mathcal{G}_\varepsilon(\psi_\varepsilon(+\infty))\Big(\frac{1+\sigma(\varepsilon)}{2} + \frac{2\varepsilon^2}{1+\sigma(\varepsilon)}\Big) = g(\psi_\varepsilon(+\infty)) = \kappa.$$

On the other hand, $\psi_\varepsilon(-\infty) = 0$ in view of $\psi_\varepsilon \in C_{\lambda_*}(\mathbb{R})$. Therefore, ψ_ε satisfies all conclusions of the third sentence of the theorem, except its positivity, which is proved below.

Let $\varepsilon_1 \in (0, \varepsilon_0)$ be such that $\lambda_* < \lambda_1(\varepsilon) < 2\lambda_* < \lambda_\infty(\varepsilon)$ for all $\varepsilon \in \mathcal{E}_1 = (-\varepsilon_1, \varepsilon_1)$. Since $\psi_{(\cdot)} : (-\varepsilon_0, \varepsilon_0) \to C_{\lambda_*}(\mathbb{R})$ is continuous, there exists a constant $C_1 > 0$ such that $|\psi_\varepsilon(t)| \le C_1 \exp(\lambda_* t),\ t \le 0$ and $|\psi_\varepsilon(t)| \le C_1$, $t > 0$, for all $\varepsilon \in \mathcal{E}_1$. Thus ψ_ε satisfies

$$\varepsilon^2\psi_\varepsilon''(t) - \psi_\varepsilon'(t) - \psi_\varepsilon(t) + p\int_{\mathbb{R}} K(w)\psi_\varepsilon(t-h-\varepsilon w)dw = \Psi_\varepsilon(t), \tag{5.26}$$

where

$$\Psi_\varepsilon(t) = \int_{\mathbb{R}} K(w)\left(p\psi_\varepsilon(t-h-\varepsilon w) - g(\psi_\varepsilon(t-h-\varepsilon w))\right)dw$$

is continuous in $(\varepsilon, t) \in \mathcal{E}_1 \times \mathbb{R}$. Since $g(s) = ps + O(s^2)$ as $s \to 0$, we have that $|ps - g(s)| \le Ms^2$, $s \in [-s_0, s_0]$ for some $s_0, M > 0$. Furthermore, since $g(s) = s\gamma(s)$ for a bounded $\gamma \in C(\mathbb{R})$, we obtain that $|ps - g(s)| \le \frac{\sup_{s\in\mathbb{R}}|p-\gamma(s)|}{s_0}s^2$, for all $|s| \ge s_0$. Hence $|ps - g(s)| \le C_2 s^2$, $s \in \mathbb{R}$, where $C_2 = \max\{M, \frac{\sup_{s\in\mathbb{R}}|p-\gamma(s)|}{s_0}\}$. Consequently,

$$|\Psi_\varepsilon(t)| \le C_2\|\psi_\varepsilon\|^2 e^{2\lambda_* t}\int_{\mathbb{R}} K(w)e^{2\lambda_*|w|}dw \le C_3 e^{2\lambda_* t}, \quad t \in \mathbb{R}.$$

Setting $y_\varepsilon(t) = \psi_\varepsilon(-t)$, we see that y_ε satisfies

$$\varepsilon^2 y''(t) + y'(t) - y(t) + p\int_{\mathbb{R}} K(-w)y(t + h - \varepsilon w)dw = f_\varepsilon(t), \quad t \ge 0, \tag{5.27}$$

where $f_\varepsilon(t) := \Psi_\varepsilon(-t)$. We observe that y_ε and f_ε satisfy the conditions of Lemma V.11 with $\gamma = \lambda_*$ and $b = 2\lambda_*$. Therefore, for $\delta > 0$ there is $\sigma' \in (0, \delta)$ such that

$$y_\varepsilon(t) = \sum_{\lambda_* \le \Re\lambda_j(\varepsilon) < 2\lambda_* - \sigma'} e^{-\lambda_j(\varepsilon)t}B_j(\varepsilon) + \tilde{w}_\varepsilon(t),$$

where $|\tilde{w}_\varepsilon(t)| + |\tilde{w}'_\varepsilon(t)| \le C_* e^{-(2\lambda_* - \delta)t}$, $t \ge 0$.

Next, we can suppose that $\lambda_* < \lambda_1(\varepsilon) < 2\lambda_* - \delta$ for all $\varepsilon \in \mathcal{E}_2 = (-\varepsilon_2, \varepsilon_2) \subset \mathcal{E}_1$. By Lemmas II.7, V.9, we have that $\Re\lambda_j(\varepsilon) < \lambda_* < \lambda < 2\lambda_* < \lambda_\infty(\varepsilon)$, $j \ge 2$, provided that ε is small (say, $\varepsilon \in \mathcal{E}_2 \subset \mathcal{E}_1$) and λ_* is sufficiently close to λ. In consequence, setting $\theta_\varepsilon(t) = \tilde{w}_\varepsilon(-t)$, we obtain

$$\psi_\varepsilon(t) = B(\varepsilon)\exp(\lambda_1(\varepsilon)t) + \theta_\varepsilon(t), \tag{5.28}$$

$$\psi'_\varepsilon(t) = B(\varepsilon)\lambda_1(\varepsilon)\exp(\lambda_1(\varepsilon)t) + \theta'_\varepsilon(t), \tag{5.29}$$

where $B_\varepsilon : \mathcal{E}_2 \to \mathbb{R}$ is continuous and $|\theta_\varepsilon(t)| + |\theta'_\varepsilon(t)| \le C_* e^{(2\lambda_* - \delta)t}$, $\varepsilon \in \mathcal{E}_2$, $t \le 0$.

Now, consider the heteroclinic solution $\psi(t) = \exp(\lambda t) + z(t)$ of equation (1.5). Observe that $z(t) = O(\exp(2\lambda_* t))$ at $t = -\infty$. Thus we get $1 - B(0) = e^{-\lambda t}(w_0(t) - z(t))$ and

$$|1 - B(0)| \le C_4(e^{(2\lambda_* - \delta - \lambda)t} + e^{(2\lambda_* - \lambda)t}), \quad t \le 0,$$

which is possible only if $B(0) = 1$.

Hence, there are $\mathcal{E}_3 = (-\varepsilon_3, \varepsilon_3) \subset \mathcal{E}_2$ and $T < 0$ (independent of ε) such that

$$\psi_\varepsilon(t) \geq e^{\lambda_1(\varepsilon)t}(0.7 - C_* e^{(2\lambda_* - \delta - \lambda_1(\varepsilon))t}) \geq 0.5 e^{\lambda_1(\varepsilon)t} > 0, \quad t < T,$$

for all $\varepsilon \in \mathcal{E}_3$. On the other hand, we know that $\lim_{\varepsilon \to 0} \psi_\varepsilon(t) = \psi(t)$ uniformly on $\mathbb{R}$ and that ψ is bounded from below by a positive constant on $[T, \infty)$. In consequence, we conclude that ψ_ε is positive on $\mathbb{R}$, if $\varepsilon \in \mathcal{E}_4 = (-\varepsilon_4, \varepsilon_4) \subset \mathcal{E}_3$.

Finally, to complete the proof of Theorem V.12 it suffices to take

$$t_0(\varepsilon) = -\lambda_1^{-1}(\varepsilon) \log(B(\varepsilon)).$$

$\square$

5.6 Non-monotonicity of wavefronts

In this section, we prove that the fast positive wavefronts are non-monotone if continuous g is differentiable at κ and

$$g'(\kappa) h e^{h+1} < -1. \tag{5.30}$$

Set $\Delta(z, \varepsilon) = \varepsilon^2 z^2 - z - 1 + g'(\kappa) \exp(-zh) \int_{\mathbb{R}} K(w) \exp(-zw\varepsilon) dw$.

Lemma V.13. *Assume condition (5.30) and that supp $K \subset [-\eta, \eta]$, for some $\eta > 0$. Then the characteristic equation $\Delta(z, \varepsilon) = 0$ has no real negative roots for all $|\varepsilon|$ sufficiently small.*

Proof: Suppose that $|\varepsilon| < \frac{h}{\eta}$. Since $g'(\kappa) < 0$, for $|\varepsilon|$ small, we have

$$\begin{aligned} \Delta''_z(z, \varepsilon) &= 2\varepsilon^2 + g'(\kappa) \int_{-\eta}^{\eta} K(w) \exp(-z(w\varepsilon + h))(\varepsilon w + h)^2 dw \\ &\leq 2\varepsilon^2 + g'(\kappa) \int_{-\eta}^{\eta} K(w)(\varepsilon w + h)^2 dw < 0, \quad z < 0, \end{aligned}$$

so that $\Delta(z, \varepsilon)$ is strictly convex with respect to $z < 0$ for each $|\varepsilon|$ small. This guaranties the existence of at most two negative roots. Moreover, as it was proved in [21, Lemma 15], $\Delta(z, 0)$ has not real roots once condition (5.30) is satisfied and $\Delta(z, 0) < 0, z \leq 0$ with $\Delta'_z(z, 0) = -1 - hg'(\kappa)e^{-zh}$. If we suppose that $hg'(\kappa) < -1$, then

$$\Delta'_z(0, \varepsilon) = -1 - hg'(\kappa) - \varepsilon g'(\kappa) \int_{-\eta}^{\eta} K(w)wdw > 0,$$

for all $|\varepsilon|$ sufficiently small. Since $\Delta(0, \varepsilon) = -1 + g'(\kappa) < 0$ we get, for $|\varepsilon|$ small, $\Delta(z, \varepsilon) < 0,\ z \leq 0$. Now suppose that $hg'(\kappa) \geq -1$. This implies that $\Delta'_z(0, 0) = -1 - hg'(\kappa) \leq 0$ and since $\lim_{z \to -\infty} \Delta(z, 0) = -\infty$ there exists $z_0 < 0$ such that $\Delta(z_0, 0)$ is a maximum point of $\Delta(z, 0)$. We have $\Delta'_z(z_0, 0) = 0$ and $\Delta''_z(z_0, 0) = -h$, so that we can use the implicit function theorem to deduce the existence of a negative root $z(\varepsilon)$ of the equation $\Delta'_z(z, \varepsilon) = 0$ with $z(0) = z_0$, when $|\varepsilon|$ is small. Moreover, $z(\varepsilon)$ is the absolute maximum point of $z \to \Delta(z, \varepsilon)$ on $]-\infty, 0]$. Finally, since $\Delta(z(\varepsilon), \varepsilon)$ depends continuously on ε, for $|\varepsilon| > 0$ small we have $\Delta(z, \varepsilon) < 0$ for all $z \leq 0$. □

