On the generalized blob algebras and the Nil-blob algebra

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Introduction

Cellular algebras were introduced by Graham-Lehrer as a general framework for studying modular representation theory. They are finite dimensional algebras endowed with a basis such that the structure constants with respect to the basis satisfy certain natural conditions. A cellular algebra $\mathcal{A}$ is always equipped with a family $\{\Delta(\lambda)\}$ of 'cell modules' for λ running over a poset Λ which is part of the cellular basis data. Each cell module $\Delta(\lambda)$ is endowed with a billinear form $\langle \cdot, \cdot \rangle$ and the irreducible modules $\{L(\lambda)\}$ all arise as quotients by the radical of the form $L(\lambda) = \Delta(\lambda)/\mathrm{rad}\langle \cdot, \cdot \rangle$. Using this, there is for a cellular algebra $\mathcal{A}$ a concrete way of obtaining the irreducible $\mathcal{A}$-modules, at least in principle.

Two of the motivating examples for cellular algebra were the Temperley-Lieb algebra TL_n with its diagram basis and the Hecke algebra $\mathcal{H}_n(q)$ with its cell basis derived from the Kazhdan-Lusztig basis. In fact, one parameter Hecke algebras of finite type are always cellular, as was shown by Geck, [13]. For Hecke algebras $\mathcal{H}(W, S)$ with unqeal parameters associated with a finite Coxeter system, Lusztig's cell theory depends on the choice of a *weight function* on W, and conjecturally it leads to a cellular basis as well, see [6]. For the cyclotomic Hecke algebra $\mathcal{H}_n(q_1, \ldots, q_l)$ there is also a concept of a *weighting function* θ, which plays a key role for the Fock space approach to the representation theory of $\mathcal{H}_n(q_1, \ldots, q_l)$, see [2], [12], [18], [44]. For $\mathcal{H}_n(Q, q)$ and for the zero weighting θ_0, Lusztig's approach *does* induce a cellular algebra structure on $\mathcal{H}_n(Q, q)$ and this was shown in [43] to be compatible with the diagram basis on blob algebra b_n.

In the first part of this thesis we make a complete review of the general concepts described above. We recall the formal definition of graded cellular algebras, given by Hu and Mathas in [17], where they provide an extension of the theory of cellular algebras given by Graham and Lehrer (see [14]). Also in the first part of the thesis, we set up the combinatorial concepts and notations that are needed for our work, including multipartitions, tableaux, and so on. We also present the various order relations on multipartitions and tableaux that play a role throughout the paper. They all depend on the choice of a weighting $\theta \in \mathbb{Z}^l$.

For $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$ we prove a version of Ehresmann's Theorem relating the order relation $\trianglelefteq_\theta$ on $\mathrm{Tab}(\boldsymbol{\lambda})$ with the Bruhat order on the symmetric group $\mathfrak{S}_n$. Although this and a few other of our results are valid for general θ we soon concentrate on the zero weighting θ_0.

The second part of this Thesis is concerned with the generalized blob algebra $\mathbb{B}_n$ introduced by Martin and Woodcock.

The original blob algebra $b_n = b_n(q, m)$, also known as the Temperley-Lieb algebra of type B, was introduced by Martin and Saleur via considerations in statistical mechanics. The usual Temperley-Lieb algebra $TL_n = TL_n(q)$ can be realized as a quotient of the Hecke algebra $\mathcal{H}_n(q)$ of finite type A and similarly it has also been known for some time that b_n is a quotient of the two-parameter Hecke algebra $\mathcal{H}_n(Q, q)$ of type B. Since $\mathcal{H}_n(Q, q)$ is the special case $l = 2$ of a cyclotomic Hecke algebra $\mathcal{H}_n(q_1, \ldots, q_l)$ one could now hope that this construction make sense for any cyclotomic Hecke algebra. Martin and Woodcock showed in [28] that this indeed is the case. They obtain b_n as the quotient of $\mathcal{H}_n(Q, q)$ by the ideal generated by the idempotents for the irreducible $\mathcal{H}_2(Q, q)$-modules associated with the bipartitions $((2), \emptyset)$ and $(\emptyset, (2))$ and showed that this idea generalizes to every $\mathcal{H}_n(q_1, \ldots, q_l)$. The quotient algebras of $\mathcal{H}_n(q_1, \ldots, q_l)$ that arise this way are the generalized blob algebras $\mathbb{B}_n = \mathbb{B}_n(q_1, \ldots, q_l)$ of the title. The parameter l is known as the level parameter and the generalized blob algebras can therefore be considered as the *Temperley-Lieb algebras at level l*.

We are interested in the modular, that is non-semisimple, representation theory of $\mathbb{B}_n$. This is the case where the ground field $\mathbb{F}$ is of positive characteristic or where the parameters q_i are roots of unity. The modular representation theories of TL_n and b_n are well understood and may be considered as approximations of the modular representation theory of $\mathbb{B}_n$. The modular representation theory of $\mathbb{B}_n$ is more complicated. In characteristic 0 it involves Kazhdan-Lusztig polynomials of type $\tilde{A}$, see [4] and [28], and in characteristic p it involves the p-canonical basis, at least conjecturally, see [23].

In the second part of this thesis we define the notation and give the necessary background for the KLR-approach to the representation theory of generalized blob algebras and then we show that $\mathbb{B}_n$ is a cellular algebra with respect to the zero weighting. There is however neither a natural Temperley-Lieb like diagram basis nor a Lusztig cell theory available for $\mathbb{B}_n$ and in fact our methods for showing cellularity of $\mathbb{B}_n$ are completely new. They are based on the seminal work by Brundan-Kleshchev and Rouquier that establishes an isomorphism between the KLR-algebra $\mathcal{R}_n$ and the cyclotomic Hecke algebra $\mathcal{H}_n(q_1, \dots, q_l)$. The KLR-algebra $\mathcal{R}_n$ is a $\mathbb{Z}$-graded algebra and our graded cellular basis on $\mathbb{B}_n$ inherits this $\mathbb{Z}$-grading, making it a graded cellular basis.

The KLR-algebra has already been used by Hu-Mathas, [17], and by Plaza and Ryom-Hansen, [38], to construct $\mathbb{Z}$-graded cellular bases for $\mathcal{H}_n(q_1, \dots, q_l)$ and for $b_n(q)$, but contrary to the present work those papers rely in a decisive way on already existing non-graded cellular bases on the algebras in question. Indeed Hu-Mathas rely in [17] on Murphy's standard basis for $\mathcal{H}_n(q_1, \dots, q_l)$, and in [38] the diagram basis for b_n is needed in order to derive the graded cellular bases. Note that Murphy's standard basis only exists for the classical dominance order on $\mathrm{Par}_{l,n}$, which is unrelated to the zero weighting.

The representation theory of $\mathcal{H}_n(q_1, \dots, q_l)$ is parametrized by l-multipartitions $\mathrm{Par}_{l,n}$ of n whereas the representation theory of $\mathbb{B}_n$ is parametrized by one-column l-multipartitions Par_n^1 of n. Our $\mathbb{Z}$-graded cellular basis

$$\mathcal{C}_n = \{m_{\mathfrak{st}} \mid \boldsymbol{\lambda} \in \mathrm{Par}_n^1, \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})\} \tag{0.0.1}$$

shares notationally several features of Murphy's standard basis and just like that basis it depends on the existence of a unique maximal $\boldsymbol{\lambda}$-tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$ for each $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$, with respect to θ_0. For $\boldsymbol{\lambda} \notin \mathrm{Par}_n^1$ there are in general many maximal $\boldsymbol{\lambda}$-tableaux and so our methods do not generalize to give a cellular basis for $\mathcal{H}_n(q_1, \dots, q_l)$, with respect to θ_0. In particular we do not recover Bowman's general results from [3] who give cellular bases on $\mathcal{H}_n(q_1, \dots, q_l)$ for any weighting θ, but at the cost of dealing with the 'fiendishly' complicated diagram combinatorics of Webster's diagrammatic Cherednik algebra, see [45].

In the third and last part of this thesis we investigate three different, although well-known, diagram algebras. The three diagram algebras arise in three quite different settings. Even so we show in this thesis that the three algebras are surprisingly closely related.

The first algebra of this algebras is a variation of the blob algebra $\mathbb{B}_n$. This is the Nil-blob algebra $\mathbb{NB}_n$. We provide its definition using a presentation on generators $\mathbb{U}_0, \mathbb{U}_1, \dots, \mathbb{U}_{n-1}$ and a series of relations that are reminiscent of the relations of the original blob algebra. (We also introduce the extended nil-blob algebra $\widetilde{\mathbb{NB}}_n$ by adding an extra generator $\mathbb{J}_n$ which is central in $\widetilde{\mathbb{NB}}_n$). We next go on to prove that $\mathbb{NB}_n$ is a diagram algebra where the diagram basis is the same as the one used for the original blob algebra, but where the multiplication rule is modified. The candidates for the diagrammatical counterparts of the generators $\mathbb{U}_i$'s are the obvious ones, but the fact that these diagrams generate the diagram algebra is not so obvious. We establish it in Theorem 8.0.5. From this Theorem we obtain the dimensions of $\mathbb{NB}_n$ (and $\widetilde{\mathbb{NB}}_n$) and we also deduce from it that $\mathbb{NB}_n$ is a cellular algebra in the sense of Graham and Lehrer. Finally, we indicate that this cellular structure is endowed with a family of JM-elements, in the sense of [32].

Our second diagram algebra has its origin in the theory of Soergel bimodules. Soergel bimodules were introduced by Soergel in the nineties, first for Weyl groups and then for general Coxeter systems (W, S). Building on the work of Elias and Khovanov in type A_n, Elias and Williamson proved that in general the category of Soergel bimodules $\mathcal{D}$ can be described diagrammatically, using generators and relations. For our second diagram algebra we choose W of type $\tilde{A}_1$ and consider a diagrammatically defined subalgebra of the endomorphism algebra $\mathrm{End}_{\mathcal{D}}(\underline{w})$, where $\underline{w}$ is a certain expression over S.

Our third diagram algebra is given by idempotent truncation, of the KLR-version of the generalized blob algebra $\mathbb{B}_n$ at level 2, with respect to a singular weight in the associated alcove geometry.

In the last part of this thesis we show that these three diagram algebras are isomorphic. We do so by giving a presentation for each of the three algebras, in terms of generators and relations. The three presentations turn out to be identical and from this we obtain the isomorphisms between the three algebras. As far as we know, the algebra defined by the common presentation of the three algebras has not appeared before in literature; it is the nil-blob algebra $\mathbb{NB}_n$.

For type $\tilde{A}_n$, it is already known that there are connections between the diagrammatical Soergel category $\mathcal{D}$ and the KLR-algebra. For example in positive characteristic, Riche and Williamson showed in [37] that $\mathcal{D}$ acts on the category of tilting modules for GL_n, via an action of the KLR-category. Our connection between the diagram algebras is however rather inspired by the categorical Blob vs. Soergel conjecture, that was recently formulated in [23], by Plaza and Libedinsky. If this conjecture were true, the representation theory of the generalized blob algebra, would be governed by the p-canonical basis for type $\tilde{A}_n$. We view the results of the last part of this thesis as evidence in favor of the categorical Blob vs. Soergel conjecture and in fact they are close to a proof of this conjecture in type $\tilde{A}_1$.

Contents

Part I

Generalities

Chapter 1

Graded cellular algebras and Jucys-Murphy elements

1.1 Graded cellular algebras

In this section we recall definitions and main results given by Hu and Mathas in [17] on graded cellular algebras, where they extend Graham and Lehrer's theory of cellular algebras [14]. We concentrate only in the case of $\mathbb{Z}-$graded cellular algebras.

Let R be a commutative integral domain with 1. A *graded* ($\mathbb{Z}-$*graded*) $R-$*module* is an $R-$module M which has a direct sum decomposition $M = \bigoplus_{d\in\mathbb{Z}} M_d$. If $m \in M_d$, for some $d \in \mathbb{Z}$, then m is *homogeneous* of *degree* d and we set $\deg m = d$. If M is a graded $R-$module let $\underline{M}$ be the ungraded $R-$module obtained by forgetting the grading on M. If M is a graded $R-$module and $s \in \mathbb{Z}$, let $M\langle s\rangle$ be the graded $R-$module obtained by shifting the grading on M up by s; that is, $M\langle s\rangle_d = M_{d-s}$, for $d \in \mathbb{Z}$.

A *graded* $R-$*algebra* is a unital associative $R-$algebra $A = \bigoplus_{d\in\mathbb{Z}} A_d$ which is graded $R-$module such that $A_d A_e \subset A_{d+e}$, for all $d, e \in \mathbb{Z}$. It follows that $1 \in A_0$ and A_0 is a graded subalgebra of A. A graded (right) $A-$module is a graded $R-$module M such that $\underline{M}$ is an $A-$module and $M_d A_e \subset M_{d+e}$, for all $d, e \in \mathbb{Z}$. Graded submodules, graded left $A-$modules and so on are all definied in the obvious way.

Definition 1.1.1. *Suppose that A is a $\mathbb{Z}-$graded $R-$algebra which is free of finite rank over R. A* graded cell datum *for A is an ordered quadruple $(\mathcal{P}, T, C, \deg)$, where $(\mathcal{P}, \rhd)$ is the* weight poset*, $T(\lambda)$ is a finite set for $\lambda \in \mathcal{P}$, and*

$$C : \coprod_{\lambda\in\mathcal{P}} T(\lambda) \times T(\lambda) \to A; \quad (\mathfrak{s}, \mathfrak{t}) \mapsto c^{\lambda}_{\mathfrak{s},\mathfrak{t}},$$

and

$$\deg : T(\mathcal{P}) \to \mathbb{Z} \quad \textit{where} \quad T(\mathcal{P}) = \coprod_{\lambda\in\mathcal{P}} T(\lambda)$$

are functions such that C is injective and

1. *$\{c^{\lambda}_{\mathfrak{s},\mathfrak{t}} : \mathfrak{s}, \mathfrak{t} \in T(\lambda), \lambda \in \mathcal{P}\}$ is an $R-$basis of A.*
2. *If $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, for some $\lambda \in \mathcal{P}$, and $a \in A$ then there exist scalars $r_{\mathfrak{t},\mathfrak{v}}(a)$, which do not depend on $\mathfrak{s}$, such that*

$$c^{\lambda}_{\mathfrak{s},\mathfrak{t}} a = \sum_{\mathfrak{v}} r_{\mathfrak{t},\mathfrak{v}}(a) c^{\lambda}_{\mathfrak{s},\mathfrak{v}} \quad (mod A^{\rhd\lambda})$$

   where $A^{\rhd\lambda}$ is the $R-$submodule of A spanned by $\{c^{\mu}_{\mathfrak{a},\mathfrak{b}} : \mu \rhd \lambda \quad and \quad \mathfrak{a}, \mathfrak{b} \in T(\mu)\}$

3. *The $R-$linear map $* : A \to A$ determined by*

$$(c^{\lambda}_{\mathfrak{s},\mathfrak{t}})^* = c^{\lambda}_{\mathfrak{t},\mathfrak{s}} \quad (\lambda \in \mathcal{P}, \mathfrak{s}, \mathfrak{t} \in T(\lambda)),$$

   is an anti-isomorphism of A.

4. *Each basis element $c^{\lambda}_{\mathfrak{s},\mathfrak{t}}$ is homogeneous of degree $\deg c^{\lambda}_{\mathfrak{s},\mathfrak{t}} = \deg \mathfrak{s} + \deg \mathfrak{t}$, for $\lambda \in \mathcal{P}$ and $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$.*

A *graded cellular algebra* is a graded algebra which has a graded cell datum. The basis $\{c^\lambda_{\mathfrak{s},\mathfrak{t}} : \mathfrak{s},\mathfrak{t} \in T(\lambda), \lambda \in \mathcal{P}\}$ is a *graded cellular basis* of A.

If we omit item 4 of definition 1.1.1 we recover Graham and Leherer's definition of an (ungraded) cellular algebra. Therefore, by forgetting the grading, any graded cellular algebra is an (ungraded) cellular algebra in the original sense of Graham and Lehrer.

Definition 1.1.2. *Suppose A is a graded cellular algebra with graded cell datum $(\mathcal{P}, T, C, \deg)$, and fix $\lambda \in \mathcal{P}$. Then the* graded cell module C^λ *is the graded right $A-$module*

$$C^\lambda = \bigoplus_{z\in\mathbb{Z}} C^\lambda_z,$$

where C^λ_z is the free $R-$module with basis $\{c^\lambda_\mathfrak{t} : \mathfrak{t} \in T(\lambda), \deg(\mathfrak{t}) = z\}$ and where the action of A on C^λ is given by

$$c^\lambda_\mathfrak{t} a = \sum_\mathfrak{v} r_{\mathfrak{t},\mathfrak{v}}(a) c^\lambda_\mathfrak{v}$$

where the scalars $r_{\mathfrak{t},\mathfrak{v}}(a)$ are the scalars appearing in item 2 in definition 1.1.1.

Similarly, let $C^{*\lambda}$ be the left graded $A-$module which, as an $R-$module is equal to C^λ, but where the action of A is given by $a \cdot x = xa^*$, for $a \in A$ and $x \in C^{*\lambda}$. It follows directly from definition 1.1.1 that C^λ and $C^{*\lambda}$ are graded $A-$modules.

Let $A^{\trianglerighteq\lambda}$ be the $R-$module spanned by the elements $\{c^\mu_{\mathfrak{a},\mathfrak{b}} : \mu \trianglerighteq \lambda \quad \text{and} \quad \mathfrak{a},\mathfrak{b} \in T(\mu)\}$. It is straightforward to check that $A^{\trianglerighteq\lambda}$ is a graded two-sided ideal of A and that

$$A^{\trianglerighteq\lambda}/A^{\triangleright\lambda} \cong C^{*\lambda} \otimes_R C^\lambda \cong \bigoplus_{\mathfrak{s}\in T(\lambda)} C^\lambda\langle \deg \mathfrak{s}\rangle$$

as graded (A, A)-bimodules for the first isomorphism and as graded right $A-$modules for the second.

Let t be an indeterminate over $\mathbb{N}_0$. If $M = \bigoplus_{z\in\mathbb{Z}} M_z$ is a graded $A-$module such that each M_z is free of finite rank over R, then its *graded dimension* is the Laurent polynomial

$$\dim_t M = \sum_{k\in\mathbb{Z}} (\dim_R M_k) t^k.$$

Corollary 1.1.3. *Suppose that A is a graded cellular algebra and $\lambda \in \mathcal{P}$. Then*

$$\dim_t C^\lambda = \sum_{\mathfrak{s}\in T(\lambda)} t^{\deg \mathfrak{s}}.$$

Consequently,

$$\dim_t A = \sum_{\lambda\in\mathcal{P}} \sum_{\mathfrak{s},\mathfrak{t}\in T(\lambda)} t^{\deg \mathfrak{s}+\deg \mathfrak{t}} = \sum_{\lambda\in\mathcal{P}} (\dim_t C^\lambda)^2.$$

Suppose that $\mu \in \mathcal{P}$. Then it follows from definition 1.1.1, exactly as in [14], that there is a bilinear form $\langle , \rangle_\mu$ on C^μ which is determined by

$$c^\mu_{\mathfrak{a}\mathfrak{s}} c^\mu_{\mathfrak{t}\mathfrak{b}} \equiv \langle c^\mu_\mathfrak{s}, c^\mu_\mathfrak{t}\rangle_\mu c^\mu_{\mathfrak{a},\mathfrak{b}} \pmod{A^{\triangleright\mu}},$$

for any $\mathfrak{s},\mathfrak{t},\mathfrak{a},\mathfrak{b} \in T(\mu)$.

Lemma 1.1.4. *Suppose that $\mu \in \mathcal{P}$. Then the radical*

$$Rad(C^\mu) = \{x \in C^\mu : \langle x, y\rangle_\mu = 0 \quad \textit{for all} \quad y \in C^\mu\}$$

is a graded submodule of C^μ.

Proof. See [17]. □

The last lemma allows us to define a graded quotient of C^μ, for $\mu \in \mathcal{P}$.

Definition 1.1.5. *Suppose that $\mu \in \mathcal{P}$. Let $D^\mu = C^\mu/Rad(C^\mu)$.*

By definition D^μ is a graded right $A-$module. Henceforth, let $R = K$ be a field and $A = \bigoplus_{z\in\mathbb{Z}} A_z$ a graded cellular $K-$algebra. Let $\mathcal{P}_0 = \{\lambda \in \mathcal{P} : D^\lambda \neq 0\}$.

Theorem 1.1.6. *Suppose that K is a field and A is a graded cellular $K-$algebra.*

1. *If $\mu \in \mathcal{P}_0$ then D^μ is an absolutely irreducible graded $A-$module.*
2. *Suppose that $\lambda, \mu \in \mathcal{P}_0$. Then $D^\lambda \cong D^\mu\langle k\rangle$, for some $k \in \mathbb{Z}$, if and only if $\lambda = \mu$ and $k = 0$.*
3. *The set $\{D^\mu\langle k\rangle : \mu \in \mathcal{P}_0 \quad \text{and} \quad k \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic graded simple $A-$modules.*

Proof. See [17]. □

In particular, just as Graham and Lehrer proved (see [14]) in the ungraded case, every field is a splitting field for a graded cellular algebra.

Corollary 1.1.7. *Suppose that K is a field and A is a graded cellular algebra over K. Then $\{\underline{D}^\mu : \mu \in \mathcal{P}_0\}$ is a complete set of pairwise non-isomorphic ungraded simple $A-$modules.*

Proof. See [17]. □

1.2 Jucys-Murphy elements

In this section we recall the definition and some main results on Jucys-Murphy elements, given by Mathas and Soriano in [32]. For the rest of this section let R be a commutative integral domain with 1 and A be a cellular $R-$algebra (in the sense of [14]) with cell datum $(\mathcal{P}, T, C)$, and where each set $T(\lambda)$ is a poset $(T(\lambda), \rhd)$. We also define a partial order $\succ$ on $T(\mathcal{P})$, given by

$$\mathfrak{s} \succ \mathfrak{t} \quad \text{if and only if} \quad (\text{shape}(\mathfrak{s}) \rhd \text{shape}(\mathfrak{t})) \quad \text{or} \quad (\text{shape}(\mathfrak{s}) = \text{shape}(\mathfrak{t}) \quad \text{and} \quad \mathfrak{s} \rhd \mathfrak{t}). \tag{1.2.1}$$

Definition 1.2.1. *A family of* Jucys-Murphy elements *(or for simplicity* JM-elements*) is a set $\{L_1, \dots, L_M\}$ of commuting elements of A together with a set of scalars $\{u_\mathfrak{t}(i) \in R : \mathfrak{t} \in T(\mathcal{P}) \quad \text{and} \quad 1 \leq i \leq M\}$, such that for every $i = 1, \dots M$ we have $L_i^* = L_i$ and, for all $\lambda \in \mathcal{P}$ and $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$,*

$$c^\lambda_{\mathfrak{s},\mathfrak{t}} L_i = u_\mathfrak{t}(i) c^\lambda_{\mathfrak{s},\mathfrak{t}} + \sum_{\mathfrak{v} \rhd \mathfrak{t}} r_{\mathfrak{t},\mathfrak{v}}(L_i) c^\lambda_{\mathfrak{s},\mathfrak{v}} \quad (mod A^{\rhd\lambda}). \tag{1.2.2}$$

We call $u_\mathfrak{t}(i)$ the content of $\mathfrak{t}$ at i.

