Carga y orden parcial de las k-formas

by

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TESIS DE DOCTORADO

Se informa a la Dirección de Postgrado del Instituto de Matemáticas y Física que la tesis de Doctorado presentada por la candidata:

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ha sido aprobada por la Comisión de Evaluación de la tesis como requisito para optar al grado de Doctor en Matemáticas en el exámen de Defensa de Tesis rendido el día...

Director de tesis:

Dr. Luc Lapointe ....

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A mis hijas.

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Introducción

La presente tesis se enmarca dentro del estudio de las funciones simétricas [24]. Estas funciones son polinomios en varias variables que permanecen invariantes bajo el intercambio de cualquiera de sus variables. Existen muchas conexiones de estas funciones, tanto en teoría de representaciones, geometría algebraíca, combinatoria e incluso en física.

Una de las bases más importantes en el espacio de la funciones simétricas Λ son las funciones de Schur $s_\lambda(x)$, indexadas por particiones, las cuales han jugado un papel trascendental en esta teoría. Las conexiones que se han producido en diferentes contextos las exhiben como una base digna de ser estudiadas. En teoría de representaciones por ejemplo, éstas se identifican como los caractereres de las representaciones polinomiales irreducibles de Gl_n y por la correspondencia de Frobenius éstas están asociada a las representaciones irreducibles de S_n. En geometría algebraíca los coeficientes entregados en la regla de Littlewood-Richardson corresponden a las constantes de estructura del anillo de cohomología del Grassmanniano.

Otra base importante son los polinomios de Madconald $H_\mu(x; q, t)$ los que dependen de dos parámetros t y q. Cabe señalar que las bases fundamentales utilizadas en la teoría de funciones simétricas, tale como, los monomiales, las funciones de Schur, Zonal, Jack y los polinomios de Hall-Littlewood, no son más que casos particulares de éstos. Así, los polinomios de Macdonald cuando $q = 0$ son las funciones de

Hall-Littlewood, las que denotaremos por $H_\mu(x,t)$.

Uno de los problemas que ha generado gran interés en las últimas 3 décadas, es el estudio de los coeficientes q,t-Kostka en la expansión de los polinomios de Macdonald (modificados plethisticamente) en término de funciones de Schur, es decir

$$H_\mu(x;q,t) = \sum_\lambda K_{\lambda\mu}(q,t)s_\lambda(x). \tag{0.0.1}$$

Macdonald conjeturó que los coeficientes $K_{\lambda\mu}(q,t)$ pertenecen a $\mathbb{N}[q,t]$, lo que luego fue demostrado por Haiman en [7]. Dado que el número de términos monomiales de $K_{\lambda\mu}(q,t)$ es igual al número de tableaux estandar de forma λ, nos lleva a pensar que es posible asociar a cada uno de estos tableaux, dos enteros no negativos, $a_{\lambda\mu}$ y $b_{\lambda\mu}$, de manera tal que $q^{a_{\lambda\mu}}t^{b_{\lambda\mu}}$ sean exactamente los monomiales que definen a $K_{\lambda\mu}(q,t)$

Para el caso particular de las funciones de Hall-Littlewood, Lascoux y Schützenberger en [22], entregaron una solución a dicho problema. De hecho, de manera combinatorial los dos autores asociaron a cada tableaux un entero no negativo que llamaron carga, y que denotaremos por $\mathrm{ch}(T)$, la que permitió descomponer las funciones de Hall-Littlewood en funciones de Schur de la siguiente manera:

$$H_\mu(x;t) = \sum_T t^{\mathrm{ch}(T)}s_\lambda(x). \tag{0.0.2}$$

En busqueda de una manera combinatorial de obtener los coeficiente del desarrollo dado en (0.0.1), Lapointe, Lascoux y Morse observaron empíricamente la existencia de una familia de funciones, las cuales formaban una base de ciertos subespacios de Λ. Estas funciones inicialmente fueron llamadas atomos [14], pero luego por su similitud con las funciones de Schur, se consolidaron con el nombre de funciones k-Schur. Los cálculos sugerían que para cada entero positivo k, existía una familia de funciones simétricas, las que eran indexadas por un conjunto de ciertos tableaux, $\mathcal{A}_\mu^k$, tales que

$$s_\mu^{(k)}(x;t) = \sum_{T \in \mathcal{A}_\mu^k} t^{\mathrm{ch}(T)} s_{\mathrm{forma}(T)}(x) \tag{0.0.3}$$

y

$$H_\mu(x;q,t) = \sum_{\mu:\mu_1 \leq k} K_{\lambda\mu}^{(k)}(q,t) s_\lambda^{(k)}(x;t), \tag{0.0.4}$$

donde $K_{\lambda\mu}^{(k)}(q,t) \in \mathbb{N}[q,t]$. Además ellas eran una generalización de las funciones de Schur, de manera que $s_\lambda^{(k)}(x;t) = s_\lambda(x;t)$ para un k suficientemente grande. Experimentalmente este nuevo objeto satisfacía propiedades clásicas de las funciones de Schur, tales como la regla de Pieri y la regla de Littlewood-Richarson.

A medida que iban siendo estudiadas, ellas tomaban forma y estructura de manera tal que al fusionarse con otras áreas de la matemática, ellas se conectaban con objetos fundamentales. Por dar un ejemplo, en [18] Lapointe y Morse mostraron que los invariante de Gromov-Witten para la cohomología cuántica del Grassmanniano son iguales al caso especial de los coeficientes “k-Littlewood-Richardson”para el caso $t = 1$.

Una base dual de las funciones k-Schur fueron introducidas en [18], las que pasaron a ser un nuevo centro de investigación. Este nuevo conjunto de funciones que llamaremos k-Schur duales y denotaremos por $\mathfrak{S}_\lambda^{(k)}(x;t)$, es el objeto central de esta tesis. Indagando sobre el rol geométrico de las funciones k-Schur y las k-Schur duales, Morse y Shimozono nuevamente para el caso $t = 1$, conjeturarón que las bases para la cohomología de Schubert y la homología del Grassmanniano afín, estabán dadas por las funciones k-Schur duales y las funciones k-Schur, respectivamente. Más tarde, Lam en [10] demuestra esta conjetura, lo que sugiere que el cálculo de Schubert afín puede desarrollarse a través de estas funciones.

La conjetura más relevante en nuestro caso, es la expansión positiva de las funciones k-Schur en las funciones k'-Schur para $k' > k$ [14]:

$$s_\lambda^{(k)}(x;t) = \sum_\mu b_{\lambda,\mu}^{(k\to k')}(t)\, s_\mu^{(k')}(x;t)\,, \qquad \text{para} \quad b_{\lambda,\mu}^{(k\to k')}(t) \in \mathbb{Z}_{\geq 0}[t]. \tag{0.0.5}$$

Cuando k' es suficientemente grande, esta conjetura establece la positividad en la expansión de las funciones k-Schur en funciones de Schur (un resultados parciales en esta dirección se obtuvo en [1]).

Por dualidad entre las funciones k-Schur y k-Schur duales la conjetura dada en (0.0.5), es equivalente a demostrar que las funciones k'-Schur duales se expanden positivamente en funciones k-Schur duales para $k' > k$. Esto fue probado en [12], para el caso $t = 1$.

Uno de los obstáculos para la generalización de los resultados dados en [12] para un valor genérico t, fue la ausencia de una definición explícita de las función k-Schur duales. En esta tesis introducimos una carga a los k-tableaux la que nos permitirán dar esa definición. Para ser más precisos, las funciones k-Schur duales son funciones generadas por unos objetos combinatoriales llamados k-tableaux. Definiremos la versión graduada de las funciones k-Schur duales como

$$\mathfrak{S}_\lambda^{(k)}(x;t) = \sum_Q t^{\mathrm{ch}(Q)} x^Q \tag{0.0.6}$$

donde la suma es sobre todos los k-tableaux Q, de forma λ y donde $\mathrm{ch}(Q)$ es una cierta generalización de la carga de un tableaux mencionada anteriormente. Conjeturamos que la carga también permite una expansión en funciones k-Schur de los polinomios de Hall-Littlewood indexados por particiones k-acotadas

$$H_\lambda(x;t) = \sum_{Q^{(k)}} t^{\mathrm{ch}(Q^{(k)})} s^{(k)}_{\mathrm{sh}(Q^{(k)})}(x;t) \tag{0.0.7}$$

donde la suma es sobre todos los k-tableaux de contenido λ y forma $\mathrm{sh}(Q^{(k)})$. Esta fórmula que demuestra el caso $q = 0$ en (0.0.4) también generaliza el resultado de Lascoux y Schützenberger dado en (0.0.2).

Daremos evidencia de la validez de la definición de la carga mostrando que el concepto de carga es compatible en el caso estandar con la biyección débil introducida en [12]. La idea original de la tesis era demostrar la compatibilidad de la carga en el caso semiestandar, pero sólo se pudo lograr en el caso estandar. La gran dificultad que tuvimos para el caso semiestandar fue encontrar una acción del grupo simétrico sobre los k-tableaux de tipo Lascoux-Schützenberger. Empíricamemete se encontró esta acción pero no pudimos demostrar la compatibilidad.

Desarrollo de la tesis.

En el capitulo I, revisaremos vocabulario y nociones básicas de las funciones simétricas, todas ellas se pueden ver en [5, 24, 26]. Además tiene como objetivo, tanto familiarizar al lector con nuevos objetos combinatoriales como el de entregar herramientas suficientes para entender y seguir la teoría que será entregada en los capítulos II y III

A lo largo del capítulo II presentaremos una idea de los orígenes de las funciones k-Schur y la construcción de una de sus definiciones, junto con las funciones k-Schur duales, ambas para el caso $t = 1$.

Luego en el capítulo III generalizaremos la definición de las funciones k-Schur duales para un t genérico, definiremos la carga de un k-tableaux y probaremos que esta definición es compatible con la biyección antes mencionada, para el caso estandar.

Por último en el capítulo IV se mostrarán resultados parciales que obtuvimos para mostrar la conmutatividad de la acción del grupo simétrico sobre los k-tableaux con la biyección débil, en el caso semiestandar.

CHAPTER I

Funciones Simétricas

Para introducir las funciones simétricas, es necesario entregar algunas definiciones de objetos combinatoriales básicos, terminologías y resultados importantes. Debemos aclarar que nuestro enfoque será el de exponer los principales resultados sin sus demostraciones, esperando que los ejemplos que son entregados, sean suficientemente claros. Todos los conceptos y demostraciones se pueden encontrar en [24] [5] y [26].

1.1 Nociones basicas

Uno de los objetos fundamentales en el estudio de las funciones simétricas son las particiones. Estas se definen como una sucesión decreciente de números enteros no negativos, denotada por $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r, \ldots)$ que contiene un número finito de términos distintos de cero. Convendremos en no hacer distinción entre dos particiones que se diferencian sólo de una cadena de ceros al final. Por ejemplo, se considera que $(3, 2, 2, 1)$, $(3, 2, 2, 1, 0)$, $(3, 2, 2, 1, 0, 0, \ldots)$ definen la misma partición.

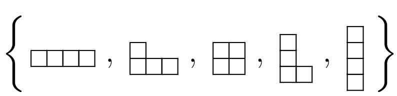
El *largo* de una partición λ, es el número de términos distintos de ceros, que denotaremos por $\ell(\lambda)$ y la suma de todos sus términos es el *tamaño*, denotado por $|\lambda|$, es decir, $|\lambda| = \sum \lambda_i$.

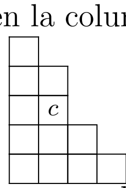
Si $|\lambda| = n$ diremos que λ es una partición de n y se denotará por $\lambda \vdash n$. El conjunto de todas las particiones de n será denotado por $\mathcal{P}_n$ y por $\mathcal{P}$ al conjunto de todas las particiones.

Se puede definir un orden $\leq$ sobre $\mathcal{P}_n$ llamado *orden de dominancia* de la siguiente manera: $\lambda \leq \mu$ si y solo si, $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ para todo i. Este orden, se usará para comparar funciones simétricas indexadas por particiones del mismo tamaño.

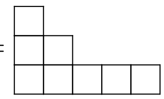
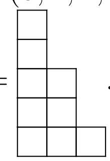
A veces es conveniente expresar una partición en la forma, $\lambda = (1^{m_1}\, 2^{m_2}\, \cdots\, r^{m_r})$ donde $m_i(\lambda)$ corresponde al número de enteros i presentes en la partición, es decir, $m_i(\lambda) = \sharp\{j : \lambda_j = i\}$. Por ejemplo, $\lambda = (3, 3, 2, 1, 1, 1, 1)$ se escribe como $\lambda = (1^4\, 2^1\, 3^2)$

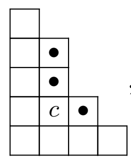
Otro objeto importante en el área de combinatoria algebraíca son los tableaux de Young. Estos están estrechamente relacionados con representaciones de grupo, especialmente en grupos lineales y simétricos. No podemos definir los tableaux de Young sin antes mencionar lo que es un *diagrama de Young*. Este es un arreglo de cajas dispuestas por filas de modo que cada fila tiene una cantidad menor o igual de cajas que la fila anterior. Utilizaremos la notación francesa para representar un diagrama de Young, dibujando las filas de abajo hacia a arriba. Se puede apreciar fácilmente que existe una correspondencia biunívoca entre particiones y diagramas de Young.

Ejemplo 1. Podemos representar $\mathcal{P}_4 = \{(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1)\}$ como $\mathcal{P}_4 =$ 

Las cajas que componen el diagrama se conocen como celda. La caja que está ubicada en la fila i y en la columna j representa a la celda (i, j). Por ejemplo, la caja c en la partición $\lambda =$ 

es la celda $(3, 2)$.

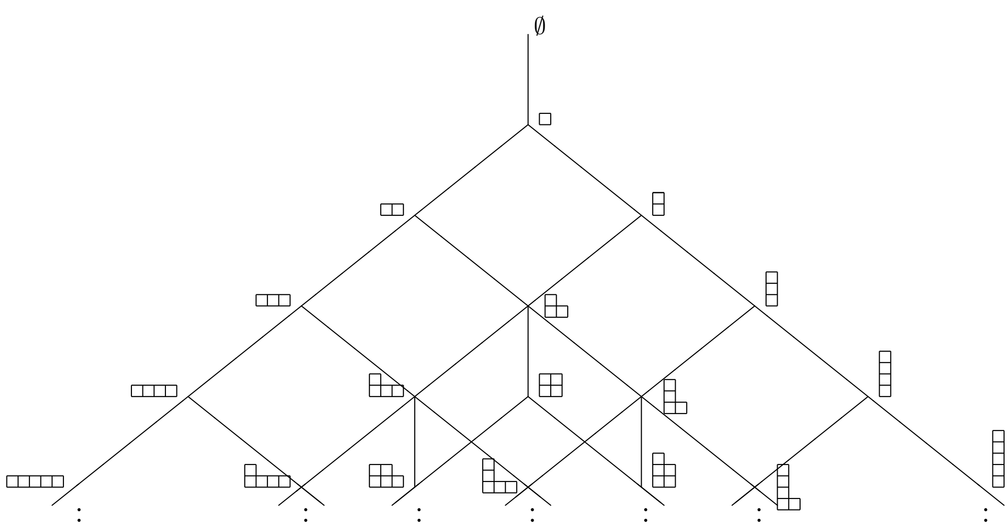
En muchos casos nos resultará de gran ayuda visualizar una partición como un diagrama de Young. Un ejemplo de ello es la definición del conjugado de una partición λ, denotado por λ', el cual es la sucesión $\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_{\lambda_l})$ donde $\lambda'_j = \sharp\{i : \lambda_i \geq j\}$. Claramente es mucho más simple mirar a λ' como el diagrama transpuesto al diagrama de Young de λ. Por ejemplo, si $\lambda = (5, 3, 1)$ entonces $\lambda' = (3, 2, 2, 1, 1)$ y visto como diagrama de Young, tenemos $\lambda =$  y $\lambda' =$  .

Lo mismo sucede con el hook de una celda $(i, j) \in \lambda$ definida como: $\mathrm{hook}(i, j) = \lambda_i + \lambda'_j - i - j + 1$. Por ejemplo, si $\lambda =$ 

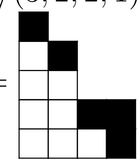
, entonces $\mathrm{hook}(c) = 3+4-2-2+1 = 4$, en cambio se puede calcular simplemente sumando las cajas que están a la derecha de c, arriba de c más la caja c en el diagrama de λ.

El conjunto de particiones $\mathcal{P}$, tiene otro orden parcial natural, llamado *orden de contención*. Diremos que μ contiene a λ, $\lambda \subseteq \mu$, si $\lambda_i \leq \mu_i$ para todo i. Representados en diagramas, una partición contiene a otra, si sus diagramas se contienen.

La siguiente figura muestra un esquema de las particiones ordenadas de menor a mayor según el orden de contención



Particiones ordenadas por contención

Existen varios tipos especiales de contenciones que veremos a continuación. Si $\lambda \subseteq \mu$, entonces μ/λ es llamado una *partición skew* y su diagrama es representado por todas las celdas que pertencen al diagrama de μ y que no pertenecen al diagrama de λ. La cantidad de celdas representadas en el diagrama skew es el tamaño de la partición que denotaremos por $|\mu/\lambda|$. Por ejemplo, la partición skew $(4,4,2,2,1)/(3,2,2,1)$ corresponde a la parte sombreada del diagrama $(4,4,2,2,1)/(3,2,2,1) =$ 

y $|(4,4,2,2,1)/(3,2,2,1)| = 5$

Por otra parte, diremos que μ/λ es un *strip horizontal (resp. vertical)* si en cada columna (resp. fila) hay a lo más una celda en μ/λ. Diremos que μ/λ es *conexo*, si para cualquier par de celdas, existe una sucesión de celdas consecutivas, desde una a la otra, donde las celdas consecutivas son las que comparten un lado. Una partición skew μ/λ se llama *ribbon* si éste es un conexo y que no contiene un subconjunto de celdas de 2×2

1.1.1 Tableaux de Young

Como mencionamos anteriormente, un objeto combinatorial importante son los tableaux de Young, los cuales nos permitirán definir las funciones simétricas de forma combinatorial. Un *tableau de Young* o simplemente un *tableau* T, de forma λ, es un relleno del diagrama λ con número enteros positivos, de forma que los números de las filas sean crecientes y de las columnas estrictamente crecientes. Si $\alpha = (1^{m_1}\, 2^{m_2}\, \ldots\, r^{m_r})$ donde m_i es igual a la cantidad de i que tiene el relleno de T, se dice que T es un tableau de contenido α.

Ejemplo 2. Para $\lambda = (5,3,2,2,1)$ hay muchos tableaux de forma λ, como por ejemplo $T_1 = \begin{array}{lllll} 6 \\ 5 & 5 \\ 3 & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 4 \end{array}$ $T_2 = \begin{array}{lllll} 9 \\ 8 & 13 \\ 6 & 12 \\ 2 & 5 & 11 \\ 1 & 3 & 4 & 7 & 10 \end{array}$ $T_3 = \begin{array}{lllll} 13 \\ 11 & 12 \\ 9 & 10 \\ 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 \end{array}$ $T_4 = \begin{array}{lllll} 6 \\ 4 & 5 \\ 3 & 3 \\ 2 & 2 & 4 \\ 1 & 1 & 2 & 4 & 6 \end{array}$

Estos tableaux son también llamados *tableaux semiestandares* y usaremos la abreviatura $SSYT$ para denotar el conjunto de todos los tableaux semiestandares. Si el contenido de un tableau semiestandar T, es la partición $\alpha = (1, 1, \ldots, 1)$, diremos que T es un *tableau estandar* y denotaremos por SYT al conjunto de todos los tableaux estandares. Dicho de otra manera un tableau estandar es aquel que no tiene repeticiones en el relleno. Podemos distinguir en el ejemplo 2, que T_2 y T_3 son los únicos tableaux estandares.

La cantidad de tableaux de forma λ y de contenido μ, se conoce como el *número de Kostka* y se denota por $K_{\lambda\mu}$. Estos números resultan ser muy utiles para determinar los coeficientes de la expansión de las funciones de Schur en otras.

Ejemplo 3. No es difícil ver que existen sólo 6 tableaux de forma $(4, 2)$ y contenido $(2, 1, 1, 1, 1)$ dados por:

$$\begin{array}{|c|c|cc}\hline 4 & 5 \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c|cc}\hline 3 & 5 \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad \begin{array}{|c|c|cc}\hline 3 & 4 \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \quad \begin{array}{|c|c|cc}\hline 2 & 5 \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad \begin{array}{|c|c|cc}\hline 2 & 4 \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \quad \begin{array}{|c|c|cc}\hline 2 & 3 \\ \hline 1 & 1 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

y sólo 4 tableaux de forma $(4, 2)$ y contenido $(2, 2, 1, 1)$;

$$\begin{array}{|c|c|cc}\hline 3 & 4 \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c|cc}\hline 2 & 4 \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c|cc}\hline 2 & 3 \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad \begin{array}{|c|c|cc}\hline 2 & 2 \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

Por lo tanto, $K_{(4,2)\,(2,1,1,1,1)} = 6, \quad K_{(4,2)\,(2,2,1,1)} = 4.$

En general se tiene que $K_{\lambda\,\lambda} = 1$ para todo $\lambda \in \mathcal{P}$. Por otra parte se puede ver facilmente que el número de tableaux estandares de forma λ está dado por el número de Kostka $K_{\lambda 1^n}$. A menudo este número es denotado por f_λ. La fórmula de Frame, Robinson y Thrall [4] establece que, para una partición $\lambda \vdash n,\ f_\lambda$ se obtiene usando la siguiente identidad

$$f_\lambda = \frac{n!}{\prod_{c\in\lambda} \mathrm{hook}(c)}.$$

1.1.2 Algoritmo de Robinson-Schensted-Knuth

El algoritmo de Robinson-Schensted-Knuth (RSK) [25] permite establecer una conección entre permutaciones y tableaux estandares. Existen muchas consecuencias útiles de este algoritmo. Por ejemplo, la identidad de Cauchy, que relaciona un producto de funciones de Schur en una serie formal, puede ser establecida usando el algortimo RSK.

Diremos que una *bipalabra* es un arreglo de enteros positivos formado por dos filas $\begin{pmatrix} u_1 & u_2 & \cdots & u_l \\ v_1 & v_2 & \cdots & v_l \end{pmatrix}$, que satisfacen la siguiente relación de orden: $u_i < u_{i+1}$ o en el caso que $u_i = u_{i+1}$ entonces $v_i \leq v_{i+1}$. Este orden es conocido como *orden lexicográfico*. Observar que una permutación es un caso particular de una bipalabra.

El algoritmo también se puede extender al caso de una bipalabra

Teorema 1. Existe una biyección entre las bipalabras w y los pares de tableaux semiestandares (P, Q) que tienen la misma forma.

Los detalles de este algoritmo y de la inserción de Schensted (operación trascendental del algoritmo), se pueden ver en [5].

Ejemplo 4. Un ejemplo simple de tal correspondencia

$$w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 1 & 4 & 2 & 3 & 1 \end{pmatrix} \quad \xrightarrow{RSK} \quad \left(\begin{array}{c} \begin{array}{|c|c|c|c|} \hline 3 \\ \cline{1-1} 2 & 2 & 4 \\ \cline{1-3} 1 & 1 & 3 & 3 \\ \hline \end{array} \\ P \end{array} \; , \; \begin{array}{c} \begin{array}{|c|c|c|c|} \hline 4 \\ \cline{1-1} 2 & 3 & 3 \\ \cline{1-3} 1 & 1 & 1 & 2 \\ \hline \end{array} \\ Q \end{array} \right)$$

Por último, para definir las funciones simétricas, usando los tableaux de Young, necesitamos un nuevo concepto. Si T es una tableaux de contenido $\alpha = (1^{m_1}\, 2^{m_2}\, \ldots r^{m_r})$ se define $x^T = x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots x_r^{m_r}$. Estos serán los términos que componen las funciones simétricas, llamados monomiales. Más aún, si $n = \Sigma_i m_i$, se dice que el monomial $x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots x_r^{m_r}$ tiene grado n. Usando el ejemplo 2, tenemos que $x^{T_1} = x_1^3 x_2^3 x_3^2 x_4^2 x_5^2 x_6$; $x^{T_4} = x_1^2 x_2^3 x_3^2 x_4^3 x_5 x_6^2$, ambos de grado 13.

1.2 El Anillo de las Funciones Simétricas

El anillo de las funciones simétricas Λ, resulta del conjunto de todas las series de potencias formales, que son invariantes bajo la acción del grupo simétrico.

Sea $x = (x_1, x_2, x_3, \ldots)$ el conjunto de infinitas variables y sea $\mathbb{C}[[x]]$ el anillo de todas las series de potencias formales. Uno de los elementos básicos en el desarrollo de las funciones simétricas es el monomial $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_l}^{\lambda_l}$.

Diremos que una función $f(x) \in \mathbb{C}[[x]]$ es homogenea de grado n, si todos los monomiales de $f(x)$ tienen grado n. Por otra parte, diremos que es simétrica si, para todo entero positivo n y $\sigma \in S_n$ (S_n es el grupo simétrico), se tiene que $f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \ldots) = f(x_1, x_2, x_3, \ldots)$.

Definición 1. La función simétrica *monomial* asociada a la partición $\lambda \vdash n$ es: $m_\lambda = \sum_\alpha x^\alpha$ donde α recorre todas las permutaciones distintas de λ.

Ejemplo 5. Para $\mathcal{P}_3$, tenemos

$$m_3 = x_1^3 + x_2^3 + x_3^3 + x_4^3 + \cdots \qquad m_{111} = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \cdots$$

$$m_{21} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + x_1^2x_4 + \cdots$$

Parece evidente que estas funciones forman una base del espacio. Definiremos al anillo de las funciones simétricas como, $\Lambda = \mathbb{C}m_\lambda$, el cual se descompone como $\Lambda = \oplus_{n\geq 0}\Lambda^n$ donde Λ^n es el espacio generado las funciones monomiales m_λ de grado n; en otras palabras, $\{m_\lambda : \lambda \vdash n\}$ define una base de Λ^n.

Existen otras bases de Λ^n que son de nuestro interés, como son las funciones simétricas homogeneas y las de potencia.

Definición 2. Sea r un número entero positivo. Se definen la r-ésima *función simétrica homogenea* h_r y la r-ésima *función simétrica de potencia* p_r como:

$$h_r = \sum_{1\leq i_1\leq\cdots\leq i_r} x_{i_1}x_{i_2}\cdots x_{i_r} = \sum_{|\lambda|=r} m_\lambda \qquad \text{y} \qquad p_r = \sum_i x_i^r = m_{(r)}.$$

Además se define $h_0 = p_0 = 1$ y $h_r = p_r = 0$ si $r < 0$. Podemos extender esta definición a una partición $\lambda = (\lambda_1, \lambda_2 \ldots \lambda_l)$, de la siguiente manera $h_\lambda = h_{\lambda_1} \cdot h_{\lambda_2} \cdots h_{\lambda_\ell}$ y $p_\lambda = p_{\lambda_1} \cdot p_{\lambda_2} \cdots p_{\lambda_\ell}$

En [24] se muestra en detalle que los conjuntos $\{h_\lambda \,|\, \lambda \vdash n\}$ y $\{p_\lambda \,|\, \lambda \vdash n\}$ también son bases de Λ^n.

Ejemplo 6. En particular el conjunto $\{h_3, h_{21}, h_{111}\}$ es una base de Λ^3. Estas funciones pueden ser expresadas en funciones monomiales de la siguiente manera

$$\begin{aligned}
h_3 &= (x_1^3 + x_2^3 + x_3^3 + \cdots) + (x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + \cdots) \\
&\quad +(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \cdots) \\
&= m_3 + m_{21} + m_{111} \\
h_{21} &= h_2h_1 \\
&= (x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots) \\
&= (x_1^3 + x_2^3 + x_3^3 + \cdots) + 2(x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + \cdots) \\
&\quad +3(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \cdots) \\
&= m_3 + 2m_{21} + 3m_{111} \\
h_{111} &= h_1h_1h_1 \\
&= (x_1 + x_2 + x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots) \\
&= (x_1^3 + x_2^3 + x_3^3 + \cdots) + 3(x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + \cdots) \\
&\quad +6(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \cdots) \\
&= m_3 + 3m_{21} + 6m_{111}
\end{aligned}$$

1.3 Funciones de Schur

Como ya hemos mencionado, una base importante en el estudio de las funciones simétricas son las funciones de Schur. Estas desempeñan un rol fundamental en combinatoria, teoría de representaciones y en geometría algebraíca.

A continuación entregaremos dos definiciones distintas de las funciones de Schur. En [24] se demuestra que esas dos definiciones son equivalentes.

• La primera definición fue dada por Jacobi en 1841. El definió los polinomios de Schur, s_λ, en n variables, indexados por un partición λ donde $\ell(\lambda) \leq n$, como

$$s_\lambda = s_\lambda(x_1, \ldots, x_n) = \frac{\det(x_i^{\lambda_j+n-j})_{1\leq i,j\leq n}}{\det(x_i^{n-j})_{1\leq i,j\leq n}}. \tag{1.3.1}$$

Jacobi probó además que $s_\lambda = \det(h_{\lambda_i+j-i}(x))_{1\leq i,j\leq \ell}$ donde h_r son las funciones simétricas homogeneas. A menudo esta identidad es llamada identidad de Jacobi-Trudi.

Ejemplo 7. Sea $\lambda = (2,1)$ y $n = 3$, entonces

$$s_{21} = \frac{\begin{vmatrix} x_1^4 & x_1^2 & x_1^0 \\ x_2^4 & x_2^2 & x_2^0 \\ x_3^4 & x_3^2 & x_3^0 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix}} = \frac{(x_1^2-x_2^2)(x_1^2-x_3^2)(x_2^2-x_3^2)}{(x_1-x_2)(x_1-x_3)(x_2-x_3)} = (x_1+x_2)(x_1+x_3)(x_2+x_3)$$

$$s_{21} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3 = m_{21} + 2m_{111}$$

De otra forma

$$s_{21} = \begin{vmatrix} h_2 & h_3 & h_4 \\ h_0 & h_1 & h_2 \\ h_{-2} & h_{-1} & h_0 \end{vmatrix} = \begin{vmatrix} h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 \\ 0 & 0 & 1 \end{vmatrix} = h_2 h_1 - h_3$$

y usando los resultados del ejemplo 6, se tiene

$$s_{21} = m_3 + 2m_{21} + 3m_{111} - m_3 - m_{21} - m_{111} = m_{21} + 2m_{1,1,1}.$$

- Una forma combinatorial de definir las funciones de Schur es

$$s_\lambda = \sum_T x^T$$

donde la suma recorre todos los tableaux semiestandares T, de forma λ.

La propiedad de simetría en esta definición no es obvia. En [26], se da una demostración, cuya idea esencial consiste en probar que existe una involución en el conjunto de todos los tableaux semiestandares de forma λ, que intercambia las cantidades de entradas i de T por las de $(i+1)$, para todo i, la cual deja invariante la forma del tableau.