Lemma V.14. *Assume I.9 and condition (5.30). Then every non-constant solution $\psi : \mathbb{R} \to \mathbb{R}$ of (1.6) satisfying $\psi(+\infty) = \kappa$, oscillates about κ.*

Proof: Consider some non-constant solution $\psi : \mathbb{R} \to \mathbb{R}$ of (1.6) such that $\psi(+\infty) = \kappa$. If for some $\eta \in \mathbb{R}$, it holds that $\psi(t) = \kappa$ for all $t \geq \eta$, then we obtain that $g(\psi(t - h)) = \kappa$, $t \geq \eta$. This yields $\psi(t) = \kappa$ for all $t \geq \eta - h$. Repeating this procedure, we find that $\psi(t) \equiv \kappa$, a contradiction.

Let $\sigma(t) := \psi(t) - \kappa \not\equiv 0$. Since $\sigma(+\infty) = 0$, it suffices to prove that σ oscillates about zero. Observe that for some $t_0 \in \mathbb{R}$, σ satisfies the following delay differential equation:

$$\sigma'(t) + \sigma(t) - q(t)\sigma(t-h) = 0,\ t \geq t_0, \tag{5.31}$$

where $q(t) \in C([t_0, +\infty), \mathbb{R}_-)$ and $\lim_{t\to+\infty} q(t) = g'(\kappa) < 0$.

Since the characteristic equation $\Delta(z, 0) = 0$ has no real roots, we obtain that every solution of the limiting equation:

$$\sigma'(t) + \sigma(t) - g'(\kappa)\sigma(t-h) = 0, \tag{5.32}$$

oscillates about zero (see [30, 35]). Now, equation (5.31) has positive and asymptotic constant coefficient. Thus we can apply Theorem 2.4.1 from [30] to conclude that every non-constant solution of (5.31) also oscillates. □

Theorem V.15. *Assume I.4 and I.9, and condition (5.30). Then for each small* $|\varepsilon| > 0$*, the positive wavefront solution* ψ_ε *of equation (1.6) is non-monotone.*

Proof: Let ψ be some positive heteroclinic solution of equation (1.6) and consider the continuous family of positive heteroclinic solutions ψ_ε of equation (1.6) obtained in Theorem V.12. By Lemma V.14, ψ oscillates about κ. This implies that there is $t_1 > 0$ such that $\psi(t_1) > \kappa$. Since $\lim_{\varepsilon\to 0} \psi_\varepsilon = \psi$ uniformly on $\mathbb{R}$, we get $\psi_\varepsilon(t_1) > \kappa$ for every small $|\varepsilon|$. Finally, since $\psi_\varepsilon(+\infty) = \kappa$ and $\psi_\varepsilon(-\infty) = 0$, we conclude that ψ_ε is non-monotone. □

Theorem V.16. *Assume I.9, condition (5.30) and let supp* $K \subset [-\eta, \eta]$*. If* ϕ_ε *is a fast positive wavefront solution of equation (1.6), then* ϕ_ε *oscillates about* κ*.*

Proof: By Lemma (V.13) equation $\Delta(z, \varepsilon) = 0$ has no real negative roots. Therefore we can apply Theorem 6 from [51] to conclude that ϕ_ε is oscillatory for all small $\varepsilon > 0$. □

5.7 Application

Corollary V.17. *Let assumptions of Corollary IV.20 . Then all conclusions of Theorem I.10 hold true.*

Now we apply Theorem I.10 and Corollary V.17 to time-delayed reaction-diffusion population model of Nicholson's and Mackey-Glass. First, we consider the non-local diffusive Nicholson's blowflies equation

$$u_t(t,x) = u_{xx}(t,x) - \delta u(t,x) + p\int_{\mathbb{R}} K(x-w)u(t-h,w)e^{-bu(t-h,w)}dw, \tag{5.33}$$

where $t, x \in \mathbb{R}$ and $\delta, b > 0$.

After a linear rescaling of both variables u and t, we can assume that $\delta = b = 1$. Equation (5.33) can therefore be written in the following normalized form

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + p\int_{\mathbb{R}} K(x-w)u(t-h,w)e^{-bu(t-h,w)}dw. \tag{5.34}$$

The case of interest is $p > 1$ where equation (5.34) has a unique positive steady state $\kappa = \ln p$. Since the birth function $g(s) = pse^{-s}, \quad s \geq 0$, satisfies all conditions of Corollary V.17, then Theorem I.10 assures the existence of positive wavefront of equation (5.34) for each sufficiently large speed c. Moreover, if $\ln p > \dfrac{he^{h+1}+1}{he^{h+1}}$, then these positive wavefront are non-monotone and are oscillating about κ if $K(s)$ has a compact support.

As a second application, let us consider the Mackey-Glass equation with non-monotone nonlinearity birth function $g(s) = \dfrac{ps}{1+s^n}$, $n \geq 1, s \geq 0$. The corresponding reaction-diffusion equation with non-local delay is

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + p\int_{\mathbb{R}} K(x-w)\frac{u(t-h,w)}{1+(u(t-h,w))^n}dw. \tag{5.35}$$

Taking $p > 1$ in equation (5.35), we find that all conditions of Corollary V.17 are satisfied with $\kappa = (p-1)^{1/n}$ and $\Gamma = 1 - n + n/p$. Then Corollary V.17 assures the existence of positive wavefronts of equation (5.35) for each sufficiently large speed c. Now if $p > \dfrac{nhe^{h+1}}{(n-1)he^{h+1}-1}$, then these fast positive wavefront are non-monotone and are oscillating about κ if $K(s)$ has a compact support.

CHAPTER VI

On the uniqueness of positive semi-wavefronts for non-local delayed reaction-diffusion equations

6.1 Introduction

In this chapter we prove the uniqueness (up to translations) of positive wave solutions $u(t,x) = \phi(x+ct)$ satisfying $\phi(-\infty) = 0$ for non-local delayed reaction-diffusion equations (1.1) where $f, g \in C(\mathbb{R}_+, \mathbb{R}_+)$ and the non-negative $K \in L^1(\mathbb{R}_+ \times \mathbb{R})$ satisfy the assumptions I.11 - I.14. However, usual Lipschitz condition $|g(s) - g(t)| \leq g'(0)|s-t|$ is not required here. The uniqueness result is proved for all speeds $c > c_\star$, where $c_\star$ is given in Definition I.15. The proof is based on the observation that, for $c > c_\star$, every two semi-wavefronts profiles to (1.1) have the same "*principal part*" in their asymptotic developments at $-\infty$. However, in difference with [49, 19, 15, 59], this "*principal part*" contains more than one term (typically, $\left[\frac{\lambda_\infty(c)}{\lambda_1(c)}\right] - 1$ terms).

We would like to emphasize that our main interest here is the uniqueness of semi-wavefronts. Therefore, in Theorem I.16 we impose only those conditions which are important for the proof of the uniqueness. It is easy to see that the assumptions of Theorem I.16 do not guarantee the existence of semi-wavefronts (E.g. take linear f, g. The same example shows that positive waves $u(t,x) = \phi(x+ct)$, with $\phi(-\infty) = 0$ and without restrictions on the growth of ϕ at $+\infty$, are generally non-unique). The relevant existence results can be found in [1, 20, 21, 37, 40, 51, 54, 55], observe that this list does not include [15].

On the other hand, integral equations for profile ϕ of semi-wavefront to (1.1) can not be written in the form of nonlinear convolution equation

$$\phi(t) = (g \circ \phi) * k(t), t \in \mathbb{R} \tag{6.1}$$

studied in [15]. As a consequence, if comparing with [15], the implementation of our idea requires new arguments, and our proof is self-contained. Moreover, our approach allows to improve the uniqueness result (Theorem 6.4) of [15] to the following form:

Theorem VI.1. *Suppose that the conditions of Theorem 6.3 from [15] are satisfied (except the condition $g(t) \leq g'(0)t$) and that, in addition, g in (6.1) is such that $|g(t_1) - g(t_2)| \leq L|t_1 - t_2|$ for all $t_1, t_2 \in [0, p]$. If*

$$L \inf_{\lambda>0} \int_{\mathbb{R}} e^{-\lambda s} k(s) ds < 1, \tag{6.2}$$

then there is at most one nontrivial solution ϕ (modulo translation) of (6.1).

The proof of Theorem VI.1 is obtained by applying our methods and following the results of [3] in the section Diekmann-Kaper theory re-visited. Theorem 6.4 in [15] assumes $L = g'(0)$, in such case (6.2) is satisfied automatically (under conditions of the theorem). This also means that the "optimal" L can be taken arbitrarily close to $\left(\inf_{\lambda>0} \int_{\mathbb{R}} e^{-\lambda s} k(s) ds\right)^{-1}$ and is bigger than $g'(0)$.