Implicity the JM-elements depends on the choice of cellular basis for A. Note that we also have a left analogue to equation (1.2.2):

$$L_i c^\lambda_{\mathfrak{s},\mathfrak{t}} = u_\mathfrak{s}(i) c^\lambda_{\mathfrak{s},\mathfrak{t}} + \sum_{\mathfrak{u} \rhd \mathfrak{s}} r_{\mathfrak{s},\mathfrak{u}}(L_i) c^\lambda_{\mathfrak{u},\mathfrak{t}} \quad (\text{mod} A^{\rhd\lambda}). \tag{1.2.3}$$

An important application of JM-elements is that they can detect when the modules D^λ are not equal to zero.

Proposition 1.2.2. *Let $R = K$ be a field, and A be a cellular $K-$algebra with a family of JM-elements $\{L_1, \dots, L_M\}$. Fix $\lambda \in \mathcal{P}$ and $\mathfrak{s} \in T(\lambda)$. Suppose that whenever $\mathfrak{t} \in T(\mathcal{P})$ and $\mathfrak{s} \succ \mathfrak{t}$ then $u_\mathfrak{s}(i) \neq u_\mathfrak{t}(i)$, for some $1 \leq i \leq M$. Then $D^\lambda \neq 0$.*

Proof. See [32]. □

The last proposition motivates the following definition

Definition 1.2.3. *Let A be a cellular $R-$algebra with JM-elemnts $\{L_1, \dots, L_M\}$, and let $\lambda \in \mathcal{P}$. We say that the JM-elements* separate *$T(\mathcal{P})$ (over R) if whenever $\mathfrak{s}, \mathfrak{t} \in T(\mathcal{P})$ and $\mathfrak{s} \succ \mathfrak{t}$ then $u_\mathfrak{s}(i) \neq u_\mathfrak{t}(i)$ for some $1 \leq i \leq M$.*

The *separation condition* (of definition 1.2.3) also provides a semisimplicity criterion for the algebra A.

Corollary 1.2.4. *Suppose that $R = K$ is a field, and A is a cellular $K-$algebra with a family of JM-elements $\{L_1, \dots, L_M\}$ that separates $T(\mathcal{P})$. Then A is (split) semisimple.*

Proof. See [32]. □

Chapter 2

Combinatorics and Tableaux

Let us recall the basic combinatorial concepts and notations associated with the representation theory of the symmetric group $\mathfrak{S}_n$ and the wreath product $C_l \wr \mathfrak{S}_n$.

We denote by $\mathbb{N}$ the positive integers and by $\mathbb{N}_0$ the non-negative integers. For $n \in \mathbb{N}_0$, a *composition* λ of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of elements of $\mathbb{N}_0$ such that $|\lambda| := \sum_k \lambda_k = n$. If k is minimal such that $\lambda_i = 0$ for all $i > k$ we also write $\lambda = (\lambda_1, \dots, \lambda_k)$ for λ. We say that a composition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n is a *partition* of n if it satisfies that $\lambda_k \geq \lambda_{k+1}$ for all $k \geq 1$.

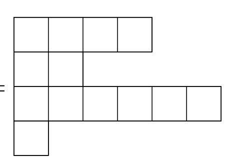
For integers $l > 0$ and $n \geq 0$, an *l-multicomposition* of n is an l-tuple of compositions $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)})$ such that $\sum_{m=1}^{l} |\lambda^{(m)}| = n$. An *l-multicomposition* $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)})$ of n is called an l-multipartition of n if all its components $\lambda^{(i)}$ are partitions. The set of all l-multicompositions of n is denoted by $\mathrm{Comp}_{l,n}$ and the set of all l-multipartitions of n is denoted by $\mathrm{Par}_{l,n}$.

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)})$ be an l-multicomposition. Then $\boldsymbol{\lambda}$ is called a *one-column* l-multicomposition if all of its components $\lambda^{(i)}$ are one-column compositions, that is each $\lambda^{(i)}$ is of the form $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_r^{(i)})$ where $\lambda_j^{(i)}$ is either 0 or 1 for all j.

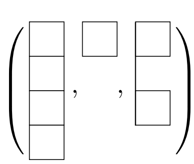
A *one-column l-multipartition* is a one-column l-multicomposition which is also an l-multipartition. For $\boldsymbol{\lambda}$ a one-column l-multipartition each of its components $\lambda^{(m)}$ is a partition of the form $\lambda^{(m)} = (1, 1, \dots, 1)$ that is $\lambda^{(m)} = (1^{a_m})$ where $a_m = |\lambda^{(m)}|$. In other words, a one-column l-multipartition is of the form $\boldsymbol{\lambda} = ((1^{a_1}), \dots, (1^{a_l}))$ for certain non-negative integers a_i. The set of all one-column l-multipartitions of n is denoted by Par_n^1.

We shall hold l fixed throughout the article, and shall therefore frequently refer to l-multicompositions (resp. l-multipartitions, etc) simply as multicompositions (resp. multipartitions, etc).

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a composition of n. Then we represent λ graphically via its *Young diagram* $[\lambda]$. We use English notation so it consists of an array of k left adjusted lines of boxes denoted the *nodes* of the diagram, the first line containing λ_1 nodes, the second line λ_2 nodes, and so on. The nodes are labelled using matrix convention, that is the j'th node of the i'th line of $[\lambda]$ is labelled (i, j) and in this case we write $(i, j) \in [\lambda]$. For example, if $\lambda = (4, 2, 6, 1)$ then the Young diagram $[\lambda]$ is

$[\lambda] =$  .

For an l-multicomposition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)})$ we define its Young diagram $[\boldsymbol{\lambda}]$ to be the l-tuple of Young diagrams $([\lambda^{(1)}], \dots, [\lambda^{(l)}])$. The nodes of $\boldsymbol{\lambda}$ are labelled by the triples (i, j, k) where (i, j) is a node of $[\lambda^{(k)}]$. For example, if $\boldsymbol{\lambda} = ((1, 1, 1, 1), (1), (1, 0, 1))$ we have that

$[\boldsymbol{\lambda}] =$  (2.0.1) or if $\boldsymbol{\mu} = ((1^4), (1^0), (1^3))$ we have that

$$[\boldsymbol{\mu}] = \left(\begin{array}{c}\Box\\\Box\\\Box\\\Box\end{array}, \emptyset, \begin{array}{c}\Box\\\Box\\\Box\end{array} \right). \tag{2.0.2}$$

For a multipartition $\boldsymbol{\lambda}$ we define the *i'th row* of $\boldsymbol{\lambda}$ as the set of nodes of the form (i, j, k).

There is a well known way to make $\mathrm{Comp}_{l,n}$ into a poset, the associated order relation being the dominance order on $\mathrm{Comp}_{l,n}$ studied for example in [8]. However, this is not the only interesting order relation on $\mathrm{Comp}_{l,n}$.

Let us fix a tuple $\theta = (\theta_1, \ldots, \theta_l) \in \mathbb{Z}^l$, called a *weighting*. Let $\gamma = (i, j, b)$ and $\gamma' = (i', j', b')$ be nodes of multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, or more generally elements of $\mathbb{N} \times \mathbb{N} \times \{1, \ldots, l\}$. Then we write $\gamma \lhd_\theta \gamma'$ if either $(\theta_b + j - i) < (\theta_{b'} + j' - i')$ or if $(\theta_b + j - i) = (\theta_{b'} + j' - i')$ and $b > b'$. (The last inequality is not an error). We write $\gamma \unlhd_\theta \gamma'$ if $\gamma \lhd_\theta \gamma'$ or if $\gamma = \gamma'$.

This defines an order on $\mathbb{N}\times\mathbb{N}\times\{1, \ldots, l\}$ that we extend to multipartitions as follows. Suppose that $\boldsymbol{\lambda} \in \mathrm{Comp}_{l,n}$ and $\boldsymbol{\mu} \in \mathrm{Comp}_{l,m}$. Then we write $\boldsymbol{\lambda} \unlhd_\theta \boldsymbol{\mu}$ if for each $\gamma_0 \in \mathbb{N} \times \mathbb{N} \times \{1, \ldots, l\}$ we have that

$$|\{\gamma \in [\boldsymbol{\lambda}] \colon \gamma \rhd_\theta \gamma_0\}| \le |\{\gamma \in [\boldsymbol{\mu}] \colon \gamma \rhd_\theta \gamma_0\}|. \tag{2.0.3}$$

This order relation $\lhd_\theta$ depends highly on the initial choice of weighting θ. When restricted to $\mathrm{Par}_{l,n}$ and choosing θ such that $\theta_i > \theta_{i+1} + n$ for all i we recover the dominance order used in [DJM] which we refer to as $\unlhd_\infty$. This is the *separated* case, but in this article we shall be mostly interested in another limit case, namely the one given by the zero weighting $\theta = (0, 0, \ldots, 0)$. We refer to the corresponding order as $\unlhd_0$.

Note that for $l = 1$, we have that $\unlhd_\theta$ is just the usual dominance order, for any θ.

In general, the order $\unlhd_\theta$ is only a partial order on the nodes of $\mathrm{Par}_{l,n}$ or $\mathbb{N} \times \mathbb{N} \times \{1, \ldots, l\}$, but it becomes a total order upon restriction to the nodes of Par^1_n or $\mathbb{N}\times\{1\}\times\{1, \ldots, l\}$. Using this we can prove the following useful Lemma that we shall use implicitly throughout the paper. It says that $\boldsymbol{\lambda} \unlhd_\theta \boldsymbol{\mu}$ if and only if $\boldsymbol{\mu}$ can be obtained from $\boldsymbol{\lambda}$ by moving nodes of $\boldsymbol{\lambda}$ upwards.

Lemma 2.0.1. *Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{Par}^1_n$. Then $\boldsymbol{\lambda} \unlhd_\theta \boldsymbol{\mu}$ if and only if there is a bijection $\Theta : [\boldsymbol{\lambda}] \to [\boldsymbol{\mu}]$ such that $\Theta(\gamma) \unrhd_\theta \gamma$ for all $\gamma \in [\boldsymbol{\lambda}]$.*

Proof. As mentioned $\unlhd_\theta$ is a total order on the nodes of $\mathbb{N} \times \{1\} \times \{1, \ldots, l\}$ and so there is an order preserving bijection from these nodes to $\mathbb{N}$, where $\mathbb{N}$ is endowed with the opposite of the natural order, that is '1' is the maximal element. Using this, we may view $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ as ordered subsets of $\mathbb{N}$. But in this situation one easily checks the equivalence of (2.0.3) with the existence of Θ. $\square$

To illustrate the difference between $\unlhd_\infty$ and $\unlhd_0$ we consider their restriction to Par^1_n. In each case there is a unique maximal element but the two maximal elements are different. The unique maximal elements with respect to $\unlhd_\infty$ is

$$\boldsymbol{\mu}_n^{max,\infty} := ((1^n), \emptyset, \emptyset, \ldots, \emptyset)) \tag{2.0.4}$$

To describe $\boldsymbol{\mu}_n^{max,0}$, the unique maximal element with respect to $\unlhd_0$, we use integer division to write $n = ql + r$ where $q, l \in \mathbb{Z}$ such that $0 \le r < l$. Then we have that $\boldsymbol{\mu}_n^{max,0}$ is given by

$$\boldsymbol{\mu}_n^{max,0} = (\overbrace{(1^{q+1}), \ldots, (1^{q+1})}^{r\, terms}, \overbrace{(1^q), \ldots, (1^q)}^{l-r\, terms}). \tag{2.0.5}$$

For example, for $n = 7$ and $l = 3$ we have that

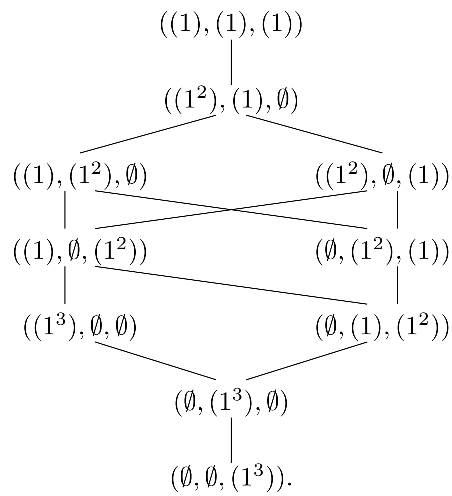
$$\boldsymbol{\mu}_n^{max,\infty} = \left(\begin{array}{c}\Box\\\Box\\\Box\\\Box\\\Box\\\Box\\\Box\end{array}, \emptyset, \emptyset \right), \quad \boldsymbol{\mu}_n^{max,0} = \left(\begin{array}{c}\Box\\\Box\\\Box\end{array}, \begin{array}{c}\Box\\\Box\end{array}, \begin{array}{c}\Box\\\Box\end{array} \right). \tag{2.0.6}$$

In general, with respect to $\unlhd_\infty$ the big multipartitions tend to have their center of mass to the left of the diagram, whereas with respect to $\unlhd_0$ the big multipartitions tend to have their center of mass in the middle of the diagram.

For $l = 2$, the restriction of $\unlhd_0$ to Par_n^1 is the total order used for example in [38] and [43]. Here is the $n = 3$ case:

$$\big(\emptyset, (1^3)\big) \unlhd_0 \big((1^3), \emptyset\big) \unlhd_0 \big((1),\, (1^2)\big) \unlhd_0 \big((1^2), (1)\big)\,. \tag{2.0.7}$$

For $l \geq 3$, the restriction of $\unlhd_0$ to Par_n^1 is only a partial order. Here we illustrate the $n = l = 3$ case:



(2.0.8)

Let λ be a composition of n. A *tableau* of shape λ or simply a λ*-tableau* is a bijection $\mathfrak{t} : \{1, \ldots, n\} \to [\lambda]$. In this case we write $\mathrm{shape}(\mathfrak{t}) = \lambda$. A λ-tableau $\mathfrak{t}$ is represented graphically via a labelling of the nodes of $[\lambda]$ using the numbers $\{1, 2, \ldots, n\}$ where the labelling of the node (i, j) is given by $\mathfrak{t}^{-1}(i, j)$. In this case we say that the (i, j)'th node of $\mathfrak{t}$ is *filled in with* $\mathfrak{t}^{-1}(i, j)$ via $\mathfrak{t}$. Let $\boldsymbol{\lambda}$ be an l-multicomposition. The concept of $\boldsymbol{\lambda}$*-tableaux* is defined the same way as for ordinary λ-tableaux, that is a $\boldsymbol{\lambda}$-tableau is a bijection $\boldsymbol{\mathfrak{t}} : \{1, \ldots, n\} \to [\boldsymbol{\lambda}]$.

A λ-tableau $\mathfrak{t}$ is called *standard* if the corresponding labelling of $[\lambda]$ has increasing numbers from left to right along rows and from top to bottom along columns. Similarly, for a tableau $\boldsymbol{\mathfrak{t}}$ of a multicomposition $\boldsymbol{\lambda}$ we say that it is standard if all its components are standard. For a composition λ, we denote by $\mathrm{Tab}(\lambda)$ and $\mathrm{Std}(\lambda)$ the set of all λ-tableaux and the set of all standard λ-tableaux and we use a similar similar notation for $\boldsymbol{\lambda}$-tableaux of a multicomposition $\boldsymbol{\lambda}$.

For a composition λ and a λ-tableau $\mathfrak{t}$ and $1 \leq k \leq n$ we denote by $\mathfrak{t}|_k$ the restriction of $\mathfrak{t}$ to the set $\{1, 2, \ldots, k\}$. A similar notation is used for tableaux for multipartitions. Let $\boldsymbol{\mu}$ be as in (2.0.2). Then the following are $\boldsymbol{\mu}$-tableaux

$$\boldsymbol{\mathfrak{t}} = \left(\begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline 5 \\ \hline 7 \\ \hline \end{array}, \emptyset, \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 6 \\ \hline \end{array} \right), \quad \boldsymbol{\mathfrak{s}} = \left(\begin{array}{|c|}\hline 1 \\ \hline 5 \\ \hline 4 \\ \hline 6 \\ \hline \end{array}, \emptyset, \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 7 \\ \hline \end{array} \right) \tag{2.0.9}$$

but only the first is standard. Note that for all $1 \leq k \leq n$ we have that $\mathrm{shape}(\mathfrak{t}\,|_k)$ is a multipartition, but in the case of $\boldsymbol{\mathfrak{s}}$ we have

$$\mathrm{shape}(\boldsymbol{\mathfrak{s}}\,|_4) = ((1, 0, 1), \emptyset, (1, 1))$$

which is not a multipartition, only a multicomposition.

We extend the order $\unlhd_\theta$ to tableaux for multipartitions n, as follows. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be multicompositions of m and n and let $\boldsymbol{\mathfrak{s}}$ and $\boldsymbol{\mathfrak{t}}$ be tableaux of shapes $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. Then we write $\boldsymbol{\mathfrak{t}} \unlhd_\theta \boldsymbol{\mathfrak{s}}$ if for all $1 \leq k \leq \min(m, n)$ we have that

$$\mathrm{shape}(\boldsymbol{\mathfrak{t}}\,|_k) \unlhd_\theta \mathrm{shape}(\boldsymbol{\mathfrak{s}}\,|_k).$$

For example, considering the tableaux $\boldsymbol{\mathfrak{s}}$ and $\boldsymbol{\mathfrak{t}}$ from (8.0.25) we have that $\boldsymbol{\mathfrak{s}} \lhd_0 \boldsymbol{\mathfrak{t}}$.

Let $\boldsymbol{\lambda} \in \mathrm{Par}_{l,n}$ be a multipartition and let $\gamma \in \mathbb{N} \times \mathbb{N} \times \{1, \ldots, l\} \setminus [\boldsymbol{\lambda}]$. Then we say that γ is an *addable node* for $\boldsymbol{\lambda}$ if $[\boldsymbol{\lambda}] \cup \gamma$ is the Young diagram of a multipartition. Dually we say that $\gamma \in [\boldsymbol{\lambda}]$ is a *removable node* for $\boldsymbol{\lambda}$ if $[\boldsymbol{\lambda}] \setminus \gamma$ is the Young diagram of a multipartition. The set of addable (removable) nodes for $\boldsymbol{\lambda}$ is totally ordered under $\unlhd_\theta$.

For $\boldsymbol{\lambda} \in \mathrm{Par}_{l,n}$ we now define multipartitions $\boldsymbol{\lambda}_{\theta,0}, \dots, \boldsymbol{\lambda}_{\theta,n} \in \mathrm{Par}_{l,n}$ recursively via $\boldsymbol{\lambda}_{\theta,0} := (\emptyset, \dots, \emptyset)$ and for $i > 0$ via $[\boldsymbol{\lambda}_{\theta,i}] := [\boldsymbol{\lambda}_{i-1}] \cup \gamma_{\theta,i}$ where $\gamma_{\theta,i} \in [\boldsymbol{\lambda}]$ satisfies the condition that it is the largest addable node for $\boldsymbol{\lambda}_{i-1}$, with respect to $\trianglelefteq_\theta$. We denote by $\mathfrak{t}_\theta^{\boldsymbol{\lambda}}$ the $\boldsymbol{\lambda}$-tableau which is given by $\mathfrak{t}_\theta^{\boldsymbol{\lambda}}(i) = \gamma_{\theta,i}$. If $\theta = \theta_\infty$ we write $\mathfrak{t}_\infty^{\boldsymbol{\lambda}}$ for $\mathfrak{t}_\theta^{\boldsymbol{\lambda}}$ and if $\theta = \theta_0$ we write $\mathfrak{t}_0^{\boldsymbol{\lambda}}$ for $\mathfrak{t}_\theta^{\boldsymbol{\lambda}}$.

Suppose that $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$. Then $\mathfrak{t}_\infty^{\boldsymbol{\lambda}}$ is the unique maximal element in $\mathrm{Tab}(\boldsymbol{\lambda})$ and $\mathrm{Std}(\boldsymbol{\lambda})$ with respect to $\trianglelefteq_\infty$. It is the $\boldsymbol{\lambda}$-tableau obtained by filling in the nodes of $[\boldsymbol{\lambda}]$ from left to right along the columns. For example, for $\boldsymbol{\lambda} = ((1^3), (1^3), (1^2))$ it is

$$\mathfrak{t}_\infty^{\boldsymbol{\lambda}} = \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline 6 \\ \hline \end{array}, \begin{array}{|c|}\hline 7 \\ \hline 8 \\ \hline \end{array} \right). \tag{2.0.10}$$

Let still $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$. Then $\mathfrak{t}_0^{\boldsymbol{\lambda}}$ is the unique maximal element in $\mathrm{Tab}(\boldsymbol{\lambda})$ and $\mathrm{Std}(\boldsymbol{\lambda})$ with respect to $\trianglelefteq_0$. It is the $\boldsymbol{\lambda}$-tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$ in which $1, \dots, n$ are filled in increasingly along the rows of $\boldsymbol{\lambda}$. For example, for $\boldsymbol{\lambda} = ((1^3), (1^3), (1^2))$ it is

$$\mathfrak{t}_0^{\boldsymbol{\lambda}} = \left(\begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 5 \\ \hline 8 \\ \hline \end{array}, \begin{array}{|c|}\hline 3 \\ \hline 6 \\ \hline \end{array} \right). \tag{2.0.11}$$

The tableau $\mathfrak{t}_\theta^{\boldsymbol{\lambda}}$ plays an important role in our paper, especially for $\theta = \theta_0$, so let us prove formally the claim on maximality of $\mathfrak{t}_\theta^{\boldsymbol{\lambda}}$.

Let first $\mathfrak{S}_n$ be the symmetric group on $\boldsymbol{n} := \{1, \dots, n\}$, and let $S = \{s_1, \dots, s_{n-1}\}$ be its subset of *simple transpositions*, i.e. for each $k = 1, \dots, n-1$ we have that $s_k = (k, k+1)$. It is well known that $\mathfrak{S}_n$ is a Coxeter group on S. For any multicomposition $\boldsymbol{\lambda}$ of n we have that $\mathfrak{S}_n$ acts on the right on $\mathrm{Tab}(\boldsymbol{\lambda})$ by permuting the entries inside a given tableaux. Thus, if $w = s_{i_1} s_{i_2} \cdots s_{i_N}$ where $s_{i_j} \in S$ and if $\mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda})$ we have that $\mathfrak{t}w = (\cdots((\mathfrak{t}s_{i_1})s_{i_2} \cdots)s_{i_N})$.

We next need to introduce yet another order on $\mathrm{Tab}(\boldsymbol{\lambda})$. Let $\boldsymbol{\lambda}$ be a multipartition and let $\mathfrak{t}, \mathfrak{s}$ be $\boldsymbol{\lambda}$-tableaux. For $s \in S$ we define $\mathfrak{t} \xrightarrow{s} \mathfrak{s}$ if $\mathfrak{s} = \mathfrak{t}s$ and $\mathfrak{s} \triangleright_\theta \mathfrak{t}$. We let $\succ_\theta$ be the order on $\mathrm{Tab}(\boldsymbol{\lambda})$ induced by $\mathfrak{t} \xrightarrow{s} \mathfrak{s}$ for all $s \in S$, that is $\mathfrak{s} \succ_\theta \mathfrak{t}$ if there is a finite sequence

$$\mathfrak{t}_0 \xrightarrow{s_{i_1}} \mathfrak{t}_1 \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_k}} \mathfrak{t}_k$$

with $\mathfrak{t}_0 = \mathfrak{t}$ and $\mathfrak{t}_k = \mathfrak{s}$. We call $\succ_\theta$ the weak order on $\mathrm{Tab}(\boldsymbol{\lambda})$. It is clear that $\mathfrak{s} \succ_\theta \mathfrak{t} \Rightarrow \mathfrak{s} \triangleright_\theta \mathfrak{t}$, but the converse is false in general. Consider for example $\boldsymbol{\mu} = ((1^3), (1^3), (1^2))$ and the $\boldsymbol{\mu}$-tableaux

$$\mathfrak{t} = \left(\begin{array}{|c|}\hline 1 \\ \hline 6 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline 8 \\ \hline \end{array}, \begin{array}{|c|}\hline 3 \\ \hline 7 \\ \hline \end{array} \right), \quad \mathfrak{s} = \left(\begin{array}{|c|}\hline 1 \\ \hline 6 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 7 \\ \hline 8 \\ \hline \end{array}, \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array} \right).$$

Then with respect to $\theta = (0, 0, \dots, 0)$ we have that $\mathfrak{t} \triangleright_\theta \mathfrak{s}$ but $\mathfrak{t} \not\succ_\theta \mathfrak{s}$.