Ejemplo 8. Existen sólo 6 tableaux de forma $(2,1)$ cuyo contenido son los números que pertenece al conjunto $\{1,2,3\}$, los cuales son: $\begin{array}{|c|c|}\hline 2 \\ \cline{1-1} 1 & 1 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 3 \\ \cline{1-1} 1 & 1 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 2 \\ \cline{1-1} 1 & 2 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 3 \\ \cline{1-1} 1 & 2 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 3 \\ \cline{1-1} 2 & 2 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 3 \\ \cline{1-1} 2 & 3 \\ \hline\end{array}$, luego, usando la última definición de las funciones de Schur, se tiene la descomposición de $s_{(2,1)}$ en monomiales

$$\begin{aligned} s_{21} &= x^{\substack{2\\ 1\,1}} + x^{\substack{3\\ 1\,1}} + x^{\substack{2\\ 1\,2}} + x^{\substack{3\\ 1\,2}} + x^{\substack{2\\ 1\,3}} + x^{\substack{3\\ 1\,3}} + x^{\substack{3\\ 2\,2}} + x^{\substack{3\\ 2\,3}} \\ &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \\ &= m_{21} + 2m_{111} \end{aligned}$$

Como consecuencia de esta definición, podemos descomponer una función de Schur en monomiales, $\quad s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu \quad$ donde μ recorre todas las particiones $\mu \leq \lambda$ y $K_{\lambda\mu}$ son los números de Kostka.

Ejemplo 9. Para $\lambda = (2,1)$, existe un único tableau de forma $(2,1)$ y contenido $(2,1)$, este es: $\begin{array}{|c|c|}\hline 2 \\ \cline{1-1} 1 & 1 \\ \hline\end{array}$ y dos tableaux de forma $(2,1)$ y contenido $(1,1,1)$, $\begin{array}{|c|c|}\hline 3 \\ \cline{1-1} 1 & 2 \\ \hline\end{array}$ y $\begin{array}{|c|c|}\hline 2 \\ \cline{1-1} 1 & 3 \\ \hline\end{array}$, luego $\quad s_{21} = m_{21} + 2m_{111}$.

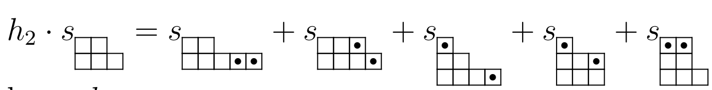
En [24], se define un producto escalar en Λ, exigiendo que las bases $\{h_\lambda\}$ y $\{m_\lambda\}$ sean duales entre sí, es decir, $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$, para todas las particiones λ, μ, donde $\delta_{\lambda\mu}$ es la delta de Kronecker. También se prueba que las funciones de potencia son ortogonales; para ser más precisos, $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$, donde $z_\lambda = \prod_{i\geq 1} m_i(\lambda)!\, i^{m_i(\lambda)}$. Igualmente se tiene que las funciones de Schur son duales entre sí, $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$, luego $\{s_\lambda\}_{\lambda\in\mathcal{P}}$ es una base ortonormal de Λ. Una forma combinatorial de probar esta afirmación, resulta de la identidad de Cauchy, $\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda s_\lambda(x)s_\lambda(y)$ la cual puede ser demostrada usando el algortimo de RSK. Esta identidad es equivalente a la ortonormalidad de las funciones de Schur.

Una de las propiedades importantes que satisfacen las funciones de Schur, es *la regla de Pieri*, la cual establece una relación combinatorial para el producto de dos funciones simétricas,

$$h_l s_\lambda = \sum_\mu s_\mu$$

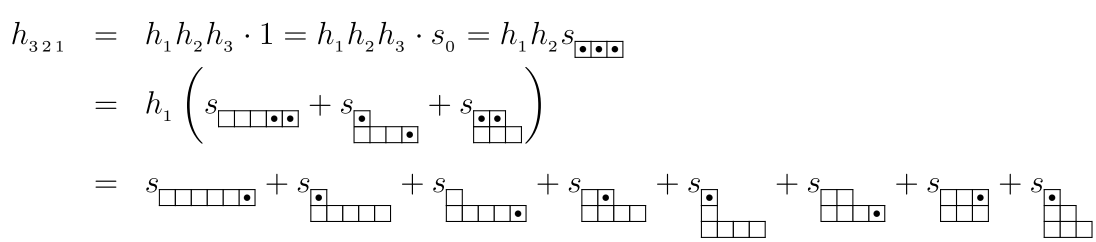
donde μ resulta de agregar l celdas a λ, todas ellas en columnas distintas.

Ejemplo 10. Para el caso particular $h_2 s_{32}$, podemos expresar las funciones de Schur indexadas por tableaux, donde las celdas marcadas con $\bullet$, son las celdas que se agregaron según esta regla.



luego $h_2 s_{32} = s_{52} + s_{43} + s_{421} + s_{331} + s_{322}$.

Ejemplo 11. También podemos representar la función h_{321} en términos de funciones de Schur, usando reiteradamente la regla de Pieri, como se muestra acontinuación



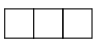
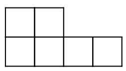
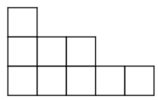
$$= s_6 + 2s_{51} + 2s_{42} + s_{411} + s_{33} + s_{321}.$$

Notar que un tableau semiestandar de forma λ puede ser representado como una sucesión de particiones

$$\lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(d)}$$

donde $\lambda^{(0)} = \emptyset$, $\lambda^{(d)} = \lambda$ y cada uno de los strip $\lambda^{(i+1)}/\lambda^{(i)}$ son horizontales. El contenido del tableau es $\alpha = (1^{m_1}, 2^{m_2}, \ldots, r^{m_r})$ en el que $m_i = \left|\lambda^i/\lambda^{(i-1)}\right|$ para todo i. Dicho de otra forma, el contenido es el resultado de poner un 1 en cada una de las celdas de $\lambda^{(1)}/\lambda^{(0)}$, un 2 en cada celda de $\lambda^{(2)}/\lambda^{(1)}$, y así sucesivamente, un i en cada una de las celda de $\lambda^{(i)}/\lambda^{(i-1)}$.

Ejemplo 12. Así, la sucesión

$\emptyset \subseteq$  $\subseteq$  $\subseteq$  es representada por el tableaux semiestandar

3				
2	2	3		
1	1	1	2	3

Basándose en la regla Pieri y usando inducción, podemos generalizar a grandes rasgos lo que hicimos en el ejemplo 11, es decir, si $\mu \vdash n$ entonces

$$h_\mu = \sum_{\lambda \vdash n} K_{\lambda\mu} s_\lambda$$

donde nuevamente $K_{\lambda\mu}$ son los números de Kostka.

1.4 Polinomios de Macdonald

Las funciones de Schur s_λ recién definidas se pueden caracterizar por las siguientes propiedades:

(a) $s_\lambda = m_\lambda +$ términos menores (usando el orden de dominancia)

(b) $\langle s_\lambda, s_\mu \rangle = 0$ si $\lambda \neq \mu$

donde el producto escalar es definido por $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$.

Lo mismo ocurre con las funciones de Hall-Littlewood $P_\lambda(x;t)$, ellas pueden ser caracterizadas por las misma dos propiedades (a) y (b), con la diferencia de que el producto escalar es ahora definido por $\langle p_\lambda, p_\mu \rangle_t = \delta_{\lambda\mu} z_\lambda(t) = \delta_{\lambda\mu} z_\lambda \prod_i (1 - t^{\lambda_i})^{-1}$.

Asi mismo, Macdonald en 1987 definió los polinomios $P_\lambda(x;q,t)$, como la familia de funciones simétricas que satisfacen las siguiente propiedades:

(a) $P_\lambda(x;q,t) = m_\lambda +$ términos menores

(b) $\langle P_\lambda(x;q,t), P_\mu(x;q,t)\rangle_{q,t} = 0$ si $\lambda \neq \mu$

con el producto escalar extendido, $\langle p_\lambda, p_\mu\rangle_{q,t} = z_\lambda \delta_{\lambda\mu} \prod_i \frac{(1+q^{\lambda_i})}{(1-t^{\lambda_i})}$.

Para probar la existencia de tales funciones, Macdonald en [24], construyó un operador autoadjunto definido en Λ, que le permitió mostrar la existencia y la unicidad de esas funciones.

Los polinomios de Macdonald generalizan a todas las funciones simétricas vistas anteriormente y otras conocidas como son las funciones simétricas de Jack, los polinomios Zonales entre otros. Por ejemplo, cuando $q = t$, ellos se reducen a las funciones de Schur s_λ, y cuando $q = 0$, se transforman en las funciones de Hall-Littlewood $P(x;t)$.

Una forma de conectar los polinomios de Macdonald con la combinatoria, es el resultado de realizar una sustitución "plethystic"(en [8] se puede ver una fórmula explícita de ella), ésta consiste en definir un homomorfismo $\rho : \Lambda \to \Lambda$ tal que $\rho(p_r) = \frac{1}{1-t^r} p_r$ y se denota por $P_\mu\left[\frac{x}{1-t};q,t\right] = \rho(P_\mu(x;q,t))$. Se probó en [24] que $H_\mu(x;q,t) = c_\mu P_\mu\left[\frac{x}{1-t};q,t\right]$ donde c_μ es una cierta constante de normalización, produce una expansión en funciones de Schur

$$H_\mu(x;q,t) = \sum_{\lambda\vdash n} K_{\lambda\mu}(q,t) s_\lambda(x), \tag{1.4.1}$$

donde $K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$ son conocidos como los polinomios q, t-Kostka.

De la ecuación 1.4.1, para $q = 0$, se obtiene la descomposición de las funciones de Hall-Littlewood en funciones de Schur, es decir, $H_\mu(x;t) = \sum K_{\lambda\mu}(t) s_\mu(x)$. Foulkes en [3] sugiere que debería existir una interpretación combinatorial para los coeficientes $K_{\lambda\mu}(t)$. Más tarde Lascoux y Schutzenberger en [21], encuentran una regla explícita, puntualmente ellos establecen que

$$K_{\lambda\mu}(t) = \sum_{T} t^{ch(T)}$$

donde T recorre todos los tableaux de forma λ y de contenido μ. En esta fórmula, $ch(T)$ es un entero no negativo, el cual está bien definido como una función $\mathbb{N}$-valuada la cual proporciona una noción de carga a cada tableaux T.

Uno de los problemas pendientes en la combinatoria algebraíca, está centrado en la ecuación 1.4.1. Una de las conjeturas postula que se puede asociar a cada partición μ un par de enteros no negativos $a_{\mu}(T)$ y $b_{\mu}(T)$ asociado a cada tableau estandar de forma λ, de manera que:

$$K_{\lambda\mu}(q,t) = \sum_{T \in SYT(\lambda)} q^{a(T)} t^{b(T)}$$

donde $SYT(\lambda)$ es el conjunto de todos los tableaux estandar de forma λ.

En la busqueda de una solución a este problema, Lapointe, Lascoux y Morse [14] encuentraron de forma experimental una base de $\Lambda_t^{(k)} = \text{span}\{H_{\lambda}(x;t) | \lambda_1 \leq k\}$, ahora llamadas, funciones k-Schur y denotaron por $s_{\lambda}^{(k)}(x;t)$; las cuales generalizaban a las funciones de Schur y al parecer verifican las relaciones:

1. $H_{\lambda}(x;t) = s_{\lambda}^{(k)}(x;t) + \sum_{\mu<\lambda} v_{\lambda\mu}^{(k)}(t) s_{\mu}^{(k)}(x;t)$ donde $v_{\lambda\mu}^{(k)}(t) \in \mathbb{N}[t]$

2. $s_{\lambda}^{(k)}(x;t) = s_{\lambda} + \sum w_{\lambda\mu}^{(k)}(x;t) s_{\mu}^{(k)}$ donde $w_{\lambda\mu}^{(k)}(t) \in \mathbb{N}[t]$

3. $s_{\lambda}^{(k)}(x;t) - s_{\mu}^{(k)}(x;t)$ no cumplen la propiedad de positividad, es decir, si $s_{\lambda}^{(k)}(x;t) - s_{\mu}^{(k)}(x;t) = \sum_{\nu} a_{\nu}(t) s_{\nu}$ entonces $a_{\nu}(t) \notin \mathbb{N}[t]$

En el siguiente capitulo, definiremos con más detalles las funciones k-Schur.

CHAPTER II

Funciones k-Schur

Las funciones k-Schur fueron introducidas por Lapointe, Lascoux y Morse en [14] en el año 2003. Estas tienen su origen en el estudio de los polinomios de Macdonald. Desde entonces, se han obtenido otras definiciones equivalentes a dichas funciones.

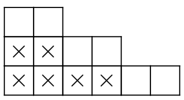
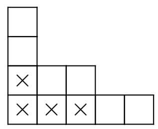
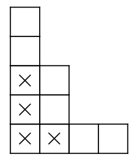
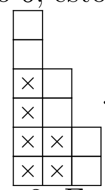
Comenzaremos con algunas nociones combinatoriales básica de la teoría de las k-Schur, incluyendo el orden del grupo simétrico afín. Este está estrechamente relacionado con la definición de los k-tableaux y los k-tableaux fuertes, quienes serán componentes esenciales en las definiciones que daremos de las funciones k-Schur y las k-Schur duales para el caso $t = 1$. En efecto, definiremos estas funciones usando una regla análoga la regla de Pieri. Cabe señalar que existe una conexión de las funciones k-Schur así definidas con la geométria algebraíca. Esta se puede ver en [10].

2.1 Particiones k-acotadas, cores y su relación con el grupo simétrico afín

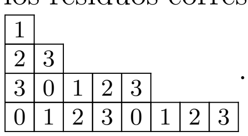
Las funciones k-Schur son funciones simétricas indexadas por particiones k-acotadas. Estas particiones se caracterizan por el hecho que sus términos son menores o iguales a k, es decir, una partición λ es k-acotada si y solo si $\lambda_1 \leq k$. El conjunto de todas las particiones k-acotadas se denotará por $\mathcal{P}^k$. Este conjunto está en biyección con varios conjuntos de objetos combinatoriales. En ciertos casos, es más cómodo expresar las funciones k-Schur en términos de estos otros objetos; uno de esos es el conjunto de los $(k+1)$-cores.

Sea $n \geq 1$, una partición se dice que es un *n-core*, si ninguna de sus celdas tiene hook igual a n. Por otra parte, el *tamaño* de un n-core es el número de celdas cuyo hook es menor que n y denotaremos por $\mathcal{C}^n$ al conjunto de todos los n-cores.

Por ejemplo, para el caso $n = 3$, existen sólo cuatro 3-core de tamaño 6, estos son:

   y  .

En este caso, se marcaron con $\times$ las celdas cuyo hook es mayor que 3. Esto nos permite observar mejor el tamaño de los 3-cores.

Un elemento importante en la construcción de un k-tableaux son los residuos. Estos son enteros no negativos asociados a cada una de las celdas de una partición. En concreto, para una partición λ, el residuo de la celda (i, j) de λ es $(j-i) \pmod{n}$. Por ejemplo, los residuos correspondientes a las celdas del 4-core de forma $\lambda = (8, 5, 2, 1)$ son: $\lambda =$ 

.

Ahora bien, para una partición λ, diremos que la celda (i, j) de λ es *extraíble* si las celdas $(i, j+1)$ y $(i+1, j)$ no pertenecen a λ. Así mismo, diremos que la celda (i, j) no pertenece a λ es *agregable* si las celdas $(i, j-1)$ y $(i-1, j)$ están en λ. Es importante notar que las celdas $(\ell(\lambda)+1, 1)$ y $(1, \lambda_1+1)$ son también consideradas como agregables.

En el caso particular de los cores, ver por ejemplo [16], se demostró la siguiente proposición.

Proposición 1. Un n-core nunca tiene una esquina extraíble y una esquina agregable con el mismo residuo.

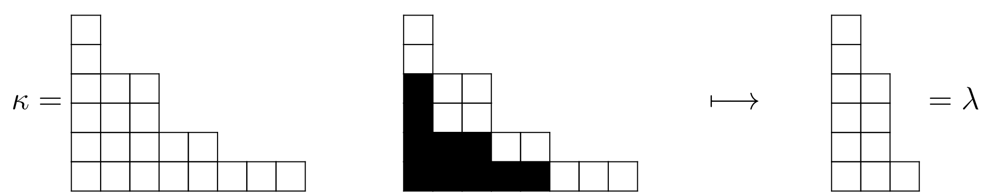
El siguiente resultado muestra la equivalencia que existe entre los $k+1$-cores y las particiones k-acotadas.

Proposición 2. Existe una biyección entre el conjunto $\mathcal{C}^{k+1}$ y el conjunto $\mathcal{P}^k$

A continuación, explicaremos brevemente en que consiste esta biyección. La demostración completa se encuentra en [16].

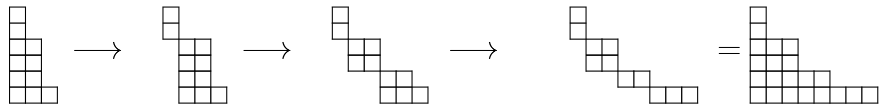
Denotaremos por $\mathfrak{p}$ a la aplicación que va de $\mathcal{C}^{k+1}$ a $\mathcal{P}^k$, es decir, para un $(k+1)$-core κ, la aplicación $\mathfrak{p} : \kappa \mapsto \lambda$, determina explícitamente a la partición λ que se define como: $\lambda_i = \sharp\{(i,j) \in \kappa \mid \text{hook}(i,j) \leq k\ \}$.

Pongamos por ejemplo, $k = 3$ y $\kappa = (8,5,3,3,1,1)$ la partición 3-acotada asociada a κ es $\lambda = \mathfrak{p}(\kappa) = (3,2,2,2,1,1)$. Esta se obtiene fácilmente del diagrama de κ eliminando todas las celdas que tienen hook mayor o igual a 4, como se puede observar en el siguiente diagrama



Por otro lado, la biyección en la dirección contraria, que denotaremos por $\mathfrak{c} : \lambda \mapsto \kappa$, se obtiene usando el siguiente algoritmo. Dada la partición k-acotada λ, se determina la fila más alta de λ que tiene una celda cuyo hook es mayor k, esta fila se desplaza a la derecha junto con todas las filas que están deabajo de ella, hasta que todas las celdas de la fila encontrada tengan hook menor o igual a k. En seguida se consideran las filas que se trasladaron como una nueva partición y se realiza el mismo prodecimiento, hasta que no existan celdas con hook mayor que k. Por último, se completa el diagrama obtiendo el $(k+1)$-core κ deseado.

Usando el mismo ejemplo anterior para $\lambda = (3,2,2,2,1,1) \in \mathcal{P}^3$, el 4-core $\kappa = \mathfrak{c}(\lambda)$ se obtiene usando el algoritmo descrito, de la siguiente manera:

$\lambda =$  $= \kappa = \mathfrak{c}(\lambda).$

Otro de los conjuntos que están en biyección con el conjunto de las particiones k-acotadas (y el conjunto de los $(k+1)$-cores) son las permutaciones afines. Estas particiones aparecen en el estudio del grupo de Weyl afín de tipo A, es decir el grupo simétrico afín $\tilde{S}_n$. Este útimo es el que vamos a definir a continuación.

2.1.1 Grupo simétrico afín

Sea n un número entero positivo, el grupo *simétrico afín* $\tilde{S}_n$ se define como el conjunto de todas las biyecciones w de $\mathbb{Z}$ en $\mathbb{Z}$, que satisfacen las siguientes condiciones: $w(i+n) = w(i)+n, \quad \forall\, i \in \mathbb{Z}$ y $\displaystyle\sum_{i=1}^{n} w(i) = \frac{n(n+1)}{2}$. Estos elementos son llamados permutaciones afines.

Las *transposiciones afines* de $\tilde{S}_n$ son elementos $t_{i,j}$, con $i,j \in \mathbb{Z}$, tales que: $t_{i,j}(i+kn) = j+kn$, $t_{i,j}(j+kn) = i+kn$, y $t_{i,j}(k) = k$ para todo $k \not\equiv i,j \pmod n$.

De igual modo, el grupo simétrico afín $\tilde{S}_n$ es un grupo Coxeter y está generado por las transposiciones adyacentes $\{s_0, s_1, \ldots, s_{n-1}\}$, donde $s_i = t_{i,i+1}$ para todo $0 \leq i < n-1$, las cuales satisfacen las siguientes relaciones:

(i) $\quad s_i^2 = 1$

(ii) $\quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

(iii) $\quad s_i s_j = s_j s_i \qquad$ para todo $(i-j) \neq 0, 1, (n-1) \pmod n$.

El *largo* de un elemento w de $\tilde{S}_n$, que denotaremos por $\ell(w)$, se define como el número mínimo de transposiciones adyacentes que descomponen a w.

Una permutación afín se dice *Grassmanniana* si es un elemento de largo mínimo de una clase lateral por la izquierda del cociente $\tilde{S}_n/S_n$ donde $S_n = \{s_1, s_2, \ldots, s_{n-1}\}$.

De aquí en adelante cuando menciones el grupo simétrico afín se entenderá que estamos en el caso $n = k+1$, puesto que éste es el que se relaciona con las funciones

k-Schur.

La relación entre el grupo simétrico afín y los cores se muestra en la siguiente proposición. Esta se puede encontrar, por ejemplo en [19].

Proposición 3. Existe una biyección entre el conjunto de las permutaciones Grassmannianas de largo m y los $(k+1)$-cores de tamaño m.

Esta biyección se describe naturalmente usando una acción del grupo simétrico afín $\tilde{S}_{k+1}$ en los $(k+1)$-cores, la que definiremos a continuación.

Definición 3. Sean κ un $(k+1)$-core, $s_i \in \tilde{S}_{k+1}$ y los conjuntos $a_i = \{$ todas las celdas agregables de κ con residuo igual a $i\}$, $e_i = \{$ todas las celdas extraíbles de κ con residuo igual a $i\}$, se define la partición $s_i \cdot \kappa$ como

$$s_i \cdot \kappa = \begin{cases} \kappa & \text{si } a_i = e_i = \emptyset \\ \kappa \cup a_i & \text{si } a_i \neq \emptyset \quad \text{y} \quad e_i = \emptyset \\ \kappa - e_1 & \text{si } e_i \neq \emptyset \quad \text{y} \quad a_i = \emptyset \end{cases}$$

Esta definición es coherente debido a la proposición 1.

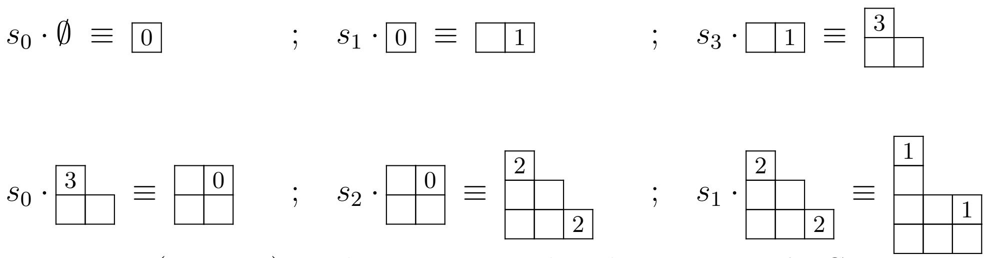
Por ejemplo, consideremos el 5-core $\kappa = (5,4,2,2,2)$, al agregar los residuos en el diagrama de κ junto con lo residuos de las celdas agregables se obtene que:

$$\kappa = \begin{array}{llllll} 0 & & & & & \\ \boxed{1} & \boxed{2} & & & & \\ \boxed{2} & \boxed{3} & & & & \\ \boxed{3} & \boxed{4} & 0 & & & \\ \boxed{4} & \boxed{0} & \boxed{1} & \boxed{2} & 3 & \\ \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & 0. \end{array} .$$

Observamos que los residuos de las celdas agregables son 0 y 3 y los de las celdas extraíbles son 2 y 4, luego tenemos que $s_0 \cdot \kappa = (6,4,3,2,2,1)$, $s_2 \cdot \kappa = (5,3,2,2,1)$ y $s_1 \cdot \kappa = (5,4,2,2,2)$.

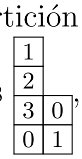
El siguiente ejemplo muestra la manera de obtener la imagen de una permutación Grassmaniana, dada por la biyección indicada en la proposición 3,

Ejemplo 13. Sea w la permutación Grassmanniana representada por la palabra reducida $w = s_1\, s_2\, s_0\, s_3\, s_1\, s_0$. Se construye el 4-core κ comenzando con el conjunto vacío y haciendo actuar los s_i consecutivamente de derecha a izquierda de la palabra w; es decir $\kappa = s_1\, s_2\, s_0\, s_3\, s_1\, s_0 \cdot \emptyset$. De forma detallada podemos ver este resultado.



Luego, $\kappa = (3, 3, 1, 1)$ es el 4-core asociado a la permutación Grassmanniana afín $w = s_1\, s_2\, s_0\, s_3\, s_1\, s_0$.

La aplicación inversa de la biyección dada en la proposición 3, que denotaremos por $\mathfrak{a} : \kappa \mapsto w$, se obtiene tomando los residuos $r_1, r_2, \ldots, r_m$ del diagrama de $\lambda = \mathfrak{p}(\kappa)$ de forma ordenada, partiendo por la fila más alta y leyendo los residuos de derecha a izquierda y continuando con las filas hacia abajo, hasta la fila más baja, obteniendose la permutación Grassmanniana $w = \mathfrak{a}(\kappa) = s_{r_1} s_{r_2} \cdots s_{r_m}$.

Podemos ilustrar esto, usando el ejemplo 13. Sabemos que el 4-core $\kappa = (3, 3, 1, 1)$ está asociado a la partición 3-acotada $\lambda = \mathfrak{p}(\kappa) = (2, 2, 1, 1)$ y su diagrama incluyendo los residuos es , luego la permutación Grassmanniana asociada a κ es $w = s_1\, s_2\, s_0\, s_3\, s_1\, s_0$.

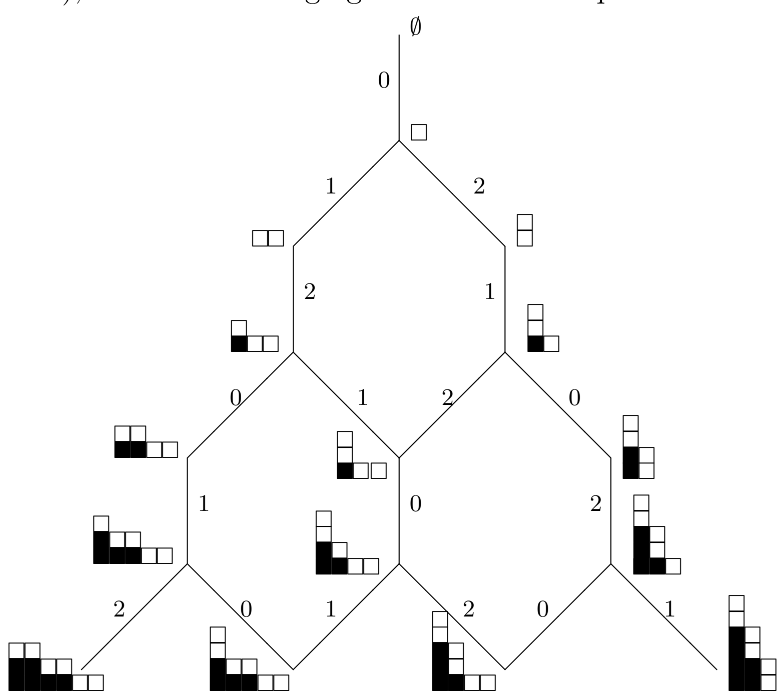
Por otra parte, $\tilde{S}_{k+1}$ como grupo coxeter es equipado con dos ordenes, el orden débil y orden fuerte (o Bruhat). Comenzaremos, caracterizando el orden débil restringido a las permutaciones Grasssmannians en el conjunto de particiones k-acotadas (o los $(k+1)$-cores). Posteriormente seguiremos con el orden fuerte. Estos serán básicos en la construcción de los k-tableaux.

El *orden débil* sobre $\tilde{S}_{k+1}$ es definido de la siguiente manera. Sean w y v elementos de $\tilde{S}_{k+1}$, se dice que, $w \prec v$ si y solo si existe un $u \in \tilde{S}_{k+1}$ tal que $uw = v$ y $\ell(u) + \ell(w) = \ell(v)$. Más aun, $w \prec\!\cdot\; v$ si y solo si existe un elemento $s_i \in \tilde{S}_{k+1}$ tal que $s_i w = v$ y $\ell(w) + 1 = \ell(v)$. Para el caso restringido a las permutaciones Grassmannianas este orden tendrá una interpretación simple en el conjunto de los cores. La siguiente proposición establece un orden en $\mathcal{C}^{k+1}$ equivalente al orden débil de $\tilde{S}_{k+1}$. Este resultado se encuentra en [19].

Proposición 4. Si κ y τ son $(k+1)$-cores tales que $|\mathfrak{p}(\kappa)| = |\mathfrak{p}(\tau)| + 1$, entonces $\tau \prec\!\cdot\; \kappa$ si y solo si existe un i tal que $\kappa = s_i \cdot \tau$.

Usando este resultado y la definicón 3 podemos deducir que, $\tau \prec\!\cdot\; \kappa$ si y sólo si todas las celdas de κ/τ tienen el mismo residuo.

El siguiente esquema muestra los 3-cores ordenados por el orden débil (hasta tamaño 6), al cual hemos agregado los residuos que intervienen en cada caso.



Los 3-cores y su correspondientes particiones 2-acotadas ordenadas por el orden débil.

Ahora bien, para los $(k+1)$-cores κ y τ diremos que κ/τ es un *strip débil horizontal* de tamaño $r \leq k$, si existe una cadena saturada de cores de τ a κ,

$$\tau \lessdot \tau^{(1)} \lessdot \tau^{(2)} \lessdot \cdots \lessdot \tau^{(r)} = \kappa,$$

tal que κ/τ es una strip horizontal y $|\mathfrak{p}(\kappa)| = |\mathfrak{p}(\tau)| + r$. Se puede demostrar que existen exactamente r residuos en el conjunto κ/τ. Estos strips, tendrán un rol combinatorial fundamental en la definición de las funciones k-Schur.

Ahora bien, del mismo modo que caracterizamos el orden débil en los $(k+1)$-cores, lo haremos para el orden fuerte (u orden de Bruhat). Definamos primero el *orden fuerte* sobre $\tilde{S}_n$. Este orden corresponde a la clausura transitiva de las relaciones: $w \lessdot v$ si y solo si existe t_{ij} tal que $t_{ij}w = u$ y $\ell(w) + 1 = \ell(v)$.