6.2 Preliminaries

Is clear that the profiles ϕ of the semi-wavefronts $u(t, x) = \phi(x + ct)$ must satisfy for $t \in \mathbb{R}$,

$$y''(t) - cy'(t) - f(y(t)) + \int_0^\infty \int_{\mathbb{R}} K(s, w) g\left(y(t - cs - w)\right) dw ds = 0. \tag{6.3}$$

Equation (6.3) can be written as

$$y''(t) - cy'(t) - \beta y(t) + f_\beta(y(t)) + \int_0^\infty \int_{\mathbb{R}} K(s,w) g(y(t-cs-w)) dw ds = 0,\ t \in \mathbb{R},$$

where $f_\beta(s) = \beta s - f(s)$ for some $\beta > 0$.

Being ϕ a positive bounded function, it should satisfy the integral equation

$$\phi(t) = \frac{1}{\sigma(c)} \left(\int_{-\infty}^t e^{\nu(t-s)} \mathcal{G}(\phi(s)) ds + \int_t^{+\infty} e^{\mu(t-s)} \mathcal{G}(\phi(s)) ds \right), \tag{6.4}$$

where $\sigma(c) = \sqrt{c^2 + 4\beta}$, $\nu < 0 < \mu$ are the roots of $z^2 - cz - \beta = 0$ and $\mathcal{G}(\phi(t)) := \int_0^\infty \int_{\mathbb{R}} K(s,w) g(\phi(t-cs-w)) dw ds + f_\beta(\phi(t))$.

Hence, in order to establish the uniqueness of semi-wavefronts, we have to prove the uniqueness of positive bounded solutions ϕ, $\phi(-\infty) = 0$ of equation (6.4). The proof will involve the following Lipschitz property of $f_\beta(s)$:

Lemma VI.2. *Suppose that f satisfies I.12. Then, for every $M > 0$ there exists $\beta = \beta(M) > 0$ sufficiently large such that $f_\beta(s) \geq 0$ for all $s \geq 0$ and*

$$|f_\beta(s_1) - f_\beta(s_2)| \leq \Big(\beta - \inf_{s \geq 0} f'(s)\Big)|s_1 - s_2|,\ \ s_1, s_2 \in [0, M].$$

Proof: Since f is continuously differentiable on $[0, M]$ and $f(0) = 0$, we can choose $\beta > \inf_{s \geq 0} f'(s)$ such that $f_\beta(s) = \beta s - f(s) \geq 0$ for all $s \in [0, M]$ and

$$\max_{s \in [0,M]} f'(s) \leq 2\beta - \inf_{s \geq 0} f'(s).$$

Take $s_1 < s_2$ in $[0, M]$, then $f(s_2) - f(s_1) = f'(s_0)(s_2 - s_1)$ for some $s_0 \in [s_1, s_2]$. Thus

$$\frac{f_\beta(s_2) - f_\beta(s_1)}{s_2 - s_1} = \beta - \frac{f(s_2) - f(s_1)}{s_2 - s_1} = \beta - f'(s_0) \leq \beta - \inf_{s \geq 0} f'(s), \tag{6.5}$$

and

$$\frac{f_\beta(s_2) - f_\beta(s_1)}{s_2 - s_1} \geq \beta - \Big(2\beta - \inf_{s \geq 0} f'(s)\Big) = -\beta + \inf_{s \geq 0} f'(s). \tag{6.6}$$

From (6.5) and (6.6) we obtain that

$$\left|\frac{f_\beta(s_2) - f_\beta(s_1)}{s_2 - s_1}\right| \le \left(\beta - \inf_{s\ge 0} f'(s)\right), \quad s_1, s_2 \in [0, M],$$

and the lemma follows. □

Lemma VI.3. *Let β be as in Lemma VI.2. If $\phi : \mathbb{R} \to (0, +\infty)$ is a bounded solution of equation (6.3), then*

$$\phi(t)e^{-\nu t} \le \phi(s)e^{-\nu s} \quad \text{and} \quad \phi(t)e^{-\mu t} \ge \phi(s)e^{-\mu s}, \quad t \le s, \tag{6.7}$$

where $\nu < 0 < \mu$ are the roots of $z^2 - cz - \beta = 0$.

Proof: Differentiating (6.4), we obtain

$$\phi'(t) = \frac{1}{\sigma(c)}\left(\nu \int_{-\infty}^{t} e^{\nu(t-s)}\mathcal{G}(\phi(s))ds + \mu \int_{t}^{+\infty} e^{\mu(t-s)}\mathcal{G}(\phi(s))ds\right), \tag{6.8}$$

so that

$$\phi'(t) - \nu\phi(t) = \frac{1}{\sigma(c)}(\mu - \nu)\int_{t}^{+\infty} e^{\mu(t-s)}\mathcal{G}(\phi(s))ds > 0, \tag{6.9}$$

and

$$\phi'(t) - \mu\phi(t) = \frac{1}{\sigma(c)}(\nu - \mu)\int_{-\infty}^{t} e^{\nu(t-s)}\mathcal{G}(\phi_c(s))ds < 0. \tag{6.10}$$

Hence, $(\phi(t)e^{-\nu t})' > 0$ and $(\phi(t)e^{-\mu t})' < 0$ for all $t \in \mathbb{R}$, which imply (6.7). □

6.3 Characteristic equations

In this section, we analyze the roots of the characteristic equation $\chi(z, c) = 0$, where $p, q \ge 0$ and

$$\chi(z, c) := z^2 - cz - q + p\int_0^\infty \int_{\mathbb{R}} K(s, w)e^{-z(cs+w)}dwds. \tag{6.11}$$

Lemma VI.4. *Assume I.13 and suppose that $p > q$. There exists a real number c_0 such that for every $c > c_0$, equation (6.11) has exactly two real roots $0 < \tilde{\lambda}_1(c) < \tilde{\lambda}_\infty(c)$. If $c < c_0$, then $\chi(z, c) > 0$ for all $z > 0$. Furthermore, the vertical strip $\tilde{\lambda}_1(c) \leq \Re z \leq \tilde{\lambda}_\infty(c)$ does not contain complex roots of (6.11) with $\Im z \neq 0$.*

Proof: Set

$$H(z, c) := q + cz - z^2, \quad G(z, c) := p \int_0^\infty \int_{\mathbb{R}} K(s, w) e^{-z(cs+w)} dwds.$$

Then $H(0, c) = q < G(0, c) = p$ and

$$G''_z(z, c) = p \int_0^\infty \int_{\mathbb{R}} K(s, w) e^{-z(cs+w)} (cs + w)^2 dwds > 0,$$

so that $G(z, c)$ is strictly concave with respect to z. As a consequence, equation $H(z, c) = G(z, c)$ has at most two real roots.

Now, note that for $z > 0$, $G(z, c)$ decreases with respect to c while $H(z, c)$ increases with respect c. Thus, we obtain there exists $c_0 > 0$ such that for every $c \in (c_0, +\infty)$, the equation (6.11) has exactly two real roots $0 < \tilde{\lambda}_1(c) < \tilde{\lambda}_\infty(c)$ and if $c < c_0$, then $\chi(z, c) > 0$ for all $z > 0$.

We now prove that the vertical strip $\tilde{\lambda}_1(c) \leq \Re z \leq \tilde{\lambda}_\infty(c)$ does not contain complex roots of (6.11) for each $c > c_0$. We first observe that if $\mu(c), \nu(c)$ are the roots of $z^2 - cz - q = 0$, then for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$|z^2 - cz - q| = |z - \mu(c)||z - \nu(c)| > |\Re z - \mu(c)||\Re z - \nu(c)| = |(\Re z)^2 - c\Re z - q|.$$

Next, since $\tilde{\lambda}_1(c)$ and $\tilde{\lambda}_\infty(c)$ are the unique real roots of equation $H(z, c) = G(z, c)$, we get $H(z, c) > G(z, c)$ for all $z \in \left(\tilde{\lambda}_1(c), \tilde{\lambda}_\infty(c)\right)$, see figure 6.1.

Thus, for all $z \in \mathbb{C} \setminus \mathbb{R}$ such that $\tilde{\lambda}_1(c) \leq \Re z \leq \tilde{\lambda}_\infty(c)$, we obtain that

$$|p \int_0^\infty \int_{\mathbb{R}} K(s, w) e^{-z(cs+w)} dwds| \leq p \int_0^\infty \int_{\mathbb{R}} K(s, w) e^{-\Re z(cs+w)} dwds$$
$$\leq q + c\Re z - \Re z^2 = |(\Re z)^2 - c\Re z - q| < |z^2 - cz - q|,$$

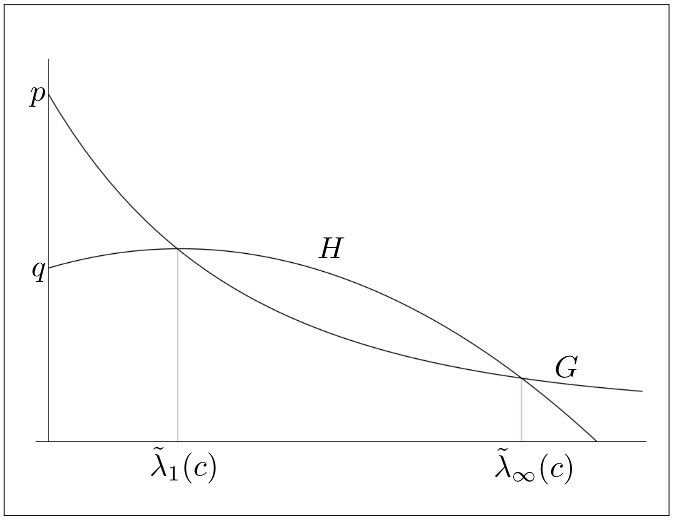

Figure 6.1: $G(z,c)$ and $H(z,c)$, $z \geq 0$, $c > c_0$.

so that the vertical strip $\tilde{\lambda}_1(c) \leq \Re z \leq \tilde{\lambda}_\infty(c)$ does not contain roots $z \in \mathbb{C} \setminus \mathbb{R}$ of (6.11) for each $c > c_0$. $\Box$

6.4 Asymptotic formulae for semi-wave profile

First, following Lemma 22 from [51], we obtain an asymptotic expansion of certain solutions y to the non-homogeneous equation

$$(6.12)\ \ y''(t) + \alpha y'(t) - qy(t) + p\int_0^\infty \int_{\mathbb{R}} K(s,w)y(t+ms+nw)dwds = h(t),\ t \in \mathbb{R},$$

where $\alpha, p, q, m, n \neq 0$. Throughout the section, we assume (I.13) and we denote by $C_1, C_2, \dots$ some positive constants and by $a, b, c, \dots$ some real numbers.