We can now prove the promised claim for $\mathfrak{t}_\theta^{\boldsymbol{\lambda}}$.

Lemma 2.0.2. *Suppose that* $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$.

a) Let $\mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda})$ *and set* $\mathfrak{s} = \mathfrak{t}s_k$. *Suppose that* $\mathfrak{t}(k) \triangleleft_\theta \mathfrak{t}(k+1)$. *Then we have that* $\mathfrak{t} \prec_\theta \mathfrak{s}$.

b) We have that $\mathfrak{t}_\theta^{\boldsymbol{\lambda}}$ *is the unique maximal element in* $\mathrm{Tab}(\boldsymbol{\lambda})$ *and* $\mathrm{Std}(\boldsymbol{\lambda})$ *with respect to* $\prec_\theta$ *and* $\triangleleft_\theta$.

Proof. The nodes of $\boldsymbol{\lambda}$ are totally ordered with respect to $\triangleleft_\theta$, and we have

$$\mathfrak{t}^{\boldsymbol{\lambda}}(i) \triangleleft_\theta \mathfrak{t}^{\boldsymbol{\lambda}}(j) \text{ iff } i > j.$$

Let ω be the one-column partition $\omega := (1^n)$. The nodes of ω are also totally ordered, with respect to the usual dominance order $\triangleleft$, and hence there is a unique order preserving bijection

$$\Phi_\theta : \mathrm{Tab}(\boldsymbol{\lambda}) \to \mathrm{Tab}(\omega). \tag{2.0.12}$$

For example, for $\theta = \theta_0$ and $\boldsymbol{\lambda} = ((1^5), (1^2), (1^6))$ we have that $\omega = (1^{13})$ and so

$$\Phi_\theta : \left(\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 3 \\ \hline 7 \\ \hline 11 \\ \hline \end{array}\, , \begin{array}{|c|}\hline 4 \\ \hline 6 \\ \hline \end{array}\, , \begin{array}{|c|}\hline 5 \\ \hline 9 \\ \hline 10 \\ \hline 13 \\ \hline 8 \\ \hline 12 \\ \hline \end{array} \right) \mapsto \left(\begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline 5 \\ \hline 1 \\ \hline 6 \\ \hline 9 \\ \hline 3 \\ \hline 10 \\ \hline 7 \\ \hline 13 \\ \hline 11 \\ \hline 8 \\ \hline 12 \\ \hline \end{array} \right). \tag{2.0.13}$$

Note that $\Phi_\theta(\mathbf{t}_\theta^{\boldsymbol{\lambda}}) = \mathfrak{t}^\omega$. Let us now prove $a)$ of the Lemma. We have that

$$\Phi_\theta(\mathbf{t}) = \left(\begin{array}{|c|}\hline \mathbf{t}^{-1}(\mathbf{t}_\theta^{\boldsymbol{\lambda}}(1)) \\ \hline \mathbf{t}^{-1}(\mathbf{t}_\theta^{\boldsymbol{\lambda}}(2)) \\ \hline \vdots \\ \hline k+1 \\ \hline \vdots \\ \hline \mathbf{t}^{-1}(\mathbf{t}_\theta^{\boldsymbol{\lambda}}(j)) \\ \hline \vdots \\ \hline k \\ \hline \vdots \\ \hline \mathbf{t}^{-1}(\mathbf{t}_\theta^{\boldsymbol{\lambda}}(n)) \\ \hline \end{array} \right), \qquad \Phi_\theta(\mathfrak{s}) = \left(\begin{array}{|c|}\hline \mathbf{t}^{-1}(\mathbf{t}_\theta^{\boldsymbol{\lambda}}(1)) \\ \hline \mathbf{t}^{-1}(\mathbf{t}_\theta^{\boldsymbol{\lambda}}(2)) \\ \hline \vdots \\ \hline k \\ \hline \vdots \\ \hline \mathbf{t}^{-1}(\mathbf{t}_\theta^{\boldsymbol{\lambda}}(j)) \\ \hline \vdots \\ \hline k+1 \\ \hline \vdots \\ \hline \mathbf{t}^{-1}(\mathbf{t}_\theta^{\boldsymbol{\lambda}}(n)) \\ \hline \end{array} \right) \tag{2.0.14}$$

and so we have

$$\Phi_\theta(\text{shape}(\mathfrak{s}|_j)) = \Phi_\theta(\text{shape}(\mathbf{t}|_j)) \tag{2.0.15}$$

for all $j \neq k$ and

$$\Phi_\theta(\text{shape}(\mathfrak{s}|_k)) \rhd \Phi_\theta(\text{shape}(\mathbf{t}|_k)) \tag{2.0.16}$$

and so $a)$ follows. In order to prove $b)$ of the Lemma, we get from $a)$ that for any $\boldsymbol{\lambda}$-tableau $\mathbf{t} \neq \mathbf{t}_\theta^{\boldsymbol{\lambda}}$ there is a sequence of simple reflections $s_{i_1}, \dots, s_{i_N}$ such that

$$\mathbf{t} \lhd_\theta \mathbf{t}s_{i_1} \lhd_\theta \mathbf{t}s_{i_1}s_{i_2} \lhd_\theta \dots \lhd_\theta \mathbf{t}s_{i_1}s_{i_2}\cdots s_{i_N} = \mathbf{t}_\theta^{\boldsymbol{\lambda}}, \tag{2.0.17}$$

that is $\mathbf{t} \prec_\theta \mathbf{t}_\theta^{\boldsymbol{\lambda}}$. Since this holds for any $\mathbf{t} \neq \mathbf{t}_\theta^{\boldsymbol{\lambda}}$ we deduce that $\mathbf{t}_\theta^{\boldsymbol{\lambda}}$ is the unique maximal tableau in $\text{Tab}(\boldsymbol{\lambda})$ with respect to both $\prec_\theta$ and $\lhd_\theta$. In order to show that $\mathbf{t}_\theta^{\boldsymbol{\lambda}}$ is also the unique maximal tableau in $\text{Std}(\boldsymbol{\lambda})$ we use that if $\mathbf{t} \in \text{Std}(\boldsymbol{\lambda})$ then each term of the chain (2.0.17) also belongs to $\text{Std}(\boldsymbol{\lambda})$. The Lemma is proved. □

We observe that if $\boldsymbol{\lambda}$ is not a one-column multipartition then there is in general not a unique maximal element in $\text{Std}(\boldsymbol{\lambda})$ with respect to $\prec_0$ or $\lhd_0$. Consider for example $\boldsymbol{\lambda} = ((1), (2))$ with its two standard $\boldsymbol{\lambda}$-tableaux

$$\mathbf{t}^{\boldsymbol{\lambda}} = \left(\begin{array}{|c|}\hline 1 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array} \right), \qquad \mathfrak{s} = \left(\begin{array}{|c|}\hline 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \right). \tag{2.0.18}$$

These are both maximal in $\text{Std}(\boldsymbol{\lambda})$ with respect to $\prec_0$ and $\lhd_0$. This observation is the main reason why the methods of our paper do not generalize in a straightforward way to general multipartitions.

Let $l(\cdot)$ be the length function on $\mathfrak{S}_n$, viewed as a Coxeter group, and let $<$ be the Bruhat order on $\mathfrak{S}_n$ with the convention that the identity element $1 \in \mathfrak{S}_n$ is the largest element. Let λ be a usual partition. For $\mathfrak{t} \in \text{Tab}(\lambda)$ we define $d(\mathfrak{t}) \in \mathfrak{S}_n$ by the condition $\mathfrak{t}^\lambda d(\mathfrak{t}) = \mathfrak{t}$. Since the action of $\mathfrak{S}_n$ is transitive and faithful we have that $d(\mathfrak{t})$ is well defined and unique. For $\boldsymbol{\lambda}$ a one-column multipartition and $\mathbf{t} \in \text{Tab}(\boldsymbol{\lambda})$ we define $d(\mathbf{t})$ in a similar way, using $\mathbf{t}_\theta^{\boldsymbol{\lambda}}$. Our next aim is to show a compatibility between the Bruhat order on $\mathfrak{S}_n$ and the order $\lhd_\theta$ on $\text{Tab}(\boldsymbol{\lambda})$. In the case of the usual dominance order $\lhd$ on $\text{Tab}(\lambda)$ this result was proved originally by Ehresmann. In fact we shall deduce our version of the Theorem from the original Ehresmann Theorem. Let us recall it.

Theorem 2.0.3. *Suppose that λ is a partition of n and that $\mathfrak{s}, \mathfrak{t} \in \mathrm{Tab}(\lambda)$ are row standard. Then we have that $d(\mathfrak{s}) < d(\mathfrak{t})$ if and only if $\mathfrak{s} \lhd \mathfrak{t}$.*

Here is our generalization of this Theorem.

Theorem 2.0.4. *Let $\boldsymbol{\lambda}$ be a one-column multipartition of n and suppose that $\mathbf{t}$ and $\mathbf{s}$ are $\boldsymbol{\lambda}$-tableaux. Then $d(\mathbf{s}) < d(\mathbf{t})$ if and only if $\mathbf{s} \lhd_\theta \mathbf{t}$.*

Proof. Again let ω be the one-column partition $\omega = (1^n)$ and let $\Phi_\theta : \mathrm{Tab}(\boldsymbol{\lambda}) \to \mathrm{Tab}(\omega)$ be the order preserving bijection that was introduced in the proof of Lemma 2.0.2. Recall that in general $\Phi(\mathbf{t}_\theta^{\boldsymbol{\lambda}}) = \mathfrak{t}^\omega$. But from this it follows that for any $\mathbf{t} \in \mathrm{Tab}(\boldsymbol{\lambda})$ we have $d(\mathbf{t}) = d(\Phi_\theta(\mathbf{t}))$. On the other hand, we have that $\mathbf{s} \lhd_\theta \mathbf{t}$ if and only if $\Phi(\mathbf{s}) \lhd \Phi(\mathbf{t})$ and so the Theorem follows from the original Ehresmann Theorem, that is Theorem 2.0.3. □

Let $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$. Then we conclude from the Theorem that the order relations $\lhd_\theta$ on $\mathrm{Tab}(\boldsymbol{\lambda})$ are all isomorphic. However, the restrictions of the order relations $\lhd_\theta$ to the relevant subsets $\mathrm{Std}(\boldsymbol{\lambda})$ are not isomorphic.

In general $\unlhd_\theta$ is not a total order on the set of tableaux, only a partial order. On the other hand, on the set of tableaux of one-column multipartitions of n there is related stronger order $<_\theta$ which is a total order. It is the lexicographical order, defined via

$$\mathbf{t} <_\theta \mathbf{s} \text{ if there is } 1 \le k \le n \text{ such that } \mathbf{t}|_j = \mathbf{s}|_j \text{ for } j < k \text{ but } \mathbf{t}|_k \lhd_\theta \mathbf{s}|_k \,. \tag{2.0.19}$$

It induces a total order on one-column multipartitions of n via

$$\boldsymbol{\lambda} <_\theta \boldsymbol{\mu} \text{ iff } \mathbf{t}_\theta^{\boldsymbol{\lambda}} <_\theta \mathbf{t}_\theta^{\boldsymbol{\mu}}. \tag{2.0.20}$$

There is an extension of $<_\theta$ to the set of all one-column multipartitions that shall be of importance to us. It is given as follows. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be one-column multipartitions of m and n and assume that $m < n$. Then we define

$$\boldsymbol{\lambda} <_\theta \boldsymbol{\mu} \text{ iff } \mathbf{t}_\theta^{\boldsymbol{\lambda}} \le_\theta \mathbf{t}_\theta^{\boldsymbol{\mu}}|_m \,. \tag{2.0.21}$$

For example if γ is an addable node for $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ is defined via $[\boldsymbol{\mu}] := [\boldsymbol{\lambda}] \cup \gamma$ then we always have that $\boldsymbol{\lambda} <_\theta \boldsymbol{\mu}$. In general for $k < n$ we define

$$\boldsymbol{\lambda}|_k = \mathrm{shape}(\mathbf{t}_\theta^{\boldsymbol{\lambda}}|_k). \tag{2.0.22}$$

Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions of m and n and that $m < n$. Then by definition $\boldsymbol{\lambda} \le_\theta \boldsymbol{\mu}|_m$ iff $\boldsymbol{\lambda} <_\theta \boldsymbol{\mu}$.

In the following we shall be mostly interested in the orders related to the zero weighting and when we write $\lhd$, $<$, $\prec$, $\mathbf{t}^{\boldsymbol{\lambda}}$, etc we refer to $\lhd_0$, $<_0$, $\prec_0$, $\mathbf{t}_0^{\boldsymbol{\lambda}}$, etc. We shall also mostly be interested in one-column multipartitions and therefore 'multipartitions' shall in the following refer to 'one-column multipartitions', unless otherwise stated.

Part II

Graded cellular basis and Jucis-Murphy elements for generalized blob algebras

Chapter 3

Generalized blob algebras

In this chapter we define the family of algebras that we are interested in. Let $\mathbb{F}$ be a field of characteristic p, where p is either a prime or zero, and suppose that $q \in \mathbb{F} \setminus \{1\}$ is a primitive e'th root of unity. (Thus if $p > 0$ we have $\gcd(e,p) = 1$). Let $I_e := \mathbb{Z}/e\mathbb{Z}$. Fix a positive integer l. The elements of $\boldsymbol{i} = (i_1, \ldots, i_n)$ of I_e^n are called *residue sequences modulo* e, or *simply residue sequences*. For $\boldsymbol{i} \in (i_1, \ldots, i_n) \in I_e^n$ and $j \in I_e$, we define the *concatenation* $\boldsymbol{i}j \in I_e^{n+1}$ via $\boldsymbol{i}j := (i_1, \ldots, i_n, j)$. The symmetric group $\mathfrak{S}_n$ acts on the left on I_e^n via permutation of the coordinates I_e^n, that is $s_k \cdot \boldsymbol{i} := (i_1, \ldots, i_{k+1}, i_k, \ldots, i_n)$.

Let $\hat{\kappa} = (\hat{\kappa}_1, \ldots, \hat{\kappa}_l) \in \mathbb{Z}^l$ where l is as before. Such a $\hat{\kappa}$ is denoted a *multicharge*. We let $\kappa_i \in I_e$ be the image of $\hat{\kappa}_i$ under the natural projection and define $\kappa := (\kappa_1, \ldots, \kappa_l) \in I_e^n$. We shall throughout choose a representative for each κ_i, also denoted by κ_i, between 0 and $e-1$.

Definition 3.0.1. *We say that* $\hat{\kappa}$ *is* strongly adjacency-free *if it satisfies*

i) $\hat{\kappa}_{i+1} - \hat{\kappa}_i \geq n$

ii) $\kappa_i - \kappa_j \neq 0, \pm 1 \mod e$ *for all* $i \neq j$

iii) $\kappa_1 \neq \kappa_l + 2 \mod e$

iv) $\kappa_1 < \kappa_2 < \ldots < \kappa_l$.

We shall in the following always assume that $\hat{\kappa}$ is strongly adjacency-free; in particular the inequality $e > 2l$ will always hold.

Our notion of a strongly adjacency-free multicharge is a generalization of the notion of an *adjacency-free multicharge*, which was introduced in [23] although already implicitly present in [28] and [38]. The difference between the two notions are the conditions *iii*) and *iv*) which are omitted in [23]. These extra conditions will be useful later on for our analysis of Garnir tableaux.

We can now define our main object of study.

Definition 3.0.2. *Given integers* $e, l, n > 1$ *and a strongly adjacency-free multicharge* $\hat{\kappa}$ *the generalized blob algebra* $\mathcal{B}_{l,n}^{\mathbb{F}}(\kappa) = \mathbb{B}_n$ *of level* l *on* n *strings is the unital, associative* $\mathbb{F}$*-algebra on generators*

$$\{\psi_1, \ldots, \psi_{n-1}\} \cup \{y_1, \ldots, y_n\} \cup \{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I_e^n\}$$

subject to the following relations

$$e(\boldsymbol{i})e(\boldsymbol{j}) = \delta_{\boldsymbol{i},\boldsymbol{j}} e(\boldsymbol{i}) \tag{3.0.1}$$

$$e(\boldsymbol{i}) = 0 \text{ if } i_1 \notin \{\kappa_1, \ldots, \kappa_l\} \tag{3.0.2}$$

$$e(\boldsymbol{i}) = 0 \text{ if } i_1 \in \{\kappa_1, \ldots, \kappa_l\} \text{ and } i_2 = i_1 + 1 \tag{3.0.3}$$

$$y_1 e(\boldsymbol{i}) = 0 \text{ if } i_1 \in \{\kappa_1, \ldots, \kappa_l\} \tag{3.0.4}$$

$$\sum_{\boldsymbol{i} \in I_e^n} e(\boldsymbol{i}) = 1 \tag{3.0.5}$$

$$y_r e(\boldsymbol{i}) = e(\boldsymbol{i}) y_r \tag{3.0.6}$$

$$\psi_r e(\boldsymbol{i}) = e(s_k \cdot \boldsymbol{i}) \psi_r \tag{3.0.7}$$

$$y_r y_s = y_s y_r \tag{3.0.8}$$

$$\psi_r y_s = y_s \psi_r \quad \textit{if} \quad s \neq r, r+1 \tag{3.0.9}$$

$$\psi_r \psi_s = \psi_s \psi_r \quad \textit{if} \quad |s-r| > 1 \tag{3.0.10}$$

$$\psi_r y_{r+1} e(\boldsymbol{i}) = (y_r \psi_r - \delta_{i_r, i_{r+1}}) e(\boldsymbol{i}) \tag{3.0.11}$$

$$y_{r+1} \psi_r e(\boldsymbol{i}) = (\psi_r y_r - \delta_{i_r, i_{r+1}}) e(\boldsymbol{i}) \tag{3.0.12}$$

$$\psi_r^2 e(\boldsymbol{i}) = \begin{cases} 0 & \textit{if} \quad i_r = i_{r+1} \\ e(\boldsymbol{i}) & \textit{if} \quad i_r \neq i_{r+1}, i_{r+1} \pm 1 \\ (y_{r+1} - y_r) e(\boldsymbol{i}) & \textit{if} \quad i_{r+1} = i_r + 1 \\ (y_r - y_{r+1}) e(\boldsymbol{i}) & \textit{if} \quad i_{r+1} = i_r - 1 \end{cases} \tag{3.0.13}$$

$$\psi_r \psi_{r+1} \psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} - 1) e(\boldsymbol{i}) & \textit{if} \quad i_{r+2} = i_r = i_{r+1} - 1 \\ (\psi_{r+1} \psi_r \psi_{r+1} + 1) e(\boldsymbol{i}) & \textit{if} \quad i_{r+2} = i_r = i_{r+1} + 1 \\ (\psi_{r+1} \psi_r \psi_{r+1}) e(\boldsymbol{i}) & \textit{otherwise.} \end{cases} \tag{3.0.14}$$

The above definition of $\mathbb{B}_n$ is the one used in [3] and [23], but it is not the original definition of the generalized blob algebra as presented in [28]. We will prove that the two definitions do coincide. The case when $l = 2$ is the original blob algebra, we will use this particular case in the last part of this thesis.

Let us take the opportunity to give the precise definition of the KLR-algebra, already mentioned above. It was introduced independently in [20] and [39].

Definition 3.0.3. *The cyclotomic KLR-algebra of type $A^{(1)}_{e-1}$, or simply the KLR-algebra, is the $\mathbb{F}$-algebra $\mathcal{R}_n$ on generators*

$$\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I_e^n\}$$

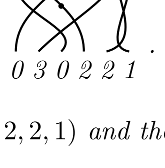
subject to the same relations as for the blob algebra $\mathbb{B}_n$ except for relation (3.0.3) which is omitted.

Let $\pi : \mathcal{R}_n \to \mathbb{B}_n$ be the projection map from the KLR-algebra to $\mathbb{B}_n$. Then, for simplicity of notation, we shall in general write x for $\pi(x)$ when $x \in \mathcal{R}_n$.

It follows from the relations that there is an antiinvolution $*$ of $\mathbb{B}_n$, and of $\mathcal{R}_n$, that fixes the generators.

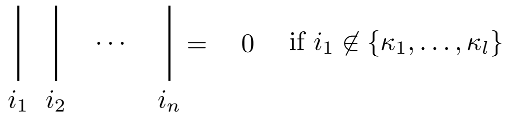
There is a diagrammatical way to view this definition which is of importance for our work. It was introduced by Khovanov and Lauda in [20]. A *Khovanov-Lauda diagram D*, or simply a *KL-diagram*, on n strings consists of n points on each of two parallel edges (the top edge and the bottom edge) and n strings connecting the points of the top edge with the points of the bottom edge. Strings may intersect, but triple intersections are not allowed. Each string may be decorated with a finite number of dots, but dots cannot be located on the intersection of two strings. Finally, each string is labelled with an element of I_e. This defines two residue sequences $t(D), b(D) \in I_e^n$ associated with the diagram D obtained by reading the residues of the extreme points from left to right. For the details concerning this definition, the reader should consult [20].

Example 3.0.4. *Let $e = 4$ and $n = 6$. Let D be the following KL-diagram:*

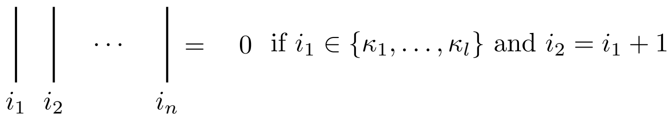


In this case the bottom sequence is $b(D) = (0, 3, 0, 2, 2, 1)$ and the top sequence is $t(D) = (2, 1, 0, 0, 2, 3)$.

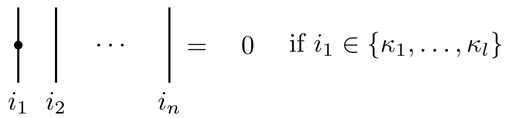
We can now define the diagrammatic algebra $\mathcal{B}^{\mathbb{F}}_{l,n}(\kappa)^{diag} = \mathbb{B}^{diag}_n$. As an $\mathbb{F}$-vector space it consists of the $\mathbb{F}$-linear combinations of KL-diagrams on n strings modulo planar isotopy and modulo the following relations:



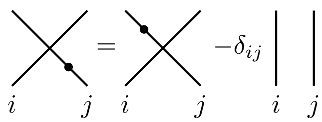
(3.0.15)



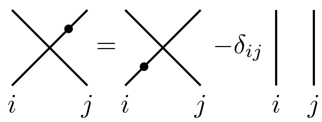
(3.0.16)



(3.0.17)

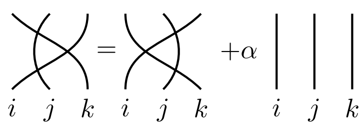


(3.0.18)



(3.0.19)

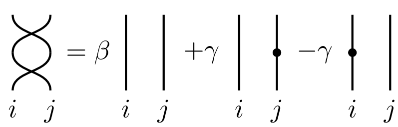
where δ_{ij} is the Kronecker delta. Moreover



(3.0.20)

where

$$\alpha = \begin{cases} -1 & \text{if} \quad i = k = j - 1 \\ 1 & \text{if} \quad i = k = j + 1 \\ 0 & \text{otherwise} \end{cases}$$



(3.0.21)

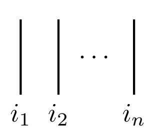
where

$$\beta = \begin{cases} 1 & \text{if} \quad |i - j| > 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\gamma = \begin{cases} 1 & \text{if} \quad j = i + 1 \\ -1 & \text{if} \quad j = i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

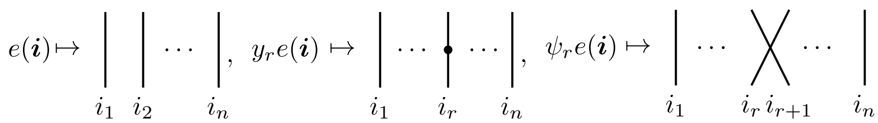
The identity element 1 of $\mathbb{B}^{diag}_n$ is the sum over all diagrams



such that $\boldsymbol{i} := (i_1, i_2, \ldots, i_n)$ belongs to I_e^n.