En [19] se dió una descripción de este orden en el conjunto de los cores. Específicamente, definieron una relación de orden en $\mathcal{C}^{k+1}$ que es equivalente al orden fuerte definido en $\tilde{S}_{k+1}$. El cual se define sobre los $(k+1)$-cores como:

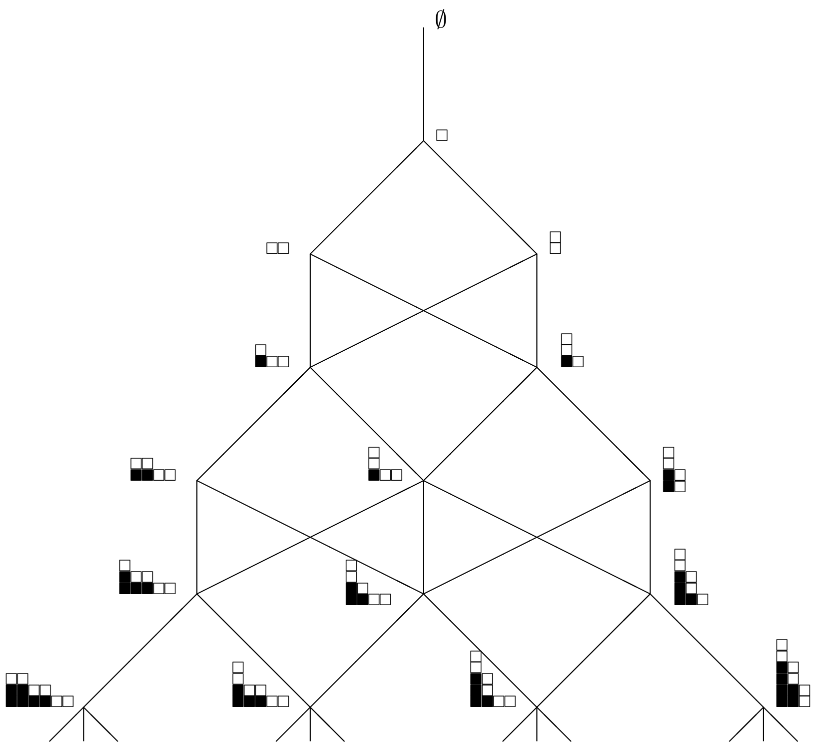
$$\tau \lessdot \kappa \Longleftrightarrow |\mathfrak{p}(\tau)| + 1 = |\mathfrak{p}(\kappa)| \quad \text{y} \quad \tau \subseteq \kappa$$

Así también, en [11] se probó que si $\tau \lessdot \kappa$, entonces el diagrama del cover fuerte κ/τ está compuesto por varios ribbons. De forma más precisa, el diagrama skew satisface las siguientes condiciones:

- Cada componente conexa de κ/τ es un ribbon y todas son idénticas entre sí;
- Los residuos de cada una de las componentes son los mismos y se encuentran en residuos consecutivos.

Cabe añadir que se entenderá por *cabeza* de un ribbon conexo, a la celda que está más al sureste del ribbon.

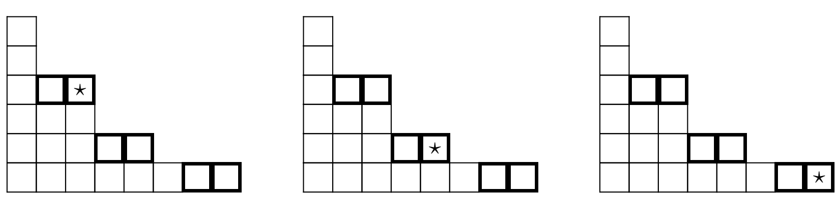
El siguiente diagrama muestra los 3-cores ordenados según el orden fuerte.



Los 3-cores (hasta tamaño 6) ordenados por el orden fuerte

Un *cover fuerte marcado* es un cover fuerte junto con un valor c, el cual indica la cabeza de uno de los ribbons. Concretamente, se define una terna marcada (κ, τ, c) donde κ y τ son $(k+1)$-cores tales que $\tau \lessdot \kappa$ y c es el número $(j-i)$ donde la celda (i, j) corresponde a la cabeza del ribbon marcado.

Ejemplo 14. Para $k = 3$ y los 4-cores $\tau = (8, 5, 3, 3, 1, 1)$ y $\kappa = (6, 3, 3, 1, 1, 1)$, se tiene que $\tau \lessdot \kappa$ y las tres cabezas marcadas son -1, 3 y 7 como se muestran en los diagramas

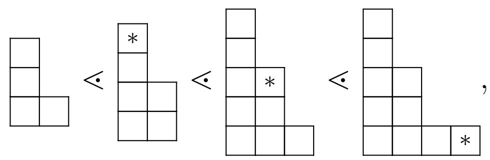


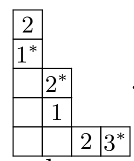
De modo semejante al orden débil, vamos a definir un *strip fuerte marcado horizontal* de tamaño $r \leq k$ como una sucesión de covers fuertes marcados,

$$\kappa^{(0)} \lessdot \kappa^{(1)} \lessdot \kappa^{(2)} \lessdot \cdots \lessdot \kappa^{(r)},$$

donde las cabezas marcadas c_i, que están asociadas al cover $\kappa^{(i)}/\kappa^{(i-1)}$ satisfacen que: $c_1 < c_2 < \cdots < c_r$.

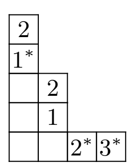
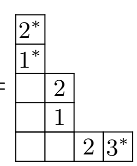
Ejemplo 15. Podemos considerar la siguiente sucesion de covers fuertes marcados, donde especificaremos la cabeza marcada con un $*$, de la siguiente manera.



al que vamos a denotar de manera más compacta por:  .

Este es un strip fuerte marcado horizontal, ya que las cabezas marcadas de cada strip están en orden creciente, específicamente $c_1 = -3,\ c_2 = -1,\ c_3 = 3$.

Observemos que para una misma sucesión de covers, pero con una marcación distinta, no siempre se obtiene un strip fuerte marcado horizontal. Por ejemplo, usando la sucesión del ejemplo anterior y marcaciones distintas, tenemos otros strips, tales como:

$T_1 =$  y $T_2 =$ 

donde T_1 es un strip fuerte marcado horizontal, mientras que T_2 no lo es.

2.2 Funciones k-Schur y k-Schur duales

Daremos la definición de los k-tablaux débiles y k-tableaux fuertes marcados, realizando la misma construcción que hicimos en el capítulo 1 para describir un tableaux semiestandar usando la regla de Pieri. Estos nos permitirán definir las funciones k-Schur y k-Schur duales, semejante al caso de las funciones de Schur.

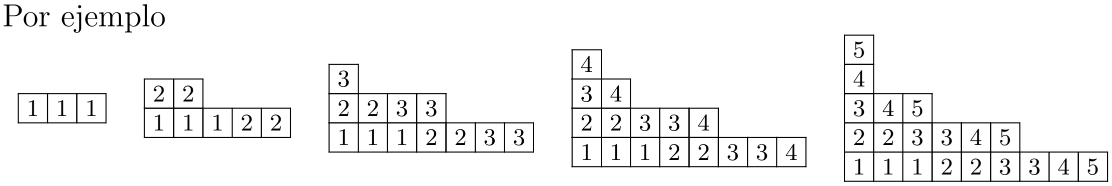
Presentaremos las funciones k-Schur, las que serán indexadas por particiones k-acotadas, como una base del espacio $\Lambda^{(k)} = \mathbb{Q}[h_1, h_2, h_3, \cdots, h_k]$. Estas se pueden definir como la única base de este anillo, tal que

$$h_r s_\lambda^{(k)} = \sum_\mu s_\mu^{(k)}, \tag{2.2.1}$$

donde μ recorre todas las particiones k-acotadas tales que $\mathfrak{c}(\mu)/\mathfrak{c}(\lambda)$ es un strip horizontal débil de tamaño r. Una forma de demostrar que las funciones k-Schur están bien definidas consiste en estudiar ciertos tableaux, al que llamaremos k-tableaux y definiremos a continuación.

Definición 4. Un k-tableau (o k-tableau débil) semiestandar de contenido $(1^{m_1} \cdots r^{m_r})$ es una sucesión de $(k+1)$-cores $\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(0)} = \lambda$ tal que $\lambda^{(i)}/\lambda^{(i-1)}$ es un strip débil horizontal y $\left|\mathfrak{p}(\lambda^{(i)})/\mathfrak{p}(\lambda^{(i-1)})\right| = m_i$, para todo $i \in \{1, 2, \ldots, r\}$.

Por ejemplo



$\lambda^{(1)}$ $\lambda^{(2)}$ $\lambda^{(3)}$ $\lambda^{(4)}$ $\lambda^{(5)}$

Ahora bien, usaremos la siguiente relación como otra definición de las funciones k-Schur

$$h_\lambda = \sum_\mu K_{\mu\lambda}^{(k)} s_\mu^{(k)}, \tag{2.2.2}$$

donde $K_{\mu\lambda}^{(k)}$ es el número de k-tableaux de forma $\mathfrak{c}(\mu)$ y de contenido λ. Estos tienen propiedades similares a los números de Kostka; De hecho, $K_{\mu\mu}^{(k)} = 1$ y $K_{\mu\lambda}^{(k)} = 0$ si $\mu \ngeq \lambda$. Lo que significa que la matriz de transición entre la base $\{h_\lambda\}_{\lambda_1 \leq k,\, \lambda \vdash m}$ y $\{s_\lambda^{(k)}\}_{\lambda_1 \leq k,\, \lambda \vdash m}$ es una matriz uni-triangular, por lo tanto el sistema de 2.2.2 es invertible y nos entrega una definición las funciones k-Schur.

Cabe señalar que no es difícil demostrar que las funciones k-Schur así definidas satisfacen 2.2.1, probando de cierta forma la equivalencia de ambas definiciones.

Para definir las funciones duales a las funciones k-Schur, las que denotaremos por $\mathfrak{S}_\lambda^{(k)}$, necesitamos un nuevo anillo. Sea $I_k = \langle m_\lambda \,|\, \lambda_1 > k \rangle$, se define $\Lambda_{(k)} = \Lambda / I_k$ el anillo cociente, donde la base natural de éste es el conjunto $\{m_\lambda\}_{\lambda_1 \leq k}$, considerando a m_λ como un representante de cada clase.

Se define entonces la forma bilineal $\langle\ ,\ \rangle^k : \Lambda^{(k)} \times \Lambda_{(k)} \longrightarrow \mathbb{Q}$, con la condición $\langle h_\lambda, m_\mu \rangle^k = \delta_{\lambda\mu}$. Esta condición implicará que $\langle s_\lambda^{(k)}, \mathfrak{S}_\mu^{(k)} \rangle^k = \delta_{\lambda\mu}$ según la definición $\mathfrak{S}_\lambda^{(k)}$ que entregaremos a continuación.

Definición 5. Para λ una partición k-acotada, se definen las funciones k-schur duales como:

$$\mathfrak{S}_\lambda^{(k)} = \sum_\mu K_{\lambda\mu}^{(k)} m_\mu. \tag{2.2.3}$$

Por otra parte, se pueden expresar las funciones k-Schur duales $\mathfrak{S}_\lambda^{(k)}$, por:

$$\mathfrak{S}_\lambda^{(k)} = \sum_\alpha K_{\lambda\alpha}^{(k)} x^\alpha.$$

La demostración de esta identidad no es en absoluto fácil. En [17] se demostró que $K_{\lambda\alpha}^{(k)} = K_{\lambda\mu}^{(k)}$ si α es un reordenamiento de μ. Consecuencia de esto, obtenemos una definición combinatorial de las funciones k-Schur duales de la siguiente manera:

$$\mathfrak{S}_\lambda^{(k)} = \sum_T x^T, \tag{2.2.4}$$

donde la suma es sobre todos los k-tableaux de forma $\mathfrak{c}(\lambda)$.

Ilustraremos estos dos resultados con el siguiente ejemplo

Ejemplo 16. Consideramos a $k = 2$ y la partición 2-acotada $\lambda = (2, 1, 1)$. Para descomponer $\mathfrak{S}_{2\,1\,1}^{(2)}$ en monomiales usando la identidad (2.2.3), necesitamos sólo

$K^{(2)}_{(2,1,1)(1,1,1,1)}$, puesto que $K^{(2)}_{(2,1,1,)(2,1,1)} = 1$. Debido a que existen sólo dos k-tableaux de forma $\mathfrak{c}(\lambda)$ y contenido (1,1,1,1), estos son: $\begin{smallmatrix}4\\3\\1&2&3\end{smallmatrix}$ y $\begin{smallmatrix}3\\2\\1&3&4\end{smallmatrix}$, podemos concluir que:

$$\mathfrak{S}^{(2)}_{211} = m_{211} + 2m_{1111}.$$

Por otro lado, usando la identidad (2.2.4) y 4 variables, se tiene que:

$$\begin{aligned}
\mathfrak{S}^{(2)}_{211} &= x^{\begin{smallmatrix}4\\3\\1&2&3\end{smallmatrix}} + x^{\begin{smallmatrix}3\\2\\1&1&2\end{smallmatrix}} + x^{\begin{smallmatrix}4\\2\\1&1&2\end{smallmatrix}} + x^{\begin{smallmatrix}4\\3\\1&1&3\end{smallmatrix}} + x^{\begin{smallmatrix}4\\3\\2&2&3\end{smallmatrix}} + x^{\begin{smallmatrix}3\\2\\1&2&2\end{smallmatrix}} + x^{\begin{smallmatrix}4\\2\\1&2&2\end{smallmatrix}} + \\
&\quad x^{\begin{smallmatrix}4\\3\\1&3&3\end{smallmatrix}} + x^{\begin{smallmatrix}4\\3\\2&3&3\end{smallmatrix}} + x^{\begin{smallmatrix}3\\2\\1&3&3\end{smallmatrix}} + x^{\begin{smallmatrix}4\\2\\1&4&4\end{smallmatrix}} + x^{\begin{smallmatrix}4\\3\\1&4&4\end{smallmatrix}} + x^{\begin{smallmatrix}4\\3\\2&4&4\end{smallmatrix}} + x^{\begin{smallmatrix}3\\2\\1&3&4\end{smallmatrix}} \\
&= x_1x_2x_3x_4 + x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4 + x_2^2x_3x_4 + x_1x_2^2x_3 + x_1x_2^2x_4 + \\
&\quad x_1x_3^2x_4 + x_2x_3^2x_4 + x_1x_2x_3^2 + x_1x_2x_4^2 + x_1x_3x_4^2 + x_2x_3x_4^2 + x_1x_2x_3x_4.
\end{aligned}$$

En [11] se demostró que las funciones k-Schur duales satisfacían una regla tipo Pieri, esto es:

$$h_l\mathfrak{S}^{(k)}_{\mu} = \sum_{\tau} \mathfrak{S}_{\mathcal{P}(\tau)}, \tag{2.2.5}$$

donde la suma es sobre todos los strips fuertes marcados.

Una vez más, utilizando de forma iterada esta regla, aparecen una familia de tableaux con nuevas condiciones, a quienes llamaremos *k-tableaux fuertes marcados* y cuya definición daremos a continuación.

Definición 6. Sean $\kappa \vdash m$ es una partición k-acotada, $\lambda = \mathfrak{c}(\kappa)$ y $\alpha = (\alpha_1, \alpha_2, \dots \alpha_d)$ una partición con $|\alpha| = m$. Un *k-tableau fuerte marcado* de forma λ y contenido α es una sucesión de strips,

$$\emptyset = \mu^{(0)} < \mu^{(1)} < \mu^{(2)} < \cdots < \mu^{(r)} = \lambda,$$

donde $\mu^{(i)}/\mu^{(i-1)}$ es un strip fuerte marcado horizontal de tamaño α_i , para todo $1 \le i \le d$.

Ejemplo 17. Veamos el caso particular $k = 2$ y la partición 2-acotada $\kappa = (2, 1, 1)$. La sucesión de strip

$$\emptyset < \begin{array}{|c|}\hline \ \\ \hline\end{array} < \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} < \begin{array}{|c|cc}\cline{1-1} \ & & \\ \hline \ & \ & \ \\ \hline\end{array} < \begin{array}{|c|cc}\cline{1-1} \ & & \\ \cline{1-1} \ & & \\ \hline \ & \ & \ \\ \hline\end{array},$$

determina dos 2-tableaux de forma $(3, 1, 1)$ y de contenido $(1, 1, 1, 1)$, los que son:

$$\begin{array}{|c|cc}\cline{1-1} 4^* & & \\ \cline{1-1} 3^* & & \\ \hline 1^* & 2^* & 3 \\ \hline\end{array} \qquad \begin{array}{|c|cc}\cline{1-1} 4^\star & & \\ \cline{1-1} 3 & & \\ \hline 1^* & 2^* & 3^* \\ \hline\end{array}$$

De manera similar se obtienen los demás 2-tableaux de forma $(3, 1, 1)$ y de contenido $(1, 1, 1, 1)$. En este caso existen seis con esas condiciones, estos son:

$$\begin{array}{|c|cc}\cline{1-1} 4^* & & \\ \cline{1-1} 3^* & & \\ \hline 1^* & 2^* & 3 \\ \hline\end{array} \qquad \begin{array}{|c|cc}\cline{1-1} 4^\star & & \\ \cline{1-1} 3 & & \\ \hline 1^* & 2^* & 3^* \\ \hline\end{array} \qquad \begin{array}{|c|cc}\cline{1-1} 3 & & \\ \cline{1-1} 3^* & & \\ \hline 1^* & 2^* & 4^* \\ \hline\end{array} \qquad \begin{array}{|c|cc}\cline{1-1} 4^* & & \\ \cline{1-1} 2^* & & \\ \hline 1^* & 3 & 3^* \\ \hline\end{array} \qquad \begin{array}{|c|cc}\cline{1-1} 3^* & & \\ \cline{1-1} 2^* & & \\ \hline 1^* & 3^* & 4^* \\ \hline\end{array} \qquad \begin{array}{|c|cc}\cline{1-1} 3 & & \\ \cline{1-1} 2^* & & \\ \hline 1^* & 3^* & 4^* \\ \hline\end{array}.$$

Ahora bien, si cambiamos el contenido por $(1, 2, 1)$, obtenemos tres 2-tableaux de forma $(3, 1, 1)$ y de contenido $(1, 2, 1)$, los cuales son:

$$\begin{array}{|c|cc}\cline{1-1} 3_1^* & & \\ \cline{1-1} 2_2 & & \\ \hline 1_1^* & 2_1^* & 2_2^* \\ \hline\end{array} \qquad \begin{array}{|c|cc}\cline{1-1} 3_1^\star & & \\ \cline{1-1} 2_1^* & & \\ \hline 1_1^* & 2_2 & 2_2^* \\ \hline\end{array} \qquad \begin{array}{|c|cc}\cline{1-1} 2_2 & & \\ \cline{1-1} 2_1^* & & \\ \hline 1_1^* & 2_2^* & 3_1^* \\ \hline\end{array}.$$

Volviendo a la identidad 2.2.5 y usando esta regla reiteradamente, podemos descomponer las funciones homogeneas h_λ (con $\lambda_1 \leq k$) en término de las funciones de k-Schur duales de la siguiente manera:

$$h_\lambda = \sum_\mu \widetilde{K}^{(k)}_{\mu\lambda} \mathfrak{S}^{(k)}_\mu,$$

donde $\widetilde{K}^{(k)}_{\mu\lambda}$ es el número de k-tableaux fuertes marcados de forma $\mathfrak{c}(\mu)$ y contenido λ. Usando esta identidad y la propiedad de dualidad entre las funciones m_μ y h_λ, obtenemos el desarrollo de las funciones k-Schur en monomiales, esto es:

$$s^{(k)}_\lambda = \sum_\mu \widetilde{K}^{(k)}_{\lambda\mu} m_\mu.$$

Además, se puede demostrar que:

$$s^{(k)}_\lambda = \sum_T x^T, \tag{2.2.6}$$

donde T recorre todos los k-tableaux fuertes marcados de forma $\mathfrak{c}(\lambda)$

Cabe añadir que la definición de las funciones k-Schur dada en 2.2.6 es equivalente a la entregada inicialmente por la relación 2.2.2. La demostración de esto consiste en probar que las funciones $s_\lambda^{(k)}$ y $\mathfrak{S}_\lambda^{(k)}$ definidas por las ecuaciones 2.2.4 y 2.2.6 son duales con respecto a la forma bilineal entregada anteriormente. Esto fue demostrado en [11], donde el ingrediente principal en esta demostración consiste en probar que existe una biyección entre las bipalabras k-acotadas y un par de tableaux $\left(P^{(k)}, Q^{(k)}\right)$ donde, una bipalabra w es k-acotada si la palabra superior tiene a lo más k copias de una letra dada, $P^{(k)}$ es un k-tableau fuerte marcado y $Q^{(k)}$ un k-tableaux , ambos de la misma forma.

Se ha probado que las funciones k-Schur y las k-Schur duales satisfacen diversas propiedades. Por ejemplo, se observa que los coeficientes de estructura $c_{\lambda\mu}^{\nu(k)}$ y $d_{\lambda\mu}^{\nu(k)}$ definidos por

$$s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu c_{\lambda\mu}^{\nu(k)} s_\nu^{(k)} \qquad y \qquad \mathfrak{S}_\mu^{(k)} \mathfrak{S}_\nu^{(k)} = \sum_\lambda d_{\lambda\mu}^{\nu(k)} \mathfrak{S}_\lambda^{(k)}$$

son enteros no negativos. En [10] se probaron ambas afirmaciones usando geometría algebraíca.

Puesto que la definición original de las funciones k-Schur dada en [14] depende de un parámetro t, se hace necesario extender esta definición al subanillo $\mathbb{Q}(t)[h_1, \ldots, h_k]$, al que denotaremos por $\Lambda_t^{(k)}$. Como consecuencia de ello, existen varias definiciones de las funciones k-Schur $s_\mu^{(k)}(x;t)$, las que son conjeturalmente equivalentes. A continuación daremos una de ellas.

Se conjetura que las funciones k-Schur extendidas al subanillo $\Lambda_t^{(k)}$, se expanden positivamente en las funciones monomiales, más aún son de la forma

$$s_\mu^{(k)}(\mathrm{x};t) = \sum_{Q^{(k)}} t^{spin(Q^{(k)})} x^{Q^{(k)}}, \tag{2.2.7}$$

donde $Q^{(k)}$ son k-tableaux duales de forma $\mathfrak{c}(\mu)$.

La motivación principal de la tesis es encontrar una definición similar a la dada en 2.2.7 para las funciones k-Schur duales. Empíricamente se obtuvo la siguiente definición de las funciones k-Schur duales.

$$\mathfrak{S}_{\mu}^{(k)}(\mathrm{x};t) = \sum_{P^{(k)}} t^{\mathrm{ch}(P^{(k)})} x^{P^{(k)}}, \tag{2.2.8}$$

donde $P^{(k)}$ son k-tableaux de forma $\mathfrak{c}(\mu)$ y $\mathrm{ch}(T)$ es una aplicación que asocia un número entero no negativo a un k-tableau.

En el siguiente capítulo entregaremos la definición de $ch(T^{(k)})$, para un k-tableau $T^{(k)}$, la que llamaremos carga de $T^{(k)}$. Además probaremos la compatibilidad de esta definición con una cierta biyección dada en [12] para el caso de los k-tableaux estandares. Esto nos permitirá concluir que la definición dada en (2.2.8) es una buena extensión de las funciones k-Schur duales.

CHAPTER III

Charge on tableaux and the poset of k-shapes

3.1 Introduction

For each integer $k \geq 1$, a family of symmetric functions $s_\mu^{(k)}(x;t)$ (now called k-Schur functions) were introduced in [14] in connection with Macdonald polynomials. To be more precise, computer evidence suggested, for each positive integer k, the existence of a family of symmetric polynomials defined by certain sets of tableaux $\mathcal{A}_\mu^{(k)}$ as:

$$s_\mu^{(k)}(x;t) = \sum_{T \in \mathcal{A}_\mu^{(k)}} t^{\mathrm{ch}(T)}\, s_{\mathrm{shape}(T)}(x) \tag{3.1.1}$$

with the property that any (plethystically modified) Macdonald polynomial $H_\lambda(x;q,t)$ indexed by a partition λ whose first part is not larger than k (a k-bounded partition), can be decomposed as:

$$H_\lambda(x;q,t) = \sum_{\mu;\mu_1 \leq k} K_{\mu\lambda}^{(k)}(q,t)\, s_\mu^{(k)}(x;t)\,, \qquad K_{\mu\lambda}^{(k)}(q,t) \in \mathbb{N}[q,t]\,. \tag{3.1.2}$$

Moreover, given that in this setting $s_\mu^{(k)}(x;t) = s_\mu(x)$ for large k, the coefficients $K_{\mu\lambda}^{(k)}(q,t)$ reduce to the usual q,t-Kostka coefficients when k is large enough [6, 24]. The study of the $s_\lambda^{(k)}(x;t)$ led to several conjecturally equivalent characterizations [14, 15, 11], but for the purpose of this article only the characterization of [11] will be relevant. Note that from now on, we will always index k-Schur functions by $k+1$-cores rather than by k-bounded partitions (see for instance [16] for the connection between the two concepts).

It was shown that the k-Schur functions at $t=1$ (in the characterization [11]) provide the natural basis to work in the quantum cohomology of the Grassmannian just as the Schur functions do for the usual cohomology [18]. Another key development was T. Lam's proof [10] that the Schubert basis of the homology of the affine Grassmannian is given by the k-Schur functions, and that the Schubert basis of the cohomology of the affine Grassmannian is given by functions dual to the k-Schur functions, called dual k-Schur functions or affine Schur functions [9, 18].

Many combinatorial conjectures about k-Schur functions have been formulated, but the conjecture especially relevant to this article is that the k-Schur functions expand positively into k'-Schur functions for $k'>k$ [14]:

$$s_\lambda^{(k)}(x;t) = \sum_\mu b_{\lambda,\mu}^{(k\to k')}(t)\, s_\mu^{(k')}(x;t)\,, \qquad \text{for } b_{\lambda,\mu}^{(k\to k')}(t) \in \mathbb{Z}_{\geq 0}[t]. \tag{3.1.3}$$

When k' is large enough, this conjecture says that k-Schur functions are Schur positive (partial results in this direction have been obtained in [1]). The conjecture also states in particular that $(k-1)$-Schur functions expand positively into k-Schur functions. We refer to the $b_{\lambda,\mu}^{(k-1,k)}(t)$ coefficients as the k-branching coefficients. We have defined in [12] a poset called the poset of k-shapes, whose maximal elements are $(k+1)$-cores and whose minimal elements are k-cores, and have a conjecture for the k-branching coefficients as enumerating maximal chains in the poset of k-shapes modulo an equivalence (see Section 3.8). Using the duality between the ungraded k-Schur and dual k-Schur functions [11], this conjecture has been shown to be valid when $t=1$ by proving that dual k'-Schur functions expand positively into dual k-Schur functions for $k'>k$ [12].

One of the obstructions to the generalization of the the results of [12] to a generic value of t was the lack of an explicit definition for the graded version of the dual k-Schur functions[1]. In this article we introduce a charge on k-tableaux that provides such a definition. To be more precise, the dual k-Schur functions are the generating function of certain combinatorial objects called k-tableaux. We define the graded version of the dual k-Schur functions as[2]

$$\mathfrak{S}_\lambda^{(k)}(x;t) = \sum_Q t^{\mathrm{ch}(Q)} x^Q \tag{3.1.4}$$

where the sum is over all k-tableaux Q of shape λ (where for simplicity the dual k-Schur function is indexed by a $k+1$-core), and where charge(Q) is a certain generalization (see Section 3.4) of the charge of a tableau [21]. We conjecture that the charge also provides the k-Schur expansion of a Hall-Littlewood polynomial indexed by a k-bounded partition

$$H_\lambda(x;t) = \sum_{Q^{(k)}} t^{\mathrm{ch}(Q^{(k)})} s^{(k)}_{\mathrm{sh}(Q^{(k)})}(x;t) \tag{3.1.5}$$

where the sum is over all k-tableaux of weight λ, and where $\mathrm{sh}(Q^{(k)})$ is the shape of the k-tableau $Q^{(k)}$. This formula, which would prove the case $q = 0$ of (3.1.2), generalizes a well-known result of Lascoux and Schützenberger providing a t-statistic on tableaux for the Kostka-Foulkes polynomials [21].

We give evidence of the validity of the definition of charge by proving that the concept of charge on k-tableaux is compatible in the standard case (the complications pertaining to the extension to the non-standard case are discussed in the conclusion) with the weak bijection introduced in [12].

[1] Another version of the graded dual k-Schur functions can be obtained from the k-Schur functions by duality with respect to the Hall-Littlewood scalar product (see [13]). This version is however not monomial positive.

[2] A different definition of graded dual k-Schur functions has been proposed in [2].

Theorem 1. *The weak bijection in the standard case*

$$\begin{aligned} \mathrm{SWTab}_\lambda^k &\longrightarrow \bigsqcup_{\mu\in\mathcal{C}^k} \mathrm{SWTab}_\mu^{k-1} \times \overline{\mathcal{P}}^k(\lambda,\mu) \\ Q^{(k)} &\longmapsto (Q^{(k-1)}, [\mathbf{p}]) \end{aligned} \tag{3.1.6}$$

where SWTab_λ^k *is the set of standard* k*-tableau (or standard weak tableau) of shape* λ*, and where* $[\mathbf{p}]$ *is a certain equivalence class of paths in the poset of* k*-shapes, is such that*

$$\mathrm{charge}(Q^{(k)}) = \mathrm{charge}(Q^{(k-1)}) + \mathrm{charge}(\mathbf{p}) \tag{3.1.7}$$

with $\mathrm{charge}(\mathbf{p})$ *the charge of the path* $\mathbf{p}$ *(see Definition 4).*

As discussed in Section 3.8, Theorem 1 is one of the key ingredients in our approach to prove the k-Schur expansion (3.1.5) of the Hall-Littlewood polynomials.

A simple consequence of the compatibility between the charge and the weak bijection is that the charge of a usual standard tableau can be given as the charge of the path associated to the tableau in our bijection. In effect, iterating the weak bijection starting from a standard tableau T of n letters,

(3.1.8)

$$T \mapsto (T^{(n-1)}, [\mathbf{p}_n]), \quad T^{(n-1)} \mapsto (T^{(n-2)}, [\mathbf{p}_{n-1}]), \quad \dots, \quad T^{(2)} \mapsto (T^{(1)}, [\mathbf{p}_2])\,.$$

we obtain a bijection that puts in correspondence T and $(T^{(1)}, [\mathbf{p}_n], [\mathbf{p}_{n-1}], \dots, [\mathbf{p}_2])$. Given that there is a unique standard 1-tableau $T^{(1)}$ we have that T is in correspondence with the equivalence of paths $([\mathbf{p}_n], [\mathbf{p}_{n-1}], \dots, [\mathbf{p}_2])$. Moreover, the charge of $T^{(1)}$ being 0, the compatibility between the charge and the weak bijection in the standard case implies

$$\mathrm{charge}(T) = \mathrm{charge}(\mathbf{p}_n) + \cdots + \mathrm{charge}(\mathbf{p}_2) \tag{3.1.9}$$

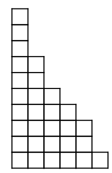
Here is the outline of the article. For the article to be self-contained, a good deal of results and definitions of [12] must be introduced or specialized to the standard case. These include for instance moves, covers, the poset of k-shapes, k-shape tableaux, the pushout algorithm, and the weak bijection. This is essentially the content of Sections 3.2, 3.3, 3.5, 3.6 and 3.7. The charge of a k-tableau is defined in Section 3.4. In Section 3.8 we describe the general context of this work, such as the Schur positivity of k-Schur functions, the k-Schur positivity of Hall-Littlewood polynomials, and finally the connection with the atoms of [14]. In Section 3.9, we define the charge and cocharge of a k-shape tableau and derive a relation between the two concepts (Proposition 5). Section 3.10 contains the proof of the main result of this article, namely the compatiblity between (co)charge and the weak bijection in the standard case (Theorem 1). Finally, we discuss in the conclusion why the non-standard case is still out of reach and how fundamental is the problem of defining a Lascoux-Schützenberger type action of the symmetric group on k-shape tableaux.