Lemma VI.5. *Let $y \in C^2(\mathbb{R},\mathbb{R})$ verify equation (6.12), where $|h(t)| \leq C_1 e^{bt}$ for all $t \geq 0$. Suppose further that $|y(t)| \leq C_2 e^{at}$, $t \geq 0$ and $|y(t)| \leq C_3 e^{dt}$, $t \leq 0$, for some a, d. If $a > b$, then given $\sigma \in (0, a-b)$ we have that*

$$y(t) = z(t) + \exp((b+\sigma)t)o(1), \quad t \to +\infty,$$

where $z(t)$ is a finite sum of eigensolutions of (6.12) associated to the eigenvalues $\tilde{\lambda}_j(c)$ such that $(b+\sigma) < \Re\tilde{\lambda}_j(c) \leq a$. The analogous result for $t \to -\infty$ also holds.

Proof: First, observe that the conditions of Lemma VI.5 imply the existence of $k \geq a$ such that $|y'(t)|, |y''(t)| \leq C_4 e^{kt}$, $t \geq 0$. Indeed, y satisfies the equation

$$y''(t) + \alpha y'(t) - qy(t) + \mathcal{H}(t) = 0, \tag{6.13}$$

where $\mathcal{H}(t) := p \int_0^{\infty} \int_{\mathbb{R}} K(s,w) y(t + ms + nw) dw ds - h(t)$ is such that

$$\begin{aligned} |\mathcal{H}(t)| \leq p \int_0^{\infty} \Bigg(\int_{-\infty}^{-\frac{t+ms}{n}} & K(s,w) |y(t + ms + nw)| dw \\ & + \int_{-\frac{t+ms}{n}}^{+\infty} K(s,w) |y(t + ms + nw)| dw \Bigg) ds + |h(t)| \\ & \leq C_5 e^{kt} \left(\int_0^{\infty} \int_{-\infty}^{+\infty} K(s,w) e^{k|ms+nw|} dw ds + 1 \right) := C_6 e^{kt}, t \geq 0, \end{aligned}$$

where $k = \max\{|a|, |d|\}$.

Now, from (6.13) for $t \geq 0$ we get that

$$(y'(t)e^{\alpha t})' = e^{\alpha t} \Big(qy(t) - \mathcal{H}(t) \Big),$$

so that

$$y'(t) = e^{-\alpha t} \Big(y'(0) + \int_0^t e^{\alpha s} (qy(s) - \mathcal{H}(s)) ds \Big).$$

Thus, we obtain that

$$|y'(t)| \leq e^{-\alpha t} \left(|y'(0)| + \frac{qC_2 + C_6}{\alpha + k} \Big(e^{(\alpha + k)t} - 1 \Big) \right), \ \alpha + k \neq 0, \ t \geq 0. \tag{6.14}$$

Hence, if $\alpha + k > 0$, then from (6.14) we have $|y'(t)| \leq C_7 e^{kt}$, $t \geq 0$, and if $\alpha + k < 0$, from (6.14) we obtain that $|y'(t)| \leq C_8 e^{-\alpha t}$, $t \geq 0$. Finally, from (6.13) we obtain easily similar estimations for $y''(t)$, $t \geq 0$, for both cases.

Applying the Laplace transform to (6.12), we can prove the assertion using the same method of proof of [51, Lemma 22]. □

The next lemma is crucial in the proof of uniqueness, it gives an asymptotic expansion of positive solutions ϕ of equation (6.3). Since $g'(0) > f'(0)$, Lemma VI.4

implies that equation (1.9) has two positive roots $\lambda_1(c) < \lambda_\infty(c)$ if and only if $c > c_*$. Moreover, the vertical strip $\lambda_1(c) \leq \Re z \leq \lambda_\infty(c)$ does not contain roots $\lambda_j(c) \in \mathbb{C}\setminus\mathbb{R}$ of (1.9).

Lemma VI.6. *Suppose that $f'(0+), g'(0+)$ are finite and let ϕ be a positive semi-wavefront solution of equation (6.3). Then for each $\rho \in (0,1)$ there exist $\Gamma > 0$ and $t_0 < 0$ such that $\sup_{t\leq t_0} \frac{\phi(t-\Gamma)}{\phi(t)} \leq \rho$.*

Proof: Suppose that, contrary to our claim, there exist $\rho \in (0,1)$ and $t_n \to -\infty$ such that for $\Gamma = -\frac{\ln\frac{\rho}{2}}{\lambda_1(c)}$, it holds $\phi(t_n - \Gamma) > \rho\phi(t_n)$. Now, since ϕ satisfies (6.3), we conclude that $\phi_n(t) := \dfrac{\phi(t+t_n)}{\phi(t_n)}$ is a positive solution of

$$\begin{aligned}&\phi_n''(t) - c\phi_n'(t) - \beta\phi_n(t) + \int_0^\infty \int_{\mathbb{R}} K_n(s,w)\phi_n(t-cs-w))dwds\\ &+ b_n(t)\phi_n(t) = 0,\end{aligned}$$

where $K_n(s,w) := K(s,w)a_n(t,w),\ a_n(t,w) := \dfrac{g(\phi(t+t_n-cs-w))}{\phi(t+t_n-cs-w)}$ and $b_n(t) := \dfrac{f_\beta(\phi(t+t_n))}{\phi(t+t_n)}$.

From (6.7), it follows that $e^{\mu t} \leq \phi_n(t) \leq e^{\nu t}$ for all $t \leq 0$ and $e^{\nu t} \leq \phi_n(t) \leq e^{\mu t}$ for all $t \geq 0$. Note also that $\phi_n(-\Gamma) > \rho$, $b_n(t) < \beta$ and since $g'(0)$ exists, $a_n(t,w) \leq C_9$ for all $n \in \mathbb{N}$ and $t,w \in \mathbb{R}$. Moreover, $\lim_{n\to\infty} K_n(s,w) = K(s,w)g'(0)$ and $\lim_{n\to\infty} b_n(t) = f_\beta'(0)$ pointwise. Set

$$G_n(t) = \int_0^\infty \int_{\mathbb{R}} K_n(s,w)\phi_n(t-cs-w)dwds + b_n(t)\phi_n(t).$$

Then for each $T > 0$ there exists $C_{10} := C_{10}(T)$ such that for all $t \in [-T,T]$ we have

$|G_n(t)| \le C_{10}$. Indeed,

$$\begin{aligned}
0 \le G_n(t) &\le C_9 \int_0^\infty \int_{\mathbb{R}} K(s,w)\phi_n(t-cs-w))dwds + \beta\phi_n(t) \\
&\le C_9 \int_0^\infty \left(\int_{-\infty}^{t-cs} K(s,w)e^{\mu(t-cs-w)}dw + \int_{t-cs}^{+\infty} K(s,w)e^{\nu(t-cs-w)}dw \right) ds \\
&\quad + \beta \max\{e^{|\nu|t}, e^{\mu t}\} \\
&\le C_9 e^{\zeta T} \int_0^\infty \int_{\mathbb{R}} K(s,w)e^{\zeta|cs+w|}dwds + \beta \max\{e^{|\nu|T}, e^{\mu T}\} := C_{10},
\end{aligned}$$

where $\zeta := \max\{\mu, |\nu|\}$.

Now, observe that (6.9) and (6.10) imply that $\nu\phi(t) \le \phi'(t) \le \mu\phi(t)$ for all $t \in \mathbb{R}$ so that

$$|\phi_n'(t)| \le \max\{|\nu|,\mu\}\phi_n(t) \le \max\{|\nu|,\mu\}e^{\max\{|\nu|,\mu\}T}.$$

In this way, we may apply the Ascoli-Arzela compactness criterion together with a diagonal argument on each of the intervals $[-T,T]$ to find a subsequence $\{\phi_{n_j}(t)\}$ converging, in the compact open topology, to a non-negative function $\phi_* : \mathbb{R} \to \mathbb{R}$. Note that $\phi_*(-\Gamma) > \rho$ and $e^{\mu t} \le \phi_*(t) \le e^{\nu t}$ for all $t \le 0$ and $e^{\nu t} \le \phi_*(t) \le e^{\mu t}$ for all $t \ge 0$. By the Lebesgue's dominated convergence theorem, we have for every fixed $t \in \mathbb{R}$

$$G_n(t) \to G_*(t) := g'(0) \int_0^\infty \int_{\mathbb{R}} K(s,w)\phi_*(t-cs-w))dwds + f_\beta'(0)\phi_*(t).$$

In consequence, integrating

$$\phi_n'(t) = \int_0^t e^{c(t-s)}(\beta\phi_n(s) - G_n(s))ds$$

between 0 and t and then taking the limit as $n_j \to \infty$, we establish that ϕ_* satisfies

$$\phi_*''(t) - c\phi_*'(t) - f'(0)\phi_*(t) + g'(0)\int_0^\infty \int_{\mathbb{R}} K(s,w)\phi_*(t-cs-w))dwds = 0. \tag{6.15}$$

By Lemma VI.5, for almost every $b < \lambda_\infty(c)$

$$\phi_*(t) = z(t) + w_+(t), \quad t \in \mathbb{R},$$

where $z(t)$ is a finite sum of eigensolutions of (6.15) associated to the eigenvalues $\lambda_j(c)$ with $b < \Re\lambda_j(c) \le \mu$ and $w_+(t) = O(e^{bt}), t \to +\infty$. Since $\phi_*(t) > 0$ for all $t \in \mathbb{R}$, then this sum does not contain eigenfunctions associated to eigenvalue with $\Re\lambda_j > \lambda_\infty(c)$ so that $b < \Re\lambda_j(c) \le \lambda_\infty(c)$. We now choose $b = \lambda_1(c) - \delta$, $\delta > 0$ small. Since the strip $\lambda_1(c) - \delta < \Re\lambda_j(c) \le \lambda_\infty(c)$ does not contain complex roots of (1.9) with $\Im\lambda_j(c) \ne 0$, we obtain

$$\phi_*(t) = A_1(c)e^{\lambda_1(c)t} + A_2(c)e^{\lambda_\infty(c)t} + w_+(t), \quad t \in \mathbb{R},$$

where $A_j(c) \in \mathbb{R}$ and $w_+(t) = O(e^{(\lambda_1(c)-\delta)t})$, $t \to +\infty$.