The multiplication DD' between two diagrams D and D' in $\mathbb{B}_n^{diag}$ is defined by vertical concatenation with D above D' if $b(D) = t(D')$. If $b(D) \neq t(D')$ the product is defined to be zero. We extend the product to all pairs of elements in $\mathbb{B}_n^{diag}$ by linearity.

The $\mathbb{F}$-linear map from $\mathbb{B}_n$ to $\mathbb{B}_n^{diag}$ given by

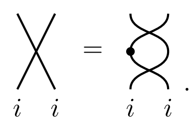


(3.0.22)

defines an isomorphism between $\mathbb{B}_n$ and $\mathbb{B}_n^{diag}$. In view of this, we shall write $\mathbb{B}_n^{diag} = \mathbb{B}_n$.

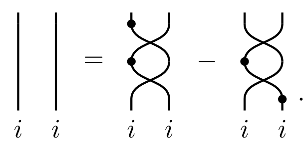
We next show some useful relations that can be derived directly from the definitions.

Lemma 3.0.5. *In $\mathbb{B}_n$ we have:*



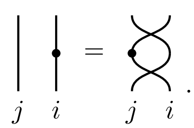
Proof. This is an immediate consequence of relations (3.0.18), (3.0.19) and (3.0.21). □

Lemma 3.0.6. *In $\mathbb{B}_n$ we have:*



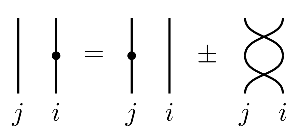
Proof. This is a consequence of relations (3.0.18), (3.0.19) and Lemma 3.0.5. □

Lemma 3.0.7. *If $|i-j| > 1$ then we have*



Proof. This is a direct consequence of the relations (3.0.18), (3.0.19) and (3.0.21). □

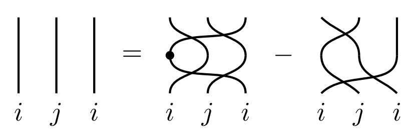
Lemma 3.0.8. *If $|i-j| = 1$ then we have*



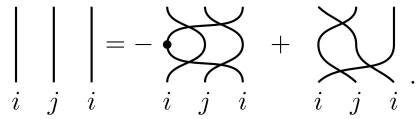
where the positive sign appears when $j = i-1$ and the negative sign when $j = i+1$.

Proof. This is a direct consequence of relation (3.0.21). □

Lemma 3.0.9. *If $j = i+1$ then we have*

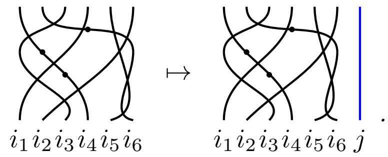


and if $j = i-1$ then we have that



Proof. This is a direct consequence of relation (3.0.20) and Lemma 3.0.5. $\square$

Lemma 3.0.10. *Let $n \geq 2$ and let $\iota_{n+1,j}$ be the concatenation on the right of a diagram in $\mathbb{B}_n$ with a through line of fixed residue j, as indicated in the following figure*



Then $\iota_{n+1,j}$ induces a (non-unital) algebra homomorphism $\iota_{n+1,j} : \mathbb{B}_n \to \mathcal{B}_{n+1}$. It satisfies $\iota_{n+1,j}(0) = 0$.

Proof. Each of the relations (3.0.15) to (3.0.21) for $\mathbb{B}_n$ maps under $\iota_{n+1,j}$ to a relation for $\mathcal{B}_{n+1}$ and so $\iota_{n+1,j}$ is well-defined. The second statement of the Lemma is obvious. $\square$

We shall use the notation $b \cdot j$ or $b\,j$ for $\iota_{n+1,j}(b)$. We remark that it can be shown that $\iota_{n+1,j}$ is an embedding.

Chapter 4

A generating set $\mathcal{C}_n$ for $\mathbb{B}_n$.

We now take the first steps towards the construction of our cellular basis for $\mathbb{B}_n$.

Let $\boldsymbol{\lambda}$ be a multipartition and let $\gamma = (r, c, m)$ be a node of $[\boldsymbol{\lambda}]$. Then we define the residue of γ via

$$\operatorname{res}(\gamma) := \kappa_m + c - r \in I_e. \tag{4.0.1}$$

Recall that a multipartition $\boldsymbol{\lambda}$ is assumed to be a one-column multipartition, unless otherwise stated. The nodes γ of a multipartition $\boldsymbol{\lambda}$ are of the form $\gamma = (r, 1, m)$ with residue $\operatorname{res}(\gamma) = \kappa_m + 1 - r$.

Any $\boldsymbol{\lambda}$-tableau $\mathfrak{t}$ gives rise to a residue sequence $\boldsymbol{i}^{\mathfrak{t}} \in I_e^n$ defined via

$$\boldsymbol{i}^{\mathfrak{t}} := (i_1, \dots, i_n) \in I_e^n \ \text{ where } i_j = \operatorname{res}(\mathfrak{t}(j)). \tag{4.0.2}$$

In the next couple of Lemmas and Corollaries we aim at showing that only the idempotents $e(\boldsymbol{i}^{\boldsymbol{\lambda}})$, with $\boldsymbol{\lambda}$ running over multipartitions, are needed in order to generate $\mathbb{B}_n$. Our proof for this is not straightforward and relies on several induction loops, all related to $\boldsymbol{\lambda}$. In essence our proofs are a chain of applications of the Lemmas 3.0.5 to 3.0.10 and could therefore have been formulated completely diagrammatically, in principle, but we choose to encode these Lemmas in an symbolic notation that we explain shortly. This symbolic notation has the advantage of enabling us to keep track of the induction parameter $\boldsymbol{\lambda}$. Our approach is therefore different from the approaches of [45], [3] that rely on manipulations of the diagrams themselves. Our proofs are rather comparable to the proofs of [21] and, in view of this, maybe surprisingly short, after all.

Let $\boldsymbol{\mu}_n^{max} = \boldsymbol{\mu}^{max}$ be the multipartition introduced in (2.0.5), which is the unique maximal multipartition of n with respect to $\lhd$, and let us denote by $\mathfrak{t}_n^{max} = \mathfrak{t}^{max}$ the unique maximal $\boldsymbol{\mu}_n^{max}$-tableau, as in Lemma 2.0.2. We denote by $\boldsymbol{i}_n^{max} = \boldsymbol{i}^{max} \in I_e^n$ the corresponding residue sequence and by $e(\boldsymbol{i}^{max}) \in \mathbb{B}_n$ the associated idempotent. We denote by $[\operatorname{res}(\mathfrak{t}^{max})]$ the corresponding residue diagram, obtained by writing $\operatorname{res}(\mathfrak{t}^{max}(k))$ in the node $\mathfrak{t}^{max}(k)$ of $[\boldsymbol{\lambda}]$. For example, for $n = 22, e = 10$ and $\kappa = (0, 2, 4, 7)$ we have the following residue diagram

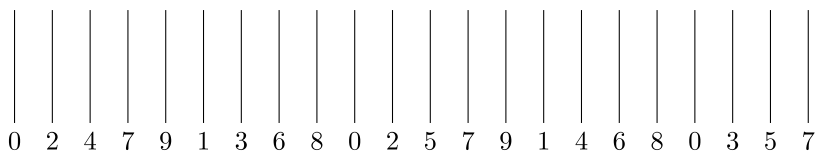
$$[\operatorname{res}(\mathfrak{t}^{max})] = \left(\begin{array}{|c|}\hline 0\\\hline 9\\\hline 8\\\hline 7\\\hline 6\\\hline 5\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline 0\\\hline 9\\\hline 8\\\hline 7\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline 3\\\hline 2\\\hline 1\\\hline 0\\\hline\end{array}, \begin{array}{|c|}\hline 7\\\hline 6\\\hline 5\\\hline 4\\\hline 3\\\hline\end{array} \right) \tag{4.0.3}$$

which gives rise to the following residue sequence

$$\boldsymbol{i}^{max} = (0, 2, 4, 7, 9, 1, 3, 6, 8, 0, 2, 5, 7, 9, 1, 4, 6, 8, 0, 3, 5, 7) \in I_{10}^{22} \tag{4.0.4}$$

and corresponding idempotent

$$e(\boldsymbol{i}^{max}) =$$



(4.0.5)

We now introduce our symbolic notation. Firstly we represent an idempotent like (4.0.5) in the following way

$$e(\boldsymbol{i}^{max}) := (0,2,4,7 \mid 9,1,3,6 \mid 8,0,2{,}5 \mid 7,1,9,4 \mid 6,8,0,3 \mid 5,7) \tag{4.0.6}$$

where the separation lines $\mid$ indicate jumps from a row to the next in $\boldsymbol{\mu}^{max}$ (although the separation lines are not always meant to have an exact meaning, but rather to be a help for the eye). Secondly we introduce the following dot notation for expressions like $y_{19}e(\boldsymbol{i}^{max})$

$$y_{19}e(\boldsymbol{i}^{max}) := (0,2,4,7 \mid 9,1,3,6 \mid 8,0,2{,}5 \mid 7,1,9,4 \mid 6,8,\overset{\bullet}{0},3 \mid 5,7). \tag{4.0.7}$$

For any $a \in \mathbb{B}_n$ we denote by $\langle a \rangle$ the two-sided ideal in $\mathbb{B}_n$ generated by a. When $a, b \in \mathbb{B}_n$ and $b \in \langle a \rangle$ we say that b *factorizes over* a.

We write $\boldsymbol{i} \overset{k}{\sim} \boldsymbol{j}$ if $\boldsymbol{i} = s_k \boldsymbol{j}$ where $i_k \neq i_{k+1} \pm 1$ and we let $\sim$ be the equivalence relation on I_e^l generated by all the $\overset{k}{\sim}$'s. If $\boldsymbol{i} \overset{k}{\sim} \boldsymbol{j}$ we say that $\boldsymbol{i}$ is obtained from $\boldsymbol{j}$ by *freely moving* the string of residue i_{k+1} past the string of residue i_k. We shall often use this concept as follows. Suppose that $\boldsymbol{i} \sim \boldsymbol{j}$. Then we have both $e(\boldsymbol{i}) \in \langle e(\boldsymbol{j}) \rangle$ and $e(\boldsymbol{j}) \in \langle e(\boldsymbol{i}) \rangle$, that is $e(\boldsymbol{i})$ factorizes over $e(\boldsymbol{j})$ and vice versa. Indeed, if $\boldsymbol{i} \overset{k}{\sim} \boldsymbol{j}$ then by relation (3.0.21) we have that $e(\boldsymbol{i}) = \psi_k e(\boldsymbol{j}) \psi_k$ as well as $e(\boldsymbol{j}) = \psi_k e(\boldsymbol{i}) \psi_k$, from which the general case follows. In particular, we have in this situation that $e(\boldsymbol{i}) = 0$ if and only if $e(\boldsymbol{j}) = 0$. The same way one sees that if $\boldsymbol{i} \sim \boldsymbol{j}$ where $\boldsymbol{i} = w\boldsymbol{j}$ for $w \in \mathfrak{S}_n$, then for all r we have $y_r e(\boldsymbol{i}) \in \langle y_s e(\boldsymbol{j}) \rangle$ and $y_s e(\boldsymbol{j}) \in \langle y_r e(\boldsymbol{i}) \rangle$ where $s = w \cdot r$.

If $\boldsymbol{i} \sim \boldsymbol{j}$ we shall also write $e(\boldsymbol{i}) \sim e(\boldsymbol{j})$ and $y_r e(\boldsymbol{i}) \sim y_s e(\boldsymbol{j})$ where r and s are related as before. When using the symbolic notation as in (4.0.6) we associate with $\sim$ a similar meaning.

We aim at proving that $y_k e(\boldsymbol{i}^{max}) = 0$ for all $k = 1, \ldots, n$. This is straightforward for small k, but gets more complicated when k grows. Let us illustrate the argument on a few small values of k, using the above example (4.0.5).

For $k = 1$ we must show that

$$y_1 e(\boldsymbol{i}^{max}) = (\overset{\bullet}{0},2,4,7 \mid 9,1,3,6 \mid 8,0,2{,}5 \mid 7,1,9,4 \mid 6,8,0,3 \mid 5,7) \tag{4.0.8}$$

is equal to zero; this is however an instance of relation (3.0.17). For $k = 2$ we must show that

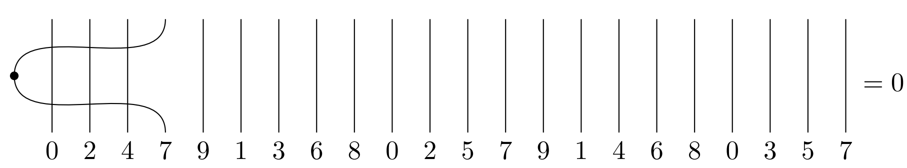
$$(0,\overset{\bullet}{2},4,7 \mid 9,1,3,6 \mid 8,0,2{,}5 \mid 7,1,9,4 \mid 6,8,0,3 \mid 5,7) = 0. \tag{4.0.9}$$

Here we may move 2 freely past 0 and so

$$(0,\overset{\bullet}{2},4,7 \mid \ldots \mid 6,8,0,3 \mid 5,7) \sim (\overset{\bullet}{2},0,4,7 \mid \ldots \mid 6,8,0,3 \mid 5,7) = 0 \tag{4.0.10}$$

where the last equality follows from (3.0.17), once again. The same kind of argument shows that $y_3 e(\boldsymbol{i}^{max}) = y_4 e(\boldsymbol{i}^{max}) = 0$. For these small values of k, one can formulate these arguments diagrammatically. Here is the case $k = 4$:

$$y_4 e(\boldsymbol{i}^{max}) = \psi_3\psi_2\psi_1\big(y_1 e(s_1 s_2 s_3 \boldsymbol{i}^{max})\big)\psi_1\psi_2\psi_3 =$$



(4.0.11)

where the last equality follows from the fact that $y_1 e(s_1 s_2 s_3 \boldsymbol{i}^{max})$, that is the middle part of the diagram (4.0.11), is equal to zero.

Let us now go on showing that $y_k e(\boldsymbol{i}^{max}) = 0$ for $k = 5, 6, 7, 8$ corresponding to the second row of the residue diagram $[\mathrm{res}(\mathbf{t}^{max})]$. For $k = 5$ we must show that

$$y_5 e(\boldsymbol{i}^{max}) = (0,2,4,7 \mid \overset{\bullet}{9},1,3,6 \mid 8,0,2,5 \mid 7,1,9,4 \mid 6,8,0,3 \mid 5,7) = 0. \tag{4.0.12}$$

But $\overset{\bullet}{9}$ moves freely past $7, 4, 2$ and so we have

$$(0,2,4,7 \mid \overset{\bullet}{9},1,3,6 \mid \ldots \mid \ldots \mid 5,7) \sim (0,\overset{\bullet}{9},2,4,7 \mid 1,3,6 \mid \ldots \mid \ldots \mid 5,7) \tag{4.0.13}$$

which we must show to be zero. But using Lemma 3.0.8 we have that

$$(0,\overset{\bullet}{9},2,4,7 \mid \dots \mid 5,7) \in \left\langle (\overset{\bullet}{0},9,2,4,7 \mid \dots \mid 5,7)\,,(9,0,2,4,7 \mid \dots \mid 5,7) \right\rangle \tag{4.0.14}$$

where $\langle\cdot\rangle$ once again denotes ideal generation. Here the first ideal generator is zero by relation (3.0.17) whereas the second ideal generator is zero by relation (3.0.16). The other cases $k = 6,7,8$ are treated essentially the same way.

Let us now consider the cases where k corresponds to the third row of $[\mathrm{res}(\mathbf{t}^{max})]$, that is we show that $y_k e(\boldsymbol{i}^{max}) = 0$ for $k = 9, 10, 11, 12$. For $k = 9$ we must show that

$$y_9 e(\boldsymbol{i}^{max}) = (0,2,4,7 \mid 9,1,3,6 \mid \overset{\bullet}{8},0,2,5 \mid 7,1,9,4 \mid 6,8,0,3 \mid 5,7) = 0. \tag{4.0.15}$$

But $\overset{\bullet}{8}$ moves freely past $6,3$ and 1 and so we have

$$y_9 e(\boldsymbol{i}^{max}) \sim (0,2,4,7 \mid 9,\overset{\bullet}{8},1,3,6 \mid 0,2,5 \mid 7,1,9,4 \mid 6,8,0,3 \mid 5,7) \tag{4.0.16}$$

which we must show to be zero. But by Lemma 3.0.8 we have that

$$(0,2,4,7 \mid 9,\overset{\bullet}{8},1,3,6 \mid \dots \mid 5,7) \in \left\langle (0,2,4,7 \mid \overset{\bullet}{9},8,1,3,6 \mid \dots \mid 5,7)\,,(0,2,4,7 \mid 8,9,1,3,6 \mid \dots \mid 5,7) \right\rangle. \tag{4.0.17}$$

Here the first generator is zero by (4.0.12) and for the second generator we have that

$$(0,2,4,7 \mid 8,9,1,3,6 \mid \dots \mid 5,7) \sim (7,8,0,2,4 \mid 9,1,3,6 \mid \dots \mid 5,7) \tag{4.0.18}$$

which is zero by relation (3.0.16). The other cases $k = 10, 11, 12$ are treated similarly. For k corresponding to the next block, the inductive argument becomes more complicated and we prefer to present it as part of the proof of the general statement $y_k e(\boldsymbol{i}^{max}) = 0$.

Lemma 4.0.1. *In $\mathbb{B}_n$ we have for all $1 \le k \le n$ the following relations*

$$y_k e(\boldsymbol{i}^{max}) = 0 = e(\boldsymbol{i}^{max}) y_k. \tag{4.0.19}$$

Proof. By (3.0.6) we know that y_k and $e(\boldsymbol{i}^{max})$ commute and so we only need to prove the first relation.

We prove it by induction on n. For $n = 1$ it is trivial. We next prove it for a fixed n, assuming that it holds for $n_1 < n$. For this fixed n, we use induction on k.

The basis step for this induction is $1 \le k \le l$, which is however easily handled using the same arguments as in the above example (4.0.6) and the case $l+1 \le k \le 2l$ where k belongs to the second row of $\boldsymbol{\mu}^{max}$ can also be treated this way.

Let us now consider the case $(m-1)l + 1 \le k \le ml$ where $m \ge 3$. Since $(m-1)l + 1 \le k \le ml$ we have that k belongs to the m'th row of $[\boldsymbol{\mu}^{max}]$. Suppose that $\kappa_1^j, \dots, \kappa_l^j$ are the residues of the j'th row of $[\mathrm{res}(\mathbf{t}^{max})]$ and that the residue of $\mathbf{t}^{max}(k)$ is α. Then we must show that

$$y_k e(\boldsymbol{i}^{max}) = (\dots \mid \kappa_1^{m-1}, \dots, \alpha+1, \dots, \kappa_l^{m-1} \mid \kappa_1^m, \dots, \overset{\bullet}{\alpha}, \dots, \kappa_l^m \mid \dots) = 0. \tag{4.0.20}$$

Here $\alpha + 1$ is the residue of the node on top of $\mathbf{t}^{max}(k)$ and so we can move $\overset{\bullet}{A}$ freely over the residues between them. Hence (4.0.21) is equivalent to

$$(\dots \mid \kappa_1^{m-1}, \dots, \alpha+1, \overset{\bullet}{\alpha}, \dots, \kappa_l^{m-1} \mid \kappa_1^m, \dots, \widehat{\alpha}, \dots, \kappa_l^m \mid \dots) = 0 \tag{4.0.21}$$

which by Lemma 3.0.8 is equivalent to the ideal

$$\begin{gathered} \big\langle (\dots \mid \kappa_1^{m-1}, \dots, \overset{\bullet}{(\alpha+1)}, \alpha, \dots, \kappa_l^{m-1} \mid \kappa_1^m, \dots, \widehat{\alpha}, \dots, \kappa_l^m \mid \dots), \\ (\dots \mid \kappa_1^{m-1}, \dots, \alpha, \alpha+1, \dots, \kappa_l^{m-1} \mid \kappa_1^m, \dots, \widehat{\alpha}, \dots, \kappa_l^m \mid \dots) \big\rangle \end{gathered} \tag{4.0.22}$$

being zero. Here the first ideal generator is zero by induction since

$$(\dots \mid \kappa_1^{m-1}, \dots, \overset{\bullet}{(\alpha+1)}) = 0 \tag{4.0.23}$$

by the inductive hypothesis on n: this is the residue sequence of a $\mathbf{t}_{n_1}^{max}$ where $n_1 < n$. Here we also used that concatenation maps zero to zero by Lemma 3.0.10. We therefore focus on the second ideal generator of (4.0.22), that is

$$(\dots \mid \kappa_1^{m-1}, \dots, \alpha, \alpha+1, \dots, \kappa_l^{m-1} \mid \kappa_1^m, \dots, \widehat{\alpha}, \dots, \kappa_l^m \mid \dots) \tag{4.0.24}$$

which is obtained from the original sequence $e(\boldsymbol{i}^{max})$ by moving α past $\alpha+1$. We have that $y_k e(\boldsymbol{i}^{max}) = 0$ if and only if this sequence (4.0.24) is zero. In (4.0.24) we now move α further to the left until it hits its first obstacle which will be $\alpha - 1$: this is so due the combinatorial structure of $[\mathbf{t}^{max}]$ and strong adjacency-freeness of $\hat{\kappa}$. On top of the node of residue α there is a node of residue $\alpha - 1$ that can be freely moved to the right until it stands next to α. Doing this we find that (4.0.24) is zero if

$$(\dots \alpha(\alpha-1)\alpha \dots \mid \kappa_1^{m-1}, \dots, \widehat{\alpha}, \alpha+1, \dots, \kappa_l^{m-1} \mid \kappa_1^m, \dots, \widehat{\alpha}, \dots, \kappa_l^m \mid \dots) \tag{4.0.25}$$

is zero. We now apply Lemma 3.0.9 to the triple $\alpha(\alpha-1)\alpha$ and get that (4.0.25) is zero if the ideal

$$\begin{aligned} \big\langle (\dots \overset{\bullet}{\alpha}\alpha(\alpha-1) \dots \mid \kappa_1^{m-1}, \dots, \widehat{\alpha}, \alpha+1, \dots, \kappa_l^{m-1} \mid \kappa_1^m, \dots, \widehat{\alpha}, \dots, \kappa_l^m \mid \dots), \\ (\dots (\alpha-1)\alpha\alpha \dots \mid \kappa_1^{m-1}, \dots, \widehat{\alpha}, \alpha+1, \dots, \kappa_l^{m-1} \mid \kappa_1^m, \dots, \widehat{\alpha}, \dots, \kappa_l^m \mid \dots) \big\rangle \end{aligned} \tag{4.0.26}$$

is zero. As before, by induction on n the first generator is here equal to zero and so $y_k e(\boldsymbol{i}^{max}) = 0$ if and only if the second term of (4.0.26) is zero. We now go on the same way, moving $\alpha - 1$ to the left, until it hits a residue $\alpha - 2$ and as before $y_k e(\boldsymbol{i}^{max}) = 0$ if the interchanging of those nodes produces a diagram which is zero. Continuing in this way, the interchanging of nodes will finally take place in the first two rows of $[\boldsymbol{\mu}^{max}]$, where by relations (3.0.15) and (3.0.16) it does produce zero. □

We have the following consequence of the Lemma.