3.2 Preliminaries

For a fixed positive integer k, the object central to our study is a family of "k-shape"partitions that contains both k and $k+1$-cores. The formula for k-branching coefficients counts paths in a poset on k-shapes. As with Young order, we will define the order relation in terms of adding boxes to a given vertex λ, but now the added boxes must form a sequence of "strings". Here we introduce k-shapes, strings, and moves – the ingredients for our poset.

3.2.1 Partitions

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of degree $|\lambda| = \sum_i \lambda_i$ is a vector of non-negative integers such that $\lambda_i \geq \lambda_{i+1}$ for $i = 1, 2, \dots$. The length $\ell(\lambda)$ of λ is the number of non-zero entries of λ. Each partition λ has an associated Ferrers' diagram with λ_i lattice squares in the i^{th} row, from bottom to top (French notation). For instance, if $\lambda = (6, 5, 5, 4, 3, 2, 2, 1, 1, 1)$, the associated Ferrers' diagram is



Any lattice square in the Ferrers diagram is called a cell (or simply a square), where the cell (i, j) is in the ith row and jth column of the diagram. Given a cell $b = (i, j)$, we let $\mathrm{row}(b) = i$ and $\mathrm{col}(b) = j$. The conjugate λ' of a partition λ is the partition whose diagram is obtained by reflecting the diagram of λ about the main diagonal. Given a cell $b = (i, j)$ in λ, we let

$$a_\lambda(b) = \lambda_i - j\,, \qquad \text{and} \qquad l_\lambda(b) = \lambda'_j - i\,. \tag{3.2.1}$$

The quantities $a_\lambda(b)$ and $l_\lambda(b)$ are respectively called the arm-length and leg-length. The hook-length of $b = (i, j) \in \lambda$ is then defined by $h_\lambda(b) = a_\lambda(b) + l_\lambda(b) + 1$. A p-core is a partition without cells of hook-length equal to p. We let $\mathcal{C}^p$ be the set of p-cores.

We say that the diagram μ is contained in λ, denoted $\mu \subseteq \lambda$, if $\mu_i \leq \lambda_i$ for all i. We also let $\lambda + \mu$ be the partitions whose entries are $(\lambda + \mu)_i = \lambda_i + \mu_i$, and $\lambda \cup \mu$ be the partition obtained by reordering the entries of the concatenation of λ and μ. The dominance ordering on partitions is such that $\lambda \geq \mu$ iff $|\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i.

A cell b is λ-addable (or b is an addable corner of λ) if adding b to λ produces a partition μ. Similarly, a cell of b is λ-removable (or b is a removable corner of λ) if removing b from λ leads to a partition μ. The diagonal index of $b = (i, j)$ is $d(b) = j - i$. From it we define the distance between cells x and y as $|d(x) - d(y)|$.

Let $D = \mu/\lambda$ be a skew shape, the difference of Ferrers diagrams of partitions $\mu \supset \lambda$. Although such a set of cells may be realized by different pairs of partitions, unless specifically stated otherwise, we shall use the notation μ/λ with the fixed pair $\lambda \subset \mu$ in mind. A *horizontal* (resp. *vertical*) strip is a skew shape that contains at most one cell in each column (resp. row).

3.2.2 k-shapes

The k-interior of a partition λ is the subpartition made out of the cells with hook-length larger than k:

$$\mathrm{Int}^k(\lambda) = \{b \in \lambda \mid h_\lambda(b) > k\} .$$

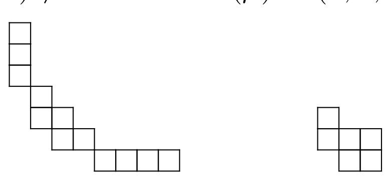
The k-boundary of λ is the skew shape of cells with hook-length bounded by k:

$$\partial^k(\lambda) = \lambda/\mathrm{Int}^k(\lambda) .$$

We define the *k-row shape* $\mathrm{rs}^k(\lambda) \in \mathbb{Z}^\infty_{\geq 0}$ (resp. *k-column shape* $\mathrm{cs}^k(\lambda) \in \mathbb{Z}^\infty_{\geq 0}$) of λ to be the sequence giving the number of cells in the rows (resp. columns) of $\partial^k(\lambda)$.

Definition 1. Let $k \geq 2$ be an integer. A partition λ is a *k-shape* if $\mathrm{rs}^k(\lambda)$ and $\mathrm{cs}^k(\lambda)$ are partitions. We let Π^k denote the set of k-shapes and $\Pi^k_N = \{\lambda \in \Pi^k : |\partial^k(\lambda)| = N\}$.

Example 1. The partition $\lambda = (8,4,3,2,1,1,1) \in \Pi^4_{12}$, since $\mathrm{rs}^4(\lambda) = (4,2,2,1,1,1,1)$ and $\mathrm{cs}^4(\lambda) = (3,2,2,1,1,1,1,1)$ are partitions and $|\partial^4(\lambda)| = 4+2+2+1+1+1+1 = 12$. The partition $\mu = (3,3,1) \notin \Pi^4$ since $\mathrm{rs}^4(\mu) = (2,3,1)$ is not a partition.



$\partial^4(\lambda)$ $\partial^4(\mu)$

The set of k-shapes includes both the k-cores and $k+1$-cores.

Proposition 1 ([12]). *$\mathcal{C}^k \subset \Pi^k$ and $\mathcal{C}^{k+1} \subset \Pi^k$.*

Remark 1. Since k remains fixed throughout, we shall often for simplicity suppress k in the notation, writing $\partial\lambda$, $\mathrm{rs}(\lambda)$, $\mathrm{cs}(\lambda)$, Π, and so forth.

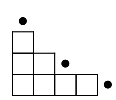
3.2.3 Strings

The primary notion to define our order on k-shapes is a string of cells lying at a diagonal distance k or $k+1$ from one another. To be precise, let b and b' be *contiguous* cells when $|d(b) - d(b')| \in \{k, k+1\}$.

Remark 2. Since λ-addable cells cannot occur on consecutive diagonals, a λ-addable corner x is contiguous with at most one λ-addable corner above (resp. below) it.

Definition 2. A *string* of *length* ℓ is a skew shape μ/λ which consists of cells $\{a_1, \ldots, a_\ell\}$, where a_i and a_{i+1} are contiguous (with a_{i+1} below a_i) for each $1 \leq i < \ell$.

Example 2. Let $k = 3$, $\lambda = (4, 2, 1)$ and $\mu = (5, 3, 1, 1)$. Then μ/λ is a string of length 3. If we denote each element of μ/λ by a $\bullet$, the string can be represented as:



Given a string $s = \mu/\lambda = \{a_1, \ldots, a_\ell\}$, of particular importance are certain columns and rows called modified rows. Define $\Delta_{\mathrm{rs}}(s) = \mathrm{rs}(\mu) - \mathrm{rs}(\lambda) \in \mathbb{Z}^\infty$. The *positively* (resp. *negatively*) *modified rows* of s are those corresponding to positive (resp. negative) entries in $\Delta_{\mathrm{rs}}(s)$. We define in a similar way $\Delta_{\mathrm{cs}}(s)$, and the positively (resp. negatively) modified columns of s. Any string $s = \mu/\lambda$ can be categorized into one of four types

- A *row-type string* if s does not have positively or negatively modified rows

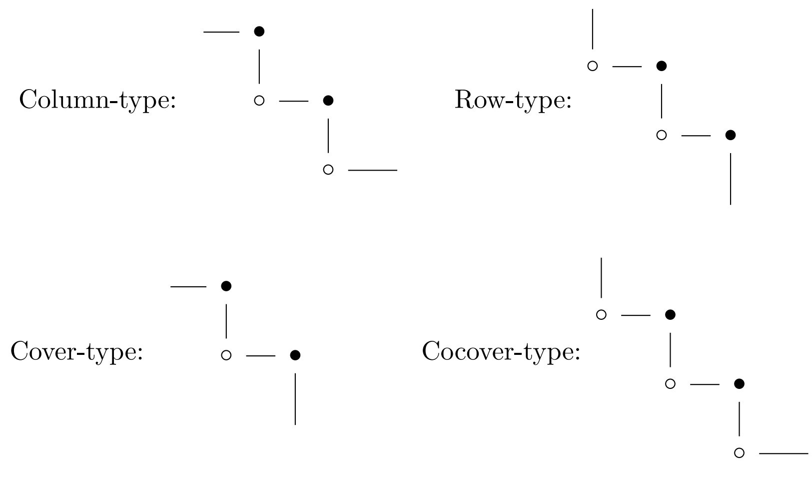


Figura 3.1: Types of string diagrams

- A *column-type string* if s does not have positively or negatively modified columns.
- A *cover-type string* if s has a positively modified row and a positively modified column
- A *cocover-type string* if s has a negatively modified row and a negatively modified column

It is helpful to depict a string $s = \mu/\lambda$ by its *diagram*, defined by the following data: cells of s are represented by the symbol $\bullet$, cells of $\partial\lambda \setminus \partial\mu$ are represented by $\circ$, and cells of $\partial\mu \cap \partial\lambda$ in the same row (resp. column) as some $\bullet$ or $\circ$ are collectively depicted by a horizontal (resp. vertical) line segment. The four possible string diagrams are shown in Figure 3.1.

3.2.4 Moves

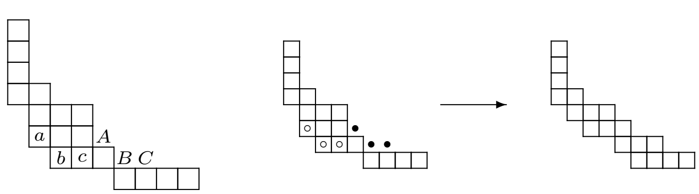
The covering relations in the poset of k-shapes will be defined by letting a k-shape λ be larger than the k-shape μ when the skew diagram μ/λ is a particular succession of strings (called a move). To this end, define two strings to be *translates* when they are translates of each other in $\mathbb{Z}^2$ by a fixed vector, and their corresponding modified rows and columns agree in size. Equivalently, two strings are translate if their diagrams have the property that $\bullet$'s and $\circ$'s appear in the same relative positions with respect to each other and the lengths of each corresponding horizontal and vertical segment are the same. We will also refer to cells a_j and b_j as translates when strings $s_1 = \{a_1, \ldots, a_\ell\}$ and $s_2 = \{b_1, \ldots, b_\ell\}$ are translates.

Definition 3. A *row move* m of rank r and length ℓ is a chain of partitions $\lambda = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r = \mu$ that meets the following conditions:

1. $\lambda \in \Pi$
2. $s_i = \lambda^i/\lambda^{i-1}$ is a row-type string consisting of ℓ cells for all $1 \leq i \leq r$
3. the strings s_i are translates of each other
4. the top cells of $s_1, \ldots, s_r$ occur in consecutive columns from left to right
5. $\mu \in \Pi$.

We say that m is a row move from λ to μ and write $\mu = m * \lambda$ or $m = \mu/\lambda$. A *column move* is the transpose analogue of a row move. A *move* is a row move or column move. It should be noted that it can be shown that moves are at most of rank $k-1$ [12].

Example 3. For $k = 5$, a row move of length 1 and rank 3 with strings $s_1 = \{A\}$, $s_2 = \{B\}$, and $s_3 = \{C\}$ is pictured below. The lower case letters are the cells that are removed from the k-boundary when the corresponding strings are added.



For $k = 3$, a row move of length 2 and rank 2 with strings $s_1 = \{A_1, A_2\}$ and $s_2 = \{B_1, B_2\}$ is:

The rationale for the "move" terminology is the following. Suppose that m is a row move from λ to μ. By definition, $\mathrm{rs}(\mu) = \mathrm{rs}(\lambda)$ (since the strings are all of row-type). Hence $\partial\mu$ can be viewed as a right-shift (or move) of certain rows of $\partial\lambda$.

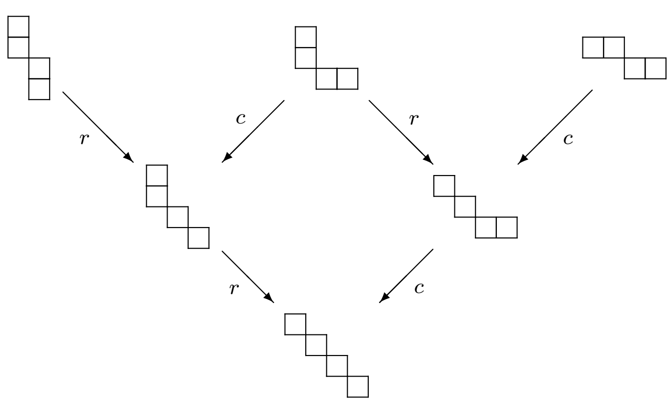
3.3 The poset of k-shapes

We now define a poset structure, called the poset of k-shapes, on the set Π_N of k-shapes of fixed size N. We say that λ dominates μ in the poset of k-shapes if there is a sequence of moves $m_1, \ldots, m_r$ such that $\mu = m_r * \cdots * m_1 * \lambda$. We let $\mathcal{P}^k(\lambda, \mu)$ denote the set of paths in the poset of k-shapes from λ to μ.

Proposition 2 ([12])**.** *An element of the poset of k-shapes is maximal (resp. minimal) if and only if it is a $(k+1)$-core (resp. k-core).*

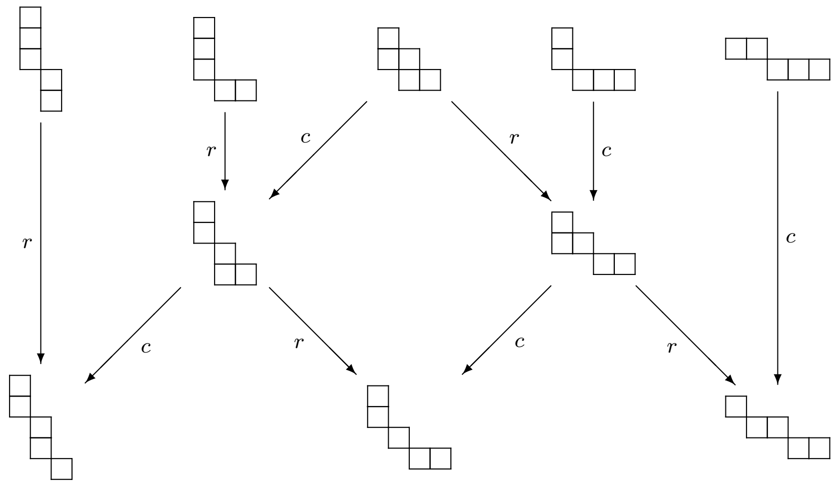
Example 4. The poset of 2-shapes of size 4 is pictured below. Only the cells of the k-boundaries are shown. Row moves are indicated by r and column moves by c.

(3.3.1)



The poset of 3-shapes of size 5 is pictured below.

(3.3.2)



Definition 4. Given a move m, the charge of m, written charge(m), is 0 if m is a row move and $r\ell$ if m is a column move of length ℓ and rank r. Notice that in the column case, $r\ell$ is simply the number of cells in the move m when viewed as a skew shape. The charge of a path $\mathbf{p} = (m_1, \ldots, m_n)$ in Π_N is charge$(m_1) + \cdots +$ charge(m_n), the sum of the charges of the moves that constitute the path.

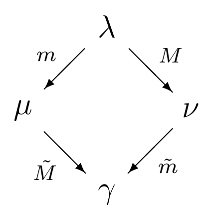
Definition 5. Given a move m, the cocharge of m, written cocharge(m), is 0 if m is a column move and $r\ell$ if m is a row move of length ℓ and rank r. The cocharge of a path is again the sum of the cocharges of the moves forming the path.

Let $\equiv$ be the equivalence relation on paths in Π_N generated by the following *diamond equivalences*:

$$\tilde{M} m \equiv \tilde{m} M \tag{3.3.3}$$

where $m, M, \tilde{m}, \tilde{M}$ are moves (possibly empty) between k-shapes such that the diagram

(3.3.4)



commutes and the charge is the same on both sides of the diamond:

$$\text{charge}(m) + \text{charge}(\tilde{M}) = \text{charge}(M) + \text{charge}(\tilde{m}). \tag{3.3.5}$$

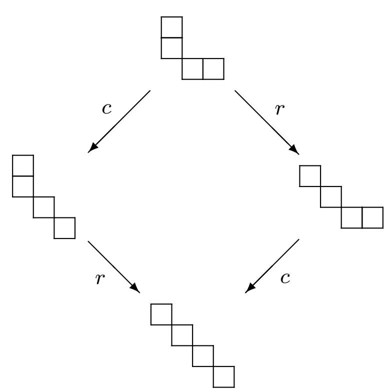
The commutation is equivalent to the equality $\tilde{M} \cup m = \tilde{m} \cup M$ where a move is regarded as a set of cells. Observe that the charge is by definition constant on equivalence classes of paths. We will let $\overline{\mathcal{P}}^k(\lambda, \mu)$ be the set of equivalences classes in $\mathcal{P}^k(\lambda, \mu)$, that is, the set of equivalences classes of paths in the poset of k-shapes from λ to μ. It is easy to see that

$$\begin{aligned} &\text{cocharge}(m) + \text{cocharge}(\tilde{M}) = \text{cocharge}(M) + \text{cocharge}(\tilde{m}) \iff \\ &\text{charge}(m) + \text{charge}(\tilde{M}) = \text{charge}(M) + \text{charge}(\tilde{m}) \end{aligned} \tag{3.3.6}$$

and thus (3.3.5) can be replaced by the left hand side of (3.3.6) in the definition of diamond equivalence.

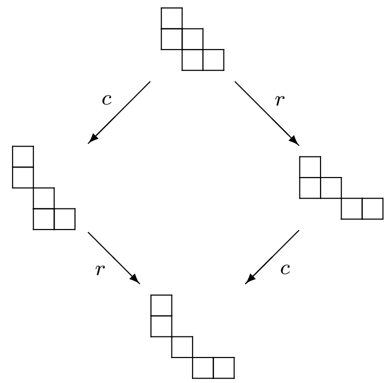
Example 5. Continuing Example 4, the two paths in the poset of 2-shapes from $\lambda = (3, 1, 1)$ to $\mu = (4, 3, 2, 1)$ have charge 2 and 3 respectively, and so are not equivalent.

(3.3.7)



The two paths in the poset of 3-shapes from $\lambda = (3, 2, 1)$ to $\nu = (4, 2, 1, 1)$ are diamond equivalent, both having charge 1.

(3.3.8)



3.4 (Co)charge of a k-tableau

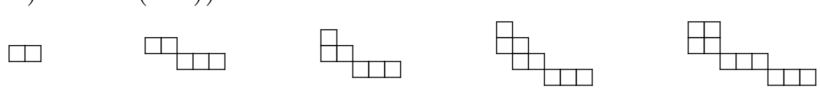
A k-tableau (or weak tableau) is a special type of tableau originally introduced to describe the Pieri-type rules that the k-Schur functions satisfy. The dual k-Schur functions are the generating series of k-tableaux of a given shape.

Definition 6. A k-tableau of weight $(\alpha_1, \ldots, \alpha_N)$ is a sequence of $k+1$-cores $\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(N)} = \lambda$ such that, for all i, $\mathrm{rs}(\lambda^{(i)})/\mathrm{rs}(\lambda^{(i-1)})$ is a horizontal strip and $\mathrm{cs}(\lambda^{(i)})/\mathrm{cs}(\lambda^{(i-1)})$ is a vertical strip, both of size α_i.

Example 6. Let $\alpha = (2, 3, 1, 2, 2)$. An example of a 3-tableau of weight α is

4	5						
3	4						
2	2	4	5	5			
1	1	2	2	2	4	5	5

The corresponding sequence of 4-cores (represented by $\partial(\lambda^{(i)})$ to view more easily $\mathrm{rs}(\lambda^{(i)})$ and $\mathrm{cs}(\lambda^{(i)})$) is



$\lambda^{(1)}$ $\lambda^{(2)}$ $\lambda^{(3)}$ $\lambda^{(4)}$ $\lambda^{(5)}$

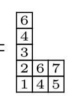
It can be checked that $\mathrm{rs}(\lambda^{(i)})/\mathrm{rs}(\lambda^{(i-1)})$ is a horizontal strip and $\mathrm{cs}(\lambda^{(i)})/\mathrm{cs}(\lambda^{(i-1)})$ is a vertical strip both of size α_i for $i = 1, 2, 3, 4, 5$.

The $k+1$-residue (or simply residue if the value of k is obvious from the context) of a cell $b=(i,j)$ is equal to $j-i \mod k+1$. It can be shown [16] that a k-tableau of weight $(\alpha_1,\ldots,\alpha_N)$ is such that the cells occupied by letter i in the k-tableau have exactly α_i distinct residues.

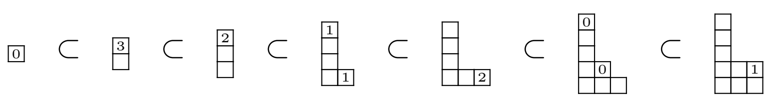
For this article, we will mostly need k-tableaux of weight $(1,1,\ldots,1)$, that is, standard k-tableaux. To be more specific:

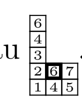
Definition 7. A standard k-tableau is a sequence of $k+1$-cores $\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(N)} = \lambda$ such that, for all i, $\lambda^{(i)}/\lambda^{(i-1)}$ is a vertical and a horizontal strip and $|\partial(\lambda^{(i)})| = |\partial(\lambda^{(i-1)})| + 1$.

Our previous observation then says that the cells occupied by letter i in a standard k-tableau all have the same residue.

Example 7. Let $k=3$. An example of a 3-tableau is $T =$ 

The corresponding sequence of 4-cores (with the residue of each letter) is



We now introduce concepts of charge and cocharge for k-tableaux. The cell corresponding to the lowermost (resp. uppermost) occurrence of a given letter n will be denoted $n^{\downarrow}$ (resp. $n^{\uparrow}$). For instance, the cell $6^{\downarrow}$ is the marked one in the tableau .
We will use $n_-^{\uparrow}$ and $n_+^{\uparrow}$ to denote respectively $(n-1)^{\uparrow}$ and $(n+1)^{\uparrow}$ (and similarly for $n_-^{\downarrow}$ and $n_+^{\downarrow}$). Given two cells b_1 and b_2 of a $(k+1)$-core λ such that b_2 is weakly below b_1, we let $\mathrm{diag}_e(b_1,b_2)$ be the number of diagonals of residue e strictly between b_1 and b_2. The charge of a standard k-tableau T on N letters is

$$\text{charge}(T) = \sum_{n=1}^{N} \text{charge}(n) \tag{3.4.1}$$

where $\text{charge}(1)=0$, and where $\text{charge}(n)$ for $n>1$ is defined recursively in the following way. Suppose that $n^{\uparrow}$ and $n_-^{\uparrow}$ have residues e and e_- respectively. Then charge(n) is defined as

$$\text{(3.4.2)} \qquad \mathrm{ch}(n) = \begin{cases} \mathrm{ch}(n-1) + \mathrm{diag}_{e_-}(n_-^{\uparrow}, n^{\uparrow}) + 1 & \text{if} \quad n_-^{\uparrow} \text{ is weakly above } n^{\uparrow} \\ \\ \mathrm{ch}(n-1) - \mathrm{diag}_{e}(n^{\uparrow}, n_-^{\uparrow}) & \text{if} \quad n_-^{\uparrow} \text{ is below } n^{\uparrow} \end{cases}$$

Example 8. Consider the 4-tableau with the cells $n^{\uparrow}$ marked

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 10 \\ \cline{1-1} 8 \\ \cline{1-1} 5 & 7 \\ \cline{1-2} 4 & 6 & 10 \\ \hline 1 & 2 & 3 & 5 & 7 & 9 & 10 \\ \hline \end{array}$$

Adding the residues to the core

$$\begin{array}{|c|c|c|c|c|c|c|} \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1} 3 & 4 \\ \cline{1-2} 4 & 0 & 1 \\ \hline 0 & 1 & 2 & 3 & 4 & 0 & 1 \\ \hline \end{array}$$

we obtain

$\mathrm{ch}(1) = 0\,; \quad \mathrm{ch}(2) = \mathrm{ch}(1) + 1 = 1\,; \quad \mathrm{ch}(3) = \mathrm{ch}(2) + 1 = 2\,; \quad \mathrm{ch}(4) = \mathrm{ch}(3) = 2$

$\mathrm{ch}(5) = \mathrm{ch}(4) = 2\,; \quad \mathrm{ch}(6) = \mathrm{ch}(5) + 1 = 3\,; \quad \mathrm{ch}(7) = \mathrm{ch}(6) = 3\,;$

$\mathrm{ch}(8) = \mathrm{ch}(7) = 3\,; \quad \mathrm{ch}(9) = \mathrm{ch}(8) + \mathrm{diag}_2(8^{\uparrow}, 9^{\uparrow}) + 1 = 3 + 1 + 1 = 5\,;$

$\mathrm{ch}(10) = \mathrm{ch}(9) - \mathrm{diag}_1(10^{\uparrow}, 9^{\uparrow}) + 1 = 5 - 1 = 4$

Hence $\mathrm{ch}(T) = 0 + 1 + 2 + 2 + 2 + 3 + 3 + 3 + 5 + 4 = 25$.

Similarly, the cocharge of a standard k-tableau T on N letters is

$$\text{(3.4.3)} \qquad \mathrm{cocharge}(T) = \sum_{n=1}^{N} \mathrm{cocharge}(n)$$

where cocharge(1) = 0, and where cocharge(n) for $n > 1$ is defined recursively as (supposing that $n^{\downarrow}$ and $n_-^{\downarrow}$ have residues e and e_- respectively)

(3.4.4)

$$\mathrm{cocharge}(n) = \begin{cases} \mathrm{cocharge}(n-1) - \mathrm{diag}_{e_-}(n_-^{\downarrow}, n^{\downarrow}) & \text{if} \quad n_-^{\downarrow} \text{ is weakly above } n^{\downarrow} \\ \\ \mathrm{cocharge}(n-1) + \mathrm{diag}_{e}(n^{\downarrow}, n_-^{\downarrow}) + 1 & \text{if} \quad n_-^{\downarrow} \text{ is below } n^{\downarrow} \end{cases}$$

Example 9. Consider the 4-tableau in Example 8 with this time the cells $n^{\downarrow}$ marked

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 10 \\ \cline{1-1} 8 \\ \cline{1-1} 5 & 7 \\ \cline{1-2} 4 & 6 & 10 \\ \cline{1-3} 1 & 2 & 3 & 5 & 7 & 9 & 10 \\ \hline \end{array}$$

We have $\text{cocharge}(1) = 0$; $\text{cocharge}(2) = \text{cocharge}(1) - \text{diag}_4(1^{\downarrow}, 2^{\downarrow}) = 0$; $\text{cocharge}(3) = \text{cocharge}(2) - \text{diag}_0(2^{\downarrow}, 3^{\downarrow}) = 0$; $\text{cocharge}(4) = \text{cocharge}(3) + \text{diag}_4(4^{\downarrow}, 3^{\downarrow}) = 1$; $\text{cocharge}(5) = \text{cocharge}(4) - \text{diag}_3(4^{\downarrow}, 5^{\downarrow}) = 1$; $\text{cocharge}(6) = \text{cocharge}(5) + \text{diag}_0(6^{\downarrow}, 5^{\downarrow}) = 2$; $\text{cocharge}(7) = \text{cocharge}(6) - \text{diag}_4(6^{\downarrow}, 7^{\downarrow}) = 2$; $\text{cocharge}(8) = \text{cocharge}(7) + \text{diag}_1(8^{\downarrow}, 7^{\downarrow}) = 4$; $\text{cocharge}(9) = \text{cocharge}(8) - \text{diag}_1(8^{\downarrow}, 9^{\downarrow}) = 3$; $\text{cocharge}(10) = \text{cocharge}(9) - \text{diag}_4(9^{\downarrow}, 10^{\downarrow}) = 3$. Hence $\text{cocharge}(T) = 0 + 0 + 0 + 1 + 1 + 2 + 2 + 4 + 3 + 3 = 16$.

Remark 3. That the charge and cocharge are given by nonnegative integers follows from their compatibility with the weak bijection. In effect, iterating the weak bijection (as was done for instance in the introduction) puts in correspondence a k-tableau $Q^{(k)}$ with a sequence of paths $([\mathbf{p}_k], [\mathbf{p}_{k-1}], \ldots, [\mathbf{p}_2])$ such that $\text{charge}(Q^{(k)}) = \text{charge}(\mathbf{p}_k) + \text{charge}(\mathbf{p}_2)$ and $\text{cocharge}(Q^{(k)}) = \text{cocharge}(\mathbf{p}_k) + \text{cocharge}(\mathbf{p}_2)$, from which the nonnegativity is immediate.

The remainder of this section is concerned with the definition of charge in the non-standard case. Our initial goal was to show that charge was also compatible with the weak bijection in the non-standard case, but as is discussed in the conclusion, we were unfortunately not able to reach that goal.

The definition of charge is given for k-tableaux of dominant weights, and as in the usual case, a Lascoux-Schützenberger type action of the symmetric group on k-tableaux allows to send a k-tableau of any weight into a k-tableau of dominant weight. We do not prove here that the definition of charge and of the Lascoux-Schützenberger type action of the symmetric group on k-tableaux are well-defined.