In a similar way, we obtain

$$\phi_*(t) = B_1(c)e^{\lambda_1(c)t} + B_2(c)e^{\lambda_\infty(c)t} + w_-(t), \quad t \in \mathbb{R},$$

where $w_-(t) = O(e^{(\lambda_\infty(c)+\sigma)t})$, $t \to -\infty$, $B_j(c) \in \mathbb{R}$ and $\sigma > 0$ small.

Since

$$w_-(t) = (A_1(c) - B_1(c))e^{\lambda_1(c)t} + (A_2(c) - B_2(c))e^{\lambda_\infty(c)t} + w_+(t), \quad t \in \mathbb{R},$$

we have $w_-(t) = O(e^{\lambda_\infty(c)t}), t \to +\infty$. Thus, for $z \in \mathbb{C}$ such that $\lambda_\infty(c) < \Re z < \lambda_\infty(c) + \sigma$, we can define the two-sided Laplace transform of w_-:

$$W(z) := \int_{\mathbb{R}} e^{-zt} w_-(t)dt.$$

Since w_- is also a solution of (6.15), applying the Laplace transform to (6.15) we obtain that $\chi_0(z,c)W(z) = 0$ for all $\lambda_\infty(c) < \Re z < \lambda_\infty(c) + \sigma$. But $\chi_0(z,c) \ne 0$ if $\lambda_\infty(c) < \Re z < \lambda_\infty(c) + \sigma$ so that $W(z) = 0$. By the Inversion Theorem [56, Theorem 6b, p.244] we get that $w_-(t) = 0$ for all $t \in \mathbb{R}$. Therefore

$$\phi_*(t) = B_1(c)e^{\lambda_1(c)t} + B_2(c)e^{\lambda_\infty(c)t}, \quad t \in \mathbb{R}, \tag{6.16}$$

where $B_1(c), B_2(c) \geq 0$.

Finally, from (6.16) we get that

$$\begin{aligned}\phi_*(t) &= e^{\lambda_1(c)(t-s)}\Big(B_1(c)e^{\lambda_1(c)s} + B_2(c)e^{\lambda_\infty(c)s}e^{(\lambda_\infty(c)-\lambda_1(c))(t-s)}\Big)\\ &\leq e^{\lambda_1(c)(t-s)}\phi_*(s), \quad t \leq s.\end{aligned}$$

This implies that $\phi_*(-\Gamma) \leq e^{-\Gamma\lambda_1(c)}\phi_*(0) = e^{-\Gamma\lambda_1(c)} = \frac{\rho}{2}$, a contradiction. □

Lemma VI.7. *Assume that $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ are functions differentiable at 0. Then any positive semi-wavefront solution ϕ of equation (6.3) satisfies that $\phi(t) = O(e^{\gamma t})$ as $t \to -\infty$, for some $\gamma = \gamma(c) > 0$.*

Proof: Let $\rho \in (0, 1)$, then Lemma VI.6 implies that there are $\Gamma > 0$ and $t_0 < 0$ such that $\phi(t - \Gamma) \leq \rho\phi(t)$ for all $t \leq t_0$. Now, we define the function $h(t) = \phi(t)e^{-\gamma t}$, $t \in \mathbb{R}$, where $\gamma = \frac{1}{\Gamma}\ln\frac{1}{\rho} > 0$. Then,

$$h(t - \Gamma) = \phi(t - \Gamma)e^{-\gamma(t-\Gamma)} \leq \rho e^{\gamma\Gamma}\phi(t)e^{-\gamma t} = h(t).$$

This implies that $\sup_{t<t_0} h(t)$ is finite. Hence, $\phi(t) = O(e^{\gamma t})$ as $t \to -\infty$.

We observed that the proof of Lemma VI.6 is a new form to prove the lemma above. See [**?**, Theorem 21] for the other form prove it. □

Lemma VI.8. *Suppose I.14. Let ϕ be a positive solution of equation (6.3) with $c > c_0$ such that $\sup_{t\geq 0}\phi(t)$ is finite. Then, there is $t_0 \in \mathbb{R}$ and a small $\delta > 0$ such that either*

$$\phi(t - t_0) = e^{\lambda_1(c)t} + w(t),\ t \in \mathbb{R},$$

or

$$\phi(t - t_0) = e^{\lambda_\infty(c)t} + \bar{w}(t),\ t \in \mathbb{R},$$

where $\lambda_1(c) < \lambda_\infty(c)$ *are the positive roots of equation (1.9),* $w(t) = O(e^{(\lambda_1(c)+\delta)t})$ *and* $\bar{w}(t) = O(e^{(\lambda_\infty(c)+\delta)t})$ *as* $t \to -\infty$*.*

Proof: By Lemma VI.7 there exists $\gamma > 0$ such that $\phi(t) = O(e^{\gamma t})$ as $t \to -\infty$. Since $\phi(t) \geq \phi(0)e^{\mu t}$ for all $t \leq 0$, ϕ has not super exponential decay at $t = -\infty$. Thus, without restricting the generality, we may assume that γ is "almost optimal" in the sense that $\phi(t) = O(e^{\gamma t})$ as $t \to -\infty$, but for all $\rho > 0$, $\phi(t) \neq O(e^{(\gamma+\rho)t}), t \to -\infty$. Being ϕ a solution of equation (6.3), it satisfies

$$y''(t) - cy'(t) - f'(0)y(t) + g'(0)\int_0^\infty \int_{\mathbb{R}} K(s,w)y(t-cs-w)dwds = h(t),\ t \in \mathbb{R}, \tag{6.17}$$

where

$$\begin{aligned} h(t) = &\int_0^\infty \int_{\mathbb{R}} K(s,w)\Big(g'(0)\phi(t-cs-w) - g(\phi(t-cs-w))\Big)dwds \\ &+ f(\phi(t)) - f'(0)\phi(t). \end{aligned}$$

Since g satisfies I.14, $|g'(0)s - g(s)| \leq Qs^{\theta+1}$ for all $s \in [0, \varepsilon]$ so that there exists C_{11} such that $|g'(0)s - g(s)| \leq C_{11}s^{\theta+1}$, $s \in [0, \sup_{t\in\mathbb{R}} \phi(t)]$. Similar arguments apply to f. Consequently, for all $t \in \mathbb{R}$

$$|h(t)| \leq C_{12}e^{\gamma(\theta+1)t}\left(\int_0^\infty \int_{\mathbb{R}} K(s,w)e^{-(\gamma(\theta+1))(cs+w)}dwds + 1\right) := C_{13}e^{\gamma(\theta+1)t}.$$

Next, consider $\psi(t) = \phi(-t)$, then ψ satisfies

$$y''(t) + cy'(t) - f'(0)y(t) + g'(0)\int_0^\infty \int_{\mathbb{R}} K(s,w)y(t+cs+w)dwds = H(t), \tag{6.18}$$

where $H(t) = h(-t)$. Note that ψ and $H(t)$ satisfy of conditions of Lemma VI.5 with $a = -\gamma$, $b = -\gamma(\theta+1)$ and $d = 0$. Hence, we have that for all $0 < \sigma < \gamma\theta$,

$$\psi(t) = \tilde{z}(t) + \tilde{w}(t), \quad t \geq 0,$$

where $\tilde{z}(t)$ is a finite sum of eigensolutions of (6.18) associated to the eigenvalues $\tilde{\lambda}_j(c)$ such that $-\gamma(\theta+1)+\sigma < \Re\tilde{\lambda}_j(c) \leq -\gamma$ and the function $\tilde{w}(t) = O(e^{(-\gamma(\theta+1)+\sigma)t})$ as $t \to +\infty$. Observe now that $\tilde{\lambda}$ is an eigenvalue of (6.18) if and only if $-\tilde{\lambda}$ is a root of (1.9). Thus,

$$\phi(t) = z(t) + w(t), \quad t \leq 0,$$

where $z(t) := \tilde{z}(-t)$ is a finite sum of eigensolutions of (6.17) associated to the eigenvalues $\lambda_j(c)$ such that $\gamma \leq \Re\lambda_j(c) < \gamma(\theta+1) - \sigma$ and $w(t) := \tilde{w}(-t) = O(e^{(\gamma(\theta+1)-\sigma)t})$ as $t \to -\infty$.