Corollary 4.0.2. *Suppose that $\iota \in I_e$ and that the concatenation $\boldsymbol{i}_n^{max}\iota$ is not of the form $\boldsymbol{i}^{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda}$ any multipartition of $n+1$. Then we have that*

$$e(\boldsymbol{i}_n^{max}\iota) = 0. \tag{4.0.27}$$

Proof. We have that

$$e(\boldsymbol{i}_n^{max}\iota) = (\dots \mid \kappa_1^{m-1}, \dots, \kappa_l^{m-1} \mid \kappa_1^m, \dots, \kappa_l^m \mid \dots \mid \iota). \tag{4.0.28}$$

By the strong adjacency-freeness ι moves here freely to the left until it hits another ι or a pair $\iota(\iota-1)$. In the first case, using Lemma 3.0.6 we replace the appearing $\iota\iota$ by $\overset{\bullet}{\iota}\,\iota$, and get by the Lemma that $e(\boldsymbol{i}_n^{max}\iota) = 0$, as claimed. In the second case, we replace $\iota(\iota-1)\iota$ by a linear combination of $\overset{\bullet}{\iota}\,\iota(\iota-1)$ and $(\iota-1)\iota\iota$. Proceeding as in the Lemma, we finally find that this is zero. □

Let us illustrate the Corollary on the example

$$\left(\begin{array}{|c|}\hline 0 \\ \hline 9 \\ \hline 8 \\ \hline 7 \\ \hline 6 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 0 \\ \hline 9 \\ \hline 8 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline \end{array}, \begin{array}{|c|}\hline 7 \\ \hline 6 \\ \hline 5 \\ \hline 4 \\ \hline 3 \\ \hline \end{array} \right) \tag{4.0.29}$$

already considered in (4.0.3). Here we can use $\iota \neq 4, 6, 9, 2$ in the Corollary. We then conclude from the Corollary that

$$e(0, 2, 4, 7, 9, 1, 3, 6, 8, 0, 2, 5, 7, 9, 1, 4, 6, 8, 0, 3, 5, 7, \iota) = 0$$

for these choices of ι.

We generalize the previous Lemma and Corollary to arbitrary multipartitions in the following way. Recall that $<$ is the total order introduced in (2.0.21).

Lemma 4.0.3. *For $\boldsymbol{\lambda}$ any multipartition of n and for $1 \le k \le n$ we have that*

$$y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = e(\boldsymbol{i}^{\boldsymbol{\lambda}}) y_k = \sum_{\boldsymbol{\mu} > \boldsymbol{\lambda}} D_{\boldsymbol{\mu}} \tag{4.0.30}$$

where the sum runs over multipartitions $\boldsymbol{\mu}$ of n and $D_{\boldsymbol{\mu}}$ factorizes over $e(\boldsymbol{i}^{\boldsymbol{\mu}})$. Suppose moreover that $D_{\boldsymbol{\lambda}}$ is any element of $\mathbb{B}_n$ and that $D_{\boldsymbol{\lambda}}$ factorizes over $e(\boldsymbol{i}^{\boldsymbol{\lambda}})$ and assume $\iota \in I_e$. Then we have that

$$D_{\boldsymbol{\lambda}} \cdot \iota = \sum_{\boldsymbol{\mu} > \boldsymbol{\lambda}} C_{\boldsymbol{\mu}} \tag{4.0.31}$$

where $\boldsymbol{\mu}$ runs over multipartitions of $n+1$ and $C_{\boldsymbol{\mu}}$ factorizes over $e(\boldsymbol{i}^{\boldsymbol{\mu}})$. Furthermore, if $\boldsymbol{i}^{\boldsymbol{\lambda}}\iota$ is not of the form $\boldsymbol{i}^{\boldsymbol{\nu}}$ for any multipartition $\boldsymbol{\nu}$ of $n+1$ then we have that

$$D_{\boldsymbol{\lambda}} \cdot \iota = \sum_{\boldsymbol{\mu}|_n > \boldsymbol{\lambda}} C_{\boldsymbol{\mu}} \tag{4.0.32}$$

where once again the sum runs over multipartitions $\boldsymbol{\mu}$ of $n+1$ and $C_{\boldsymbol{\mu}}$ factorizes over $e(\boldsymbol{i}^{\boldsymbol{\mu}})$.

Proof. We first give an example which might be useful to have in mind while going through the arguments of the actual proof. For $n = 28$, $e = 9$ and $\boldsymbol{\lambda} = ((1^6), (1^4), (1^9), (1^9))$ we have the following residue diagram for $\mathbf{t}^{\boldsymbol{\lambda}}$

$$\left(\begin{array}{|c|}\hline 0\\\hline 8\\\hline 7\\\hline 6\\\hline 5\\\hline 4\\\hline \\ \\ \\ \end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline 0\\\hline 8\\\hline \\ \\ \\ \\ \\ \end{array}, \begin{array}{|c|}\hline 4\\\hline 3\\\hline 2\\\hline 1\\\hline 0\\\hline 8\\\hline 7\\\hline 6\\\hline 5\\\hline \end{array}, \begin{array}{|c|}\hline 6\\\hline 5\\\hline 4\\\hline 3\\\hline 2\\\hline 1\\\hline 0\\\hline 8\\\hline 7\\\hline \end{array} \right). \tag{4.0.33}$$

In this case, in order to prove (4.0.30) we must show for $1 \leq i \leq 27$ that $y_i e(\boldsymbol{i}^{\boldsymbol{\lambda}})$ is a linear combination $\sum_{\boldsymbol{\mu} > \boldsymbol{\lambda}} D_{\boldsymbol{\mu}}$ as indicated and for (4.0.32) we must show that for $\iota \in I_e \setminus \{4, 6\}$ we have that $D_{\boldsymbol{\lambda}} \cdot \iota$ is a linear combination $\sum_{\boldsymbol{\mu}|_n > \boldsymbol{\lambda}} C_{\boldsymbol{\mu}}$ as indicated.

We now prove all statements of the Lemma by induction on n, the basis case $n = 1$ being straightforward. We first prove (4.0.30) by induction on k. For $k < n$ we use the inductive hypothesis on n to write $y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}|_k)$ in the form

$$y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}|_k) = \sum_{\boldsymbol{\mu} > \boldsymbol{\lambda}|_k} D_{\boldsymbol{\mu}} \tag{4.0.34}$$

where the sum runs over multipartitions $\boldsymbol{\mu}$ of k and $D_{\boldsymbol{\mu}} \in \langle e(\boldsymbol{i}^{\boldsymbol{\mu}}) \rangle$. Let $\boldsymbol{i}^{\boldsymbol{\lambda}} = (i_1, i_2, \ldots, i_n)$. We then get $y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}|_k \, i_{k+1} \cdots i_n)$ in the form

$$y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = \sum_{\boldsymbol{\tau} > \boldsymbol{\mu} > \boldsymbol{\lambda}|_k} D_{\boldsymbol{\tau}} \tag{4.0.35}$$

by concatenating each $D_{\boldsymbol{\mu}}$ on the right with $i_{k+1} \cdots i_n$ and using in each step the inductive hypothesis for (4.0.31). Here $\boldsymbol{\mu}$ is as in (4.0.34) whereas $\boldsymbol{\tau}$ runs over multipartitions of n. But $\boldsymbol{\tau} > \boldsymbol{\mu} > \boldsymbol{\lambda}|_k$ implies $\boldsymbol{\tau} > \boldsymbol{\lambda}$ and so (4.0.35) has the form indicated in (4.0.30).

In order to show (4.0.30) for $k = n$ we return to our symbolic notation. We have

$$e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = (\kappa_1^1, \ldots, \kappa_{l_1}^1 \mid \kappa_1^2, \ldots, \kappa_{l_2}^2 \mid \cdots \mid \kappa_1^r, \ldots, \kappa_{l_r}^r) \tag{4.0.36}$$

where $\kappa_1^j, \ldots, \kappa_{l_j}^j$ are the residues of the j'th row of $[\boldsymbol{\lambda}]$. In this notation, in order to show (4.0.30) we must show that

$$y_n e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = (\kappa_1^1, \ldots, \kappa_{l_1}^1 \mid \kappa_1^2, \ldots, \kappa_{l_2}^2 \mid \cdots \mid \kappa_1^r, \ldots, \overset{\bullet}{\alpha}) = \sum_{\boldsymbol{\mu} > \boldsymbol{\lambda}} D_{\boldsymbol{\mu}} \tag{4.0.37}$$

where $\alpha = \kappa_{l_r}^r$.

We now move $\overset{\bullet}{\alpha}$ freely to the left until it meets its first obstacle, which by strong adjacency-freeness is $\alpha + 1$ coming from the node on top of the node of $\overset{\bullet}{\alpha}$. We next use Lemma 3.0.8 to replace our sequence involving $(\alpha+1)\overset{\bullet}{\alpha}$ by a linear combination of sequences involving $(\overset{\bullet}{\alpha+1})\alpha$ and $\alpha(\alpha+1)$. As in the proof of (4.0.35) the first term involving $(\alpha \overset{\bullet}{+} 1)\alpha$ is of the indicated form by induction hypothesis and we must therefore consider the second term $\alpha(\alpha+1)$. We here move α freely to the left until it meets its first obstacle which must be α, $\alpha+1$ or $\alpha-1$. If it is α we use Lemma 3.0.6 to replace $\alpha\alpha$ by $\overset{\bullet}{\alpha}\alpha$ and can once again use the induction hypothesis. If it is $\alpha-1$, the situation gives rise to a triple $\alpha(\alpha-11)A$ where the first α comes from the residue on top of the node of $\alpha-1$. On this triple, we use Lemma 3.0.9 to rewrite $\alpha(\alpha-1)\alpha$ as a linear combination of $\overset{\bullet}{\alpha}\alpha(\alpha-1)$ and $(\alpha-1)\alpha\alpha$. Here the first term is dealt with using the induction hypothesis for (4.0.30), whereas the second term is dealt with using the induction hypothesis for (4.0.31).

We now consider the third case where α meets $\alpha+1$. (In the previous Lemma 4.0.1, this case did not occur). But this case corresponds to a *gap* in the diagram, where α can be positioned giving rise to the diagram $\boldsymbol{\mu}$ of a multipartition that satisfies $\boldsymbol{\mu} > \boldsymbol{\lambda}$. Summing up, this proves the inductive step of (4.0.30). The $\boldsymbol{\mu}$'s that appear in the final expansion (4.0.30) are exactly those that arise from this last case.

Let us now focus on the claims (4.0.31) and (4.0.32). Clearly it is enough to show them for $D_{\boldsymbol{\lambda}} = e(\boldsymbol{i}^{\boldsymbol{\lambda}})$ so let us do that. We first note that (4.0.31) is a consequence of (4.0.32). Indeed, if $\boldsymbol{i}^{\boldsymbol{\lambda}}\iota$ is not of the form $\boldsymbol{i}^{\boldsymbol{\nu}}$ for any multipartition $\boldsymbol{\nu}$ we have from (4.0.32) that

$$D_{\boldsymbol{\lambda}} \cdot \iota = \sum_{\boldsymbol{\mu}|_n > \boldsymbol{\lambda}} C_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\mu} > \boldsymbol{\lambda}} C_{\boldsymbol{\mu}} \tag{4.0.38}$$

where we for the last equality used that in general $\boldsymbol{\mu} > \boldsymbol{\mu}|_n$, see the definition of $>$ given in (2.0.21). On the other hand, if $\boldsymbol{i}^{\boldsymbol{\lambda}}\iota = \boldsymbol{i}^{\boldsymbol{\nu}}$ for a multipartition $\boldsymbol{\nu}$ of $n+1$, then we have that $\boldsymbol{\nu} > \boldsymbol{\lambda}$ and $e(\boldsymbol{i}^{\boldsymbol{\lambda}}\iota) = e(\boldsymbol{\nu}) = C_{\boldsymbol{\nu}}$ and so (4.0.31) also holds in this case.

Let us now prove (4.0.32) by downwards induction on $<$. For $\boldsymbol{i}^{\boldsymbol{\lambda}} = \boldsymbol{i}^{max}$, it holds by Corollary 4.0.2. We now fix an arbitrary multipartition $\boldsymbol{\lambda}$ and assume that (4.0.32) has been proved for multipartitions $\boldsymbol{\nu}$ such that $\boldsymbol{\nu} > \boldsymbol{\lambda}$. Then in the above sequence notation, and writing α for ι, for (4.0.32) we must show that

$$e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \cdot \iota = (\kappa_1^1, \dots, \kappa_{l_1}^1 \mid \kappa_1^2, \dots, \kappa_{l_2}^2 \mid \cdots \mid \kappa_1^r, \dots, \kappa_{l_r}^r \mid \alpha) = \sum_{\boldsymbol{\mu}|_n > \boldsymbol{\lambda}} C_{\boldsymbol{\mu}} \tag{4.0.39}$$

where α is positioned in the $n+1$'st position. Since we assume that the sequence is not of the form $\boldsymbol{i}^{\boldsymbol{\nu}}$ for $\boldsymbol{\nu}$ for any multipartition we can move α to the left until it meets its first obstacle, which must be α, $\alpha-1$ or $\alpha+1$. If it is α we proceed essentially as before: we use Lemma 3.0.6 to replace $\alpha\alpha$ by $\overset{\bullet}{\alpha}\alpha$ and can now use the induction hypothesis. Indeed, if $\overset{\bullet}{\alpha}$ is situated in the k'th position we are dealing with $y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}|_k\, i_{k+1} \cdots i_n i_{n+1})$ where $i_{n+1} = \kappa_{l_r}^r$ and so on for the other i_j's. Using the inductive hypothesis for n on (4.0.30) and (4.0.31) we get, arguing as in connection with (4.0.35), that

$$y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}|_k\, i_{k+1} \cdots i_n) = \sum_{\boldsymbol{\tau} > \boldsymbol{\lambda}} D_{\boldsymbol{\tau}} \tag{4.0.40}$$

where $\boldsymbol{\tau}$ runs over multipartitions of n. Finally, we use the inductive hypothesis for $<$ to write

$$e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \cdot \iota = y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}|_k\, i_{k+1} \cdots i_n i_{n+1}) = \sum_{\boldsymbol{\tau} > \boldsymbol{\lambda}} D_{\boldsymbol{\tau}} \cdot i_{n+1} = \sum_{\boldsymbol{\mu} > \boldsymbol{\tau} > \boldsymbol{\lambda}} D_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\mu}|_n > \boldsymbol{\lambda}} D_{\boldsymbol{\mu}} \tag{4.0.41}$$

where the last equality follows from the fact that $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ run over multipartitions of n and $n+1$. Hence (4.0.41) has the form required for (4.0.32).

If the first obstacle is $\alpha-1$ we essentially argue as before: the situation gives rise to a triple $\alpha(\alpha-1)\alpha$ which we rewrite, using Lemma 3.0.9, as a linear combination of $\overset{\bullet}{\alpha}\alpha(\alpha-1)$ and $(\alpha-1)\alpha\alpha$. Arguing as for (4.0.40) and (4.0.41) we get the term involving $\overset{\bullet}{\alpha}\alpha(\alpha-1)$ in the form indicated in (4.0.31), whereas for the term involving $(\alpha-1)\alpha\alpha$ we use the inductive hypothesis for (4.0.31).

Finally, if the first obstacle is $\alpha+1$ we also argue as before, essentially. Indeed, in this situation there is a gap where α can be placed. This gives rise to a multipartition $\boldsymbol{\tau}$ of k such that $\boldsymbol{\tau} > \boldsymbol{\lambda}|_k$ where k is the position of α and so we get, arguing as before, that

$$e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \cdot \iota = e(\boldsymbol{i}^{\boldsymbol{\tau}} i_{k+1} \cdots i_n i_{n+1}) = \sum_{\boldsymbol{\mu}|_n > \boldsymbol{\lambda}} D_{\boldsymbol{\mu}}. \tag{4.0.42}$$

This finishes the proof of the Lemma. □

Corollary 4.0.4. *For each $\boldsymbol{i} \in I_e^n$ there is an expansion in $\mathbb{B}_n$ of the form*

$$e(\boldsymbol{i}) = \sum_{\boldsymbol{\mu}} D_{\boldsymbol{\mu}} \tag{4.0.43}$$

where the sum runs over multipartitions $\boldsymbol{\mu}$ of n and $D_{\boldsymbol{\mu}}$ factorizes over $e(\boldsymbol{i}^{\boldsymbol{\mu}})$.

Proof. We argue by induction on n, the base case $n = 1$ being trivial. Assuming that (4.0.43) holds for $n-1$ we prove it for n. Suppose that $\boldsymbol{i} = (i_1, \ldots, i_{n-1}, i_n)$ and set $\boldsymbol{i}_{n-1} = (i_1, \ldots, i_{n-1})$. Then by induction we have that

$$e(\boldsymbol{i}_{n-1}) = \sum_{\boldsymbol{\mu}_{n-1}} D_{\boldsymbol{\mu}_{n-1}} \tag{4.0.44}$$

where $\boldsymbol{\mu}_{n-1}$ runs over multipartitions of $(n-1)$ and where $D_{\boldsymbol{\mu}_{n-1}}$ factorizes over $e(\boldsymbol{i}^{\boldsymbol{\mu}_{n-1}})$. Using (4.0.31) of the previous Lemma 4.0.3 we then get

$$e(\boldsymbol{i}) = e(\boldsymbol{i}_{n-1})i_n = \sum_{\boldsymbol{\mu}_{n-1}} D_{\boldsymbol{\mu}_{n-1}} i_n = \sum_{\boldsymbol{\mu}_{n-1}} \sum_{\boldsymbol{\nu} > \boldsymbol{\mu}_{n-1}} D_{\boldsymbol{\nu}} \tag{4.0.45}$$

and so $e(\boldsymbol{i})$ is of the form claimed in (4.0.43). □

For any $w \in \mathfrak{S}_n$ we choose once and for all a reduced expression $s_{i_1} s_{i_1} \cdots s_{i_N}$ and define $\psi_w \in \mathbb{B}_n$ via this expression

$$\psi_w := \psi_{i_1} \psi_{i_1} \cdots \psi_{i_N}. \tag{4.0.46}$$

Note that ψ_w depends on the choice of reduced expression, not just on w. We denote by *official reduced expression for w* the expression used in (4.0.47). If $w_1 = s_{j_1} s_{j_1} \cdots s_{j_N}$ is another, *'unofficial'*, reduced expression for w then the error term in using w_1 instead of w can be controlled, in the sense that we have that

$$\psi_w - \psi_{j_1} \psi_{j_1} \cdots \psi_{j_N} = \sum_{\underline{k} \in \mathbb{N}_0^n, v \in \mathfrak{S}_n, w < v} c_{\underline{k},v}\, y^{\underline{k}}\, \psi_v = \sum_{\underline{k} \in \mathbb{N}_0^n, v \in \mathfrak{S}_n, w < v} d_{\underline{k},v}\ \psi_v\, y^{\underline{k}} \tag{4.0.47}$$

where $c_{\underline{k},v}, d_{\underline{k},v} \in \mathbb{F}$ and where for $\underline{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ we define $y^{\underline{k}} := y_1^{k_1} \cdots y_n^{k_n} \in \mathbb{B}_n$.

Let $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$ be a one-column multipartition and suppose that $\mathfrak{s}, \mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda})$. For the associated group elements $d(\mathfrak{s}), d(\mathfrak{t}) \in \mathfrak{S}_n$ we have $\psi_{d(\mathfrak{s})}, \psi_{d(\mathfrak{t})} \in \mathbb{B}_n$ defined via the official reduced expression for $d(\mathfrak{s})$ and $d(\mathfrak{t})$. We then set

$$m_{\mathfrak{s}\mathfrak{t}} = \psi^*_{d(\mathfrak{s})} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \psi_{d(\mathfrak{t})} \in \mathbb{B}_n \tag{4.0.48}$$

and define $\mathcal{C}_n \subseteq \mathbb{B}_n$ via

$$\mathcal{C}_n := \{ m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathrm{Par}_n^1 \}. \tag{4.0.49}$$

A main goal of our thesis is to show that $\mathcal{C}_n$ is a cellular basis for $\mathbb{B}_n$. Our first step towards this goal is to show that $\mathcal{C}_n$ is a generating set for $\mathbb{B}_n$. We start with the following Lemma.

Lemma 4.0.5. *Suppose that $D_{\boldsymbol{\lambda}} \in \mathbb{B}_n$ factorizes over $e(\boldsymbol{\lambda})$. Then there is an expansion of the form*

$$D_{\boldsymbol{\lambda}} = \sum_{\mathfrak{s}, \mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\mu}),\, \boldsymbol{\mu} \geq \boldsymbol{\lambda}} c_{\mathfrak{s}\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}} \tag{4.0.50}$$

where $c_{\mathfrak{s}\mathfrak{t}} \in \mathbb{F}$.

Proof. It is known that

$$\mathcal{S} := \{ e(\boldsymbol{i})\, y^{\underline{k}}\, \psi_w \mid \boldsymbol{i} \in I_e^n, \underline{k} \in \mathbb{N}_0^n, w \in \mathfrak{S}_n \} \tag{4.0.51}$$

spans the KLR-algebra $\mathcal{R}_n$ over $\mathbb{F}$, see (2.7) of [7] and section 2.3 of [20]. In fact, any permutation of the three factors of $\mathcal{S}$ also gives an $\mathbb{F}$-spanning set for $\mathcal{R}_n$ over $\mathbb{F}$. But by definition $\mathbb{B}_n$ is a quotient of $\mathcal{R}_n$ and so these sets also span $\mathbb{B}_n$ over $\mathbb{F}$.

We now prove (4.0.50) using downwards induction on $<$. The induction basis is given by the multipartition $\boldsymbol{\lambda} := \boldsymbol{\mu}_n^{max}$, introduced in (2.0.5). We may assume that $D_{\boldsymbol{\lambda}} = a\, e(\boldsymbol{i}^{\boldsymbol{\lambda}})\, b$ where $a, b \in B$, since $D_{\boldsymbol{\lambda}}$ is a linear combination of such expressions. We now expand a in terms of the variation of $\mathcal{S}$ that uses the product order $\psi_w y^{\underline{k}} e(\boldsymbol{i})$ and then expand b in terms of $\mathcal{S}$. Inserting, we find expressions of the form

$$D_{\boldsymbol{\lambda}} = \sum_{v,w,\underline{k}_1,\underline{k}_2} c_{v,w,\underline{k}_1,\underline{k}_2} \psi_v y^{\underline{k}_1} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) y^{\underline{k}_2} \psi_w = \sum_{v,w} c_{v,w} \psi_v e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \psi_w \tag{4.0.52}$$

where we used Lemma 4.0.1 for the second equality. For each appearing v, w we must now show that $\psi_v e(\boldsymbol{i}^{\boldsymbol{\lambda}})\psi_w$ is a linear combination of $m_{\mathfrak{st}}$ where $\mathfrak{s}, \mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda})$. We set $\mathfrak{s} := \mathfrak{t}^{\boldsymbol{\lambda}} v^{-1}$ and $\mathfrak{t} := \mathfrak{t}^{\boldsymbol{\lambda}} w$. Then we have by definition that $d(\mathfrak{s}) = v^{-1}$ and $d(\mathfrak{t}) = w$ and so

$$D_{\boldsymbol{\lambda}} = \sum_{v,w} c_{v,w} \psi_v e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \psi_w = \sum_{\mathfrak{s},\mathfrak{t}} c_{\mathfrak{st}} m_{\mathfrak{st}} \tag{4.0.53}$$

and so we obtain the required expansion for $D_{\boldsymbol{\lambda}}$, at least in the basis case $\boldsymbol{\lambda} = \boldsymbol{\mu}_n^{max}$.