We first define the charge of a k-tableau T in the non-standard case. First suppose that the weight $(\alpha_1, \ldots, \alpha_N)$ of the k-tableau is dominant, that is, that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N$. As mentioned earlier, the letter n in T occupies exactly α_n residues $i_1, \ldots, i_{\alpha_n}$. We can thus denote the letters n in T as $n_{i_1}, \ldots, n_{i_{\alpha_n}}$. We now form α_1 words in the following way. Start with $1_{j_1} = 1_{\alpha_1 - 1}$ (the rightmost 1 in T) and construct a word recursively by appending to $1_{j_1} 2_{j_2} \cdots n_{j_n}$ the letter $(n+1)_{j_{n+1}}$, where j_{n+1} is the largest element in the total order $j_n+1 < j_n+2 < \cdots < j_n-1$ (the residues are taken modulo $k+1$). Note that it can be shown that there is such a letter. Once $w_1 = 1_{j_1} 2_{j_2} \cdots N_{j_N}$ has been constructed, remove all the letters $1_{j_1}, 2_{j_2}, \ldots, N_{j_N}$ from T, and construct w_2 in the same manner starting this time with $1_{\alpha_1 - 2}$ (the second rightmost 1 in T) and stopping at the largest letter. After all the words $w_1, \ldots, w_{\alpha_1}$ have been constructed, the charge of T is the sum of the charges of the w_i's, where the charge of w_i is the charge of the subtableau of T obtained by considering only the letters in w_i. For instance, let $k = 4$ and consider the 4-tableau of evaluation $(2, 2, 2, 2, 2, 2, 1)$

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 7 \\ \cline{1-1} 6 \\ \cline{1-2} 5 & 6 \\ \cline{1-3} 3 & 4 & 7 \\ \cline{1-5} 2 & 3 & 5 & 5 & 6 \\ \hline 1 & 1 & 2 & 3 & 4 & 4 & 5 & 5 & 6 \\ \hline \end{array}$$

The letters are $1_0, 1_1, 2_2, 2_4, 3_0, 3_3, 4_0, 4_4, 5_1, 5_2, 6_1, 6_3, 7_0$, from which we extract $w_1 = 1_1 2_4 3_3 4_0 5_2 6_1 7_0$ and $w_2 = 1_0 2_2 3_0 4_4 5_1 6_3$. The charge of w_1 is computed using the cells

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \mathbf{7} \\ \cline{1-1} \mathbf{6} \\ \cline{1-2} \mathbf{5} & 6 \\ \cline{1-3} \mathbf{3} & 4 & 7 \\ \cline{1-5} \mathbf{2} & 3 & 5 & 5 & 6 \\ \hline 1 & \mathbf{1} & 2 & 3 & 4 & \mathbf{4} & 5 & 5 & 6 \\ \hline \end{array}$$

Hence charge$(w_1) = 0 + 0 + 0 + 2 + 1 + 1 + 1 = 5$. Similarly, the charge of w_2 is computed using the cells

7								
6								
5	**6**							
3	**4**	7						
2	**3**	**5**	5	6				
1	1	**2**	3	4	4	5	5	6

and is such that $\text{charge}(w_2) = 0+1+1+1+2+2 = 7$. The charge of T is therefore equal to $5+7=12$.

If the weight of T is not dominant, we define the charge of T to be the charge of $\sigma(T)$, where $\sigma(T)$ is the unique k-tableau of dominant weight obtained by the following (conjectural) Lascoux-Schützenberger type action of the symmetric group on k-tableaux. Consider the elementary transposition σ_i which sends a tableau of weight $(\alpha_1,\dots,\alpha_N)$ to a tableau of weight $(\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\alpha_i,\alpha_{i+2},\dots\alpha_N)$. Let i and $i+1$ be denoted respectively as a and b. Suppose that the a's and b's in T occupy residues $i_1,\dots,i_r$ and $j_1,\dots,j_s$ respectively. We say that b_j lies on the floor if there is a b_j in T without an a below it (that is, one unit downward). If b_j does not lie on the floor then let $b_j = \tilde b_{j+1}$, otherwise let $b_j=\tilde b_j$. Order the a's and $\tilde b$'s using the total order

$$\tilde b_0 < a_0 < \tilde b_1 < a_1 < \cdots < \tilde b_k < a_k$$

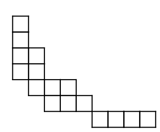
and let the word corresponding to the the ordered a's and $\tilde b$'s be w. Then do the usual pairing of the letters a's and $\tilde b$'s followed by a permutation of their weight to obtain a word w'. That is, pair every factor $\tilde b a$ of w, and let w_1 be the subword of w made out of the unpaired letters. Pair every factor $\tilde b\, a$ of w_1, and let w_2 be the subword made out of the unpaired letters. Continue in this fashion as long as possible. When all factors $\tilde b\, a$ are paired and unpaired letters of w are of the form $a^r\tilde b^s$, send $a^r\tilde b^s$ to $a^s\tilde b^r$ (keeping track of the residues). Now the a's in w' will have residue $i'_1,\dots,i'_s$. Take $T'_{i-1}=T_{i-1}$ and let T'_i be the unique tableau such that letter i occupies residues $i'_1,\dots,i'_s$. Let T'_{i+1} be the unique tableau of the same shape as

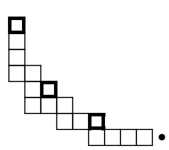
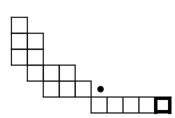
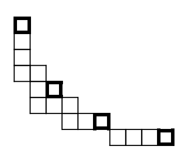
T_{i+1} with subtableau T_i'. Finally, let T' be obtained by filling the rest of the tableau as in T. We then define $\sigma_i(T) = T'$. Since the σ_i's generate the symmetric group, this defines an action of the symmetric group on k-tableaux. It is proven in chapter II (in a more general context) that σ_i is an involution that sends a k-tableau into a k-tableau. The fact that the σ_i's obey the Coxeter relations was always in our mind a consequence of the yet unproven compatibility of the Lascoux-Schützenberger type action of the symmetric group on k-tableaux with the weak bijection (in which case the Coxeter relations would follow from the known Coxeter relations when k is large), and as such we never intended to prove them.

3.5 Standard k-shape tableaux

An generalization of k-tableaux (fillings of k+1-cores) to certain fillings of k-shapes called k-shape tableaux was introduced in [12]. For the purposes of this article, we will only need to describe explicitly the standard case.

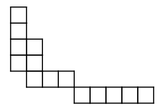
Definition 8. We say that a string $s = \mu/\lambda = \{a_1, \ldots, a_\ell\}$ can be continued below (resp. above) if there is an addable corner of λ below (resp. above) the string s that is contiguous to a_ℓ (resp. a_1). We say that a cover-type string is maximal if it cannot be continued above or below.

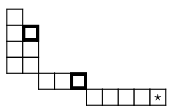
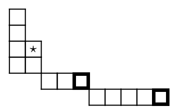
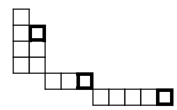
Example 10. Let $k = 5$ and consider $\lambda =$ 

$\mu =$  $\nu =$  and $\gamma =$ 

In this case μ/λ (indicated by framed boxes in the diagram) is a string that can be continued below, ν/λ is a string that can be continued above and γ/λ is a maximal cover-type string. We denoted by $\bullet$ the contiguous addable corner below or above.

Definition 9. We say that a string $s = \mu/\lambda = \{a_1, \ldots, a_\ell\}$ can be *reverse*-continued below (resp. above) if there is a removable corner of μ below (resp. above) the string s that is contiguous to a_ℓ (resp. a_1). We say that a cover-type string is reverse-maximal if it cannot be reverse-continued above or below.

Example 11. Let $k = 5$ and consider $\lambda =$ 

$\mu =$  $\nu =$  and $\gamma =$ 

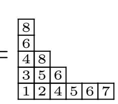
We have that μ/λ is a string that can be reverse-continued below, ν/λ is a string that can be reverse-continued above and γ/λ is a reverse-maximal cover-type string. We denoted by $\star$ the contiguous removable corners below or above.

Definition 10. We say that μ/λ is a cover if λ and μ are k-shapes and μ/λ is a cover-type string. It is maximal (resp. reverse-maximal) if μ/λ is a maximal string (resp. reverse-maximal string).

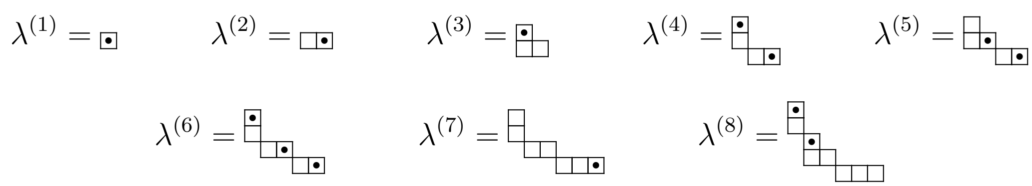
A standard k-shape tableau of shape λ is a sequence of k-shapes

$$\emptyset = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n-1)}, \lambda^{(n)} = \lambda$$

such that $\lambda^{(i)}/\lambda^{(i-1)}$ is a cover for all $i = 1, \ldots, n$. A standard k-shape tableau is maximal (resp. reverse-maximal) if every cover composing the tableau is maximal (resp. reverse-maximal). A standard k-shape tableau is naturally associated to a filling of λ such that letter i occupies the cells $\lambda^{(i)}/\lambda^{(i-1)}$. Given a k-shape tableau T of n letters, we let T_i be the subtableau of T obtained by removing all letter $i+1, \ldots, n$ from T.

Example 12. For $k = 3$, and $n = 8$ the standard 3-shape tableau $T =$ 

corresponds to the sequence of 3-shapes



It can be checked easily that $\lambda^{(i)}/\lambda^{(i-1)}$ (represented by $\bullet$ in the diagrams) is a cover for all i from 1 to 8.

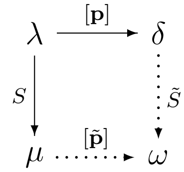
For a given k, the following proposition allows to connect sequences of maximal and reverse-maximal covers to standard $k-1$-tableaux and k-tableaux respectively.

Proposition 3 ([12]). *A sequence* $\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(N)} = \lambda$ *is a standard* k*-tableau if and only if* λ *is a* $k+1$*-core and* $\lambda^{(i)}/\lambda^{(i-1)}$ *is a reverse-maximal cover for all* i*. Similarly, a sequence* $\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(N)} = \lambda$ *is a* $k-1$*-tableau if and only if* $\lambda^{(i)}/\lambda^{(i-1)}$ *is a maximal cover for all* i*.*

3.6 Pushout algorithm in the standard case

The main result of [12] is the construction of a bijection between pairs $(S, [\mathbf{p}])$ and $(\tilde{S}, [\tilde{\mathbf{p}}])$, where $S = \mu/\lambda$ (resp. $\tilde{S} = \omega/\delta$) is a certain reverse-maximal strip (resp. maximal strip), where $\mathbf{p}$ (resp. $\tilde{\mathbf{p}}$) is a path in the poset of k-shapes from λ to δ (resp. μ to ω), and where $[\mathbf{p}]$ denotes the equivalence class of the path $\mathbf{p}$. The bijection can be described diagrammatically by the commuting diagram

(3.6.1)



In the standard case, that is, when the strips S and $\tilde{S}$ are of rank 1, the bijection is between pairs $(c, [\mathbf{p}])$ and $(\tilde{c}, [\tilde{\mathbf{p}}])$ where $c = \mu/\lambda$ (resp. $\tilde{c}$) is a reverse-maximal cover

(resp. maximal cover)

(3.6.2)
$$\begin{array}{ccc} \lambda & \xrightarrow{[\mathbf{p}]} & \delta \\ {\scriptstyle c}\downarrow & & \vdots {\scriptstyle \tilde{c}} \\ \mu & \overset{[\tilde{\mathbf{p}}]}{\cdots\cdots\blacktriangleright} & \omega \end{array}$$

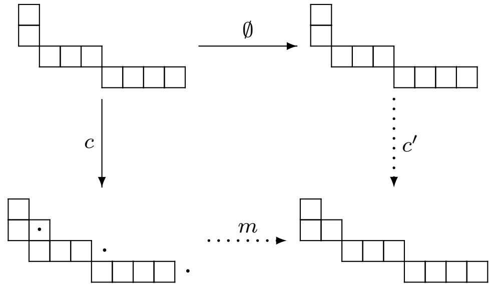
The map $(c, [\mathbf{p}]) \mapsto (\tilde{c}, [\tilde{\mathbf{p}}])$ is given by a certain pushout algorithm. We will now describe the canonical form of the pushout algorithm, which given the pair $(c, \mathbf{p})$ outputs a pair $(\tilde{c}, \tilde{\mathbf{p}})$. Note that the inverse algorithm associates to $(\tilde{c}, \tilde{\mathbf{p}})$ a pair $(c, \mathbf{p}')$, where $[\mathbf{p}'] = [\mathbf{p}]$. This explains the need to work modulo equivalences.

The basic ingredients of the algorithm are the maximization (below and above) of a cover c and the pushout of the pair (c, m) when c is a maximal cover and m is a move. Repeated application of the following three steps will produce the pair $(\tilde{c}, \tilde{\mathbf{p}})$ out of $(c, \mathbf{p})$.

Step 1 (maximization below). If the cover $c = \mu/\lambda = \{a_1, \ldots, a_\ell\}$ can be continued below, then let $c' = c \cup m = (\mu \cup m)/\lambda$, where $m = \{a_{\ell+1}, \ldots, a_{\ell+n}\}$ is the longest sequence of contiguous addable corners of λ such that $a_{\ell+1}$ is contiguous to a_ℓ and below it. In this case, it is immediate that c' is a cover and that m is a row-type string. Moreover, it can be shown [12] that m is in fact a row move from μ to $\mu \cup m$. Diagrammatically, this gives

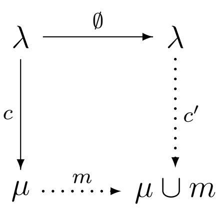
(3.6.3)
$$\begin{array}{ccc} \lambda & \xrightarrow{\emptyset} & \lambda \\ {\scriptstyle c}\downarrow & & \vdots {\scriptstyle c'} \\ \mu & \overset{m}{\cdots\cdots\blacktriangleright} & \mu \cup m \end{array}$$

Example 13. Here is an example of maximization below when $k = 4$ (the cover and its maximization below are highlighted with dots):

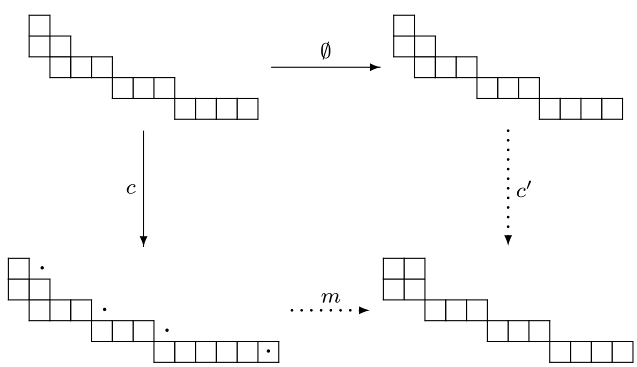


Step 2 (maximization above). If the cover c cannot be continued below but can be continued above, then let $c' = c \cup m = (\mu \cup m)/\lambda$, where $m = \{a_{-n}, \ldots, a_0\}$ is the longest sequence of contiguous addable corners of λ such that a_0 is contiguous to a_1 and above it. It is again immediate that c' is a cover and that m is a column-type string. Moreover, it is shown [12] that m is in fact a column move from μ to $\mu \cup m$. Diagrammatically, this gives

(3.6.4)



Example 14. Here is an example of a maximization above when $k = 4$ (the cover and its maximization above are highlighted with dots):



Step 3 (maximal pushout). If c is a maximal cover and m is any move then the maximal pushout of (c, m) produces the pair $(\tilde{c}, \tilde{m})$ (which we will describe explicitly below), where $\tilde{m}$ is a move:

$$\begin{array}{ccc} \lambda & \xrightarrow{m} & \nu \\ {\scriptstyle c}\downarrow & & \vdots{\scriptstyle \tilde{c}} \\ \mu & \xrightarrow{\tilde{m}} & \eta \end{array} \tag{3.6.5}$$

We say that the cover $c = \mu/\lambda$ and the row move m from λ to μ are *interfering* if c and m do not intersect and $\mu \cup m$ is not a k-shape (that is, if $\mathrm{cs}(\mu) + \Delta_{\mathrm{cs}}(m)$ is not a partition). Similarly, we say that the cover $c = \mu/\lambda$ and the column move m from λ to μ are interfering if c and m do not intersect and $\mu \cup m$ is not a k-shape (that is, if $\mathrm{rs}(\mu) + \Delta_{\mathrm{rs}}(m)$ is not a partition).

For a set of cells A, we let A_U and A_R be the set of cells obtained by translating the cells of A by one unit respectively upward and to the right.

When m is a row move, the pushout is of one of the 4 possible types [12].

I If m and c do not intersect and are not interfering then $\tilde{m} = m$ and $\tilde{c} = c$.

II If m and c do not intersect but are interfering, then $\tilde{c} = c \cup m_{\mathrm{comp}}$ and $\tilde{m} = m \cup m_{\mathrm{comp}}$, where m_{comp} (the completion of m) is the string obtained by translating to the right (by one unit) the rightmost string of m.

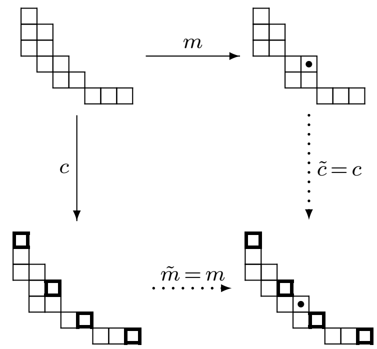
III If c and m intersect, and c continues above m (but not below), then $\tilde{c} = c \setminus (c \cap m)$ and $\tilde{m} = m \setminus (c \cap m)$.

IV If c and m intersect and c continues above and below m, then $\tilde{c} = (c \setminus (c \cap m)) \cup (c \cap m)_U$ and $\tilde{m} = (m \setminus (c \cap m)) \cup (c \cap m)_U$.

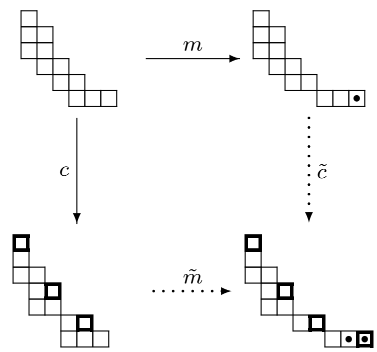
Observe that to the exception of type II, η always corresponds to the union of the cells of λ, c and m (there is interference when this union is not a k-shape).

Example 15. Here are examples of the 4 possible types of pushout when $k = 4$. We indicate cells of the moves with $\bullet$ and cells of the covers by framed boxes.

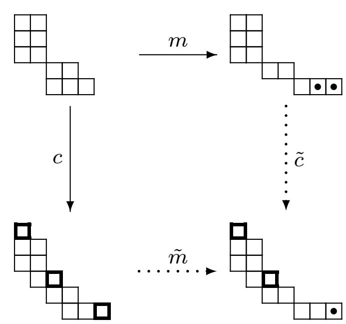
I m and c do not intersect and are not interfering



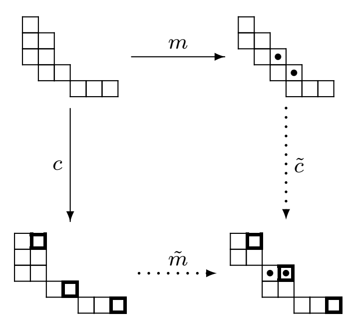
II m and c do not intersect but are interfering



III c and m intersect, and c continues above m (but not below)



IV c and m intersect and c continues above and below m



Similarly, if m is a column move the possible types are:

I If m and c do not intersect and are not interfering, then $\tilde{m} = m$ and $\tilde{c} = c$.

II If m and c do not intersect but are interfering, then then $\tilde{c} = c \cup m_{\text{comp}}$ and $\tilde{m} = m \cup m_{\text{comp}}$, where m_{comp} is the string obtained by translating by one unit upward the uppermost string of m.

III If c and m intersect, and c continues below m (but not above), then $\tilde{c} = c \setminus (c \cap M)$ and $\tilde{M} = M \setminus (c \cap M)$.

IV If c and m intersect and c continues above and below m, then $\tilde{c} = (c \setminus (c \cap m)) \cup (c \cap m)_R$ and $\tilde{m} = (m \setminus (c \cap m)) \cup (c \cap m)_R$.

Using the three steps repeatedly, the pushout algorithm produces from any pair (c, m) a pair $(\tilde{c}, \mathbf{q})$, where $\tilde{c}$ is a maximal cover and $\mathbf{q}$ is a path

(3.6.6)
$$\begin{array}{ccc} \lambda & \xrightarrow{m} & \nu \\ {\scriptstyle c}\downarrow & & \vdots\,{\scriptstyle \tilde{c}} \\ \mu & \overset{\mathbf{q}}{\cdots\cdots\to} & \gamma \end{array}$$

Applying this algorithm for every move in a given path $\mathbf{p}$, the pushout algorithm thus produces from any pair $(c, \mathbf{p})$ a pair $(\tilde{c}, \tilde{\mathbf{p}})$, where $\tilde{c}$ is a maximal cover and $\tilde{\mathbf{p}}$ is a path

(3.6.7)
$$\begin{array}{ccc} \lambda & \xrightarrow{\mathbf{p}} & \delta \\ {\scriptstyle c}\downarrow & & \vdots\,{\scriptstyle \tilde{c}} \\ \mu & \overset{\tilde{\mathbf{p}}}{\cdots\cdots\to} & \omega \end{array}$$

In the special case where c is a reverse-maximal cover, this corresponds to the bijection in (3.6.2) when equivalences of paths are considered.

3.7 Weak bijection in the standard case

The main reason to construct bijection (3.6.2) is to obtain the following bijection (weak bijection in the standard case):

$$
\begin{aligned}
\mathrm{SWTab}^k_\lambda &\longrightarrow \bigsqcup_{\mu \in \mathcal{C}^k} \mathrm{SWTab}^{k-1}_\mu \times \overline{\mathcal{P}}^k(\lambda, \mu) \\
Q^{(k)} &\longmapsto (Q^{(k-1)}, [\mathbf{p}])
\end{aligned}
\tag{3.7.1}
$$

where SWTab^k_λ is the set of standard k-tableau (or standard weak tableau) of shape λ. This bijection proceeds as follows. From Proposition 3, a k-tableau $Q^{(k)}$ of shape λ is a sequence of reverse-maximal covers $c_1 = \lambda^{(1)}/\emptyset, \ldots, c_n = \lambda/\lambda^{(n-1)}$. Starting with the pair $(c_1, [\emptyset])$, where $\emptyset$ is the empty move from the empty partition to itself, the bijection (3.6.2) gives a pair $(\tilde{c}_1, [\mathbf{p}_1])$ with $\tilde{c}_1$ a maximal cover. Then, the pair $(c_2, [\mathbf{p}_1])$ leads to the pair $(\tilde{c}_2, [\mathbf{p}_2])$, where $\tilde{c}_2$ is again a maximal cover. Continuing this way we obtain that $Q^{(k)}$ is in correspondence with a sequence of maximal covers $\tilde{c}_1, \ldots, \tilde{c}_n$ and an equivalence class of paths $[\mathbf{p}]$. This is illustrated in the following diagram:

$$
\begin{array}{ccc}
\emptyset & \xrightarrow{\emptyset} & \emptyset \\
c_1 \downarrow & & \downarrow \tilde{c}_1 \\
\lambda^{(1)} & \xrightarrow{[\mathbf{p}_1]} & \mu^{(1)} \\
c_2 \downarrow & & \downarrow \tilde{c}_2 \\
\lambda^{(2)} & \xrightarrow{[\mathbf{p}_2]} & \mu^{(2)} \\
\vdots & & \vdots \\
\lambda^{(n-1)} & \xrightarrow{[\mathbf{p}_{n-1}]} & \mu^{(n-1)} \\
c_n \downarrow & & \downarrow \tilde{c}_n \\
\lambda & \xrightarrow{[\mathbf{p}]} & \mu
\end{array}
\tag{3.7.2}
$$

The sequence of maximal covers $\tilde{c}_1, \ldots, \tilde{c}_n$ starting at the empty partition is a certain standard $k-1$-tableau $Q^{(k-1)}$ by Proposition 3. Hence $Q^{(k)}$ is in correspondence with $(Q^{(k-1)}, [\mathbf{p}])$ as claimed.

3.8 Missing bijections and what they would entail

To give some perspective to the present work, we explain our general approach to prove the Schur positivity of k-Schur functions and dual k-Schur functions. We also describe how this approach would provide a combinatorial formula for the k-Schur expansion of Hall-Littlewood polynomial indexed by k-bounded partitions, and how it relates to the atoms of [14].

Certain weak and strong tableaux (a.k.a. k-tableaux and dual k-tableaux) related respectively to the weak and strong order on Grassmannian permutations of the affine symmetric group $\tilde{S}_{k+1}$ were introduced in [16, 11]. These tableaux have a certain weight (just as the usual tableaux do) which tells how many times a given letter appears in the tableau.

The graded k-Schur functions (depending on a parameter t and indexed by $k+1$-cores) are defined as [11]

$$s_\lambda^{(k)}(x;t) = \sum_P t^{\mathrm{spin}(P)} x^P \tag{3.8.1}$$

where the sum is over all strong tableaux P of shape λ, where spin is a certain statistic on strong tableaux, and where $x^P = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ if the weight of P is $(\alpha_1, \ldots, \alpha_n)$.

We now provide a similar definition for the graded dual k-Schur functions (indexed again by $k+1$-cores). Let

$$\mathfrak{S}_\lambda^{(k)}(x;t) = \sum_Q t^{\mathrm{charge}(Q)} x^Q \tag{3.8.2}$$

where the sum is over all weak tableaux Q of shape λ, and where we recall that the charge of a k-tableau was defined in Section 3.4.

Computational evidence suggests the conjecture that the k-Schur functions expand positively into k'-Schur functions for $k' > k$:

$$s_\mu^{(k)}(x;t) = \sum_\lambda b_{\mu\lambda}^{(k\to k')}(t)\, s_\lambda^{(k')}(x;t)\,, \qquad \text{for } b_{\lambda\mu}^{(k\to k')}(t) \in \mathbb{Z}_{\geq 0}[t]. \tag{3.8.3}$$

For sufficiently large k', it is known that k'-Schur functions and Schur functions coincide. The conjecture thus implies in particular the Schur positivity of k-Schur functions.

In order to prove the previous conjecture, it is sufficient to understand the case when k and k' differ by one. An explicit conjecture concerning this case was stated in [12] (and proven in the case $t = 1$). It relates the coefficients $b_{\mu\lambda}^{(k-1\to k)}(t)$ (the branching coefficients) to certain paths in the poset of k-shapes weighted by charge.

Let

$$b_{\mu\lambda}^{(k)}(t) := \sum_{[\mathbf{p}]\in\overline{\mathcal{P}}^k(\lambda,\mu)} t^{\mathrm{charge}(\mathbf{p})}. \tag{3.8.4}$$

Conjecture 1. *For all $\lambda \in \mathcal{C}^{k+1}$ and $\mu \in \mathcal{C}^k$, the special case $k \mapsto k-1$ and $k' \mapsto k$ of* (3.8.3) *is given by*

$$b_{\mu\lambda}^{(k-1\to k)}(t) = b_{\mu\lambda}^{(k)}(t) \tag{3.8.5}$$

It is also conjectured that the dual k-Schur functions expand positively into k'-Schur functions, but this time for $k' < k$:

$$\mathfrak{S}_{\mu}^{(k)}(x;t) = \sum_{\lambda} b_{\lambda\mu}^{(k'\to k)}(t)\,\mathfrak{S}_{\lambda}^{(k')}(x;t) \quad \text{mód } I_{k'} \tag{3.8.6}$$

where $I_{k'}$ is the ideal generated by the monomial symmetric functions m_ρ such that $\rho_1 > k'$. We stress that the conjecture makes the stronger claim that the coefficients are the same (up to transposition) as those in the decompositions (3.8.3). This conjecture, also shown to hold when $t = 1$ in [12], would follow again from the case where k and k' differ by one.

Conjecture 2. *For all $\mu \in \mathcal{C}^{k+1}$ and $\lambda \in \mathcal{C}^k$, the special case $k' \mapsto k-1$ of* (3.8.3) *is given by*

$$b_{\lambda\mu}^{(k-1\to k)}(t) = b_{\lambda\mu}^{(k)}(t) \tag{3.8.7}$$

We now describe an approach to prove Conjectures 1 and 2. Let WTab_λ^k and STab_λ^k stand respectively for the set of k-tableaux and dual k-tableaux of shape λ. As we will see, it would be immediate that the conjectures hold if one could find two weight-preserving bijections (respectively the weak and strong bijections)

$$\begin{aligned} \mathrm{WTab}_\lambda^k &\longrightarrow \bigsqcup_{\mu\in\mathcal{C}^k} \mathrm{WTab}_\mu^{k-1} \times \overline{\mathcal{P}}^k(\lambda,\mu) \\ Q^{(k)} &\longmapsto (Q^{(k-1)}, [\mathbf{p}]) \end{aligned} \tag{3.8.8}$$

and

$$\begin{aligned} \mathrm{STab}_\lambda^{k-1} &\longrightarrow \bigsqcup_{\mu\in\mathcal{C}^{k+1}} \mathrm{STab}_\mu^k \times \overline{\mathcal{P}}^k(\mu,\lambda) \\ P^{(k-1)} &\longmapsto (P^{(k)},[\mathbf{p}]) \end{aligned} \tag{3.8.9}$$

such that

$$\mathrm{ch}(Q^{(k)}) = \mathrm{ch}(\mathbf{p}) + \mathrm{ch}(Q^{(k-1)}) \tag{3.8.10}$$

and

$$\mathrm{spin}(P^{(k-1)}) = \mathrm{ch}(\mathbf{p}) + \mathrm{spin}(P^{(k)}) \tag{3.8.11}$$

We should add that in the weak bijection the weight of $Q^{(k)}$ in (3.8.8) needs to be $k-1$-bounded (that is no letter can occur more than $k-1$ times). The weak bijection was constructed in [12]. The aim of this article is to show (Theorem 1) that it satisfies (3.8.10) in the standard case (we will discuss in the conclusion why the semi-standard case is still out of reach). The obtention of the strong bijection (3.8.9) satisfying (3.8.11) is a wide open problem.

We now show how the weak bijection implies Conjecture 2 if it satisfies (3.8.10) (the proof that the strong bijection implies Conjecture 1 if it satisfies (3.8.11) is practically identical and will thus be omitted). We have from (3.8.8) and (3.8.10) that

(3.8.12)

$$\sum_{Q^{(k)}\in \mathrm{WTab}_\lambda^k} t^{\mathrm{ch}(Q^{(k)})} x^{Q^{(k)}} = \sum_{\mu} \sum_{[\mathbf{p}]\in\overline{\mathcal{P}}^k(\lambda,\mu)} \sum_{Q^{(k-1)}\in\mathrm{WTab}_\mu^{k-1}} t^{\mathrm{ch}(Q^{(k-1)})+\mathrm{ch}(\mathbf{p})} x^{Q^{(k-1)}} \mod I_{k-1}$$

where the equality only holds modulo I_{k-1} since the leftmost sum is only over k-tableaux $Q^{(k)}$ of $k-1$-bounded weight. Therefore, it immediately follows that

$$\mathfrak{S}_\lambda^{(k)}(x;t) = \sum_{\mu} \sum_{[\mathbf{p}]\in\overline{\mathcal{P}}^k(\lambda,\mu)} t^{\mathrm{ch}(\mathbf{p})} \mathfrak{S}_\mu^{(k-1)}(x;t) \mod I_{k-1} \tag{3.8.13}$$

which is equivalent to Conjecture 2.

Another interesting consequence of the weak and strong bijections would be the following explicit expression for the k-Schur expansion of Hall-Littlewood polynomials indexed by k-bounded partitions:

$$H_\lambda(x;t) = \sum_{Q^{(k)}} t^{\mathrm{ch}(Q^{(k)})} s^{(k)}_{\mathrm{sh}(Q^{(k)})}(x;t) \tag{3.8.14}$$

where the sum is over all k-tableaux of weight λ. This formula generalizes a well-known result of Lascoux and Schützenberger providing a t-statistic on tableaux for the Kostka-Foulkes polynomials [21].