Now, being $\phi(t) > 0$ observe that $[\gamma, \gamma(\theta+1)] \cap \{\lambda_1(c), \lambda_\infty(c)\} \neq \emptyset$ so that for some σ small we have

$$\phi(t) = B_1(c)e^{\lambda_1(c)t} + B_2(c)e^{\lambda_\infty(c)t} + O(e^{(\lambda_\infty(c)+\delta)t}),\ t \in \mathbb{R},$$

where $B_1(c), B_2(c) \in \mathbb{R}$ can not both be zero and $\delta > 0$ is small. Next, if $B_1(c) \neq 0$, then the positivity of ϕ implies that $B_1(c) > 0$ and the first affirmation of lemma follows. Otherwise, if $B_1(c) = 0$, then $B_2(c) > 0$ and the second affirmation of lemma follows. $\square$

6.5 Uniqueness of positive semi-wavefront

In this section we establish the uniqueness of the positive semi-wavefront of equation (6.3) for each speed $c > c_\star$, where $c_\star$ is given in (1.8). In the sequel, we will assume that I.11- I.14 hold.

In the following lemma we will prove that the asymptotic formula with $\lambda_\infty(c)$ in Lemma VI.8 can not happen when $c > c_\star$.

Lemma VI.9. *If ψ is a positive semi-wavefront of equation (6.3) with $c > c_\star$, then* *there are* $t_0 \in \mathbb{R}$ *and small* $\delta > 0$ *such that*

$$\psi(t - t_0) = e^{\lambda_1(c)t} + w(t),\ t \in \mathbb{R},$$

where $\lambda_1(c)$ *is the smallest positive root of equation (1.9) and* $w(t) = O(e^{(\lambda_1(c)+\delta)t})$ *as* $t \to -\infty$.

Proof: Suppose that the assertion of the lemma is false. Then Lemma VI.8 assures that for all $z \in \mathbb{C}$ such that $0 < \Re z < \lambda_\infty(c)$ the two-side Laplace transform of ψ,

$$\Psi(z) = \int_{\mathbb{R}} e^{-zs}\psi(s)\,ds, \quad \text{is well defined.}$$

Next, we observe that ψ satisfies the equation

(6.19)
$$\psi''(t) - c\psi'(t) - \Big[\inf_{s\geq 0} f'(s)\Big]\psi(t) + L\int_0^\infty \int_{\mathbb{R}} K(s,w)\psi(t - cs - w)dwds = h(t).$$

Here

$$\begin{aligned} h(t) = & \int_0^\infty \int_{\mathbb{R}} K(s,w)(L\psi - g\circ\psi)(t - cs - w)dwds \\ & + \Big(\beta - \inf_{s\geq 0} f'(s)\Big)\psi(t) - f_\beta(\psi(t)) \end{aligned}$$

is a non-negative bounded function such that $h(t) \leq Ce^{\lambda_\infty(c)t}, t \leq 0$, for some $C > 0$. Indeed,

$$Ls - g(s) \leq s\left(L + \sup_{s\in A}\frac{g(s)}{s}\right), \quad \text{where} \quad A := (0, \sup_{t\in\mathbb{R}}\psi(t)].$$

The same reasoning applies to $\Big(\beta - \inf_{s\geq 0} f'(s)\Big)\psi(t) - f_\beta(\psi(t))$. Therefore, for some $C_1 > 0$,

$$0 \leq h(t) \leq C_1 e^{\lambda_\infty(c)t}\left(\int_0^\infty \int_{\mathbb{R}} K(s,w)e^{-\lambda_\infty(c)(cs+w)}dwds + 1\right), \quad t \leq 0.$$

Next, from equations (6.3), (6.9) and (6.10) we obtain easily that $\psi'(t)$ and $\psi''(t)$ are bounded in $\mathbb{R}$ and $\psi'(t), \psi''(t) = O(e^{\lambda_\infty(c)t})$ at $t \to -\infty$. Applying the two-side Laplace transform to (6.19), we get that $\chi_L(z,c)\Psi(z) = H(z)$, where $H(z) = \int_{\mathbb{R}} e^{-zs}h(s)\,ds$. Moreover, $\Psi(z)$ and $H(z)$ are analytic in $0 < \Re z < \lambda_\infty(c)$. As a consequence, the function $\Psi(z) = \frac{H(z)}{\chi_L(z,c)}$ has removable singularities in $0 < \Re z < \lambda_\infty(c)$.

Now, if we suppose that $h(t) = 0, t \in \mathbb{R}$, then $H(z) = 0$ for all $z \in \mathbb{C}$ so that $\Psi(z) = 0$ in $0 < \Re z < \lambda_\infty(c)$. By the Inversion Theorem we have $\psi(t) = 0$ for all $t \in \mathbb{R}$, a contradiction. Hence, $h(t) > 0$ on some subinterval of $\mathbb{R}$ and $H(\gamma_1(c)), H(\gamma_\infty(c)) > 0$, where $\gamma_1(c)$ and $\gamma_\infty(c)$ are the positive roots of $\chi_L(z,c) = 0$. Then we get that $\gamma_1(c)$ and $\gamma_\infty(c)$ are simple poles of $\Psi(z)$, a contradiction.

Theorem VI.10. *Suppose that ϕ and ψ are two different positive semi-wavefront solutions of (6.3) and $c > c_\star$. Then there exists $t_0(c) \in \mathbb{R}$ such that $\phi(t - t_0) = \psi(t)$ for all $t \in \mathbb{R}$.*

Proof: Due to the Lemma VI.9, we can assume that ϕ, ψ have the same asymptotic representation $\phi(t), \psi(t) = e^{\lambda_1(c)t} + O(e^{(\lambda_1(c)+\delta)t})$ at $-\infty$, for some $\delta > 0$ small. We will divide the proof into three step.

Step I. By way of contradiction, suppose that $\Omega(t) := \phi(t) - \psi(t) \neq 0$. Then, $\Omega(-\infty) = 0$, $\sup_{t>0} \Omega(t)$ is finite and $\Omega(t) = O(e^{(\lambda_1(c)+\delta)t})$ as $t \to -\infty$. Moreover, Ω satisfies the equation

$$y''(t) - cy'(t) - f'(0)y(t) + g'(0)\int_0^\infty \int_{\mathbb{R}} K(s,w)y(t - cs - w)dwds = h(t),$$

where

$$h(t) = \int_0^\infty \int_{\mathbb{R}} K(s,w)(g'(0)\Omega - g\circ\phi + g\circ\psi))(t - cs - w)dwds$$
$$+ \quad f(\phi(t)) - f(\psi(t)) - f'(0)\Omega(t).$$

Since g satisfies (I.14), for all $s_1, s_2 \in [0,\varepsilon]$ we get that

$$|g(s_1) - g(s_2) - g'(0)(s_1 - s_2)| \le \left|\frac{g(s_1) - g(s_2)}{s_1 - s_2} - g'(0)\right| |s_1 - s_2|$$
$$\le |g'(s) - g'(0)||s_1 - s_2| \le Qs^\theta |s_1 - s_2|,$$

where s is a number between s_1 and s_2. Moreover, (I.11) implies that

$$|g(s_1) - g(s_2) - g'(0)(s_1 - s_2)| \le \frac{L + g'(0)}{(2\varepsilon)^\theta}|s_1 - s_2|(s_1 + s_2)^\theta, s_1, s_2 > \varepsilon.$$

Similar arguments apply to f. Consequently, we obtain easily that the function h satisfies $h(t) = O(e^{(\lambda_1(c)(\theta+1)+\delta)t})$ as $t \to -\infty$.

Since the characteristic equation (1.9) does not have roots for $\lambda_1(c) < \Re z < \lambda_\infty(c)$, a proceed analogously to the prove of Lemma VI.8 enables us to write $\Omega(t) = O(e^{\gamma t})$ as $t \to -\infty$, where $\gamma \ge \lambda_\infty(c)$ is almost optimal.

Step II. Let $\gamma_1(\varepsilon) < \gamma_\infty(\varepsilon)$ be the positive roots of equation $\chi_L(z,c) = 0$ and fix $m > 0$ such that $\gamma_1(c) < m < \gamma_\infty(c)$. Then

$$m^2 - cm - \inf_{s\ge 0} f'(s) + L\int_0^\infty \int_{\mathbb{R}} K(s,w)e^{-m(cs+w)}dwds < 0, \tag{6.20}$$

and if we define $P := \beta - \inf_{s\ge 0} f'(s)$, then (6.20) implies that

$$\frac{P + L\displaystyle\int_0^\infty \int_{\mathbb{R}} K(s,w)e^{-m(cs+w)}dwds}{\beta + cm - m^2} < 1.$$

Note that $\lambda_1(c) \le \gamma_1(c) < m < \gamma_\infty(c) \le \lambda_\infty(c)$.

Step III. We now define $\bar\Omega(t) := |\Omega(t)|e^{-mt} \ge 0$, $t \in \mathbb{R}$. Then step I implies that $\bar\Omega(\pm\infty) = 0$ and $\bar\Omega(\tau) = \max_{s\in\mathbb{R}} \bar\Omega(s) > 0$ for some $\tau \in \mathbb{R}$.