We next show the existence of the expansion (4.0.50) for $D_{\boldsymbol{\lambda}}$ for a general $\boldsymbol{\lambda}$, assuming that it exists for all $\boldsymbol{\mu} > \boldsymbol{\lambda}$. Once again we may assume that $D_{\boldsymbol{\lambda}} = a\, e(\boldsymbol{i}^{\boldsymbol{\lambda}})\, b$ where $a, b \in \mathbb{B}_n$ and once again we expand a in terms of the variation of $\mathcal{S}$ that uses the product order $\psi_w y^{\underline{k}} e(\boldsymbol{i})$ and b in terms of $\mathcal{S}$. Inserting, we now get an expression of the form

$$D_{\boldsymbol{\lambda}} = \sum_{v,w,\underline{k}_1,\underline{k}_2} c_{v,w,\underline{k}_1,\underline{k}_2} \psi_v y^{\underline{k}_1} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) y^{\underline{k}_2} \psi_w = \sum_{v,w} c_{v,w} \psi_v e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \psi_w + \sum_{\boldsymbol{\mu} > \boldsymbol{\lambda}} D_{\boldsymbol{\mu}} \tag{4.0.54}$$

where we this time used Lemma 4.0.3 for the last equality. Arguing as we did in the inductive basis step we now rewrite $\sum_{v,w} c_{v,w} \psi_v e(\boldsymbol{i}^{\boldsymbol{\lambda}})\psi_w$ as a linear combination of $m_{\mathfrak{st}}$'s and then get

$$D_{\boldsymbol{\lambda}} = \sum_{\mathfrak{s},\mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda})} c_{\mathfrak{st}} m_{\mathfrak{st}} + \sum_{\boldsymbol{\mu} > \boldsymbol{\lambda}} D_{\boldsymbol{\mu}}. \tag{4.0.55}$$

We now use the inductive hypothesis on the terms $D_{\boldsymbol{\mu}}$ to conclude the proof of the Lemma. $\square$

Lemma 4.0.6. *The subset of $\mathbb{B}_n$ given by*

$$\{ m_{\mathfrak{st}} \mid \boldsymbol{\lambda} \in \mathrm{Par}_n^1,\ \mathfrak{s}, \mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda}) \} \tag{4.0.56}$$

spans $\mathbb{B}_n$ over $\mathbb{F}$.

Proof. Choose $b \in \mathbb{B}_n$ and expand it in terms of $\mathcal{S}$ as follows

$$b = \sum c_{\boldsymbol{i},\underline{k},w}\, e(\boldsymbol{i})\, y^{\underline{k}}\, \psi_w \tag{4.0.57}$$

where $c_{\boldsymbol{i},\underline{k},w} \in \mathbb{F}$. Using Corollary 4.0.4 we write each appearing $e(\boldsymbol{i})$ as a linear combination of $D_{\boldsymbol{\mu}}$'s where $\boldsymbol{\mu}$ runs over multipartitions and $D_{\boldsymbol{\mu}}$ factorizes over $e(\boldsymbol{i}^{\boldsymbol{\mu}})$. Inserting this in (4.0.57) we find that any $b \in \mathbb{B}_n$ is a linear combination of $D_{\boldsymbol{\mu}}$'s. We can then apply the previous Lemma 4.0.5 to conclude the proof of the Lemma. $\square$

Our next goal is to show that the non-standard tableaux are not needed in (4.0.56). Our method for proving this is an adaption of Murphy's method using Garnir tableaux, see [31] and [35].

Let $\boldsymbol{\lambda}$ be a multipartition and $\mathfrak{g}$ a $\boldsymbol{\lambda}$-tableau. We say that $\mathfrak{g}$ is a *Garnir tableau* if there is an $1 \leq i < n$ such that

a) $\mathfrak{g}$ is not standard, but $\mathfrak{g}s_i$ is standard.

b) If $s \in S$ and $\mathfrak{g}s \rhd \mathfrak{g}$ then $s = s_i$.

Here are some examples

$$\left(\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}, \begin{array}{|c|}\hline 3 \\ \hline 5 \\ \hline 6 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|}\hline 4 \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline 6 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|}\hline 1 \\ \hline 6 \\ \hline 12 \\ \hline 15 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 7 \\ \hline 13 \\ \hline \end{array}, \begin{array}{|c|}\hline 3 \\ \hline 11 \\ \hline 10 \\ \hline \end{array}, \begin{array}{|c|}\hline 4 \\ \hline 8 \\ \hline 14 \\ \hline \end{array}, \begin{array}{|c|}\hline 5 \\ \hline 9 \\ \hline \end{array} \right). \tag{4.0.58}$$

In order to get a better description of Garnir tableaux we introduce some further notation. Let $\boldsymbol{\lambda}$ be a one-column multipartition and let $\gamma = (r, 1, m)$ be a node of $[\boldsymbol{\lambda}]$, which does not belong to the first row of $[\boldsymbol{\lambda}]$. We then denote by γ^+ the node $(r-1,1,m)$ of $[\boldsymbol{\lambda}]$, that is γ^+ is the node of $[\boldsymbol{\lambda}]$ that is situated on top of γ in $[\boldsymbol{\lambda}]$. We then define the *Garnir snake* of γ as the following interval in $[\boldsymbol{\lambda}]$ with respect to $\lhd$

$$\mathrm{Snake}(\gamma) := [\gamma, \gamma^+] = \{\tau \in [\boldsymbol{\lambda}] \mid \gamma \unlhd \tau \unlhd \gamma^+\}. \tag{4.0.59}$$

We also define

$$\boldsymbol{n}_{Snake(\gamma)} := \{i \in \boldsymbol{n} \mid \mathfrak{t}^{\boldsymbol{\lambda}}(i) \in [\gamma, \gamma^+]\} \tag{4.0.60}$$

that is $\boldsymbol{n}_{Snake(\gamma)}$ is the set of numbers that are used to fill in $\mathrm{Snake}(\gamma)$ for $\mathfrak{t}^{\boldsymbol{\lambda}}$.

For $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$ and $\gamma = (r,1,m)$ a node of $[\boldsymbol{\lambda}]$, not belonging to the first row, we define *the classical Garnir tableau* $\mathfrak{g}_{clas,\gamma}$ by setting $\mathfrak{g}_{clas,\gamma}(i) := \mathfrak{t}^{\boldsymbol{\lambda}}(i)$ for $i \notin \boldsymbol{n}_{Snake(\gamma)}$ and by requiring that the numbers from $\boldsymbol{n}_{Snake(\gamma)}$ are filled in consecutively from left to right in $\mathrm{Snake}(\gamma)$ except for an upwards jump from γ to γ^+. Here is an example with $\gamma = (3,1,3)$

$$\mathfrak{g}_{clas,\gamma} = \left(\begin{array}{|c|}\hline 1\\\hline 6\\\hline 8\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 7\\\hline 9\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 11\\\hline 10\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 12\\\hline 13\\\hline\end{array} \right). \tag{4.0.61}$$

It should be noted that $\mathfrak{g}_{clas,\gamma}$ is *not* a Garnir in the classical sense, as considered for example by Murphy and Mathas. On the other hand, it is similar to the classical Garnir tableaux in the sense that if we view the components of $\boldsymbol{\lambda}$ as the columns of an ordinary partition (possibly with 'missing' nodes as in the example) then $\mathfrak{g}_{clas,\gamma}$ becomes a Garnir tableau in the classical sense.

We need another class of Garnir tableaux that we denote $\tilde{\mathfrak{g}}_\gamma$. They are defined by filling in the numbers from $\boldsymbol{n}_{Snake(\gamma)}$ into $\mathrm{Snake}(\gamma)$ in increasing order, beginning with γ, then γ^+ and the other nodes of the row of γ^+ and finally the remaining nodes of the row of γ. Here is an example with $\gamma = (3,1,3)$

$$\tilde{\mathfrak{g}}_\gamma = \left(\begin{array}{|c|}\hline 1\\\hline 6\\\hline 11\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 7\\\hline 12\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 9\\\hline 8\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 10\\\hline 13\\\hline\end{array} \right). \tag{4.0.62}$$

Recall the weak order $\succ$ on $\mathrm{Tab}(\boldsymbol{\lambda})$. The following Lemma relates it to Garnir tableaux. Set first $\mathrm{NStd}(\boldsymbol{\lambda}) := \mathrm{Tab}(\boldsymbol{\lambda}) \setminus \mathrm{Std}(\boldsymbol{\lambda})$, that is $\mathfrak{s} \in \mathrm{NStd}(\boldsymbol{\lambda})$ if and only if $\mathfrak{s}$ is a non-standard $\boldsymbol{\lambda}$-tableau.

Lemma 4.0.7. *Suppose that* $\mathfrak{t} \in \mathrm{NStd}(\boldsymbol{\lambda})$. *Then*

a) The tableau $\mathfrak{t}$ *is a maximal in* $\mathrm{NStd}(\boldsymbol{\lambda})$ *with respect* $\succ$ *if and only if* $\mathfrak{t}$ *is a Garnir tableau.*

b) If $\mathfrak{t}$ *is a maximal in* $\mathrm{NStd}(\boldsymbol{\lambda})$ *with respect* $\rhd$ *then* $\mathfrak{t}$ *is a Garnir tableau.*

Proof. Let us first prove *a*) of the Lemma. Assume that $\mathfrak{t}$ is a maximal tableau in $\mathrm{NStd}(\boldsymbol{\lambda})$ with respect to $\succ$. Then for all $s_i \in S$ we have that either $\mathfrak{t}s_i \lhd \mathfrak{t}$ or $\mathfrak{t}s_i \in \mathrm{Std}(\boldsymbol{\lambda})$. If $\mathfrak{t}s_i \lhd \mathfrak{t}$ for all i we have that $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$ which contradicts that $\mathfrak{t} \in \mathrm{NStd}(\boldsymbol{\lambda})$. Hence there is an s_{i_0} such that $\mathfrak{t}s_{i_0} \rhd \mathfrak{t}$ and for this s_{i_0} we have $\mathfrak{t}s_{i_0} \in \mathrm{Std}(\boldsymbol{\lambda})$ by maximality of $\mathfrak{t}$ in $\mathrm{NStd}(\boldsymbol{\lambda})$. On the other hand, there can only be one s_{i_0} with this property. Indeed, suppose that also $\mathfrak{t}s_{j_0} \rhd \mathfrak{t}$. Setting $\mathfrak{u} := \mathfrak{t}s_{i_0}$ and $\mathfrak{v} := \mathfrak{t}s_{j_0}$ we have that $\mathfrak{u}$ and $\mathfrak{u}s_{i_0}s_{j_0}$ are standard tableaux, whereas $\mathfrak{u}s_{i_0}$ is non-standard. This is only possible if $i_0 = j_0$ and so $\mathfrak{t}$ is a Garnir tableau, as claimed.

Now assume that $\mathfrak{t}$ is not a maximal tableau in $\mathrm{NStd}(\boldsymbol{\lambda})$ with respect to $\succ$. Then there is an $s \in S$ such that $\mathfrak{t}s \rhd \mathfrak{t}$ and $\mathfrak{t}s \in \mathrm{NStd}(\boldsymbol{\lambda})$. This implies that $\mathfrak{t}$ is not a Garnir tableau.

We now show *b*) of the Lemma. If $\mathfrak{t}$ is a maximal tableau in $\mathrm{NStd}(\boldsymbol{\lambda})$ with respect to $\rhd$ then $\mathfrak{t}$ is also a maximal tableau in $\mathrm{NStd}(\boldsymbol{\lambda})$ with respect to $\succ$, since $\succ$ is a weaker order than $\rhd$, and so $\mathfrak{t}$ must be a Garnir tableau by *a*). This proves *b*) of the Lemma. □

The converse of *b*) of the Lemma does not hold as can be seen in the following example. Let $\boldsymbol{\lambda} = (1^2, 1^2, 1^2, 1^2, 1)$ and define

$$\mathfrak{g}_1 = \left(\begin{array}{|c|}\hline 1\\\hline 7\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 8\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 6\\\hline 9\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline\end{array} \right), \mathfrak{g}_2 = \left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 8\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline 4\\\hline\end{array}, \begin{array}{|c|}\hline 6\\\hline 9\\\hline\end{array}, \begin{array}{|c|}\hline 7\\\hline\end{array} \right). \tag{4.0.63}$$

Then both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Garnir tableaux, and it is easy to see that $\mathfrak{g}_1 \rhd \mathfrak{g}_2$ and so $\mathfrak{g}_2$ is not a maximal tableau in $\mathrm{NStd}(\boldsymbol{\lambda})$ with respect to $\rhd$.

Corollary 4.0.8. *Let $\mathfrak{t}$ be a $\boldsymbol{\lambda}$-tableau which is non-standard. Then there exists a Garnir tableau $\mathfrak{g}$ and a $w \in \mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{g}w$ and $l(d(\mathfrak{t})) = l(d(\mathfrak{g})) + l(w)$.*

Proof. This is a consequence of *a*) of Lemma 4.0.7. □

Let us now give our characterization of Garnir tableaux.

Lemma 4.0.9. *Given a multipartition $\boldsymbol{\lambda}$ of n and let $\mathfrak{g}$ be a $\boldsymbol{\lambda}$-tableau. Then $\mathfrak{g}$ is a Garnir tableau if and only if there is a node $\gamma \in [\boldsymbol{\lambda}]$, not belonging to the first row, and an $i_0 \in \boldsymbol{n}$ such that*

(1) $\mathfrak{g}(i_0) = \gamma$ and $\mathfrak{g}(i_0+1) = \gamma^+$.

(2) For all $i \neq i_0$ we have $\mathfrak{g}(i) \rhd \mathfrak{g}(i+1)$.

(3) For all $i \in \boldsymbol{n} \setminus \boldsymbol{n}_{\mathrm{Snake}(\gamma)}$ we have that $\mathfrak{g}(i) = \mathfrak{t}^{\boldsymbol{\lambda}}(i)$.

Proof. Suppose first that $\mathfrak{g}$ is a Garnir tableau. Then $\mathfrak{g}$ is not standard and maximal with respect to $\prec$ and hence there is an $i_0 \in \boldsymbol{n}$ such that $\mathfrak{g}s_{i_0}$ is standard. The entries i_0 and i_0+1 belong to the same component (column) of $[\boldsymbol{\lambda}]$ and $\mathfrak{g}(i_0+1) \rhd \mathfrak{g}(i_0)$. Let $\gamma = \mathfrak{g}(i_0+1)$ and $\beta = \mathfrak{g}(i_0)$. Suppose that $\beta^+ \neq \gamma$ and choose $a \in \boldsymbol{n}$ such that $\mathfrak{g}(a) = \beta^+$. Then $\gamma \rhd \beta^+$ and since $\mathfrak{g}s_{i_0}$ is standard we have that $i_0 < a < i_0+1$, a contradiction. Therefore $\beta = \gamma^+$ and by definition $\mathfrak{g}(i_0)^+ = \mathfrak{g}(i_0+1)$.

Since $\mathfrak{g}$ is a Garnir tableaux, we have for $i \neq i_0$ that $\mathfrak{g} \rhd \mathfrak{g}s_i$ and then $\mathfrak{g}(i) \rhd \mathfrak{g}(i+1)$, see *a*) of Lemma 2.0.2.

Let us say that $i \in \boldsymbol{n}$ defines a *simple non-inversion* if $\mathfrak{g}(i) \rhd \mathfrak{g}(i+1)$ and that $i \in \boldsymbol{n}$ defines a *simple inversion* if $\mathfrak{g}(i) \lhd \mathfrak{g}(i+1)$. With this terminology we have so far proved that i_0 is the only simple inversion of $\boldsymbol{n}$, all other elements are simple non-inversions.

Let $k_0 = \min(\mathfrak{g}^{-1}(\mathrm{Snake}(\gamma)))$ and $k_1 = \max(\mathfrak{g}^{-1}(\mathrm{Snake}(\gamma)))$. Since i_0 is the only inversion of $\boldsymbol{n}$ we have that $k_0 - 1$ appears before k_0 in $\mathfrak{g}$ whereas $k_0 - 2$ appears before $k_0 - 1$ and so on until 1. On the other hand, no $j > k_0$ can appear before k_0 in $\mathfrak{g}$, since for the smallest such j we would have that $j-1$ is a inversion distinct from i_0. We have thus showed that for $i = 1, 2, \ldots, k_0 - 1$ we have that $\mathfrak{g}(i) = \mathfrak{t}^{\boldsymbol{\lambda}}(i)$. Similarly, one shows that also for $i = k_1+1, k_1+2, \ldots, n$ we have that $\mathfrak{g}(i) = \mathfrak{t}^{\boldsymbol{\lambda}}(i)$. Thus we have that $\mathfrak{g}^{-1}(\mathrm{Snake}(\gamma)) = \boldsymbol{n}_{\mathrm{Snake}(\gamma)}$ and that $\mathfrak{g}$ verifies the conditions (1), (2) and (3) of the Lemma.

Finally, if $\mathfrak{g}$ is a $\boldsymbol{\lambda}$-tableau verifying the conditions (1), (2) and (3) of the Lemma, then clearly $\mathfrak{g}$ is a Garnir tableau. □

For the next Lemma we need condition *iii*) from Definition 3.0.1 of strong adjacency-freeness.

Corollary 4.0.10. *Let $\boldsymbol{\lambda}$ be a multipartition and let $\gamma \in [\boldsymbol{\lambda}]$. Suppose that $\mathfrak{g}_1$ and $\mathfrak{g}_1$ are Garnir tableaux of the same shape $\boldsymbol{\lambda}$ with respect to the same γ as in part* (1) *of the previous Lemma 4.0.9. Then $e(\boldsymbol{i}^{\mathfrak{g}_1}) \sim e(\boldsymbol{i}^{\mathfrak{g}_2})$.*

Proof. It is enough to prove that for any Garnir tableau $\mathfrak{g} = \mathfrak{g}_1$, satisfying the conditions of the Corollary, we have that $\mathfrak{g}_1 \sim \mathfrak{g}_{clas,\gamma}$. Let $\mathfrak{g}$ be the one line (ordinary) partition $\mathfrak{g} = (|\boldsymbol{n}_{Snake(\gamma)}|)$. Then we can view $\mathfrak{g}\,|_{\boldsymbol{n}_{Snake(\gamma)}}$ as a $\mathfrak{g}$-tableau $\mathfrak{t}(\mathfrak{g})$ by reading the numbers in $Snake(\gamma)$ from left to right. The Garnir tableaux from (4.0.63) correspond for example to the $\mathfrak{g}$-tableaux

$$\mathfrak{t}(\mathfrak{g}_1) = \boxed{7 \,|\, 8 \,|\, 4 \,|\, 5 \,|\, 6 \,|\, 3}\,, \qquad \mathfrak{t}(\mathfrak{g}_2) = \boxed{3 \,|\, 8 \,|\, 4 \,|\, 5 \,|\, 6 \,|\, 7} \tag{4.0.64}$$

where $\mathfrak{g} = (6)$, whereas $\mathfrak{g}_{clas,\gamma}$ in general corresponds to $\mathfrak{t}^{\mathfrak{g}}$ (on the numbers $\boldsymbol{n}_{Snake(\gamma)}$), that is

$$\mathfrak{t}^{\mathfrak{g}} = \boxed{3 \,|\, 4 \,|\, 5 \,|\, 6 \,|\, 7 \,|\, 8} \tag{4.0.65}$$

in this case. Since $\hat{\kappa}$ is strongly adjacency free, we have on the other hand that the residues of all of the nodes of $Snake(\gamma)$, except γ and γ^+, differ by 2 or more. Let now $w \in \mathfrak{S}_n$ be such that $\mathfrak{t}(\mathfrak{g})w = \mathfrak{t}^{\mathfrak{g}}$ and choose a reduced expression $w = s_{i_1} \cdots s_{i_N}$ for w. Then, for all j, we have that $s_{i_{j+1}}$ does not interchange the numbers appearing in the nodes corresponding to γ and γ^+ in $\mathfrak{t}_j := \mathfrak{t}(\mathfrak{g})s_{i_1} \cdots s_{i_j}$. For example, for $\mathfrak{t}(\mathfrak{g}_1)$ in (4.0.64) the sequence $s_{i_1}, \ldots, s_{i_N}$ never interchanges two numbers in the positions colored with red, and similarly for $\mathfrak{t}(\mathfrak{g}_2)$. The Corollary follows from this. □

We have the following Lemma.

Lemma 4.0.11. *Suppose that $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$ and that $\mathfrak{s}, \mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda})$. If $\mathfrak{t} \in \mathrm{NStd}(\boldsymbol{\lambda})$ then there is an expansion*

$$m_{\mathfrak{s}\mathfrak{t}} = \sum_{\mathfrak{t}_1 \in \mathrm{Std}(\boldsymbol{\lambda}), \mathfrak{t}_1 \rhd \mathfrak{t},} c_{\mathfrak{s}\mathfrak{t}_1} m_{\mathfrak{s}\mathfrak{t}_1} + \sum_{\boldsymbol{\mu} > \boldsymbol{\lambda}, \mathfrak{s}_2, \mathfrak{t}_2 \in \mathrm{Std}(\boldsymbol{\mu})} c_{\mathfrak{s}_2\mathfrak{t}_2} m_{\mathfrak{s}_2\mathfrak{t}_2} \tag{4.0.66}$$

where $c_{\mathfrak{s}\mathfrak{t}_1}, c_{\mathfrak{s}_2\mathfrak{t}_2} \in \mathbb{F}$.

Remark 4.0.12. *A similar statement holds for* $\mathfrak{s}$.

Proof. We shall argue via downwards induction on $\boldsymbol{\lambda}$ with respect to $<$. Let us first consider the case $\boldsymbol{\lambda} = \boldsymbol{\mu}_n^{max}$. We consider $m_{\mathfrak{st}}$ for $\mathfrak{s}, \mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda})$ and suppose that $\mathfrak{t} \in \mathrm{NStd}(\boldsymbol{\mu}^{max})$. We show using downwards induction on $\mathfrak{t}$ with respect to $\lhd$ that $m_{\mathfrak{st}}$, for $\mathfrak{t} \in \mathrm{NStd}(\boldsymbol{\mu}^{max})$, can be written in the form given by (4.0.66).

In view of *b*) of Lemma 4.0.7 the basis step for this induction is given by $\mathfrak{t} = \mathfrak{g}$ a Garnir tableau. Let us do it. By relation (3.0.7) we have that

$$m_{\mathfrak{sg}} = \psi^*_{d(\mathfrak{s})} e(\boldsymbol{i}^{max}) \psi_{d(\mathfrak{g})} = \psi^*_{d(\mathfrak{s})} \psi_{d(\mathfrak{g})} e(\boldsymbol{i}^{\mathfrak{g}}) \tag{4.0.67}$$

and so for the basis step to work it is enough to prove that $e(\boldsymbol{i}^{\mathfrak{g}}) = 0$. Let $\gamma \in [\boldsymbol{\mu}^{max}]$ be the node associated with $\mathfrak{g}$ as in Lemma 4.0.9. Using Lemma 4.0.10 we may assume that

$$e(\boldsymbol{i}^{\mathfrak{g}}) \sim e(\boldsymbol{i}^{\tilde{\mathfrak{g}}_\gamma}). \tag{4.0.68}$$

Let $j = \tilde{\mathfrak{g}}_\gamma^{-1}(\gamma)$. Applying Corollary 4.0.2 to the restriction of $\tilde{\mathfrak{g}}_\gamma$ to the numbers $\{1, 2, \dots, j-1\}$ and $\iota = \mathrm{res}(\gamma)$ we now get that $e(\boldsymbol{i}^{\tilde{\mathfrak{g}}_\gamma}) = 0$, and so also $e(\boldsymbol{i}^{\mathfrak{g}}) = 0$ which proves the claim in this case.