This is shown in the following way. Let $n = |\lambda|$. If we iterate the weak bijection (3.8.15)

$$Q \mapsto (Q^{(n-1)}, [\mathbf{p}_n]), \quad Q^{(n-1)} \mapsto (Q^{(n-2)}, [\mathbf{p}_{n-1}]), \quad \ldots, \quad Q^{(k+1)} \mapsto (Q^{(k)}, [\mathbf{p}_{k+1}])$$

we can establish a bijective correspondence

$$\begin{aligned} \mathrm{WTab}^n_\lambda &\longrightarrow \bigsqcup_{\mu \in \mathcal{C}^{k+1}} \mathrm{WTab}^k_\mu \times \overline{\mathcal{P}}^{n\to k}(\lambda,\mu) \\ Q &\longmapsto (Q^{(k)}, [\mathbf{p}_n], \ldots, [\mathbf{p}_{k+1}]) \end{aligned} \tag{3.8.16}$$

where $\overline{\mathcal{P}}^{n\to k}(\lambda,\mu)$ is the set of all sequences $[\mathbf{p}_n], \ldots, [\mathbf{p}_{k+1}]$ such that

$$\mathbf{p}_n \in \mathcal{P}^n(\lambda, \mu^{(n-1)}), \mathbf{p}_{n-1} \in \mathcal{P}^{n-1}(\mu^{(n-1)}, \mu^{(n-2)}), \ldots, \mathbf{p}_{k+1} \in \mathcal{P}^{k+1}(\mu^{(k+1)}, \mu)$$

Moreover, the previous correspondence satisfies

$$\mathrm{ch}(Q) = \mathrm{ch}(Q^{(k)}) + \mathrm{ch}(\mathbf{p}_n) + \cdots + \mathrm{ch}(\mathbf{p}_{k+1})$$

which implies that

$$H_\lambda(x;t) = \sum_Q t^{\mathrm{ch}(Q)} s_{\mathrm{sh}(Q)} = \sum_{Q^{(k)}} t^{\mathrm{ch}(Q^{(k)})} \sum_\mu \sum_{[\mathbf{p}] \in \overline{\mathcal{P}}^{n\to k}(\mu, \mathrm{sh}(Q^{(k)}))} t^{\mathrm{ch}(\mathbf{p})} s_\mu \tag{3.8.17}$$

where for short we use $[\mathbf{p}]$ for $[\mathbf{p}_n], \ldots, [\mathbf{p}_{k+1}]$, and $\text{charge}(\mathbf{p}) = \text{charge}(\mathbf{p}_n) + \cdots + \text{charge}(\mathbf{p}_{k+1})$. Equation (3.8.14) then follows since repeated applications of Conjecture 1 gives

$$\sum_{\mu} \sum_{[\mathbf{p}] \in \overline{\mathcal{P}}^{n\to k}(\mu, \text{sh}(Q^{(k)}))} t^{\text{ch}(\mathbf{p})} s_{\text{sh}(Q)}(x) = s^{(k)}_{\text{sh}(Q^{(k)})}(x;t) \tag{3.8.18}$$

Finally, we discuss the connection between the set $\mathcal{A}^{(k)}_\mu$ (called atoms in [14]) and the weak bijection (3.8.8). When $Q \mapsto (Q^{(k)}, [\mathbf{p}_n], \ldots, [\mathbf{p}_{k+1}])$, we say that $Q^{(k)}$ is the k-tableau associated to Q or that Q maps to $Q^{(k)}$. It was conjectured in [12] that the set $\mathcal{A}^{(k)}_\mu$ appearing in (3.1.1) is given by the set of tableaux that map to a certain k-tableaux $T^{(k)}_\mu$:

Conjecture 3. *Let ρ be the unique element of $\mathcal{C}^{k+1}$ such that $\text{rs}(\rho) = \mu$, and let $T^{(k)}_\mu$ be the unique k-tableau of weight μ and shape ρ (see [16]). Then*

$$\mathcal{A}^{(k)}_\mu = \left\{ T \text{ of weight } \mu \,\middle|\, T^{(k)}_\mu \text{ is the } k\text{-tableau associated to } T \right\} . \tag{3.8.19}$$

Our results would imply that if $Q^{(k)}_1$ and $Q^{(k)}_2$ are two k-tableaux of the same shape, then the sets of tableaux $\mathcal{A}$ and $\mathcal{B}$ that respectively map to $Q^{(k)}_1$ and $Q^{(k)}_2$ are each in correspondence with the elements $[\mathbf{p}_n], \ldots, [\mathbf{p}_{k+1}]$ of $\overline{\mathcal{P}}^{n\to k}(\lambda, \mu)$. Hence

$$t^{-\text{charge}(Q^{(k)}_1)} \sum_{T \in \mathcal{A}} t^{\text{charge}(T)} s_{\text{shape}(T)} = t^{-\text{charge}(Q^{(k)}_2)} \sum_{T' \in \mathcal{B}} t^{\text{charge}(T')} s_{\text{shape}(T')} \tag{3.8.20}$$

given that

$$\text{ch}(T) - \text{ch}(Q^{(k)}_1) = \text{ch}(T') - \text{ch}(Q^{(k)}_2) = \text{ch}(\mathbf{p}_n) + \cdots + \text{ch}(\mathbf{p}_{k+1}) \tag{3.8.21}$$

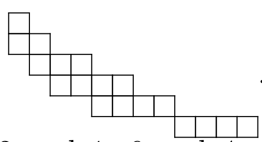
The two symmetric functions $\sum_{T \in \mathcal{A}} t^{\text{charge}(T)} s_{\text{shape}(T)}$ and $\sum_{T' \in \mathcal{B}} t^{\text{charge}(T')} s_{\text{shape}(T')}$ thus only differ by a power of t. In the language of [14], these are instances of copies of atoms (which in [14] where only conjectured to exist).

3.9 (Co)charge of a standard k-shape tableau

We now generalize the notions of charge and cocharge to standard k-shape tableau. We will then establish a relation between charge and cocharge (see Proposition 5) that will prove very useful in the proof of the compatibility between (co)charge and the weak bijection.

3.9.1 k-connectedness

Let r and r' (with $r > r'$) be rows of the k-shape λ that each have an addable corner. We say that r' is the k-connected row below row r (or simply that r and r' are k-connected rows) if r' is the lowest row such that the distance between the addable corners in row r and r' is not larger than $k+1$. If the distance between the addable corners in rows r and r' is k or $k+1$ then r and r' are said to be contiguously connected. We say that $r_1, \ldots, r_m$ form a sequence of k-connected rows of length m if r_{i+1} is the k-connected row below row r_i for all $i = 1, \ldots, m-1$.

Example 16. Let $k = 5$ and $\lambda =$  . The pairs of 5-connected rows are: 7 and 5; 5 and 3; 3 and 2; 2 and 1; 6 and 4; 4 and 2. Moreover rows 4 and 2 are contiguously connected, as are rows 5 and 3. Rows $7, 5, 3, 1$ form a sequence of 5-connected rows of length 4. Observe that two distinct rows can have the same k-connected row below: for instance row 2 is the 5-connected row below rows 3 and 4.

Let r and r' (with $r > r'$) be rows of the k-shape λ that each have an addable corner. Define $[r, r']_k$ to be equal to the length of the longest sequence of k-connected rows $r_1, r_2, \ldots, r_m$ such that $r_1 = r$ and $r_m \geq r'$. We define $(r, r']_k$, $[r, r')_k$ and $(r, r')_k$ in the same fashion (not counting row r or r' according to whether the interval is closed or open).

3.9.2 Definition of charge and cocharge

The charge of a k-shape tableau T on N letters is

$$\text{charge}(T) = \sum_{n=1}^{N} \text{charge}(n) \tag{3.9.1}$$

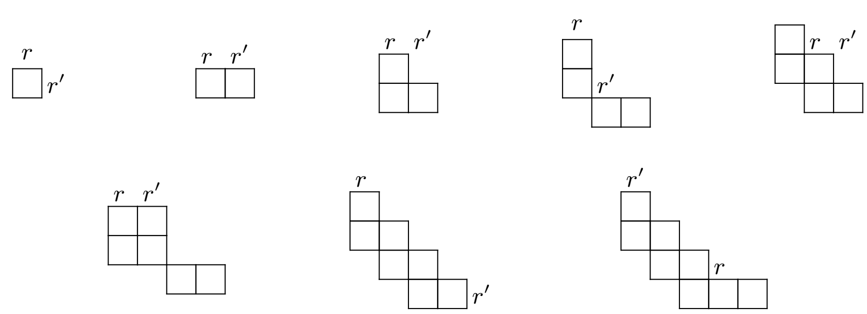
where $\text{charge}(1) = 0$, and where $\text{charge}(n)$ for $n > 1$ is defined recursively in the following way. Let r be the row above that of $n_-^{\uparrow}$ and r' be the row of $n^{\uparrow}$. Then

$$\text{charge}(n) = \begin{cases} \text{charge}(n-1) + [r, r')_k & \text{if} \quad r \geq r' \\ \\ \text{charge}(n-1) - (r', r]_k & \text{if} \quad r < r' \end{cases} \tag{3.9.2}$$

where $[r, r')_k$ and $(r', r]_k$ are calculated using the k-shape corresponding to the shape of T_{n-1}. We also stress that $[r, r')_k = 0$ if $r = r'$.

Example 17. Let $k = 4$ and $\quad T = \begin{array}{|c|c|c|c|c|c|} \hline 9 \\ \cline{1-1} 7 \\ \cline{1-3} 4 & 6 & 9 \\ \cline{1-3} 3 & 5 & 7 \\ \hline 1 & 2 & 4 & 6 & 8 & 9 \\ \hline \end{array}$.

The sequence of k-shapes with the corresponding rows r and r' are



Therefore $\text{charge}(1) = 0$; $\text{charge}(2) = \text{charge}(1) + [2, 1)_4 = 1$; $\text{charge}(3) = \text{charge}(2) - (2, 2]_4 = 1$; $\text{charge}(4) = \text{charge}(3) - (3, 3]_4 = 1$; $\text{charge}(5) = \text{charge}(4) + [4, 2)_4 = 2$; $\text{charge}(6) = \text{charge}(5) - (3, 3]_4 = 2$; $\text{charge}(7) = \text{charge}(6) - (4, 4]_4 = 2$; $\text{charge}(8) = \text{charge}(7) + [5, 1)_4 = 4$; $\text{charge}(9) = \text{charge}(8) - (5, 2]_4 = 3$. This gives $\text{charge}(T) = \sum_{n=1}^{9} \text{charge}(n) = 0 + 1 + 1 + 1 + 2 + 2 + 2 + 4 + 3 = 16$.

Similarly, the cocharge of a k-shape tableau T on N letters is

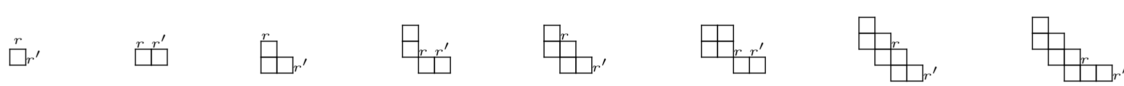
$$\text{cocharge}(T) = \sum_{n=1}^{N} \text{cocharge}(n) \tag{3.9.3}$$

where $\text{cocharge}(1) = 0$, and where $\text{cocharge}(n)$ for $n > 1$ is defined recursively in the following way. Let r be the row above that of $n^{\downarrow}_{-}$ and r' be the row of $n^{\downarrow}$. Then

$$\text{cocharge}(n) = \begin{cases} \text{cocharge}(n-1) - (r, r')_k & \text{if} \quad r > r' \\ \\ \text{cocharge}(n-1) + [r', r]_k & \text{if} \quad r \leq r' \end{cases} \tag{3.9.4}$$

where $(r, r')_k$ and $[r', r]_k$ are again calculated using the k-shape corresponding to the shape of T_{n-1}.

Example 18. Using the same tableau as in Example 17, we get the following sequence of k-shapes (with rows r and r' identified)



We thus have $\text{cocharge}(1) = 0$; $\text{cocharge}(2) = \text{cocharge}(1) - (2,1)_4 = 0$; $\text{cocharge}(3) = \text{cocharge}(2) + [2,2]_4 = 1$; $\text{cocharge}(4) = \text{cocharge}(3) - (3,1)_4 = 1$; $\text{cocharge}(5) = \text{cocharge}(4) + [2,2]_4 = 2$; $\text{cocharge}(6) = \text{cocharge}(5) - (3,1)_4 = 2$; $\text{cocharge}(7) = \text{cocharge}(6) + [2,2]_4 = 3$; $\text{cocharge}(8) = \text{cocharge}(7) - (3,1)_4 = 3$; $\text{cocharge}(9) = \text{cocharge}(8) - (2,1)_4 = 3$. Therefore, $\text{cocharge}(T) = \sum_{n=1}^{9} \text{cocharge}(n) = 0+0+1+1+2+2+3+3+3 = 15$.

We now show that the definition of (co)charge of a k-shape tableau actually extends that of (co)charge of a k-tableau.

Proposition 4. *Let T be a standard k-shape tableau that is also a k-tableau. Then definitions* (3.4.2) *and* (3.9.2) *of charge coincide (and similarly for cocharge).*

Demostración. Let r and r' be respectively the row above that of $n_-^{\uparrow}$ and the row of $n^{\uparrow}$. Let also e and e' be the residues of $n^{\uparrow}$ and $n_-^{\uparrow}$ respectively.

Suppose that $r \geq r'$. Then we need to show that

$$[r, r')_k = \mathrm{diag}_{e'}(n_-^{\uparrow}, n^{\uparrow}) + 1 \tag{3.9.5}$$

or more simply, that

$$(r, r')_k = \mathrm{diag}_{e'}(n_-^{\uparrow}, n^{\uparrow}) \tag{3.9.6}$$

It is an elementary fact of k-tableaux (see for instance [16]) that $e \neq e'$. This implies that $\mathrm{diag}_{e'}(n_-^{\uparrow}, n^{\uparrow}) = \mathrm{diag}_{e'-1}(n_-^{\uparrow}, n^{\uparrow})$. Since there is a residue $e' - 1$ in row r, (3.9.6) follows from Lemma 1.

Now suppose that $r < r'$. We need to show that

$$(r', r]_k = \mathrm{diag}_{e}(n^{\uparrow}, n_-^{\uparrow}) \tag{3.9.7}$$

or equivalently, that

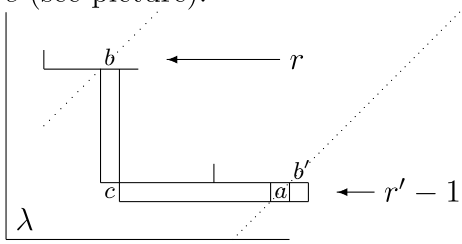
$$(r', r-1)_k = \mathrm{diag}_{e}(n^{\uparrow}, n_-^{\uparrow}) \tag{3.9.8}$$

Since there is a residue e in row r', the equation follows again from Lemma 1.

The proof when charge is replaced by cocharge is similar. □

Lemma 1. *Let λ be a $k+1$-core. Let r' be the k-connected row below a certain row r of λ. If there is a residue e on the ground in row r of λ (that is, there is a cell $b = (i, j)$ of residue e in row r such that $(i, j) \notin \lambda$ and $(i-1, j) \in \lambda$), then there is also a residue e on the ground in row r'. Moreover, there is no diagonal of residue e between rows r and r'.*

Demostración. Let b be the position of the cell of residue e on the ground in row r. Let c be the position of the uppermost cell of $\mathrm{Int}^k(\lambda)$ in the column of b. It is easy to see that c lies in row $r'-1$. Since $h_c(\lambda) > k$, there is a cell a of residue e in the row of c (see picture).



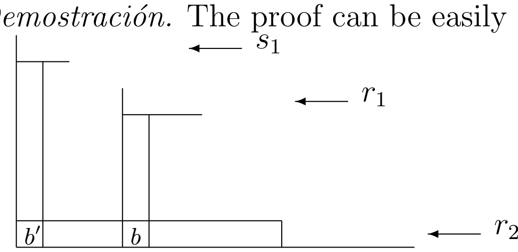
Observe that by definition of c, the cell b' in the picture does not belong to λ. It thus suffices to show that the cell b' has a cell of λ immediately below. But this has to be the case since λ is a $k+1$-core (otherwise we would have $h_c(\lambda) = k+1$). □

3.9.3 Relation between charge and cocharge

Before establishing the relation between charge and cocharge given in Proposition 5 below, we need to prove a series of technical lemmas that will culminate with Lemma 7 (which is essentially equivalent to Proposition 5).

Lemma 2. *Let r_1 y r_2 be two contiguously connected rows of a given k-shape λ, with $r_1 > r_2$. If s_1 is a row of λ such that $s_1 > r_1$, then the k-connected row s_2 below s_1 is such that $s_2 > r_2$.*

Demostración. The proof can be easily visualized with the following diagram:

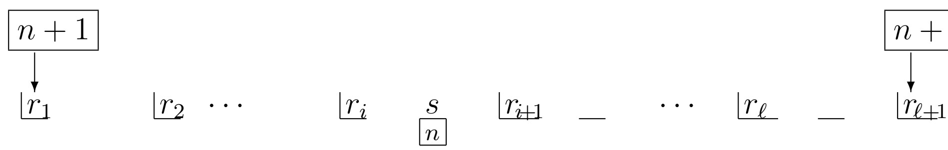


Obviously $h_{b'}(\lambda) \geq h_b(\lambda)+2$. By hypothesis, $h_b \geq k-1$, which implies $h_{b'}(\lambda) \geq k+1$. Hence, from the definition of k-connected rows, we have $s_1 > r_2$. □

The next lemma is immediate.

Lemma 3. *Let r_1 and r_2 be two k-connected rows of a given k-shape λ, with $r_1 > r_2$. If s_1 is a row of λ such that $s_1 \le r_1$, then the k-connected row s_2 below s_1 is such that $s_2 \le r_2$.*

Lemma 4. *Let T be any k-shape tableau, and let* $\text{cocharge}'(n+1)$ *be the hypothetical cocharge in T of the letter $n+1$ if $n_+^{\downarrow}$ were located in the position of $n_+^{\uparrow}$. Then* $\text{cocharge}'(n+1) - \text{cocharge}(n+1) = \ell$, *where $\ell = |c_{n+1}| - 1$.*



Demostración. Let T_n be the shape of λ. Let $r_1, \ldots, r_{\ell+1}$ be the rows where the letter $n+1$ occurs in T (r_1 and $r_{\ell+1}$ are thus the rows of $n_+^{\uparrow}$ and $n_+^{\downarrow}$ respectively). These correspond by definition of c_{n+1} to a sequence of contiguously connected rows of λ.

Suppose that s_1 is the row of λ above that of $n^{\downarrow}$. We will only prove the case where s_1 lies between r_1 and $r_{\ell+1}$ (the other cases are simplified versions of that case). We thus suppose that s_1 lies between r_1 and $r_{\ell+1}$. Let r_i be the lowest among $r_1, \ldots, r_{\ell+1}$ such that $r_i \ge s_1$. Let also $s_1, \ldots, s_{\ell-i+1}$ be the sequence of k-connected rows weakly below row s_1. From Lemma 2 and 3, we have

$$r_i \ge s_1 > r_{i+1} \ge s_2 > \cdots \ge s_{\ell-i} > r_{\ell+1} \ge s_{\ell-i+1}$$

and it is then immediate that $(s_1, r_{\ell+1})_k = (r_i, r_{\ell+1})_k$.

We can now proceed with the proof of the lemma. We have $\text{cocharge}'(n+1) = \text{cocharge}(n) + [r_1, s_1]_k$ and $\text{cocharge}(n+1) = \text{cocharge}(n) - (s_1, r_{\ell+1})$. Thus $\text{cocharge}'(n+1) - \text{cocharge}(n+1) = [r_1, s_1]_k + (s_1, r_{\ell+1})_k$. But $[r_1, s_1]_k = [r_1, r_i]_k$ and, as we have seen, $(s_1, r_{\ell+1})_k = (r_i, r_{\ell+1})_k$. This implies that $\text{cocharge}'(n+1) - \text{cocharge}(n+1) = [r, r_{\ell+1})_k = \ell$. $\square$

The next lemma, which is similar to the previous one, is stated without proof.

Lemma 5. *Let T be any k-shape tableau, and let* $\text{charge}'(n)$ *be the hypothetical charge in T of the letter n if $n^{\uparrow}$ were located instead in the position of $n^{\downarrow}$. Then* $\text{charge}'(n) - \text{charge}(n) = \ell$*, where $\ell = |c_n| - 1$.*

Lemma 6. *Let T be any k-shape tableau, and let* $\text{charge}'(n)$ *and* $\text{charge}'(n+1)$ *be the hypothetical charge in T of the letters n and $n+1$ respectively if $n^{\uparrow}$ were located instead in the position of $n^{\downarrow}$. Then* $\text{charge}(n+1) = \text{charge}'(n+1)$.

Demostración. Let T_n be the shape of λ. Let also $r_1, \ldots, r_{\ell+1}$ be the rows where the letter n occurs in T_n, and define $\bar{r}_i = r_i + 1$ for $i = 1, \ldots, \ell + 1$. Because of the presence of the letter n in rows $r_1, \ldots, r_{\ell+1}$, the k-shape λ has addable corners in rows $\bar{r}_1, \ldots, \bar{r}_{\ell+1}$. Furthermore, since by definition of c_n the rows $r_1, \ldots, r_{\ell+1}$ form a sequence of contiguously connected rows, we get that $\bar{r}_1, \ldots, \bar{r}_{\ell+1}$ are k-connected rows below row $\bar{r}_1$. Observe also that $\bar{r}_1$ is the row above that of $n^{\uparrow}$.

Now, let s_1 be the row of λ where $n_+^{\uparrow}$ lies, and let $s_1, \ldots, s_{\ell-i+1}$ be the sequence of k-connected rows weakly below row s_1. We will again only treat the case where s_1 is between $\bar{r}_1$ and $\bar{r}_{\ell+1}$. Define r_i to be the lowest among $r_1, \ldots, r_{\ell+1}$ such that $r_i \geq s_1$. Then from Lemmas 2 and 3, we have

$$r_i \geq s_1 > r_{i+1} \geq s_2 > \cdots \geq s_{\ell-i} > r_{\ell+1} \geq s_{\ell-i+1}$$

which is equivalent to

$$\bar{r}_i > s_1 \geq \bar{r}_{i+1} > s_2 \geq \cdots > s_{\ell-i} \geq \bar{r}_{\ell+1} > s_{\ell-i+1}$$

Observe that $(s_1, \bar{r}_{\ell+1}]_k = (\bar{r}_{i+1}, \bar{r}_{\ell+1}]_k$.

We can now finalize the proof of the lemma. We have $\text{charge}'(n+1) = \text{charge}'(n) - (s_1, \bar{r}_{\ell+1}]_k$ and $\text{charge}(n+1) = \text{charge}(n) + [\bar{r}_1, s_1)_k$. Thus $\text{charge}(n+1) - \text{charge}'(n+1) = \text{charge}(n) - \text{charge}'(n) + [\bar{r}_1, s_1)_k + (s_1, \bar{r}_{\ell+1}]_k$. But $[\bar{r}_1, s_1)_k = [\bar{r}_1, \bar{r}_i]_k$ and, as we have seen, $(s_1, \bar{r}_{\ell+1}]_k = (\bar{r}_{i+1}, \bar{r}_{\ell+1}]_k$. This implies from Lemma 5 that $\text{charge}(n+1) - \text{charge}'(n+1) = \text{charge}(n) - \text{charge}'(n) + [\bar{r}_1, \bar{r}_i]_k + (\bar{r}_{i+1}, \bar{r}_{\ell+1}]_k = -\ell + \ell = 0$. □

The charge and cocharge of the letter n are related in the following way:

Lemma 7. *Let c_n be the cover corresponding to letter n in the k-shape tableau T. Then the charge and cocharge of the letter n satisfy the relation*

$$\text{charge}(n) = n - \text{cocharge}(n) - |c_n| \tag{3.9.9}$$

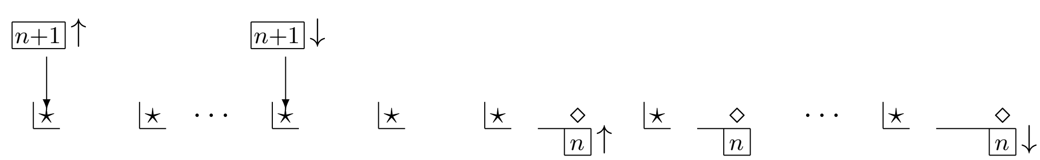
Demostración. We proceed by induction. The case $n = 1$ is easily seen to hold. We need to establish that

$$\text{charge}(n+1) = n + 1 - \text{cocharge}(n+1) - |c_{n+1}| \tag{3.9.10}$$

Using the induction hypothesis, it suffices to prove that

$$\text{charge}(n+1) - \text{charge}(n) + \text{cocharge}(n+1) - \text{cocharge}(n) = |c_n| - |c_{n+1}| + 1 \tag{3.9.11}$$

We first consider the case where $n_+^{\uparrow}$ and $n_+^{\downarrow}$ are above $n^{\uparrow}$ and $n^{\downarrow}$. We can visualize this case with a diagram. The symbol $\star$ denotes the rows that are k-connected with the row of $n_+^{\uparrow}$, while $\diamond$ denote the rows above the letters n.



Let $r_1, \ldots, r_\ell$ be the rows of the $\diamond$'s in the picture (r_1 and r_ℓ are the rows of the $\diamond$ above $n^{\uparrow}$ and $n^{\downarrow}$ respectively). Let also s_1 be the row of the lowest $\star$ such that

$s_1 \geq r_1$. By Lemma 3 the sequence $s_1, \ldots, s_{\ell+1}$ of connected row starting from s_1 is such that

$$s_1 \geq r_1 > s_2 \geq r_2 > \cdots > s_\ell \geq r_\ell > s_{\ell+1}$$

We then get

$$(n_+^{\uparrow}, n^{\uparrow}]_k + (n^{\uparrow}\,,\, n^{\downarrow}]_k \quad = \quad (n_+^{\uparrow}\,,\, n_+^{\downarrow})_k + [n_+^{\downarrow}\,,\, n^{\downarrow}]_k$$

where we identified the cells with their rows. Therefore, the definition of charge and cocharge imply

$$\text{(3.9.12)}\quad \text{charge}(n) - \text{charge}(n+1) + |c_n| - 1 = |c_{n+1}| - 2 + \text{cocharge}(n+1) - \text{cocharge}(n)$$

which is equivalent to (3.9.11).

We have established that the result holds when $n_+^{\uparrow}$ and $n_+^{\downarrow}$ are above $n^{\uparrow}$ and $n^{\downarrow}$. In all the other cases we have either:

I) $n_+^{\uparrow}$ or $n_+^{\downarrow}$ lies between $n^{\uparrow}$ and $n^{\downarrow}$

II) $n^{\downarrow}$ or $n^{\uparrow}$ lies between $n_+^{\uparrow}$ and $n_+^{\downarrow}$

Consider case **I)**. From Lemmas 5 and 6, if we hypothetically shift the position of $n^{\uparrow}$ to that of $n^{\downarrow}$, we obtain new $\text{ch}'(n)$ and $\text{ch}'(n+1)$ such that $\text{charge}'(n) - \text{charge}(n) = |c_n| - 1$ and $\text{ch}'(n+1) = \text{ch}(n+1)$. From

$$\text{(3.9.13)}\quad \text{charge}'(n+1) - \text{charge}'(n) + \text{cocharge}'(n+1) - \text{cocharge}'(n) = |c'_n| - |c'_{n+1}| + 1$$

we get using the extra relations $|c'_n| = 1$, $|c'_{n+1}| = |c_{n+1}|$, $\text{cocharge}'(n+1) = \text{cocharge}(n+1)$ and $\text{cocharge}'(n) = \text{cocharge}(n)$, that (3.9.11) holds if (3.9.13) holds. Note that shifting $n^{\uparrow}$ changes its position relative to either $n_+^{\uparrow}$ or $n_+^{\downarrow}$.

Consider case **II)**. From Lemma 4, if we hypothetically shift the position of $n_+^{\downarrow}$ to that of $n_+^{\uparrow}$, we obtain a new $\text{cocharge}'(n+1)$ such that $\text{cocharge}'(n+1) - \text{cocharge}(n+$

$1) = |c_{n+1}|-1$. Using (3.9.13) with $|c'_n| = |c_n|$, $|c'_{n+1}| = 1$, $\text{cocharge}'(n) = \text{cocharge}(n)$, $\text{charge}'(n) = \text{charge}(n)$, and $\text{charge}'(n+1) = \text{charge}(n+1)$, we obtain again that (3.9.11) holds if (3.9.13) holds. Observe that shifting $n_+^{\downarrow}$ changes its position relative to either $n^{\uparrow}$ or $n^{\downarrow}$.

Applying cases **I)** and **II)** again and again, we get that the general case follows from the previously established case where $n_+^{\uparrow}$ and $n_+^{\downarrow}$ are above $n^{\uparrow}$ and $n^{\downarrow}$. □

An immediate consequence of Lemma 7 is the following relation between the charge and the cocharge of a standard k-shape tableau, which generalizes the usual relation between charge and cocharge of a standard tableau.

Proposition 5. *Let T be a standard k-shape tableau of shape λ. Then*

$$\text{charge}(T) = \frac{n(n-1)}{2} - \text{cocharge}(T) - |\text{Int}^k(\lambda)| \tag{3.9.14}$$

where we recall that $\text{Int}^k(\lambda)$ was defined at the beginning of Subsection 3.2.2.

Demostración. Summing (3.9.9) from 1 to n, we get

$$\text{charge}(T) = \frac{n(n+1)}{2} - \text{cocharge}(T) - (|c_1| + \cdots + |c_n|) \tag{3.9.15}$$

Adding the cover c_i increases by $|c_i| - 1$ the number of hooks larger than k. Hence $|c_1| + \cdots + |c_n| - n = \text{Int}^k(\lambda)$, and the corollary follows. □

Example 19. Using the standard k-shape tableau T in Examples 17 and 18, we get

$$\text{charge}(T) = \frac{9(9-1)}{2} - \text{cocharge}(T) - |\text{Int}^k(\lambda_9)| = 36 - 15 - 5 = 16$$

as wanted.

3.10 Compatibility between (co)charge and the weak bijection

Our goal is to show that the weak bijection (3.8.8) satisfies the conditions

$$\text{cocharge}_k(Q^{(k)}) = \text{cocharge}(\mathbf{p}) + \text{cocharge}_{k-1}(Q^{(k-1)}) \qquad \text{and}$$
$$\text{charge}_k(Q^{(k)}) = \text{charge}(\mathbf{p}) + \text{charge}_{k-1}(Q^{(k-1)}) \tag{3.10.1}$$

when $Q^{(k)}$ (resp. $Q^{(k-1)}$) is standard k-tableau (resp. $k-1$-tableau). We have emphasized that the (co)charge of $Q^{(k)}$ and $Q^{(k-1)}$ are computed considering that they are sequences of k and $k-1$-shapes respectively. The next proposition shows that this distinction is not necessary.