Next, take $M := \max\{\sup_{t\in\mathbb{R}} \psi(t), \sup_{t\in\mathbb{R}} \phi(t)\}$ in Lemma VI.2. This lemma and integral equation (6.4) imply that

$$\begin{aligned}
|\Omega(t)| &\le \frac{1}{\sigma(c)} \left(\int_{-\infty}^{t} e^{\nu(t-s)} |\mathcal{G}(\phi(s)) - \mathcal{G}(\psi(s))| ds \right. \\
&\quad \left. + \int_{t}^{+\infty} e^{\mu(t-s)} |\mathcal{G}(\phi(s)) - \mathcal{G}(\psi(s))| ds \right) \\
&\le \frac{1}{\sigma(c)} \left(\int_{-\infty}^{t} e^{\nu(t-s)} \left(L \int_0^\infty \int_{\mathbb{R}} K(r,w) |\Omega(s-cr-w)| dwdr + P|\Omega(s)| \right) ds \right. \\
&\quad \left. + \int_{t}^{+\infty} e^{\mu(t-s)} \left(L \int_0^\infty \int_{\mathbb{R}} K(r,w) |\Omega(s-cr-w)| dwdr + P|\Omega(s)| \right) ds \right),
\end{aligned}$$

so that

$$\begin{aligned}
\bar{\Omega}(\tau) &\le \frac{\bar{\Omega}(\tau)}{\sigma(c)} \left(L \int_0^\infty \int_{\mathbb{R}} K(r,w) e^{-m(cr+w)} dwdr + P \right) \left(\int_{-\infty}^{\tau} e^{(\nu-m)(\tau-s)} ds \right. \\
&\quad \left. + \int_{\tau}^{+\infty} e^{(\mu-m)(\tau-s)} ds \right) \\
&= \frac{\bar{\Omega}(\tau)}{\sigma(c)} \left(L \int_0^\infty \int_{\mathbb{R}} K(r,w) e^{-m(cr+w)} dwdr + P \right) \frac{\nu-\mu}{(\nu-m)(\mu-m)} \\
&= \bar{\Omega}(\tau) \left(L \int_0^\infty \int_{\mathbb{R}} K(r,w) e^{-m(cr+w)} dwdr + P \right) \frac{1}{\beta+cm-m^2} < \bar{\Omega}(\tau), \qquad (6.21)
\end{aligned}$$

which is impossible. Hence, $\bar{\Omega}(\tau) = 0$ and the lemma follows. □

Remark VI.11. Some estimations of $c_\star$ can be found in [5, 53, 57].

CHAPTER VII

On the minimal speed of traveling waves for a non-local delayed reaction-diffusion equation

7.1 Estimation of the minimal speed of propagation

In this chapter, we estimate the minimal speed of propagation of positive traveling wave solutions for non-local delayed reaction-diffusion equation (1.6), which is widely used in applications, e.g. see [28, 36, 47, 51, 55] and references wherein. It is assumed that the birth function g is of the monostable type, $p := g'(0) > 1$ and $h \geq 0$. The non-negative kernel K is such that $K(s) = K(-s)$ for $s \in \mathbb{R}$, $\int_{\mathbb{R}} K(s)ds = 1$ and $\int_{\mathbb{R}} K(s)\exp(\lambda s)ds$ is finite for all $\lambda \in \mathbb{R}$. Consider

$$\psi(z,\varepsilon) = \varepsilon z^2 - z - 1 + p\exp(-zh)\int_{\mathbb{R}} K(s)\exp(-\sqrt{\varepsilon}zs)ds, \tag{7.1}$$

which determines the eigenvalues of equation (1.6) at the trivial steady state. From [40, 51], we know that there is $\varepsilon_0 = \varepsilon_0(h) > 0$ such that $\psi(z,\varepsilon_0) = 0$ has a unique multiple positive root $z_0 = z_0(h)$. Furthermore, if $g(s) \leq g'(0)s$ for $s \geq 0$, then the minimal speed c_* is equal to $c_* = 1/\sqrt{\varepsilon_0}$. Note that z_0 and ε_0 are the unique solutions of the system

$$\psi(z,\varepsilon) = 0, \quad \psi_z(z,\varepsilon) = 0. \tag{7.2}$$

Let us state our main result.

Notation VII.1. Set

$$
\begin{aligned}
k_1 &= 2\sqrt{\frac{p-1}{1+\frac{p}{2}\int_{\mathbb{R}} s^2K(s)ds}} - p\int_{\mathbb{R}} sK(s)\exp\left(-s\sqrt{\frac{p-1}{1+\frac{p}{2}\int_{\mathbb{R}} s^2K(s)ds}}\right)ds,\\
k_2 &= \frac{1}{\sqrt{\ln p}}\ln\left(p\int_{\mathbb{R}} K(s)\exp(-\sqrt{\ln p}s)ds\right).
\end{aligned}
$$

It is clear that $k_2 > 0$ and bellow we will show that k_1 is positive.

Theorem VII.2. *Assume that $K(s) \geq 0$ is such that $K(s) = K(-s)$ for $s \in \mathbb{R}$, $\int_{\mathbb{R}} K(s)ds = 1$ and $\int_{\mathbb{R}} K(s)\exp(\lambda s)ds$ is finite for all $\lambda \in \mathbb{R}$. Then $c_* = c_*(h) = 1/\sqrt{\varepsilon_0(h)}$ is a C^∞-smooth decreasing function of variable $h \in \mathbb{R}_+$. Moreover,*

1. $\max\left\{2\sqrt{\frac{p-1}{p(2h+h^2)+1}}, \frac{2\sqrt{\ln p}}{1+h}\right\} < c_* < \min\left\{\frac{k_1}{1+h}, \frac{k_2}{h}\right\}$, $h \in [0,1]$,

2. $\max\left\{2\sqrt{\frac{p-1}{p(2h+h^2)+1}}, \frac{\sqrt{\ln p}}{h}\right\} < c_* < \min\left\{\frac{k_1}{2}, \frac{k_2}{\sqrt{h}}\right\}$, $h \in [1,+\infty)$.

Furthermore, $\frac{C_1}{h} \leq c_*(h) \leq \frac{C_2}{h}$, $h \geq 1$, for some positive $C_1 < C_2$.

Observe that Theorem VII.2 implies that $c_*(h) = O(h^{-1})$, $h \to +\infty$, in this way we improve the estimation $c_*(h) = O(h^{-1/2})$, $h \to +\infty$, proved in [53, 57].

Proof: It follows from [40, 51] that the functions $z_0 = z_0(h)$ and $\varepsilon_0 = \varepsilon_0(h)$ are well defined for all $h \geq 0$. Set $F(h,z,\varepsilon) = (\psi(z,\varepsilon), \psi_z(z,\varepsilon))$. It is easy to see $F \in C^\infty(\mathbb{R}_+ \times \mathbb{R} \times (0,\infty), \mathbb{R}^2)$, $F(h, z_0, \varepsilon_0) = 0$, and

$$
\begin{aligned}
\left|\frac{\partial F(h,z_0,\varepsilon_0)}{\partial(z_0,\varepsilon_0)}\right| &= \psi_{zz}(z_0,\varepsilon_0)\psi_\varepsilon(z_0,\varepsilon_0)\\
&= (2\varepsilon_0 + p\int_{\mathbb{R}} K(s)\exp(-z_0(h+\sqrt{\varepsilon_0}s))(h+\sqrt{\varepsilon_0}s)^2ds)\\
&\times \frac{z_0}{2\varepsilon_0}(1 + hp\int_{\mathbb{R}} K(s)\exp(-z_0(h+\sqrt{\varepsilon_0}s))ds) > 0.
\end{aligned}
$$

Applying the Implicit Function Theorem we find that $z_0, \varepsilon_0 \in C^\infty(0,+\infty)$.

On the other hand, after introducing a new variable $w = \sqrt{\varepsilon}z$ we find that system (7.2) takes the following form:

$$\left(1 + \frac{w}{\sqrt{\varepsilon}} - w^2\right) \exp\left(\frac{wh}{\sqrt{\varepsilon}}\right) = p \int_{\mathbb{R}} K(s) \exp(-ws) ds. \tag{7.3}$$

$$\left(\frac{h}{\sqrt{\varepsilon}} w^2 + \left(2 - \frac{h}{\varepsilon}\right) w - \frac{1+h}{\sqrt{\varepsilon}}\right) \exp\left(\frac{wh}{\sqrt{\varepsilon}}\right) = p \int_{\mathbb{R}} sK(s) \exp(-ws) ds. \tag{7.4}$$

Let $G(w) = \left(1 + \frac{w}{\sqrt{\varepsilon_0}} - w^2\right)$, $H(w) = \left(1 + \frac{w}{\sqrt{\varepsilon_0}} - w^2\right) \exp\left(\frac{wh}{\sqrt{\varepsilon_0}}\right)$ and $R(w) = p \int_{\mathbb{R}} K(s) \exp(-ws) ds$. Set also $w_0 = w_0(h) = \sqrt{\varepsilon_0(h)} z_0(h)$. First, note that $G(w_0) = \exp\left(\frac{-wh}{\sqrt{\varepsilon_0}}\right) R(w_0) > 0$ and $G(w) \geq 1$ when $0 \leq w \leq 1/\sqrt{\varepsilon_0}$. As can be checked directly, H has a unique positive local extremum (maximum) at some $\bar{w}$. Since $K(s) = K(-s)$, $s \in \mathbb{R}$, it is easy to see that R increases on $\mathbb{R}_+$.

Differentiating equation (7.3) with respect to h and using (7.4) we get the following differential equation

$$\varepsilon_0'(h) = \frac{2\varepsilon_0(h) G(w_0(h))}{1 + hG(w_0(h))} > 0. \tag{7.5}$$

The remainder of the proof will be divided in several steps.