Let us now consider the case of a general non-standard $\boldsymbol{\mu}_n^{max}$-tableau $\mathfrak{t}$. Using Corollary 4.0.8 there exists a Garnir tableau $\mathfrak{g}$ and a $w \in \mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{g}w$ and $l(d(\mathfrak{t})) = l(d(\mathfrak{g})) + l(w)$. Hence there exists a reduced expression for $d(\mathfrak{t})$ of the form $d(\mathfrak{t}) = s_{i_i} \cdots s_{i_N} s_{j_i} \cdots s_{j_M}$ where $d(\mathfrak{g}) = s_{i_i} \cdots s_{i_N}$ and $w = s_{j_i} \cdots s_{j_M}$. If this reduced expression is the official one for $d(\mathfrak{t})$ we have that

$$m_{\mathfrak{st}} = \psi^*_{d(\mathfrak{s})} e(\boldsymbol{i}^{max}) \psi_{d(\mathfrak{g})} \psi_w = 0 \tag{4.0.69}$$

by the inductive basis, proved above. If it is not the official expression for $d(\mathfrak{t})$ we have by (4.0.47) that the error term that occurs when changing to the official expression is given by a linear combination of terms of the form $y^{\underline{k}} \psi_v$ where $\underline{k} \in \mathbb{N}_0^n$ and $v > d(\mathfrak{t})$. Now for any non-trivial factor $y^{\underline{k}}$ we have that $e(\boldsymbol{i}^{max}) y^{\underline{k}}$ is zero by Lemma 4.0.1 and for the terms ψ_v we have by Theorem 2.0.4 that $v = d(\mathfrak{t}_1)$ with $\mathfrak{t}_1 \rhd \mathfrak{t}$, and so we may use the inductive hypothesis on the non-standard $\mathfrak{t}_1$'s that may occur.

Let us now consider a general multipartition $\boldsymbol{\lambda} \neq \boldsymbol{\mu}_n^{max}$. We consider $m_{\mathfrak{st}}$ for $\mathfrak{s} \in \mathrm{Tab}(\boldsymbol{\lambda}), \mathfrak{t} \in \mathrm{NStd}(\boldsymbol{\lambda})$ and once again use downwards induction on $\mathfrak{t}$ with respect to $\lhd$ to show that $m_{\mathfrak{st}}$, for $\mathfrak{t} \in \mathrm{NStd}(\boldsymbol{\mu}^{max})$, can be written in the form given by (6.0.9). For $\mathfrak{t}$ maximal in $\mathrm{NStd}(\boldsymbol{\lambda})$ we have that $\mathfrak{t} = \mathfrak{g}$ is a Garnir tableau for $\boldsymbol{\lambda}$ and so, arguing the same way as we did for (4.0.67), we get

$$m_{\mathfrak{sg}} = \psi^*_{d(\mathfrak{s})} e(\boldsymbol{i}^{max}) \psi_{d(\mathfrak{g})} = \psi^*_{d(\mathfrak{s})} \psi_{d(\mathfrak{g})} e(\boldsymbol{i}^{\mathfrak{g}}). \tag{4.0.70}$$

Passing to $\tilde{\mathfrak{g}}_\gamma$ as we did get in the inductive basis case, and using (4.0.31) and (4.0.32) of Lemma 4.0.3, we then get

$$m_{\mathfrak{sg}} = \sum_{\boldsymbol{\mu} > \boldsymbol{\lambda}} D_{\boldsymbol{\mu}} = \sum_{\mathfrak{s}, \mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\mu}),\, \boldsymbol{\mu} > \boldsymbol{\lambda}} c_{\mathfrak{st}} m_{\mathfrak{st}} \tag{4.0.71}$$

where we used Lemma 4.0.5 for the second equality. We then use the inductive hypothesis on each appearing $m_{\mathfrak{st}}$, to rewrite in terms of $m_{\mathfrak{s}_1 \mathfrak{t}_1}$ for $\mathfrak{s}_1$ and $\mathfrak{t}_1$ standard tableaux. This concludes the case $\mathfrak{t} = \mathfrak{g}$.

Finally, for the general non-standard $\boldsymbol{\lambda}$-tableau $\mathfrak{t}$ we have that

$$m_{\mathfrak{st}} = \psi^*_{d(\mathfrak{s})} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \psi_{d(\mathfrak{g})} \psi_w = \sum_{\mathfrak{t}_1 \in \mathrm{Std}(\boldsymbol{\lambda}), \mathfrak{t}_1 \rhd \mathfrak{t}} c_{\mathfrak{st}} m_{\mathfrak{st}_1} + \sum_{\boldsymbol{\mu} > \boldsymbol{\lambda}} D_{\boldsymbol{\mu}} \tag{4.0.72}$$

where the second equality arises from the error terms $\psi^*_{d(\mathfrak{s})} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) y^{\underline{k}} \psi_v$. But as before we can apply the induction hypothesis on each $D_{\boldsymbol{\mu}}$ rewriting it in terms of $m_{\mathfrak{s}_1 \mathfrak{t}_1}$ where $\mathfrak{s}_1$ and $\mathfrak{t}_1$ are standard tableaux. This concludes the general $\mathfrak{t}$-case. Finally the $\mathfrak{s}$-case follows from the $\mathfrak{t}$-case by applying $*$ and so the Lemma is proved. ☐

From the Lemma we deduce the following Corollary. It is the main result of this chapter.

Corollary 4.0.13. *The subset* $\mathcal{C}_n$ *of* $\mathbb{B}_n$ *given by*

$$\mathcal{C}_n := \{ m_{\mathfrak{st}} \mid \boldsymbol{\lambda} \in \mathrm{Par}_n^1,\, \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \} \tag{4.0.73}$$

spans $\mathbb{B}_n$ *over* $\mathbb{F}$.

Proof. This is a consequence of Lemma 4.0.5 and Lemma 4.0.11. ☐

Chapter 5

Linear Independence of $\mathcal{C}_n$.

In this chapter we show that the set $\mathcal{C}_n$ constructed in (4.0.49) is a linearly independent set. Our methods used so far, essentially being manipulations with the defining relations for $\mathbb{B}_n$, are not sufficient for proving this and in fact it cannot even be proved that $m_{\mathfrak{st}}$ is non-zero with these methods.

To show the linear independence of $\mathcal{C}_n$ we shall rely on the seminal work by Brundan-Kleshchev and Rouquier, see [7], [39] that establishes an isomorphism between the cyclotomic KLR-algebra $\mathcal{R}_n$ and the cyclotomic Hecke algebra $\mathcal{H}_n$.

Let us give the precise definition of the relevant cyclotomic Hecke algebra.

Definition 5.0.1. *Let $\mathbb{F}$, e and $\hat{\kappa} \in \mathbb{Z}^l$ be as above, and let $q \in \mathbb{F} \setminus \{1\}$ be an e'th primitive root of unity. The* cyclotomic Hecke algebra $\mathcal{H}_n(q, \kappa)$ *is the $\mathbb{F}$-algebra with generators $L_1, \ldots, L_n$, $T_1, \ldots, T_{n-1}$ and relations*

$$(L_1 - q^{\kappa_1}) \cdots (L_1 - q^{\kappa_l}) = 0 \tag{5.0.1}$$

$$(T_r + 1)(T_r - q) = 0 \tag{5.0.2}$$

$$T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1} \tag{5.0.3}$$

$$L_r L_s = L_s L_r,\ T_r L_r = L_{r+1}(T_r - q + 1) \tag{5.0.4}$$

$$T_r L_s = L_s T_r \text{ if } |r - s| > 1 \text{ and } T_r T_s = T_s T_r \text{ if } s \neq r, r+1 \tag{5.0.5}$$

for all admissible r, s.

It follows from the relations that there is antiinvolution $*$ of $\mathcal{H}_n$, fixing the generators T_i and L_i. We have that T_r is invertible with $T_r^{-1} = q^{-1}(T_r - q + 1)$. From this one gets that

$$L_{r+1} = q^{-1} T_r L_r T_r \tag{5.0.6}$$

and so $L_2, \ldots, L_n$ are actually redundant for generating $\mathcal{H}_n$. The elements L_i are called *Jucys-Murphy elements* for $\mathcal{H}_n$.

Let $\hat{q}$ be a variable and let $\mathcal{K}$ be the quotient field of the polynomial ring $\mathbb{F}[\hat{q}]$. Let $\mathcal{O}$ be the subring of $\mathcal{K}$ given by $\mathcal{O} := \{\frac{f(\hat{q})}{g(\hat{q})} \mid f(\hat{q}), g(\hat{q}) \in \mathbb{F}[\hat{q}], g(q) \neq 0\}$. Then $\mathcal{O}$ is a local ring with maximal ideal $\mathfrak{m} := (\hat{q} - q) = \{\frac{f(\hat{q})}{g(\hat{q})} \in \mathcal{O} \mid f(q) = 0\}$. The evaluation map $\mathcal{O} \to \mathbb{F}$, $\frac{f(\hat{q})}{g(\hat{q})} \mapsto \frac{f(q)}{g(q)}$ induces an isomorphism $\mathcal{O}/\mathfrak{m} \cong \mathbb{F}$ and so the triple $(\mathcal{O}, \mathbb{F}, \mathcal{K})$ is a modular system.

Let $\mathcal{H}_n^{\mathcal{O}} = \mathcal{H}_n^{\mathcal{O}}(\hat{q}, \kappa)$ be the $\mathcal{O}$-algebra given by the same presentation as $\mathcal{H}_n$, but replacing q by $\hat{q} \in \mathcal{O}$, and let similarly $\mathcal{H}_n^{\mathcal{K}} = \mathcal{H}_n^{\mathcal{K}}(\hat{q}, \kappa)$ be the $\mathcal{K}$-algebra given by the same presentation used for $\mathcal{H}_n$, but replacing q by $\hat{q} \in \mathcal{K}$. It is known that $\mathcal{H}_n^{\mathcal{O}}$ is free over $\mathcal{O}$ of rank $l^n n!$. Furthermore, we have that $\mathcal{H}_n^{\mathcal{O}} \otimes_{\mathcal{O}} \mathbb{F} \cong \mathcal{H}_n$ where $\mathbb{F}$ is made into an $\mathcal{O}$-algebra via evaluation in q, and that $\mathcal{H}_n^{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{K} \cong \mathcal{H}_n^{\mathcal{K}}$, via extension of scalars. It follows that $\mathcal{H}_n$ and $\mathcal{H}_n^{\mathcal{K}}$ both have dimension $l^n n!$.

The representation theory of $\mathcal{H}_n$ is governed by $\mathrm{Par}_{l,n}$, that is l-multipartitions of n. Let $\boldsymbol{\lambda}$ be an element of $\mathrm{Par}_{l,n}$ and let $\mathfrak{s} \in \mathrm{Tab}(\boldsymbol{\lambda})$. Then we define the content function of $\mathfrak{s}$ via the formula

$$c_{\mathfrak{s}}(i) = q^{\mathrm{res}(\mathfrak{s}(i))} \in \mathbb{F} \tag{5.0.7}$$

where res is as in (4.0.1). Note that since q is an e'th primitive root of unity, this makes sense. The content function for $\mathcal{H}_n^{\mathcal{O}}$ and $\mathcal{H}_n^{\mathcal{K}}$ is defined via

$$c_{\mathfrak{s}}^{\mathcal{O}}(i) = c_{\mathfrak{s}}^{\mathcal{K}}(i) = \hat{q}^{\hat{\kappa}_k + c - r} \in \mathcal{O} \subseteq \mathcal{K} \tag{5.0.8}$$

where $\mathfrak{s}(i) = (r, c, k)$. By the condition $i)$ on the multicharge $\hat{\kappa}$, the content function satisfies the separability condition given in [32] and so $\mathcal{H}_n^{\mathcal{K}}$ is a semisimple algebra.

The following concepts and results have their origin in Murphy's papers. Let $\mathrm{Std}(n) := \cup_{\boldsymbol{\lambda} \in \mathrm{Par}_n} \mathrm{Std}(\boldsymbol{\lambda})$. For $\mathfrak{s}$ any element of $\mathrm{Std}(n)$ we define

$$F_{\mathfrak{s}} := \prod_{k=1}^{n} \prod_{\substack{\mathfrak{t} \in \mathrm{Std}(n) \\ c_{\mathfrak{s}}^{\mathcal{K}}(k) \neq c_{\mathfrak{t}}^{\mathcal{K}}(k)}} \frac{L_k - c_{\mathfrak{t}}^{\mathcal{K}}(k)}{c_{\mathfrak{s}}^{\mathcal{K}}(k) - c_{\mathfrak{t}}^{\mathcal{K}}(k)} \in \mathcal{H}_n^{\mathcal{K}}. \tag{5.0.9}$$

It is known that the $F_{\mathfrak{s}}$'s form a complete system of orthogonal idempotents. The $F_{\mathfrak{s}}$'s are simultaneous eigenvectors for the action of the L_i's and the corresponding eigenvalues are given by the contents:

$$L_i F_{\mathfrak{s}} = F_{\mathfrak{s}} L_i = c_{\mathfrak{s}}^{\mathcal{K}}(i) F_{\mathfrak{s}}. \tag{5.0.10}$$

Unfortunately, a construction in $\mathcal{H}_n$ similar to (5.0.9) does not lead to idempotents in $\mathcal{H}_n$. Note also that $F_{\mathfrak{s}} \notin \mathcal{H}_n^{\mathcal{O}}$ because of the denominators. In order to get idempotents in $\mathcal{H}_n^{\mathcal{O}}$ and $\mathcal{H}_n$, we consider the sum over the $F_{\mathfrak{s}}$'s for $\mathfrak{s}$ belonging to a class of a certain equivalence relation on tableaux, that we now explain. Let $\mathfrak{s}$ and $\mathfrak{t}$ be tableaux for multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. Then we set $\mathfrak{s} \sim_e \mathfrak{t}$ if $\mathrm{res}(\mathfrak{s}(i)) = \mathrm{res}(\mathfrak{t}(i)) \bmod e$ for all i, or equivalently $c_{\mathfrak{s}}(i) = c_{\mathfrak{t}}(i)$ for all i. This indeed defines an equivalence class on the set of all tableaux. We denote by $[\mathfrak{s}] = [\mathfrak{s}]_e$ the class under $\sim_e$ represented by $\mathfrak{s}$ and set

$$E_{[\mathfrak{s}]} := \sum_{\mathfrak{t} \in [\mathfrak{s}] \cap \mathrm{Std}(n)} F_{\mathfrak{t}}. \tag{5.0.11}$$

Then Mathas has proved in [30], building on Murphy's ideas in the symmetric group case, that $E_{[\mathfrak{s}]}$ belongs to $\mathcal{H}_n^{\mathcal{O}}$ and hence $E_{[\mathfrak{s}]} \otimes_{\mathcal{O}} 1$ belongs to $\mathcal{H}_n$. We shall write $E_{[\mathfrak{s}]}$ for $E_{[\mathfrak{s}]} \otimes_{\mathcal{O}} 1$ as well. Clearly the $E_{[\mathfrak{s}]}$'s are orthogonal idempotents in both $\mathcal{H}_n$ and $\mathcal{H}_n^{\mathcal{O}}$.

Any equivalence class $[\mathfrak{s}]$ gives rise to a residue sequence $\boldsymbol{i}^{\mathfrak{s}} := (i_1, i_2, \ldots, i_n) \in I_e^n$ via $i_j := c_{\mathfrak{s}}(j)$. By construction, $\boldsymbol{i}^{\mathfrak{s}}$ is independent of the choice of representative of $[\mathfrak{s}]$.

The Brundan-Kleshchev and Rouquier isomorphism Theorem establishes an isomorphism of $\mathbb{F}$-algebras $f : \mathcal{R}_n \cong \mathcal{H}_n$. We need to explain the images of the generators under f.

In the case of $f(e(\boldsymbol{i}))$, Brundan and Kleshchev describe it as the idempotent for the generalized eigenspace for the joint action of the L_i's, that is

$$f(e(\boldsymbol{i}))\mathcal{H}_n = \{h \in \mathcal{H}_n \mid (L_k - i_k)^m h = 0 \text{ for some } m > 1\}. \tag{5.0.12}$$

There is however a more concrete description of $f(e(\boldsymbol{i}))$ due to Hu-Mathas, see [17]. It is of importance to us because it allows us to lift $f(e(\boldsymbol{i}))$ to $\mathcal{H}_n^{\mathcal{K}}$, via (5.0.11). It is given by the formula

$$f(e(\boldsymbol{i})) = \begin{cases} E_{[\mathfrak{s}]} & \text{if } \boldsymbol{i} = \boldsymbol{i}^{\mathfrak{s}} \text{ for some } \mathfrak{s} \in \mathrm{Std}(n) \\ 0 & \text{otherwise.} \end{cases} \tag{5.0.13}$$

In order to describe $f(y_i)$ and $f(\psi_i)$ it is enough to describe $f(y_i)E_{[\mathfrak{s}]}$ and $f(\psi_i)E_{[\mathfrak{s}]}$, since we have that $\sum_{[\mathfrak{s}]} E_{[\mathfrak{s}]} = 1$. In [7] $f(y_i)$ is described as the 'nilpotent part of the Jucys-Murphy element L_i', or more precisely

$$f(y_i)E_{[\mathfrak{s}]} = \left(1 - \frac{1}{c_{\mathfrak{s}}(i)} L_i\right) E_{[\mathfrak{s}]}. \tag{5.0.14}$$

We have a lift of this to $\mathcal{H}_n^{\mathcal{K}}$ as well. Supposing that $c_{\mathfrak{s}}(i) = q^{\kappa_m + c - r} \in \mathbb{F}$ we let $\widehat{c_{\mathfrak{s}}}(i) := \hat{q}^{\hat{\kappa}_m + \hat{c} - \hat{r}}$ where $\hat{c} - \hat{r} \in \mathbb{Z}$ is any preimage of $c - r$ mod e. Then our lift of (5.0.14) is

$$\left(1 - \frac{1}{c_{\mathfrak{s}}(i)} L_i\right) \sum_{\mathfrak{t} \in [\mathfrak{s}]} F_{\mathfrak{t}} = \sum_{\mathfrak{t} \in [\mathfrak{s}]} \left(1 - \frac{c_{\mathfrak{t}}^{\mathcal{K}}(i)}{\widehat{c_{\mathfrak{s}}}(i)}\right) F_{\mathfrak{t}} \in \mathcal{H}_n^{\mathcal{K}}. \tag{5.0.15}$$

The y_i's are nilpotent elements of $\mathcal{R}_n$. Using this, Brundan and Kleshchev define in [7] formal power series $P_i(\boldsymbol{i}), Q_i(\boldsymbol{i})$ in $\mathbb{F}[[y_i, y_{i+1}]]$. They give the formula

$$\psi_i e(\boldsymbol{i}) = (T_i + P_r(\boldsymbol{i})) Q_i(\boldsymbol{i})^{-1} e(\boldsymbol{i}) \tag{5.0.16}$$

which defines $f(\psi_i)$ since we already know $f(y_i)$ and $f(e(\boldsymbol{i}))$.

To make use of these formulas we shall rely on $\{f_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathrm{Par}_n\}$, the *seminormal basis* for $\mathcal{H}_n^{\mathcal{K}}$, constructed by Mathas in [30]. We have that

$$F_{\mathfrak{s}} f_{\mathfrak{s}_1 \mathfrak{t}_1} F_{\mathfrak{t}} = \delta_{\mathfrak{s}, \mathfrak{s}_1} \delta_{\mathfrak{t}, \mathfrak{t}_1} f_{\mathfrak{st}} \tag{5.0.17}$$

where $\delta_{\mathfrak{s}, \mathfrak{s}_1}$ and $\delta_{\mathfrak{t}, \mathfrak{t}_1}$ are Kronecker delta functions, and so $\{f_{\mathfrak{st}}\}$ is a $\mathcal{K}$-basis for $\mathcal{H}_n^{\mathcal{K}}$ consisting of eigenvectors for the action of the L_i's.

We need the following analog of the classical formulas for the action of s_i on the seminormal basis of the group algebra of the symmetric group. In this particular case, they are due to Mathas, see Proposition 2.7 of [30].

Proposition 5.0.2. *Let $\mathfrak{s}$ and $\mathfrak{u}$ be standard $\boldsymbol{\lambda}$-tableaux and let $\mathfrak{t} = \mathfrak{s} s_i$. If $\mathfrak{t}$ is standard then*

$$f_{\mathfrak{us}} T_i = \begin{cases} \frac{(q-1)c_{\mathfrak{t}}^{\mathcal{K}}(i)}{c_{\mathfrak{t}}^{\mathcal{K}}(i) - c_{\mathfrak{s}}^{\mathcal{K}}(i)} f_{\mathfrak{us}} + f_{\mathfrak{ut}} & \text{if } \mathfrak{s} \rhd_\infty \mathfrak{t} \\ \frac{(q-1)c_{\mathfrak{t}}^{\mathcal{K}}(i)}{c_{\mathfrak{t}}^{\mathcal{K}}(i) - c_{\mathfrak{s}}^{\mathcal{K}}(i)} f_{\mathfrak{us}} + \frac{(qc_{\mathfrak{s}}^{\mathcal{K}}(i) - c_{\mathfrak{t}}^{\mathcal{K}}(i))(c_{\mathfrak{s}}^{\mathcal{K}}(i) - qc_{\mathfrak{t}}^{\mathcal{K}}(i))}{(c_{\mathfrak{t}}^{\mathcal{K}}(i) - c_{\mathfrak{s}}^{\mathcal{K}}(i))^2} f_{\mathfrak{ut}} & \text{if } \mathfrak{s} \lhd_\infty \mathfrak{t} \end{cases} \tag{5.0.18}$$

whereas if $\mathfrak{t}$ is non-standard then

$$f_{\mathfrak{us}} T_i = \begin{cases} q f_{\mathfrak{us}} & \text{if } i \text{ and } i+1 \text{ are in the same row of } \mathfrak{s} \\ -f_{\mathfrak{us}} & \text{if } i \text{ and } i+1 \text{ are in the same column of } \mathfrak{s}. \end{cases} \tag{5.0.19}$$

There are versions of (5.0.18) and (5.0.19), with T_i multiplying on the left.

Actually there are some minor sign errors at this point in [30]. In fact, our formulas (5.0.18) are completely identical with the formulas used by Mathas in [30], but only our formulas are correct since Mathas' quadratic relations take the form $(T_r - 1)(T_r + q) = 0$ whereas ours are $(T_r + 1)(T_r - q) = 0$, see (5.0.2).

Note that the formulas of the Proposition depend on the order $\unlhd_\infty$, although we believe that it is possible to obtain similar formulas depending on $\unlhd_0$. Note also that it follows from the formulas that $\mathrm{span}_{\mathcal{K}}\{f_{\mathfrak{st}} \mid \mathrm{shape}(\mathfrak{s}) = \boldsymbol{\lambda}_0\}$ is a two-sided ideal of $\mathcal{H}_n^{\mathcal{K}}$ where $\boldsymbol{\lambda}_0$ is any fixed multipartition. Finally, note that all coefficients appearing in the formulas are nonzero. In the case of the second coefficient of (5.0.18), this is a consequence of the condition $i)$ on the multicharge $\hat{\kappa}$.