Proposition 6. *The cocharge (resp. charge) of a standard k-tableau T is the same whether it is considered as a sequence of k-shapes or as a sequence of $k+1$-shapes. That is,*

$$\text{cocharge}_k(T) = \text{cocharge}_{k+1}(T) \qquad \text{and} \qquad \text{charge}_k(T) = \text{charge}_{k+1}(T)$$

if T is a standard k-tableau.

Demostración. Since a $k+1$-core does not have hooks of length $k+1$, two rows of a $k+1$-core are $k+1$-connected if and only if they are k-connected. In the case of a k-tableau, the corresponding sequence of k-shapes is a sequence of $k+1$-cores. Hence the result follows immediately from the definition of cocharge (resp. charge). □

It thus suffices to show that the compatibility between the (co)charge and the weak bijection (in the standard case) is true when the $k-1$-tableau is considered as a sequence of k-shapes. The remainder of this section will be devoted to showing the next proposition, which from Proposition 6 implies Theorem 1.

Proposition 7. *Let $Q^{(k)}$ and $Q^{(k-1)}$ be standard k and $k-1$-tableaux respectively such that $Q^{(k)} \longleftrightarrow (Q^{(k-1)}, [\mathbf{p}])$ in the weak bijection* (3.8.8)*. Then*

$$\text{cocharge}(Q^{(k)}) = \text{cocharge}(\mathbf{p}) + \text{cocharge}(Q^{(k-1)}) \qquad \text{and}$$
$$\text{charge}(Q^{(k)}) = \text{charge}(\mathbf{p}) + \text{charge}(Q^{(k-1)}) \tag{3.10.2}$$

where the (co)charge is computed considering that both $Q^{(k)}$ and $Q^{(k-1)}$ are sequences of k-shapes.

In order to show Proposition 7, we will proceed locally using the pushout algorithm. We say that the standard k-shape tableaux T and U differ by moves if there exists a sequence of row moves (resp. column moves) $m_1, \ldots, m_N$ such that the following diagram commutes:

$$\begin{array}{ccc}
\emptyset & \xrightarrow{\emptyset} & \emptyset \\
\downarrow c_1 & & \downarrow \tilde{c}_1 \\
\lambda^{(1)} & \xrightarrow{m_1} & \mu^{(1)} \\
\downarrow c_2 & & \downarrow \tilde{c}_2 \\
\lambda^{(2)} & \xrightarrow{m_2} & \mu^{(2)} \\
\vdots & & \vdots \\
\lambda^{(N-1)} & \xrightarrow{m_{N-1}} & \mu^{(N-1)} \\
\downarrow c_N & & \downarrow \tilde{c}_N \\
\lambda & \xrightarrow{m_N} & \mu
\end{array} \tag{3.10.3}$$

where $c_1, \ldots, c_N$ and $\tilde{c}_1, \ldots, \tilde{c}_N$ correspond respectively to T and U, and where every commutative square in the diagram corresponds to one of the 3 steps described in

Section 3.6. The pushout algorithm ensures that one can obtain a sequence

$$Q^{(k)} = T^{(0)}, T^{(1)}, \ldots, T^{(r-1)}, T^{(r)} = Q^{(k-1)} \tag{3.10.4}$$

where, for all i, $T^{(i)}$ and $T^{(i+1)}$ either differ by row moves or column moves. Therefore, in order to prove Proposition 7, it suffices to show that T and U described in (3.10.3) are such that

(3.10.5)

$$\text{cocharge}(T) = \text{cocharge}(m_N) + \text{cocharge}(U) \qquad \text{and} \qquad \text{charge}(T) = \text{charge}(m_N) + \text{charge}(U)$$

Using the definition of charge and cocharge, is is straightforward to see that this is equivalent to proving that

$$\text{cocharge}(N) - \text{cocharge}(\bar{N}) = \text{cocharge}(m_N) - \text{cocharge}(m_{N-1}) \tag{3.10.6}$$

and

$$\text{charge}(N) - \text{charge}(\bar{N}) = \text{charge}(m_N) - \text{charge}(m_{N-1}) \tag{3.10.7}$$

where for simplicity we denote the charge of the letter N in T by charge(N), and the charge of the letter N in U by charge($\bar{N}$) (and similarly for cocharge).

The next lemma shows that both problems are equivalent.

Lemma 8. *We have*

$$\begin{gathered}\text{cocharge}(N) - \text{cocharge}(\bar{N}) = \text{cocharge}(m_N) - \text{cocharge}(m_{N-1}) \iff \\ \text{charge}(N) - \text{charge}(\bar{N}) = \text{charge}(m_N) - \text{charge}(m_{N-1})\end{gathered} \tag{3.10.8}$$

Demostración. Suppose that T and U differ by a row move. We have to show that

(3.10.9)

$$\text{cocharge}(N) - \text{cocharge}(\bar{N}) = |m_N| - |m_{N-1}| \iff \text{charge}(N) = \text{charge}(\bar{N})$$

From Lemma 7, we get

(3.10.10)

$$\text{cocharge}(N) - \text{cocharge}(\bar{N}) = (N - \text{charge}(N) - |c_N|) - (N - \text{charge}(\bar{N}) - |\tilde{c}_N|) =$$
$$\text{charge}(\bar{N}) + |\tilde{c}_N| - \text{charge}(N) - |c_N|$$

which leads to

$$\begin{aligned} &\text{cocharge}(N) - \text{cocharge}(\bar{N}) = |m_N| - |m_{N-1}| \iff \\ &\text{charge}(\bar{N}) + |\tilde{c}_N| - \text{charge}(N) - |c_N| = |m_N| - |m_{N-1}| \end{aligned} \tag{3.10.11}$$

By inspection of Step 1 and Step 3 (in the row case), it is easy to deduce that in all cases

$$|\tilde{c}_N| - |c_N| = |m_N| - |m_{N-1}| \tag{3.10.12}$$

and thus (3.10.9) follows from (3.10.11). The proof when T and U differ by a column move is identical. □

It thus suffices to prove (3.10.6) when T and U differ by a column move, and (3.10.7) when T and U differ by a row move. Observe that in both cases the right hand side of the equations is then equal to zero. We will proceed by induction. We will suppose that the relations hold for all N up to $N = n$ and then show that the case $N = n + 1$ also holds, that is, that

- $\text{cocharge}(n + 1) = \text{cocharge}(\bar{n} + 1)$ when T and U differ by a column move
- $\text{charge}(n + 1) = \text{charge}(\bar{n} + 1)$ when T and U differ by a row move

The proof will rely on commutative diagrams of the type

$$\begin{array}{ccc} T_n & \xrightarrow{m_n} & U_n \\ \downarrow{\scriptstyle c_{n+1}} & & \downarrow{\scriptstyle \tilde{c}_{n+1}} \\ T_{n+1} & \xrightarrow{m_{n+1}} & U_{n+1} \end{array} \tag{3.10.13}$$

where T_n and U_n denote standard k-shape tableaux of n letters, and where, for simplicity, we use T_{n+1} instead of $\mathrm{sh}(T_{n+1})$. We will also keep denoting by $\bar{n}$ the letter n in U. For instance $\bar{n}^{\uparrow}$ denotes the highest occurrence of the letter n in U, while $n^{\uparrow}$ denotes the highest occurrence of the letter n in T.

We now proceed to analyze all the possible cases.

3.10.1 Row and column maximization

We first consider the situation where c_{n+1} is not maximal. The maximization described in Step 1 and 2 is such that

(3.10.14)
$$\begin{array}{ccc} T_n & \xrightarrow{\emptyset} & T_n \\ \big\downarrow{\scriptstyle c_{n+1}} & & \big\downarrow{\scriptstyle \tilde{c}_{n+1}} \\ T_{n+1} & \xrightarrow{m} & U_{n+1} \end{array}$$

where m is a maximization below or above of the cover c_{n+1}. Suppose that m is a row move (maximization below). We have that $n_+^{\uparrow}$ and $\bar{n}_+^{\uparrow}$ lie in the same position (above the move), as do obviously $n^{\uparrow}$ and $\bar{n}^{\uparrow}$ (given that $T_n = U_n$). Since $\mathrm{charge}(n+1)$ and $\mathrm{charge}(\bar{n}+1)$ are computed using the k-shape corresponding to the shape of T_n, it is immediate that $\mathrm{charge}(n+1) - \mathrm{charge}(n) = \mathrm{charge}(\bar{n}+1) - \mathrm{charge}(\bar{n}) = \mathrm{charge}(\bar{n}+1) - \mathrm{charge}(n)$. Therefore, we get $\mathrm{charge}(n+1) = \mathrm{charge}(\bar{n}+1)$.

If m is a column move, we can use a similar analysis (this time $n_+^{\downarrow}$ and $\bar{n}_+^{\downarrow}$ lie in the same position below the move), to show that $\mathrm{cocharge}(n+1) = \mathrm{cocharge}(\bar{n}+1)$.

3.10.2 Maximal pushout (row case)

We now show that the charge is conserved in the pushout of a maximal cover c_{n+1} and a row move m.

Lemma 9. *Consider the following situation*

$$\begin{array}{ccc} T_{n-1} & \xrightarrow{m'} & U_{n-1} \\ \downarrow{\scriptstyle c_n} & & \downarrow{\scriptstyle \tilde{c}_n} \\ T_n & \xrightarrow{m} & U_n \\ \downarrow{\scriptstyle c_{n+1}} & & \downarrow{\scriptstyle \tilde{c}_{n+1}} \\ T_{n+1} & \xrightarrow{\tilde{m}} & U_{n+1} \end{array} \tag{3.10.15}$$

where c_{n+1} is maximal, and where either c_n is maximal or m is is the maximization below of c_n (in which case m' is empty). Then $\mathrm{charge}(n+1) = \mathrm{charge}(\bar{n}+1)$ *if* $\mathrm{charge}(n) = \mathrm{charge}(\bar{n})$.

Before proceeding to the proof of the lemma, we need to establish a few elementary results. Let m be a row move from λ to μ originating, as in Lemma 9, either from a maximization below of c_n or from the maximal pushout of the pair (c_n, m'). The following observations follow easily from the definition of maximal pushout in the row case.

(i) $n^{\uparrow}$ and $\bar{n}^{\uparrow}$ are in the same position

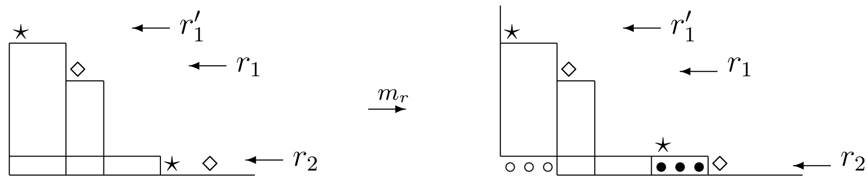
(ii) $n_{+}^{\uparrow}$ and $\bar{n}_{+}^{\uparrow}$ are in the same position

(iii) $n^{\uparrow}$ is never in a row that belongs to the move m.

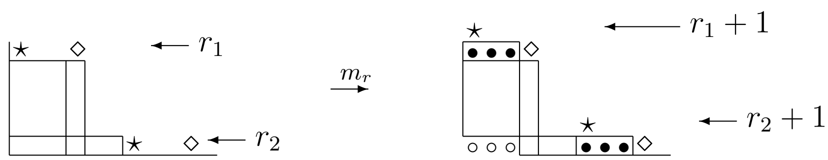
(iv) $n_{+}^{\uparrow}$ is never in a row that belongs to the move m.

We also need to describe how the move m affects k-connectedness between rows. We will always consider that r_1 and r_2 ($r_1 > r_2$) are two k-connected rows in λ. There are essentially 3 cases to consider.

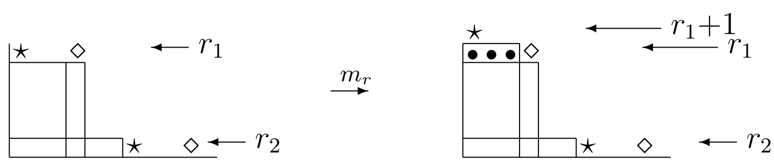
1. If the move m intersects row r_2 but does not continue above, then as seen in the picture, the row that corresponds to the negatively modified column of m is now k-connected to row r_2+1, while the other rows remain connected in the same way:



2. If both rows r_1 and r_2 belong to m, then rows r_1+1 and r_2+1 are k-connected in μ, while row r_1 and r_2 remain connected (if there is still an addable corner in row r_1 of μ)



3. If row r_1 belongs to the move m but r_2 does not then r_1+1 now connects with r_2 in μ, while row r_1 and r_2 remain connected (if there is still an addable corner in row r_1 of μ)



We are now in a position to prove Lemma 9.

Proof of Lemma 9. Suppose that $n^{\uparrow}$ is above $n^{\uparrow}_{+}$. Let r_1 be the row above that of $n^{\uparrow}$ in λ, and let $r_1,\dots,r_{\ell+1}$ be the connected rows below row r_1 such that $r_\ell > s$ and $r_{\ell+1} \leq s$, where s is the row of $n^{\uparrow}_{+}$ in λ. From observations (i) and (ii), in μ row r_1 is still the row above that of $\bar{n}^{\uparrow}$ and s is still the row of $\bar{n}^{\uparrow}_{+}$. From the analysis of the 3 cases considered above, in μ the connected rows below r_1 will be $r_1,\dots,r_{i-1},r_i+1,\dots,r_j+1,r_{j+1},\dots,r_{\ell+1}$ if rows r_i up to r_j belong to the move m. Note that $i>1$ since, as we just mentioned, r_1 did not change. We have to show that $[r_1,s)_k$ is the same in λ and in μ. In λ, we have $[r_1,s)_k=\ell$. In μ we will still have $[r_1, s)_k = \ell$ unless $r_{\ell+1} = s$ and s is a row of the move. But this is impossible from (iv).

Now suppose that $n^\uparrow$ is weakly below $n^\uparrow_+$. Let r_1 be the row of $n^\uparrow_+$ in λ, and let $r_1, \ldots, r_{\ell+1}$ be the connected rows below row r_1 such that $r_\ell \geq s$ and $r_{\ell+1} < s$, where s is the row above that of $n^\uparrow$ in λ. From observation (i) and (ii), in μ row r_1 is still the row of $\bar{n}^\uparrow_+$ and s is still the row above that of $\bar{n}^\uparrow$. In μ, the connected rows below r_1 will again be $r_1, \ldots, r_{i-1}, r_i + 1, \ldots, r_j + 1, r_{j+1}, \ldots, r_{\ell+1}$, with $i > 1$, if rows r_i up to r_j belong to the move m. We have to show that $(r_1, s]_k$ is the same in λ and in μ. In λ, we have $(r_1, s]_k = \ell - 1$. In μ we will still have $(r_1, s]_k = \ell - 1$ unless $r_{\ell+1} = s - 1$ and $s - 1$ is a row of the move. But this is impossible by (iii) since $s - 1$ is the row of $n^\uparrow$. □

3.10.3 Maximal pushout (column case)

We need to show that the cocharge is conserved in the pushout of a maximal cover c_{n+1} and a column move m.

Lemma 10. *Consider the following situation*

(3.10.16)
$$\begin{array}{ccc} T_{n-1} & \xrightarrow{m'} & U_{n-1} \\ \downarrow{\scriptstyle c_n} & & \downarrow{\scriptstyle \tilde{c}_n} \\ T_n & \xrightarrow{m} & U_n \\ \downarrow{\scriptstyle c_{n+1}} & & \downarrow{\scriptstyle \tilde{c}_{n+1}} \\ T_{n+1} & \xrightarrow{\tilde{m}} & U_{n+1} \end{array}$$

where c_{n+1} is maximal, and where either c_n is maximal or m is is the maximization above of c_n (in which case m' is empty). Then $\mathrm{cocharge}(n+1) = \mathrm{cocharge}(\bar{n}+1)$ *if* $\mathrm{cocharge}(n) = \mathrm{cocharge}(\bar{n})$.

Before proceeding to the proof of the lemma, we need as in the previous subsection to establish a few elementary results. Let m be a row move from λ to μ originating, as in Lemma 10, either from a maximization above of c_n or from the maximal pushout of the pair (c_n, m'). The following observations follow easily from the definition of maximal pushout in the column case.

(i) $n^{\downarrow}$ and $\bar{n}^{\downarrow}$ are in the same position

(ii) $n^{\downarrow}_{+}$ and $\bar{n}^{\downarrow}_{+}$ are in the same position

(iii) $n^{\downarrow}$ is never in a row that belongs to the move m.

(iv) $n^{\downarrow}_{+}$ is never in a row that belongs to the move m.

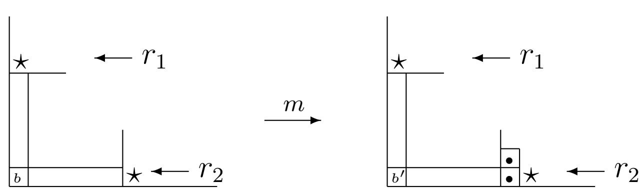
We also need to describe how the move m affects k-connectedness between rows. The main claim is the following.

Lemma 11. *Let m be a column move from λ to μ. Suppose that λ and μ have an addable corner in row r_1. Then the k-connected row r_2 below row r_1 is the same in λ and in μ.*

Demostración. We consider all possible cases and see that the result always hold.

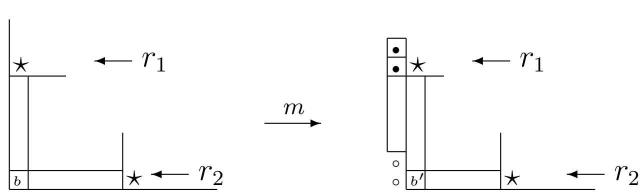
1. If neither r_1 nor r_2 belong to the move m, then the result is immediate.

2. If only one of r_1 and r_2 belong to m, we have either



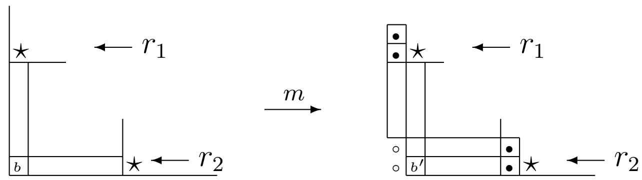
$h_b(\lambda) \leq k-1$ $\qquad$ $h_{b'}(\mu) = h_b(\lambda) + 1 \leq k$

or



$h_b(\lambda) = k$ $\qquad$ $h_{b'}(\mu) = h_b(\lambda) - 1 \leq k-1$

3. If r_1 and r_2 both belong to m



$h_b(\lambda) = k$ or $k-1$ $\qquad$ $h_{b'}(\mu) = h_b(\lambda) = k$ or $k-1$

□

We now proceed to the proof of Lemma 10.

Proof of Lemma 10. Suppose that $n^{\downarrow}$ is above $n_+^{\downarrow}$. By (i) and (ii), we have that $n^{\downarrow}$ coincides with $\bar{n}^{\downarrow}$ and that $n_+^{\downarrow}$ coincides with $\bar{n}_+^{\downarrow}$. The row r above that of $n^{\downarrow}$ still has an addable corner in μ from (i). Using Lemma 11 again and again we get that the string of k-connected rows below r is the same in λ and μ. It is thus immediate that cocharge$(\bar{n}+1) =$ cocharge$(n+1)$ if cocharge$(\bar{n}) =$ cocharge(n).

Suppose that $n^{\downarrow}$ is above $n_+^{\downarrow}$. By (i) and (ii), we still have that $n^{\downarrow}$ coincides with $\bar{n}^{\downarrow}$ and that $n_+^{\downarrow}$ coincides with $\bar{n}_+^{\downarrow}$. The row r of $n_+^{\downarrow}$ thus has an addable corner in λ and μ by definition. Using Lemma 11 again and again we get that the string of k-connected rows below r is the same in λ and μ. It is then again immediate that cocharge$(\bar{n}+1) =$ cocharge$(n+1)$ if cocharge$(\bar{n}) =$ cocharge(n).

□

3.11 Conclusion

The compatibility between charge and the weak bijection was only established in the standard case. We discuss here briefly the obstruction to extending this compatibility to the semi-standard case. In order to extend the charge of a k-tableau of dominant weight given at the end of Section 3.4 to arbitrary k-shape tableaux of dominant weight, we would need a way to order the various occurrences of the letters in the tableau (such as is done when computing the charge). Finding this order is essentially equivalent to defining a Lascoux-Schützenberger-type action of the symmetric group on k-shape tableaux [23, 20] that would extend that on k-tableaux defined at the end of Section 3.4. Unfortunately, we have been able to define such an action only on maximal (and reverse-maximal) tableaux (see IV), which does not appear sufficient to prove the compatiblity with the weak bijection. One of the reasons why the extension to the non-maximal case is non-trivial is that the number of k-shape tableaux of a given shape and weight depends in general on the weight (the number does not depend on the weight only after a notion of equivalence classes on k-shape tableaux of a given weight and shape has been defined).

Our ultimate goal would be to show that our Lascoux-Schützenberger-type action of the symmetric group on k-tableaux commutes with the weak bijection. To be more precise, recall that the Lascoux-Schützenberger action of the symmetric group on words associates to every given permutation an operator σ that permutes the weight of a word (or tableau) according to the permutation. We conjecture that the weak bijection is such that (using the language of (3.8.16))

$$T \longleftrightarrow (T^{(k)}, [\mathbf{p}]) \qquad \Longleftrightarrow \qquad \sigma(T) \longleftrightarrow (\sigma(T^{(k)}), [\mathbf{p}])$$

where we still denote by σ the corresponding operator in the Lascoux-Schützenberger action of the symmetric group on k-tableaux. That is, the pushout algorithm appears to commute with the action of σ (observe that $[\mathbf{p}]$ is left unchanged):

$$\begin{array}{ccc} T & \longrightarrow & \sigma(T) \\ \big\downarrow [\mathbf{p}] & & \big\downarrow [\mathbf{p}] \\ T^{(k)} & \longrightarrow & \sigma(T^{(k)}) \end{array} \tag{3.11.1}$$

Apart from providing us a tool to demonstrate the compatibility of the charge and the weak bijection in the non-standard case, proving the commutativity (3.11.1) would imply that the pushout is only necessary in the standard case (which is technically much simpler than in the non-standard case). In effect, a natural standardization Std follows from the Lascoux-Schützenberger action of the symmetric group on words. The standardization Std would then immediately commute with the pushout:

$$T \longleftrightarrow (T^{(k)}, [\mathbf{p}]) \qquad \Longleftrightarrow \qquad \mathrm{Std}(T) \longleftrightarrow (\mathrm{Std}(T^{(k)}), [\mathbf{p}])$$

Since the standardization has a left inverse Std^{-1}, to obtain the non-standard pushout $T \longleftrightarrow (T^{(k)}, [\mathbf{p}])$, one would simply need to compute the standard pushout $U = \mathrm{Std}(T) \longleftrightarrow (U^{(k)}, [\mathbf{p}])$ and then get $T^{(k)}$ from the relation $T^{(k)} = \mathrm{Std}^{-1} \circ U^{(k)}$.

CHAPTER IV

The Lascoux-Schützenberger action of the symmetric group on maximal k-shape tableaux

We present in this chapter a Lascoux-Schützenberger type action of the symmetric group on maximal k-shape tableaux. We prove that the generators σ_i are well-defined involutions sending maximal k-shape tableaux into maximal k-shape tableaux. We do not prove however that they satisfy the Coxeter relations. These relations would have followed from our initial goal, discussed in the conclusion of Chapter III, of showing the commutativity of this action with the weak bijection. But unfruitful attempts to define a Lascoux-Schützenberger type action of the symmetric group on arbitrary k-shape tableaux prevented us from reaching that goal.

The content of this chapter represents only a small part of all the effort that was put into showing the commutativity of the action with the weak bijection. The main unsuccessful approach was to associate, to every double strip (see Section 4.2), a certain word which we called the invariant of the double strip and which is in correspondence with the maximal double strip associated to the double strip (that is, the invariant gives us the maximization of the double strip without having to do the maximization and pushouts described in [12]). This approach, when completed, would have provided a proof of the commutativity

$$\begin{array}{ccc} T^{(k)} & \longrightarrow & \sigma(T^{(k)}) \\ \big\downarrow [\mathbf{p}] & & \big\downarrow [\mathbf{p}'] \\ T^{(k-1)} & \longrightarrow & \sigma(T^{(k-1)}) \end{array} \tag{4.0.1}$$

but without showing the equality between the class of paths $[\mathbf{p}]$ and $[\mathbf{p}']$. This defect eventually led us to abandon the approach.

The results of this chapter are highly dependent of the concept of k-coloring, which interpolates between k and $k+1$-residues. The charge and cocharge defined in Chapter III were also originally given in terms of k-coloring, but it was later realized that the concept of k-connectedness was more appropriate. We believe that using k-connectedness would probably also greatly simplify the proofs contained in this chapter. As such, no great effort was put into polishing them. We felt nevertheless that they had to be included in this thesis to give a flavor of the work that was done over the last few years.

4.1 k-coloring

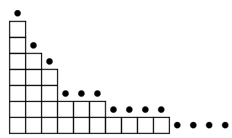
We will need the following elementary property of k-shapes:

Lemma 12 ([12])**.** *Let λ be a k-shape. Suppose that r and r' are k-connected rows of λ with $r > r'$. Then*

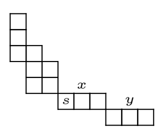
$$\lambda_{r-1} - \lambda_r \leq \lambda_{r'-1} - \lambda_{r'} \tag{4.1.1}$$

Note that if $r' = 1$, the inequality is still valid if we consider that λ_0 is infinite. In words, the inequality says that the portion of row $r-1$ that sticks out is not longer than the portion of $r'-1$ that sticks out.

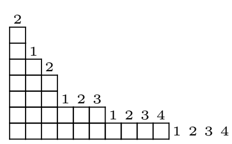
Let λ be a k-shape. We say that μ/λ is the *complete strip of* λ if μ/λ is the horizontal strip with a cell in each column of λ and k additional cells in the first column of λ. In other words, μ/λ is the unique horizontal strip with cells in columns 1 to $\lambda_1 + k$. In the example that follows, with $k = 4$ and $\lambda = (10, 6, 3, 3, 2, 1, 1)$, the complete strip is indicated with bullets:



We now describe how to associate a color (an element of the set $\{1, 2, \ldots, k\}$) to every cell in the complete strip μ/λ. The cells in the first row of the complete strip are colored from left to right as $1, 2, \ldots, k$. The rest of the complete strip is colored recursively in the following way. Let x be a cell of μ/λ in row $r > 1$, and let r' be the k-connected row below row r. We define the *color-hook* of x to be size of the hook between the addable corners in row r and r'. To be more precise, let s be the lowest cell of $\partial\lambda$ in the column of the addable corner in the row of x. If $h_s = k$ (resp. $h_s < k$), then the color associated to x is the color of the cell y in μ/λ that lies in the row of s and whose distance from x is $k+1$ (resp. k). We say in this case that x inherits its color from y. For instance, if $k = 4$, the cell x in the following example inherits its color from the cell y:

$\partial\lambda =$ 

In the following example, where again $k = 4$, we give the colors associated to the cells of the complete strip of $\lambda = (10, 6, 3, 3, 2, 1, 1)$.



For simplicity, the cells of the complete strip of λ will be called the *colorable cells of* λ.

Lemma 13. *The coloring of the complete strip of a k-shape λ is well-defined.*

Demostración. Suppose that x and y are such as in the definition of the coloring. We need to prove that y belongs to the complete strip μ/λ. If $h_s = k$, this immediate from Lemma 12. Suppose otherwise that $h_s < k$, and let r and c be respectively the row of y and the column of x. Observe that by definition of s, the cell $(r-1, c)$ cannot belong to $\partial\lambda$ (otherwise the column of s would be smaller than that of x in $\partial\lambda$). Since $h_s < k$, it is easy to see that if cell y does not belong to to μ/λ then we have the contradiction that the hook-length of the cell $(r-1, c)$ is not larger than k in λ. □

Remark 4. Two distinct cells x and x' cannot inherit their colors from the same cell y. If this were to occur, x and x' would need to belong to different rows by definition of the coloring. But then the distance between x and x' would be at least 2, and thus their distances from y could not both belong to $\{k, k+1\}$. *This implies that the cells of any given color form a string starting from the first row.*

Remark 5. Suppose the addable corner in a given row r has color i. Then the colorable cells in row r have, from left to right, colors $i, i+1, ..., j$, for some $j \leq k$.

Remark 6. Suppose row r of the complete strip μ/λ contains colors i and j. Then every row of μ/λ below row r that contains a color in $\{i, j\}$ contains both colors i and j.

Remark 7. Suppose row r of the complete strip μ/λ contains color i but not color $i+1$. Suppose also that the k-connected row r' below row r contains colors i and $i+1$. Then the the string $i+1$ continues above row r' only if rows r and r' are strongly connected and λ has an addable corner in row $r-1$.

4.2 Consistency in the colorings

We will call a *double strip* a sequence of two consecutive weak strips $S_A = \mu/\lambda$ and $S_B = \nu/\mu$. We will say that the double strip is *maximal* (resp. *reverse-maximal*) if both strips S_A and S_B are maximal (resp. reverse-maximal).

Let $S_A = \mu/\lambda$ and $S_B = \nu/\mu$ be a double strip, and let $C_1 = \rho/\lambda$ and $C_2 = \delta/\mu$ be the complete strips of λ and μ respectively. It is easy to show that $C_1 \cap C_2$ intersect in the first row (recall that strips of rank k are not allowed). We say that the colorings of λ and μ have been made *consistent* if the colors of the cells of C_2 have been translated by a fixed integer (modulo k) in such a way that the cells of $C_1 \cap C_2$ in the first row have matching colors. Note that this does not imply that the colorings of λ and μ have matching colors in every cells of $C_1 \cap C_2$.

Suppose that row r' is the k-connected row below row r. In the case where the distance between the colorable cells of r and r' is $k+1$, we say that r and r' are *strongly connected*. Otherwise, we say that they are *weakly connected*. Note that it is easy to see that r and r' are strongly connected if and only if the hook-length of the cell (r', c), where c is the column of the addable corner in row r of λ, is equal to k.

Lemma 14. *Suppose that $S = \mu/\lambda$ is a maximal strip with an addable corner in row r of λ. Let r' be the k-connected row below row r in λ. Suppose μ still has an addable corner on the ground (that is, such that the cell immediately below belongs to λ) in row r. Then r and r' are strongly (resp. weakly) connected in μ if they are strongly (resp. weakly) connected in λ.*

Demostración. Suppose r and r' are strongly connected in λ. In this case, as we observed earlier, the hook-length of the cell (r', c), where c is the column of the addable corner in row r of λ is equal to k. We first show that S contains the same number of cells in row r and r'. Suppose that S has more cells in row r', and let c' be the column of the addable corner in row r of μ. Then it is easy to see that $\mathrm{cs}(\mu)_{c'}$ is shorter than $\mathrm{cs}(\lambda)_{c'}$, which is a contradiction. It was shown in [**?**] that S cannot have more cell in r than in r' and we thus conclude that S has the same number of cells in r and in r'. In this case, the hook-length of the cell (r', c') is equal to k in μ. Therefore row r and r' are also strongly connected in λ.