Step I. If $h \in [0,1]$, then $H'(1/\sqrt{\varepsilon_0}) = \left(\frac{h-1}{\sqrt{\varepsilon_0}}\right) e^{h/\varepsilon_0} \leq 0$. Hence, $\bar{w} \leq 1/\sqrt{\varepsilon_0}$. In addition, if $w \in (0, \bar{w})$ then $H'(w) > 0$. As $R'(w) > 0$ for $w > 0$, we have $w_0 < \bar{w} \leq 1/\sqrt{\varepsilon_0}$. Thus, we get $G(w_0) \geq 1$. In this way, $\varepsilon_0'(h) \geq 2\varepsilon_0(h)/(1+h)$ for $h \in [0,1]$ that yields $(1+h)^2 \varepsilon_0(0) \leq \varepsilon_0(h) \leq (1+h)^2 \varepsilon_0(1)/4$ (equivalently, $2c_*(1)/(1+h) \leq c_*(h) \leq c_*(0)/(1+h)$, for $h \in [0,1]$). Next, taking $h = 0$ in equations (7.3) and (7.4) we obtain that

$$\frac{1}{\sqrt{\varepsilon_0(0)}} = 2w_0(0) - p \int_{\mathbb{R}} sK(s) \exp\left(-w_0(0)s\right) ds, \tag{7.6}$$

$$\begin{aligned} 1 + w_0^2(0) &= p \int_{\mathbb{R}} K(s)(1 + w_0(0)s) \exp(-w_0(0)s) ds \\ &= p\left(1 - \frac{\int_{\mathbb{R}} s^2 K(s) ds}{2} w_0^2(0) - \frac{\int_{\mathbb{R}} s^4 K(s) ds}{8} w_0^4(0) - \ldots\right). \end{aligned}$$

As a consequence of the latter formula, we get

$$w_0(0) < \sqrt{\frac{p-1}{1+\frac{p}{2}\int_{\mathbb{R}} \ s^2K(s)ds}} \, .$$

Then (7.6) implies that $c_*(0) < k_1$ so that $c_*(h) < k_1/(1+h)$ for $h \leq 1$. Note that $k_1 > 0$ since R is increasing for $w > 0$. Finally, since $c_*(h)$ is decreasing, we have that $c_*(h) < k_1/2$ for $h \geq 1$.

Step II. If $h \geq 1$, then $\bar{w} \geq 1/\sqrt{\varepsilon_0}$. As consequence, $G(\bar{w}) \leq 1 = G(1/\sqrt{\varepsilon_0})$ so that $G(w) \geq G(\bar{w})$ for all $w \in [0, \bar{w}]$ (see Figure 7.1). Additionally, $G(\bar{w}) = (2\bar{w}\sqrt{\varepsilon_0} - 1)\frac{1}{h} \geq \frac{1}{h}$, therefore we conclude that $G(w_0) \geq 1/h$. Hence, we have $\varepsilon_0'(h) \geq \varepsilon_0(h)/h$, so that $\varepsilon(h) \geq \varepsilon(1)h$ (equivalently, $c_*(h) \leq c_*(1)/\sqrt{h}$) for $h \geq 1$. Now, if $h = 1$ we have $\bar{w} = 1/\sqrt{\varepsilon_0(1)}$. Thus, taking $h = 1$ and $w = \bar{w}$ in (7.3) we get $\exp(1/\varepsilon_0(1)) = R(\bar{w}) > R(0) = p$ that yields $\sqrt{\ln p} < 1/\sqrt{\varepsilon_0(1)} = c_*(1)$. On the other hand, for all $0 \leq w < 1/\sqrt{\varepsilon_0}$, we have

$$\exp\left(\frac{wh}{\sqrt{\varepsilon_0}}\right) < \left(1 + \frac{w}{\sqrt{\varepsilon_0}} - w^2\right)\exp\left(\frac{wh}{\sqrt{\varepsilon_0}}\right) \leq p\int_{\mathbb{R}} K(s)\exp(-ws)ds. \tag{7.7}$$

In particular, taking $h = 1$ and $w = \sqrt{\ln p}$ in (7.7) we conclude that $c_*(1) < k_2$ so that $c_*(h) < k_2/\sqrt{h}$ for $h \geq 1$. Additionally, using $c_*(h) \geq 2c_*(1)/(1+h)$ obtained in step I, we also concluded that $c_*(h) > 2\sqrt{\ln p}/(1+h)$, for $h \in [0, 1]$.

Step III. For $h > 0$, it is evident that $\varepsilon_0'(h) \leq 2\varepsilon_0(h)/h$. Integrating the latter inequality on $[h, 1]$ we obtain $\varepsilon(h) \geq \varepsilon(1)h^2$ (equivalently, $c_*(h) \leq c_*(1)/h$), for $0 < h \leq 1$ so that $c_*(h) < k_2/h$, for $h \in (0, 1]$. Analogous, by integrating $\varepsilon_0'(h) \leq 2\varepsilon_0(h)/h$ on $[1, h]$ we have $\varepsilon_0(h) \leq \varepsilon_0(1)h^2$ (equivalently, $c_*(h) \geq c_*(1)/h$), for $h \geq 1$. Thus, we obtain $c_*(h) > \sqrt{\ln p}/h, \ h \geq 1$.

On the other hand, for all $h \geq 0$, we have $G(w_0) \leq 1 + 1/(4\varepsilon_0)$. As consequence, $\varepsilon_0'(h) \leq (4\varepsilon_0(h)+1)/(2(1+h))$ for all $h \geq 0$ so that $\varepsilon_0(h) \leq ((4\varepsilon_0(0)+1)(1+h)^2-1)/4$. Taking $h = 0$ in (7.3), we get $1 + 1/(4\varepsilon_0(0)) > G(w_0(0)) = R(w_0(0)) > p$ so that

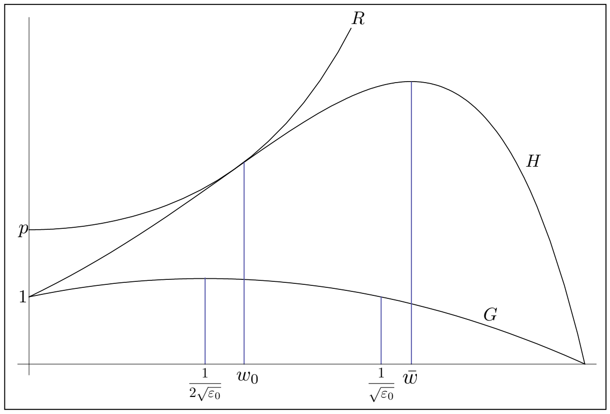

Figure 7.1: G, H and R for $h > 1$.

$c_*(0) > 2\sqrt{p-1}$. In consequence,

$$c_*(h) > 2\sqrt{\frac{p-1}{p(2h+h^2)+1}}, \quad h \geq 0. \tag{7.8}$$

Step IV. Setting $w = r$, $r \in (0,1)$, in the second inequality of (7.7) we obtain

$$\left(1-r^2\right)\exp\left(\frac{rh}{\sqrt{\varepsilon_0(h)}}\right) < p\int_{\mathbb{R}} K(s)\exp(-rs)ds,$$

from which we get that

$$\frac{1}{\sqrt{\varepsilon_0(h)}} < \frac{1}{hr}\ln\left(\frac{p}{1-r^2}\int_{\mathbb{R}} K(s)\exp(-rs)ds\right), \quad h > 0. \tag{7.9}$$

Considering (7.8) and (7.9) we get $\dfrac{C_1}{h} \leq c_*(h) \leq \dfrac{C_2}{h}$ for $h \geq 1$. This completes the proof. $\square$

7.2 An example

Consider the heat kernel $K_\alpha(s) = (4\pi\alpha)^{-1/2}\exp\left(-s^2/(4\alpha)\right)$. Then Theorem VII.2 applies with

$$k_1 = 2\sqrt{p-1}\left(\frac{1+\alpha p\exp\left(\frac{\alpha(p-1)}{1+\alpha p}\right)}{\sqrt{1+\alpha p}}\right), \quad k_2 = (1+\alpha)\sqrt{\ln p}.$$

In fact, in this case we can plot graphs of c_* against h using standard numerical

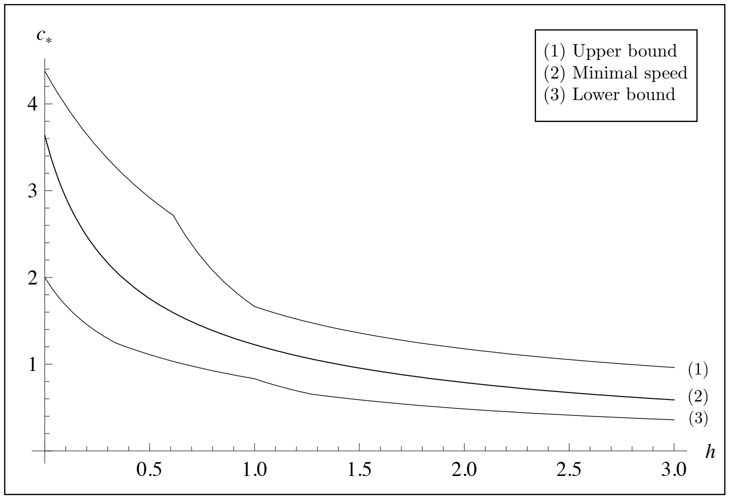

Figure 7.2: The minimal speed and its bounds ($p = 2$ and $\alpha = 1$).

methods to solve some appropriately chosen initial value problem $\varepsilon_0(h_0) = \rho_0$ for differential equation (7.5). For example, if we take $h_0 = \alpha$ then ρ_0 coincides with positive solution of the equation $1 + \frac{1}{4\rho} = p\exp\left(-\frac{\alpha}{4\rho}\right)$. Next, we can explicitly find $G(w_0)$ in (7.5) by using Cardano's formulas to solve the cubic equation $(w_0^2 - w_0/\sqrt{\varepsilon_0} - 1)(2\sqrt{\varepsilon_0}\alpha w_0 - h) + 1 - 2\sqrt{\varepsilon_0}w_0 = 0$. It is easy to see that this equation has three real roots for all $h \geq 0$ and $\alpha > 0$, and that w_0 is the leftmost positive root.

Figure 7.2 shows the minimal speed c_* and its estimations when $p = 2$ and $\alpha = 1$. Remark that we do not need the restriction $\alpha \leq h$ required in [57].

Finally, note that letting $\alpha \to 0^+$ in (1.6) and (7.1) we recover the characteristic equation for the delayed reaction-diffusion equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h,x)),$$

which was studied by various authors (e.g. see [4, 53] and references therein). In this case, our results complete and partially improve the estimations of [53].

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