Now suppose that r and r' are weakly connected in λ, and let again c' be the column of the addable corner in row r of μ. Suppose by contradiction that the hook-length of the cell (r', c') in μ is equal to k. We will see that we get that the addable corner a in row r of μ is an upper addable corner of the strip S, in contradiction to the maximality of S. To see that a is an upper addable corner, we simply need to show that row r' is a modified row of S. Suppose that row r' is not a modified row of S. Then there are as many cells in row r' of $\partial\lambda \setminus \partial\mu$ as there are cells in row r' of S. Let c'' be the column of the leftmost cell in row r' of $\partial\lambda$. Then the topmost cell in column c'' of λ lies either $i)$ weakly above row r or $ii)$ just below row r. In case $i)$, the hook-length of (r', c') in μ is strictly smaller than the hook-length of the cell (r', c'') in λ. Since the latter hook-length is not larger than k, we get the contradiction that the hook-length of (r', s') in μ is smaller than k. In case $ii)$, the hook-length of (r', c') in μ is equal to the hook-length of the cell (r', c'') in λ. But since r and r' are weakly connected in λ, we again have the contradiction that the hook-length of (r', s') in μ is smaller than k. □

Lemma 15. *Let $S = \mu/\lambda$ be a maximal strip, and suppose that the colorings of λ and μ have been made consistent. If a column c of λ does not intersect S then the k-colors in column c in λ and μ are the same.*

Demostración. Let c belong to row r of μ and λ. Suppose by induction that the

k-colors coincide below row r in every column of μ that does not intersect S. The result is then immediate since by Lemma 14 the k-color of the cell (r, c) is inherited from the same cell in a certain row r' of λ and μ. $\square$

4.3 Maximal strips and sets of k-colors

When a strip S of rank r is maximal (resp. reverse-maximal), it can be characterized by a set of k-colors (resp. reverse k-colors) of cardinality r.

Proposition 8. *The cells of a maximal strip S of rank r occupy exactly r colors. Furthermore, the map $S \mapsto A = \{i_1, \ldots, i_r\}$, where $i_1 < \cdots < i_r$ are the k-colors occupied by the cells of S, is invertible.*

Demostración. From Lemmata 90 and 91 of [12], we can decompose S as a sequence of r cover-type strings $s_1, \ldots, s_r$ such that the modified columns of $s_1, \ldots, s_r$ go from left to right, $\lambda^{(\ell)} = \lambda \cup s_1 \cdots \cup s_\ell$ is a k-shape and $\mu/\lambda^{(\ell)}$ is a strip for $\ell = 1, \ldots, r$. Note that in this process, the strings s_ℓ are added maximally such that $s_\ell \subseteq S$. We first show that $\lambda^{(\ell)}/\lambda$ is a maximal strip. It is immediate that $\lambda^{(\ell)}/\lambda$ is a horizontal strip since μ/λ is a strip. Since the s_i's are cover-type strings and there are exactly r of them, we must have $\mathrm{cs}(\lambda) \subseteq \mathrm{cs}(\lambda^{(\ell)}) \subseteq \mathrm{cs}(\mu)$ given that $|\mathrm{cs}(\mu)| - |\mathrm{cs}(\lambda)| = r$. Therefore, the verticality of $\mathrm{cs}(\mu)/\mathrm{cs}(\lambda)$ implies that $\mathrm{cs}(\lambda^{(\ell)})/\mathrm{cs}(\lambda)$ is a vertical strip. We can conclude similarly that $\mathrm{rs}(\lambda^{(\ell)})/\mathrm{rs}(\lambda)$ is a horizontal strip. Hence, we have proved that $\lambda^{(\ell)}/\lambda$ is a strip. Since the modified columns of $\lambda^{(\ell)}/\lambda$ are modified columns of S, the lower augmentable corners of the strip $\lambda^{(\ell)}/\lambda$ are also lower augmentable corners of S. By the maximality of S, $\lambda^{(\ell)}/\lambda$ does not have lower augmentable corners. Suppose that $\lambda^{(\ell)}/\lambda$ has an upper augmentable corner u and suppose by induction that $\lambda^{(\ell-1)}/\lambda$ does not have upper augmentable corners. Since the modified column of the string s_ℓ is to the right of $\lambda^{(\ell-1)}/\lambda$, the upper augmentable corner u needs to be associated to the modified row of s_ℓ. By the maximality described above in the construction of the string s_ℓ, the upper augmentable corner u does not belong to S. Therefore, we have the contradiction that u is also an upper augmentable corner of S. Hence, $\lambda^{(\ell)}/\lambda$ is a maximal strip.

By induction we can suppose that the strings $s_1, \dots, s_{\ell-1}$ occupy exactly $\ell - 1$ colors of λ. It is obvious that the string s_ℓ occupies a single k-color of $\lambda^{(\ell-1)}$. From Lemma 15, the the string s_ℓ also occupies a single k-color of λ. By the maximality of $\lambda^{(\ell-1)}/\lambda$ this k-color cannot coincide with any of the $\ell - 1$ colors of $\lambda^{(\ell-1)}/\lambda$. It thus follows by induction that the maximal strip S occupies exactly r colors.

Now suppose that the k-colors occupied by S are $A = \{i_1, \dots, i_r\}$ with $i_1 < i_2 < \cdots < i_r$. Let $\lambda^{(0)} = \lambda$ and construct $\lambda^{(\ell)}$ recursively by adding to $\lambda^{(\ell-1)}$ all its addable corners of k-color i_ℓ that do not sit above cells of $\lambda^{(\ell-1)}/\lambda$. By induction we suppose that $\lambda^{(\ell-1)}/\lambda$ coincides with the cells of S whose color belongs to $\{i_1, \dots, i_{\ell-1}\}$. Since i_ℓ is a k-color of S and all k-colors of S smaller than i_ℓ belong to $\lambda^{(\ell-1)}/\lambda$, we have by Remark 5 that all the cells of S of color i_ℓ are addable corners. Therefore $\lambda^{(\ell)}/\lambda$ contains the cells of S whose color belongs to $\{i_1, \dots, i_\ell\}$. Suppose there is a cell of $\lambda^{(\ell)}/\lambda$ that does not belong to S. Then the cover-type string of S of color i_ℓ, as constructed above, could be extended above or below. In this case, S would then have either a lower augmentable corner or an upper augmentable corner, in contradiction with the maximality of S. Hence, $\lambda^{(\ell)}/\lambda$ coincides with cells of S whose color belongs to $\{i_1, \dots, i_\ell\}$. It thus follows by induction that S can be recovered from A. □

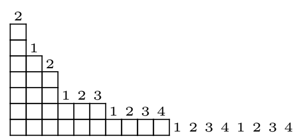
Here are some important observations about the colors belonging to a maximal strip $S = \mu/\lambda$ occupying colors $A = \{i_1, \dots, i_r\}$.

Remark 8. Let a be a colorable cell of λ of color i, where $i \in A$. If a does not belong to S, then every cell of S of color i lies above cell a.

Remark 9. Let a be a colorable cell of λ of color i, where $i \in A$. If a belongs to S, then every colorable cell of λ above cell a belongs to S.

4.4 Maximal double strips

From Proposition 8, it is obvious that a maximal double strip $S_A = \mu/\lambda$ and $S_B = \nu/\mu$ can be characterized by two sets of colors A and B, where the colors of A and B belong to the colorings of λ and μ respectively. We will show in this section that, remarkably, the coloring of λ suffices to characterize the maximal double strip S_A and S_B. Let the extending coloring of λ be the usual coloring of λ with extra colors 1 to k added in the first row. For instance, here follows the extended coloring of the 4-shape $\lambda = (10, 6, 3, 3, 2, 1, 1)$:



We say that a cell of S_B *lies on the ground* if it does not lie above a cell of S_A. Otherwise, we say that the cell *lies above the ground*.

Lemma 16. *Suppose that a cell of S_A of color i lies beneath a cell of S_B. Then every cell of color i in S_A above it also lies beneath a cell of S_B.*

Lemma 17. *Suppose that a modified column c of S_B lies above the ground, and let i be the color in A corresponding to the cell of S_A in column c. Then no cell of S_B on the ground has color i in the extended coloring of λ.*

Demostración. Let b be the cell of S_B in column c. Then no cell of S_B on the ground above b has color i in the extended coloring of λ since by Remark 9 all those cells are occupied by S_A.

Suppose there is a cell of S_B on the ground below cell b that has color i in the coloring of λ (not the extended coloring). By Remark 9, this implies that there is a

b_i on the ground below the lowest a_i and k-connected to it. The cell above the lowest a_i needs to contain a cell of S_B since the column of a_i lost a cell when S_B was added (the cell of λ in the column of the lowest a_i and the row of the b_i below it belongs to $\partial\lambda \setminus \partial\mu$). Therefore, by Lemma 16, all the a_i's have a cell of S_B above them and we have the contradiction that c is not a modified column of S_B.

Finally consider the case where there is a cell of S_B on the ground below cell b that has color i in the extended coloring of λ but that does not belong to the coloring of λ. By definition, this cell b_i lies in the first row. Since the first row cannot contain two b_i's this implies that there is also an a_i in the first row. But this implies that the cell a_i in the first row needs to have a cell of S_B above it since the cell corresponding to a_i does not belong to $\partial\nu$. Again by Lemma 16 we have the contradiction that c is not a modified column of S_B. □

Lemma 18. *Let $S_A = \mu/\lambda$ be a maximal strip. Let x be a cell that lies above a cell y of S_A of color i.*

1. *If y is the lowest a_i then x has color $i-1$ or i in the coloring of μ.*
2. *If y is not the lowest a_i, then x has color $j-1$ or j in the coloring of μ, where j is the color of the cell above the a_i immediately below y in the string of color i.*

If the color of x is $i-1$ (resp. $j-1$) then the string i (resp. j) in the coloring of μ stops before the row of x unless there is a cell of S_A immediately to the right of x. In the latter case, we say that i (resp. j) jumped from a_i to a_{i+1}.

An immediate consequence of the previous lemma:

Lemma 19. *Let $S_A = \mu/\lambda$ be a maximal strip. Let x be a cell that lies above a cell y of S_A of color i. If $j < i$ is the color of the cell x in the coloring of μ then, every* *row weakly below the row of x that contains a cell of color $j+1$ also contains a cell of color j (in the coloring of μ).*

Lemma 20. *Suppose that a modified column c of S_B lies above the ground, and let i be the color in A corresponding to the cell of S_A in column c. If the b in column c has color $i-\ell$ in the coloring of μ, then no cell of S_B on the ground has color $i-\ell, i-\ell+1, \ldots, i$ in the coloring of μ or in the extended coloring of λ.*

Demostración. We show the result for the coloring of μ, the result will then also apply to the extended coloring of λ by Lemma 15. In what follows the colors always refer to those in the coloring of μ.

First, there can be no $b_{i-\ell}$ on the ground since they would have to appear below the b in column c and by Remark 9 column c would not be a modified column of S_B. By Lemma 19 applied again and again, we have that every row below the row of b that contains a cell of a color that belongs to $\{i-\ell+1, \ldots, i\}$ also contains a cell of color $i-\ell$. Hence no $b_{i-\ell+1}, \ldots, b_i$ can occur below the row of b. If any $b_{i-\ell+1}, \ldots, b_i$ occur weakly above the row of b they have to be above the ground by Lemma 18. □

Lemma 21. *Suppose that both colors i and $i+1$ belong to A. If there is a free a_{i+1} (that is, not lying next to an a_i) above the lowest a_i, then the cell above it is an addable corner of μ. Furthermore, if there is no row with an a_i and an a_{i+1}, then the lowest a_i lies above the lowest a_{i+1}.*

Lemma 22. *Suppose that color i belongs to A. Then color i in the coloring of μ can only lie above a cell of S_A of another color if i jumps from a_i to a_{i+1} at some point.*

Proposition 9. *Let $S_A = \mu/\lambda$, $S_B = \nu/\mu$ be a maximal double strip. Let $\{i_1, \ldots, i_s\}$ be the colors in the extended coloring of λ occupied by S_B, and let $\{j_1, \ldots, j_t\}$ be the colors of A belonging to modified columns of S_B above the ground. Then $s+t$ is equal* *to the rank of S_B, and the map $S_B \mapsto B = \{i_1, \ldots, i_s\} \cup \{j_1, \ldots, j_t\}$ is invertible. Furthermore, if n_A is the number of cells of S_A in the first row and $B = \{\ell_1, \ldots, \ell_r\}$ is such that $\ell_1, \ldots, \ell_r$ is increasing in the cyclical order $n_A + 1 < n_A + 2 < \cdots < k < 1 < \cdots < n_A$, then S_B can be reconstructed from B by adding the maximal string whose modified column is of color ℓ_1, then the maximal string whose modified column is of color ℓ_2, and so on.*

Demostración. By the proof of Proposition 8, if the rank of S_B is r then S_B can be decomposed as r strings each occupying a distinct color of the coloring of μ. In particular, the modified columns of S_B must occupy exactly r colors.

We first show that $s + t$ is the rank of S_B. By Lemma 16 we easily get that no two modified columns above the ground can sit above cells in S_A of the same color. Therefore t corresponds to the number of modified columns of S_B above the ground. By Lemma 17, we have immediately that $\{i_1, \ldots, i_s\}$ and $\{j_1, \ldots, j_t\}$ do not intersect. Hence, $s+t \geq r$ since the modified columns of S_B on the ground correspond to *distinct* colors in $\{i_1, \ldots, i_s\}$ (otherwise Remark 9 would have to be violated). We have left to show that every color $i \in \{i_1, \ldots, i_s\}$ corresponds to a modified column of S_B. Since the lowest b_i sits on the ground, it either lies in the first row or inherited its color from a cell y below it. In the first case, it is obvious that the column of the lowest b_i is a modified column of S_B. In the other case, the lowest b_i inherited its color from a cell y below it which lies on the ground by definition. The cell y cannot belong to S_A by Remark 9 or to S_B since there would otherwise be a lower b_i. If the distance between the lowest b_i and y is k, the column of the lowest b_i is a modified column of S_B. If the distance between the lowest b_i and y is $k + 1$, the cell immediately to the left of y cannot belong to μ by maximality of S_B, and we have again that the column of the lowest b_i is a modified column of S_B. By the proof of Proposition 8, S_B can be recovered from the colors in B since they identify the modified columns of S_B.

Suppose that the colorings of λ and μ have been made consistent. The first color that appears in the first row in the coloring of μ is thus $n_A + 1$. Suppose that o_i is the color in the μ-coloring that corresponds to ℓ_i. By the proof of Proposition 8, the last statement will follow if we show that $o_i < o_{i+1}$ for $i = 1, \ldots, r-1$ in the cyclical order $n_A + 1 < n_A + 2 < \cdots < k < 1 < \cdots < n_A$.

Suppose that o_i and o_{i+1} lie on the ground. Then $o_i = \ell_i$ and $o_{i+1} = \ell_{i+1}$ and we have immediately $o_i < o_{i+1}$.

Suppose that o_i lies on the ground and o_{i+1} does not. By Lemma 18, $o_{i+1} = \ell_{i+1} - \ell$ for some $\ell \geq 0$. By Lemma 20, we have that $o_i \notin \{\ell_{i+1} - \ell, \ldots, \ell_{i+1}\}$, and thus $o_i = \ell_i < \ell_{i+1}$ implies $o_i < o_{i+1}$.

Suppose that o_i lies above the ground and o_{i+1} does not. Again by Lemma 18, $o_i = \ell_i - \ell$ for some $\ell \geq 0$. By Lemma 19, the colors o_i and ℓ_i belong to the same row at some point and thus by Remark 5 we have that $o_i \leq \ell_i$ in the cyclical order. Since $\ell_i < \ell_{i+1} = o_{i+1}$ we have that $o_i < o_{i+1}$.

Finally suppose that o_i and o_{i+1} lie above the ground. As in the previous paragraph, we have that $o_i \leq \ell_i$ and $o_{i+1} \leq \ell_{i+1}$ in the cyclical order. First consider the case $\ell_{i+1} = \ell_i + 1$. If there is a free a_{ℓ_i+1} then by Lemma 21 the lowest a_{ℓ_i} lies above the lowest a_{ℓ_i+1}. Lemma 21 also implies that every a_{ℓ_i+1} above the lowest a_{ℓ_i} has a b above it. Let j be the color of the cell (in the coloring of μ) above every such a_{ℓ_i+1}. It is easy to see that the color of the cell above the lowest a_{ℓ_i} is not equal to j unless the lowest a_{ℓ_i} lies next ot an $a_{\ell_{i+1}}$. In the latter case it is easy to see that we have immediately $o_i < o_{i+1} = o_i + 1$. In the former case, by Lemma 18, $j \leq o_{i+1} = \ell_i + 1 - \ell$ for some $\ell \geq 0$. By Lemma 20, we have that $o_i \notin \{\ell_i - \ell + 2, \ldots, \ell_i + 1\}$, and thus $o_i \neq j$

and $o_i \le \ell_i$ imply $o_i < o_{i+1}$. We can thus suppose that there is a no free a_{ℓ_i+1}. But then the result is immediate since in this case we necessarily have $o_i < o_{i+1} = o_i + 1$. Consider finally the case where $\ell_i + 1 \ne \ell_{i+1}$. By Lemma 18, color ℓ_i in μ can only lie above an a_{ℓ_i} in S_A since it cannot jump from a_{ℓ_i} to a_{ℓ_i+1} (there is no such cell). Lemma 18 then implies that $\ell_i < o_{i+1} \le \ell_{i+1}$, which gives $o_i < o_{i+1}$. □

4.5 Bijection on maximal double strips

Let $S_A = \mu/\lambda$ and $S_B = \nu/\mu$ be a maximal double strip, where the k-colors of $B = \{\ell_1, \dots, \ell_r\}$ are such as in Proposition 9, and those of $A = \{i_1, \dots, i_s\}$ are the usual ones. We associate a word to the maximal double strip by taking the sequence $a_{i_1}, \dots a_{i_s}, b_{\ell_1}, \dots b_{\ell_r}$ and reordering according to the total order $b_1 < a_1 < b_2 < a_2 < \cdots < b_k < a_k$. This word will be denoted $w_{A,B}$. Note that from Proposition 9, $w_{A,B}$ uniquely determines the double strip S_A and S_B.

Our goal is to give a bijection that sends the maximal double strip S_A and S_B to a maximal double strip $S_{A'} = \gamma/\lambda$ and $S_{B'} = \nu/\gamma$, where the cardinality of A' and B' is equal to r and s respectively. This bijection will correspond when k is sufficiently large to the Lascoux-Schützenberger action of the symmetric group on words.

Lemma 23. *Suppose that $|A| < k-1$, and that there is an a_i but no b_i in $w_{A,B}$. Suppose that a b is above an a_ℓ with $\ell > i$ and in the row of an a_i. Then the string b_j containing the b is such that $i < j \le \ell$.*

We first need to establish certain properties.

Lemma 24. *Let $j < i$. Suppose that the rightmost (resp. leftmost) color in the coloring of row r (resp. $r-1$) of λ is a j (resp. an i). If there is a b on the ground in row r of the maximal double strip $S_A = \mu/\lambda$ and $S_B = \nu/\mu$ then A does not contain $j+1, \dots, i-1$.*

Demostración. There cannot be any $a_{j+1}, ..., a_{i-1}$ above row r since the strings of the corresponding k-colors stop below row r. Suppose there is a b on the ground in row r of color ℓ. Below row r, any row that contains the k-color $j+1$ must also contain the k-color ℓ. Since there is a b_ℓ on the ground in row r, there cannot be any a_ℓ below row r by Remark 9. Hence neither can $a_{j+1}, ..., a_{i-1}$ appear below row r. □

Lemma 25. *Suppose there is an a_i with a b above it in the maximal double strip S_A, S_B. Then a_i is married in the word $w_{A,B}$.*

Demostración. If a given column of an a_i is a modified column of S_B then there is a b_i in $w_{A,B}$ and then a_i is automatically married in $w_{A,B}$. We can thus suppose that no column containing an a_i is a modified column of S_B.

Suppose first that the lowest a_i, in a certain row r, has a b above it. Let $a_{i-\ell}$ be the leftmost a in row r. The b's above $a_{i-\ell}, \ldots, a_i$ that continue below in row r' are either $b_{i-\ell}, \ldots, b_i$ or $b_{i-\ell-1}, \ldots, b_{i-1}$. In the former case, $a_{i-\ell}, \ldots, a_i$ are married in $w_{A,B}$ and we are done. In the latter case, $a_{i-\ell}, \ldots, a_i$ are also married in $w_{A,B}$ if there is no $a_{i-\ell-1}$. Hence we suppose that there is an $a_{i-\ell-1}$ and that the b's above $a_{i-\ell}, \ldots, a_i$ that continue below are $b_{i-\ell-1}, \ldots, b_{i-1}$. Since there is an $a_{i-\ell-1}$, the k-color of the rightmost color in row $r+1$ of λ is $i-\ell-1$. Furthermore, it is easy to see that rows $r+1$ and r' are strongly connected. Since $a_{i-\ell}, \ldots, a_i$ do not continue in row r', this implies that there is no $a_{i-\ell-1}$ in row $r+1$, and we thus have a $b_{i-\ell-1}$ in row $r+1$ by construction. But this is a contradiction since it implies that the b's above $a_{i-\ell}, \ldots, a_i$ continue as $b_{i-\ell}, \ldots, b_i$.

Now suppose that rows r and r' are k-connected rows containing an a_i such that the a_i in row r has a b above it while that in row r' does not. Let $a_{i-\ell}$ be the leftmost a in row r, so that $a_{i-\ell}, \ldots, a_i$ also appear in row r'. Note the if $\ell \geq 1$ then the a_{i-1} in row r' has a b above it. Let j be the rightmost color in row $r'+1$ of λ. Note that there is a b_j in row $r'+1$ since the a_i in row r' does not have a b above it and by hypothesis the number of b's in row $r'+1$ is at least $i-\ell+1$. Also observe that $j<i$. Since there is a b_j in row r', there is a $s \leq j$ such that s does not belong to A, and let s be the largest among those. By construction we have that $b_s, \ldots, b_j$ belong to row $r'+1$. Hence we have $b_s, \ldots, b_j$ and ℓ extra b's (those above $a_{i-\ell}, \ldots, a_{i-1}$) whose index is by Lemma 24 between $j+1$ and $i-1$. By Lemma 24 again, there is no $a_{j+1}, \ldots, a_{i-1}$ and thus we have at most $a_{s+1}, \ldots, a_j$ and $a_{i-\ell}, \ldots, a_i$ in that range. Therefore all those a's, and in particular a_i, are married in $w_{A,B}$. □

Let w be a word in a and b such that after all the pairings have been done, the remaining word is $a^r b^s$. If $r \geq 1$, we let f be the operator that sends w to w', where w' is obtained from w by sending the remaining word $a^r b^s$ to $a^{r-1} b^{s+1}$ [20]. Similarly, if $s \geq 1$, we let e be the operator that sends w to w', where w' is obtained from w by sending the remaining word $a^r b^s$ to $a^{r+1} b^{s-1}$.

Lemma 26. *Let $S_A = \mu/\lambda$ and $S_B = \nu/\mu$ be a maximal double strip, and let $w_{A,B}$ be its corresponding word. Suppose that a_i goes to b_i when "f" is applied to $w_{A,B}$. If the string i in S_A can be continued above a certain a_j such that $i<j$ then there is a b above every a in the row of that a_j.*

Demostración. Let r be the row of the a_j in question and let r' be the row of the uppermost a_i in S_A. Observe that by supposition the distance between the a_j in row r and the a_i in row r' is $k-1$. First consider the case where there is an a_j in row r'. By supposition we have $a_j = a_{i+1}$, and by hypothesis there needs to be a b_{i+1} in $w_{A,B}$. That b_{i+1} cannot be on the floor since there is no b_i and every row below r' that contains a color $i+1$ also contains a color i (observe that if there is an a_i in a certain row below r' then there is also an a_{i+1} by maximality). So there is a modified column of S_B that lies in the column of an a_{i+1}. By assumption that the string can be continued above), the $a_j = a_{i+1}$ in row r has an addable corner above it in μ. Hence by maximality there is a b above the a_j in row r. If the a_{i+1} is followed by $a_{i+2}, \ldots, a_{i+s}$ in row r, the same argument implies that they also have a b above them.

Now suppose that there is no a_j in row r'. By maximality this implies that there is an a_j in row $r'-1$ in the column next to that of the a_i in row r'. If $j = i+1$ then the analysis is as in the previous paragraph so we can suppose that $j > i+1$. We now show that there is no a_{i+1}. If there is an a_{i+1} it needs to appear below row r' since the string of k-color $i+1$ does not reach row r'. By hypothesis ($a_i \mapsto b_i$) there also needs to be a b_{i+1} if there is an a_{i+1}. As before, b_{i+1} cannot be on the ground and there has to be a modified column of S_B that lies in the column of an a_{i+1}. But this is impossible since no a_i has a b above it (Lemma 25) and every a_{i+1} lies to the right of an a_i by our assumptions. Therefore there is no a_{i+1}. Since a_j is married, there needs to be a b_ℓ where $i < \ell \leq j$. If this b_ℓ touches the ground, there needs to be a b_{i+1} since every row on the ground that has a color j also has the colors $i, i+1, \ldots, j-1$ (recall that there is no a_{i+1}). In this case the cell to the right of the a_i in row r' is occupied by a b by maximality. We have then immediately that the cell above the a_j in row r is also occupied by a b. If a_j is followed by $a_{j+1}, \ldots, a_{j+s}$ (married ones) in row r, it is not hard to deduce that in this case there needs to be $b_{i+2}, \ldots, b_{i+s+1}$, which implies that all the a's in row r have b's above them. Finally suppose that the b_ℓ never touches the ground. In this case, the lowest occurrence of b_ℓ is above an a_ℓ and both letters are immediately married. Hence the only way for a_j to be married is if there is a modified column of S_B that lies in a column of an a_j. But then by maximality the a_j in row r has a b above it and so does all the a's in row r.

□

Theorem 2. *Let $w_{A,B}$ be the word associated to the maximal double strip $S_A = \mu/\lambda$ and $S_B = \nu/\mu$, and let $w_{A',B'} = \sigma(w_{A,B})$ be the result of the Lascoux-Schützenberger action of the symmetric group on $w_{A,B}$. Then $S_{A'} = \gamma/\lambda$, $S_{B'} = \nu/\gamma$ is a maximal double strip where, by Lascoux-Schützenberger action of the symmetric group on words, we have $|A| = |B'|$ and $|B| = |A'|$.*

Demostración. We only consider the case where $|A| > |B|$ since the other case is similar. If $|A| = k-1$, there is a unique ρ such that ν/ρ is a maximal strip of rank $k-1$. We thus let A' and B' be such that $S_{A'} = \rho/\lambda$ and $S_{B'} = \nu/\rho$. Suppose otherwise that $|A| < k-1$. We have that $\sigma(w_{A,B}) = f^\ell(w_{A,B})$ for some ℓ. It thus suffices to show that if $f(w_{A,B}) = w_{A',B'}$ then $S_{A'} = \gamma/\lambda$, $S_{B'} = \nu/\gamma$ is a maximal double strip such that $|A'| = |A| - 1$ and $|B'| = |B| + 1$. We will see that if f is such that $a_i \mapsto b_i$ then the maximal double strip $S_{A'}, S_{B'}$ is obtained from S_A, S_B by sending every a_i to a b_i and every a_j weakly to the right of an a_i (in the same row) to a b_j.

Let $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ be the strips obtained from S_A, S_B by sending every a_i to a b_i and every a_j weakly to the right of an a_i (in the same row) to a b_j. First, by Lemma 25, there is never a b above an a_i and so $S_{\mathcal{B}}$ is a horizontal strip (it is obvious that $S_{\mathcal{A}}$ is horizontal). Now, if $a_{i+1}, \ldots, a_{i+s}$ is to the right of an a_i (in the same row), then there has to be $b_{i+1}, \ldots, b_{i+s}$ in $w_{A,B}$ since $a_{i+1}, \ldots, a_{i+s}$ are married. These $b_{i+1}, \ldots, b_{i+s}$ can never be on the ground since every row below that contains a color $i+1$ also contains a color i and there is no b_i There is thus a modified column of S_B that lies in the column of an $a_{i+1}, \ldots, a_{i+s}$. Thus changing every a_j weakly to the left of an a_i (in the same row) to a b_j does not affect the evaluation (the only real change is

$a_i \mapsto b_i$), and we have that $w_{\mathcal{A},\mathcal{B}}$ is such that $|\mathcal{A}| = |A| - 1$ and $|\mathcal{B}| = |B| + 1$. We now have to make sure that $S_{\mathcal{A}}, S_{\mathcal{B}}$ is a maximal double strip. It is easy to see that $S_{\mathcal{A}}$ is maximal. We will now see that $S_{\mathcal{B}}$ is also a maximal strip. Let γ be the outer shape of $S_{\mathcal{A}}$. Consider the order used in the proof of Proposition 9 (the n_A order). The b_j's with $j < i$ occupy the same cells in S_B and in $S_{\mathcal{B}}$ and are still maximal. Recall that the only difference between S_A in $S_{\mathcal{A}}$ is that every a_i and the a's to its right (in the same row) are not in $S_{\mathcal{A}}$. A b_j with $j < i$ can never lie on the ground in the row of an a_i. Neither can it lie to the right of an a_i and above an a. And it cannot lie on top of an a_ℓ to the right of an a_i (in the same row) since there is no b above the a_i. Finally the b_j's are maximal: if a b on the ground can be completed in one of the new empty cells, then S_A would not be maximal. If a b above an a can be can be completed (below) in one of the new empty cells, then the a_ℓ on which it sits is necessarily such that $i < \ell$.

The b_i occupies the cells of the a_i plus maybe some extra cells above. It is maximal below since otherwise a_i would need to be married (if not maximal there is no a_{i-s} for some s and $b_{i-s}, \ldots, b_{i-1}$ are there). The cells that the b_i's occupy above the a_i belong to S_B by Lemma 26, and were occupied by a b_ℓ with $i < \ell$ (the maximality of the string above is obvious since S_A, S_B and $S_{\mathcal{A}}, S_{\mathcal{B}}$ are equal above the highest a_i). Suppose that there are b's above the $a_j, \ldots, a_{j+s}$ in row r. Then the b_ℓ above a_j is the next string to be inserted. It will occupy the same places in S_A, S_B and in $S_{\mathcal{A}}, S_{\mathcal{B}}$ except that in S_A, S_B it will continue above the a_{j+1} in row r. The same will happen for all the b's above the $a_j, \ldots, a_{j+s}$ in row r except the rightmost one. In this case, the b will occupy the same places in S_A, S_B and in $S_{\mathcal{A}}, S_{\mathcal{B}}$ except that it will not continue above row r. It will be maximal above since by assumption there is no addable corner a distance $k+1$ from the uppermost b below row r. If $j = i+1$, it is easy to see that after the b's above the a's in row r have been added then the remaining strings are added in the same manner in S_A, S_B and in $S_{\mathcal{A}}, S_{\mathcal{B}}$.

$\square$

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