We have the following formula relating the seminormal basis to the $F_{\mathfrak{t}}$'s

$$F_{\mathfrak{t}} = \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{tt}} \tag{5.0.20}$$

where $\mathfrak{t}$ is any standard tableau of a multipartition $\boldsymbol{\lambda}$ and where $\gamma_{\mathfrak{t}} \in \mathcal{K}^\times$ is a known constant.

We need the following Lemma.

Lemma 5.0.3. *Let $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$ be a one-column multipartition and let $\mathfrak{t}^{\boldsymbol{\lambda}}$ be the maximal $\boldsymbol{\lambda}$-tableau, as above. Suppose that $\mathfrak{s} \in [\mathfrak{t}^{\boldsymbol{\lambda}}] \backslash \{\mathfrak{t}^{\boldsymbol{\lambda}}\}$ and that shape$(\mathfrak{s}) \in \mathrm{Par}_n^1$. Then $\mathfrak{s} > \mathfrak{t}^{\boldsymbol{\lambda}}$.*

Proof. Let $\mathfrak{s} \in [\mathfrak{t}^{\boldsymbol{\lambda}}] \setminus \{\mathfrak{t}^{\boldsymbol{\lambda}}\}$ and let $i \in \boldsymbol{n}$ be minimal such that $\mathfrak{s}(i) \neq \mathfrak{t}^{\boldsymbol{\lambda}}(i)$. The nodes $\mathfrak{s}(i)$ and $\mathfrak{t}^{\boldsymbol{\lambda}}(i)$ have the same residues since $\mathfrak{s} \sim_e \mathfrak{t}^{\boldsymbol{\lambda}}$ and so strong adjacency-freeness of $\hat{\kappa}$, together with the fact that $\mathfrak{s}$ is standard, implies that i is situated higher in $\mathfrak{s}$ than in $\mathfrak{t}^{\boldsymbol{\lambda}}$, that is $\mathfrak{s}(i) \rhd \mathfrak{t}^{\boldsymbol{\lambda}}(i)$. But then we have either $\mathfrak{s} > \mathfrak{t}^{\boldsymbol{\lambda}}$ or shape$(\mathfrak{s}) \notin \mathrm{Par}_n^1$ which proves the Lemma. ☐

With these preparations, we can now prove the linear independence of our proposed basis.

Theorem 5.0.4. *The set $\mathcal{C}_n = \{m_{\mathfrak{s}\mathfrak{t}} \mid \boldsymbol{\lambda} \in \mathrm{Par}_n^1,\ \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})\}$ introduced in (4.0.73) is linearly independent over $\mathbb{F}$ and hence it is a basis for $\mathbb{B}_n$.*

Proof. Let us assume that there is a non-trivial linear dependence between the elements of $\mathcal{C}_n$

$$\sum_{\mathfrak{s},\mathfrak{t}} \lambda_{\mathfrak{s}\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}} = 0. \tag{5.0.21}$$

Letting $\pi : \mathcal{R}_n \to \mathbb{B}_n$ be the projection map from the KLR-algebra to the blob-algebra and taking inverse images on both sides of (5.0.21) we then get

$$\sum_{\mathfrak{s},\mathfrak{t}} \lambda_{\mathfrak{s}\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}} + p = 0 \tag{5.0.22}$$

for some $p \in \ker \pi$ and so

$$\sum_{\mathfrak{s},\mathfrak{t}} \lambda_{\mathfrak{s}\mathfrak{t}} f(m_{\mathfrak{s}\mathfrak{t}}) + f(p) = 0. \tag{5.0.23}$$

We now note that any $f(m_{\mathfrak{s}\mathfrak{t}}) = f(\psi^*_{d(\mathfrak{s})} e(\boldsymbol{i}^{\boldsymbol{\lambda}})\psi_{d(\mathfrak{t})})$ can be written as a linear combination of terms of the form $T_v^* g_v(y) E_{[\mathfrak{t}^{\boldsymbol{\lambda}}]} f_w(y) T_w$ where $g_v(y), f_w(y) \in \mathbb{F}[y_1,\ldots,y_n]$ for some $v, w \in \mathfrak{S}_n$ with $v \geq d(\mathfrak{s})$ and $w \geq d(\mathfrak{t})$ and where $g_{d(\mathfrak{s})}(y)$ and $f_{d(\mathfrak{t})}(y)$ are invertible, that is of nonzero constant terms. That this is possible follows from (5.0.13) and an observation due to Hu and Mathas, see the proof of Lemma 5.4 of [17]. Combining this expansion with Lemma 4.0.3 we get that

$$f(m_{\mathfrak{s}\mathfrak{t}}) = T^*_{d(\mathfrak{s})} E_{[\mathfrak{t}^{\boldsymbol{\lambda}}]} T_{d(\mathfrak{t})} + \sum_{v > d(\mathfrak{s}), w > d(\mathfrak{t})} \mu_{v,w}\, T_v^* E_{[\mathfrak{t}^{\boldsymbol{\lambda}}]} T_w + \sum_{\boldsymbol{\mu} > \boldsymbol{\lambda}} f(D_{\boldsymbol{\mu}}) + f(p_1) \tag{5.0.24}$$

where $D_{\boldsymbol{\mu}} \in \langle e(\boldsymbol{i}^{\boldsymbol{\mu}})\rangle$, $\mu_{v,w} \in \mathbb{F}$ and $p_1 \in \ker \pi$. This expression for $f(m_{\mathfrak{s}\mathfrak{t}})$ takes place in $\mathcal{H}_n$, but can be lifted to $\mathcal{H}_n^{\mathcal{O}}$ via (5.0.13) and then embedded in $\mathcal{H}_n^{\mathcal{K}}$. Let us now analyse the various ingredients of (5.0.24), starting with $f(p_1)$. We have that

$$\ker \pi = \langle e(\boldsymbol{i}) \mid i_1 \in \{\kappa_1,\ldots,\kappa_l\}, i_2 = i_1 + 1 \bmod e\rangle \subseteq \mathcal{R}_n \tag{5.0.25}$$

corresponding to the omission of relation (3.0.3). Using (5.0.11) and (5.0.13) we then get that

$$f(p_1) = \sum_{\mathfrak{s} \in \mathrm{Std}(n)} \sum_{\mathfrak{t} \in [\mathfrak{s}]} a^{\mathfrak{s}}_{\mathfrak{t},1} F_{\mathfrak{t}}\, a^{\mathfrak{s}}_{\mathfrak{t},2} \tag{5.0.26}$$

where $a^{\mathfrak{s}}_{\mathfrak{t},1}, a^{\mathfrak{s}}_{\mathfrak{t},2} \in \mathcal{H}_n^{\mathcal{K}}$ and where $\mathfrak{s} \in \mathrm{Std}(n)$ satisfies $\mathrm{res}(\mathfrak{s}(1)) \in \{\kappa_1,\ldots,\kappa_l\}$ and $\mathrm{res}(\mathfrak{s}(2)) = \mathrm{res}(\mathfrak{s}(1)) + 1 \bmod e$. These conditions, together with the conditions on $\hat{\kappa}$, imply that for each $\mathfrak{t} \in [\mathfrak{s}]$ we have $\mathrm{shape}(\mathfrak{t}) \notin \mathrm{Par}_n^1$. Combining this with Proposition 5.0.2 and (5.0.20) we get that

$$f(p_1) \in \mathrm{span}_{\mathcal{K}}\{f_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda} \notin \mathrm{Par}_n^1\}. \tag{5.0.27}$$

Let us now consider the terms $f(D_{\boldsymbol{\mu}})$ of (5.0.24). We have that

$$f(D_{\boldsymbol{\mu}}) = \sum_{\mathfrak{t} \in [\mathfrak{t}^{\boldsymbol{\mu}}]} a_{\mathfrak{t},1} F_{\mathfrak{t}}\, a_{\mathfrak{t},2} \tag{5.0.28}$$

where $a_{\mathfrak{t},1}, a_{\mathfrak{t},2} \in \mathcal{H}_n^{\mathcal{K}}$. For each appearing $\mathfrak{t}$ we have $\mathfrak{t} > \mathfrak{t}^{\boldsymbol{\mu}}$ by Lemma 5.0.3. Combining this with $\boldsymbol{\mu} > \boldsymbol{\lambda}$, that is $\mathfrak{t}^{\boldsymbol{\mu}} > \mathfrak{t}^{\boldsymbol{\lambda}}$, we get that $\mathfrak{t} > \mathfrak{t}^{\boldsymbol{\lambda}}$ and so there is a k such that $\mathfrak{t}|_k = \mathfrak{t}^{\boldsymbol{\lambda}}|_k$ and $\mathfrak{t}(k+1) \rhd \mathfrak{t}^{\boldsymbol{\lambda}}(k+1)$. But then $\mathfrak{t}(k+1) \notin [\boldsymbol{\lambda}]$, which implies that $\mathrm{shape}(\mathfrak{t}) > \boldsymbol{\lambda}$. Hence we have that

$$f(D_{\boldsymbol{\mu}}) \in \mathrm{span}_{\mathcal{K}}\{f_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\nu}), \boldsymbol{\nu} > \boldsymbol{\lambda}\}. \tag{5.0.29}$$

Similarly, for all tableaux $\mathfrak{t}$ in $[\mathfrak{t}^{\boldsymbol{\lambda}}]$ we have that $\mathrm{shape}(\mathfrak{t}) > \boldsymbol{\lambda}$. Hence from (5.0.24), (5.0.27) and (5.0.29) we get that

$$f(m_{\mathfrak{s}\mathfrak{t}}) \in T^*_{d(\mathfrak{s})} F_{\boldsymbol{\lambda}} T_{d(\mathfrak{t})} + \sum_{v > d(\mathfrak{s}), w > d(\mathfrak{t})} \mu_{v,w}\, T_v^* F_{\boldsymbol{\lambda}} T_w + \mathrm{span}_{\mathcal{K}}\{f_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\nu}), \boldsymbol{\nu} > \boldsymbol{\lambda} \text{ or } \boldsymbol{\nu} \notin \mathrm{Par}_n^1\} \tag{5.0.30}$$

where $\boldsymbol{\lambda}$ as a subscript refers to $\mathfrak{t}^{\boldsymbol{\lambda}}$.

Let us now focus on $T^*_{d(\mathfrak{s})}F_{\boldsymbol{\lambda}}T_{d(\mathfrak{t})}$. Let $d(\mathfrak{t}) = s_{i_1}s_{i_2}\cdots s_{i_N}$ be a reduced expression for $d(\mathfrak{t})$. When calculating $f_{\boldsymbol{\lambda}}T_{d(\mathfrak{t})}$ using this expression and Proposition 5.0.2, we obtain an expression for $f_{\boldsymbol{\lambda}}T_{d(\mathfrak{t})}$ as a $\mathcal{K}$-linear combination of certain $f_{\boldsymbol{\lambda}\mathfrak{u}}$'s. But by the formulas of the Proposition, for each appearing $\mathfrak{u}$ we have that $d(\mathfrak{u})$ is a subexpression of $s_{i_1}s_{i_2}\cdots s_{i_N}$ and so by our version of the Ehresmann Theorem, that is Theorem 2.0.4, we have that $\mathfrak{t}\trianglelefteq\mathfrak{u}$ for each occurring $f_{\boldsymbol{\lambda}\mathfrak{u}}$. Letting $\mathfrak{t}_k := \mathfrak{t}^{\boldsymbol{\lambda}}s_{i_1}\dots s_{i_k}$ we have $\mathfrak{t}_{k+1}\triangleleft\mathfrak{t}_k$ for all $k = 1,\dots,N-1$ and so in the above expansion of $f_{\boldsymbol{\lambda}}T_{d(\mathfrak{t})}$ the term $f_{\boldsymbol{\lambda}\mathfrak{t}}$ corresponds exactly to the subexpression of $s_{i_1}s_{i_2}\cdots s_{i_N}$ where no s_i is omitted. By the remarks following the Proposition, the corresponding coefficient $\alpha_{\mathfrak{t}}$ is nonzero and so we have

$$f_{\boldsymbol{\lambda}}T_{d(\mathfrak{t})} = \alpha_{\mathfrak{t}}f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \sum_{\mathfrak{u}\triangleright\mathfrak{t}}\alpha_{\mathfrak{u}}f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{u}} \tag{5.0.31}$$

where $\alpha_{\mathfrak{s}}, \alpha_{\mathfrak{u}} \in \mathcal{K}$ and where $\alpha_{\mathfrak{t}} \neq 0$. Acting on the left with $T^*_{d(\mathfrak{s})}$, and arguing the same way as we did for (5.0.31), we obtain an expansion

$$T^*_{d(\mathfrak{s})}f_{\boldsymbol{\lambda}}T_{d(\mathfrak{t})} = \alpha_{\mathfrak{s}\mathfrak{t}}f_{\mathfrak{s}\mathfrak{t}} + \sum_{\mathfrak{u},\mathfrak{v}\triangleright\mathfrak{t}}\alpha_{\mathfrak{u}\mathfrak{v}}f_{\mathfrak{u}\mathfrak{v}} \tag{5.0.32}$$

where $\alpha_{\mathfrak{v}\mathfrak{u}}, \alpha_{\mathfrak{s}\mathfrak{t}} \in \mathcal{K}$ and where $\alpha_{\mathfrak{s}\mathfrak{t}} \neq 0$. Let us now focus on the term $T^*_vF_{\boldsymbol{\lambda}}T_w$ of (5.0.30). But arguing as was done for $T^*_{d(\mathfrak{s})}F_{\boldsymbol{\lambda}}T_{d(\mathfrak{t})}$, we can write $T^*_vF_{\boldsymbol{\lambda}}T_w$ as a linear combination of $f_{\mathfrak{v}\mathfrak{u}}$'s. Moreover, since $v > d(\mathfrak{s})$ and $w > d(\mathfrak{t})$ we get for each appearing $\mathfrak{u}$ and $\mathfrak{v}$ the relations $\mathfrak{u}\triangleright\mathfrak{s}$ and $\mathfrak{v}\triangleright\mathfrak{t}$.

All together we can now write (5.0.30) in the form

$$f(m_{\mathfrak{s}\mathfrak{t}}) \in \alpha_{\mathfrak{s}\mathfrak{t}}f_{\mathfrak{s}\mathfrak{t}} + \sum_{\mathfrak{u}\triangleright\mathfrak{s},\mathfrak{v}\triangleright\mathfrak{t}}\alpha_{\mathfrak{u}\mathfrak{v}}f_{\mathfrak{u}\mathfrak{v}} + \operatorname{span}_{\mathcal{K}}\{f_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t}\in \operatorname{Std}(\boldsymbol{\nu}), \boldsymbol{\nu} > \boldsymbol{\lambda} \text{ or } \boldsymbol{\nu}\notin \operatorname{Par}^1_n\} \tag{5.0.33}$$

where $\alpha_{\mathfrak{s}\mathfrak{t}} \in \mathcal{K}^\times$ and $\alpha_{\mathfrak{u}\mathfrak{v}} \in \mathcal{K}$.

Let us finally return to the linear dependency (5.0.23). Let us extend the order $\triangleleft$ to pairs $\{(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}(\boldsymbol{\lambda})^2 \mid \boldsymbol{\lambda} \in \operatorname{Par}^1_n\}$ via $(\mathfrak{s},\mathfrak{t})\triangleleft(\mathfrak{s}_1,\mathfrak{t}_1)$ if $\mathfrak{s}\triangleleft\mathfrak{s}_1$ and $\mathfrak{t}\triangleleft\mathfrak{t}_1$ and let us choose $(\mathfrak{s}_0,\mathfrak{t}_0)$ minimal such that $\lambda_{\mathfrak{s}_0\mathfrak{t}_0} \neq 0$. Let $\boldsymbol{\lambda}_0 = \operatorname{shape}(\mathfrak{s}_0)$. Using (5.0.33) we can rewrite (5.0.23) in terms of the $f_{\mathfrak{s}\mathfrak{t}}$'s. In this expression, there are no cancellations for the coefficient of $f_{\mathfrak{s}_0\mathfrak{t}_0}$'s which is therefore $\lambda_{\mathfrak{s}_0\mathfrak{t}_0}\cdot\alpha_{\mathfrak{s}_0\mathfrak{t}_0} \neq 0$. But this is in contradiction with the fact that the $f_{\mathfrak{s}\mathfrak{t}}$'s form a basis for $\mathcal{H}^{\mathcal{K}}_n$ and so the Theorem is proved. □

Chapter 6

Cellularity of $\mathcal{C}_n$ and JM-elements

In this chapter we obtain our main results of this part of the thesis, showing that $\mathcal{C}_n$ is a cellular basis for $\mathbb{B}_n$ with respect to $\lhd$, endowed with a family of JM-elements.

In the previous chapters we have proved that $\mathcal{C}_n$ is a basis for $\mathbb{B}_n$ and in fact one can even deduce from the results of these sections that $\mathcal{C}_n$ is a graded cellular basis for $\mathbb{B}_n$, with respect to $<$. However, we aim at proving the stronger statement that $\mathcal{C}_n$ is a graded cellular basis with respect to $\lhd$. The key combinatorial ingredient that allows us to pass from $<$ to $\lhd$ is given by the following two Lemmas.

Lemma 6.0.1. *Let $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$ be a one-column multipartition and let $\mathfrak{t}^{\boldsymbol{\lambda}}$ be the maximal $\boldsymbol{\lambda}$-tableau, as before. Suppose that $\mathfrak{t} \in [\mathfrak{t}^{\boldsymbol{\lambda}}] \setminus \{\mathfrak{t}^{\boldsymbol{\lambda}}\}$ and that shape$(\mathfrak{s}) \in \mathrm{Par}_n^1$. Then shape$(\mathfrak{t}) \rhd \boldsymbol{\lambda}$.*

Proof. Set $\boldsymbol{\mu} := \mathrm{shape}(\mathfrak{t})$. By Lemma 2.0.1 it is enough to find a bijection $\Theta : [\boldsymbol{\lambda}] \to [\boldsymbol{\mu}]$ such that $\Theta(\gamma) \unrhd \gamma$ for all $\gamma \in [\boldsymbol{\lambda}]$. Our candidate for this bijection is $\Theta := \mathfrak{t} \circ (\mathfrak{t}^{\boldsymbol{\lambda}})^{-1}$. Surely Θ is a bijection so let us check that Θ satisfies the order condition. Assume to the contrary that there is $\gamma = \mathfrak{t}^{\boldsymbol{\lambda}}(k) \in [\boldsymbol{\lambda}]$ such that $\Theta(\gamma) \lhd \gamma$, or equivalently $\mathfrak{t}(k) \lhd \mathfrak{t}^{\boldsymbol{\lambda}}(k)$, and let k_0 be the minimal such k. Let $\mathfrak{t}^{\boldsymbol{\lambda}}(k_0) = (r_0, 1, j_0)$ and $\mathfrak{t}(k_0) = (r, 1, j)$. By strong adjacency-freeness of κ, and the fact that $\mathfrak{t}^{\boldsymbol{\lambda}}(k_0)$ and $\mathfrak{t}(k_0)$ have the same residue, we have that $r > r_0 + 1$, that is $\mathfrak{t}(k_0)$ is located at least two rows below $\mathfrak{t}^{\boldsymbol{\lambda}}(k_0)$. But by minimality of k_0 we have that $\mathfrak{t}(k)$ is located above $\mathfrak{t}^{\boldsymbol{\lambda}}(k)$ for all $k < k_0$. This is impossible since $\mathfrak{t}$ is standard. □

For the next Lemma we need the conditions *iii)* and *iv)* from Definition 3.0.1 of strong adjacency-freeness.

Lemma 6.0.2. *Let $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$ be a one-column multipartition and let $\mathfrak{g}$ be Garnir tableau of shape $\boldsymbol{\lambda}$. Let $\mathfrak{t} \in [\mathfrak{g}] \setminus \{\mathfrak{g}\}$ and suppose that shape$(\mathfrak{t}) \in \mathrm{Par}_n^1$. Then shape$(\mathfrak{t}) \rhd \boldsymbol{\lambda}$.*

Proof. We shall follow the same approach as in the proof of the previous Lemma. Set $\boldsymbol{\mu} := \mathrm{shape}(\mathfrak{t})$. As in the previous Lemma it is enough to find a bijection $\Theta : [\boldsymbol{\lambda}] \to [\boldsymbol{\mu}]$ such that $\Theta(\beta) \unrhd \beta$ for all $\beta \in [\boldsymbol{\lambda}]$. This time the candidate for the bijection is $\Theta := \mathfrak{t} \circ (\mathfrak{g})^{-1}$. This Θ is also clearly a bijection so we must check that Θ satisfies the order condition. Assume to the contrary that there is $\beta = \mathfrak{g}(k) \in [\boldsymbol{\lambda}]$ such that $\Theta(\beta) \lhd \beta$, or equivalently $\mathfrak{t}(k) \lhd \mathfrak{g}(k)$, and let k_0 be the minimal such k. Let $\mathfrak{g}(k_0) = (r_0, 1, j_0)$ and $\mathfrak{t}(k_0) = (r, 1, j)$. Using the previous Lemma, and part (3) of the characterization of Garnir tableaux given in Lemma 4.0.9, we conclude that $\mathfrak{g}(k_0) \in \mathrm{Snake}(\gamma)$, where γ is the special node for the Garnir tableau $\mathfrak{g}$, according to Lemma 4.0.9. But then from strong adjacency-freeness of $\hat{\kappa}$ we conclude that $r = r_0 + 2$, since there are no nodes of the same residue in consecutive rows of $\boldsymbol{\lambda}$, that is $\mathfrak{t}(k_0) = (r, 1, j)$ is situated two rows below $\mathfrak{g}(k_0) = (r_0, 1, j_0)$. On the other hand, using condition *iv)* of the Definition 3.0.1 of strong adjacency-freeness, we get that those nodes in the r'th row of $[\mathrm{res}(\mathfrak{t}^{\boldsymbol{\lambda}})]$ that have the same residues as nodes in the r_0'th row, are all shifted one to the right. In other words, we have that $j = j_0 + 1$. But this produces a gap between $\mathfrak{t}(k_0)$ and $\mathrm{Snake}(\gamma)$ and so $\mathfrak{t}$ cannot be standard. The Lemma is proved.

Let us illustrate this last point on the following example with $\boldsymbol{\lambda} = ((1^{11}), (1^{11}), (1^{11}), (1^{10}), (1), (1^2))$, $e = 13$ and

$\gamma = (9, 1, 2)$:

$$\mathfrak{g} = \left(\begin{array}{|c|}\hline 1\\\hline 7\\\hline 12\\\hline 16\\\hline 20\\\hline 24\\\hline 28\\\hline 32\\\hline {\color{red}33}\\\hline 40\\\hline 44\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 8\\\hline 13\\\hline 17\\\hline 21\\\hline 25\\\hline 29\\\hline {\color{red}35}\\\hline {\color{red}34}\\\hline 41\\\hline 45\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 9\\\hline 14\\\hline 18\\\hline 22\\\hline 26\\\hline 30\\\hline {\color{red}36}\\\hline 38\\\hline 42\\\hline 46\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline 10\\\hline 15\\\hline 19\\\hline 23\\\hline 27\\\hline 31\\\hline {\color{red}37}\\\hline 39\\\hline 43\\\hline 47\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline\end{array}, \begin{array}{|c|}\hline 6\\\hline 11\\\hline\end{array} \right), [\mathrm{res}(\mathfrak{t}^{\boldsymbol{\lambda}})] = \left(\begin{array}{|c|}\hline 0\\\hline 12\\\hline 11\\\hline 10\\\hline 9\\\hline 8\\\hline 7\\\hline 6\\\hline {\color{red}5}\\\hline 4\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline 0\\\hline 12\\\hline 11\\\hline 10\\\hline 9\\\hline {\color{red}8}\\\hline {\color{red}7}\\\hline 6\\\hline 5\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline 3\\\hline 2\\\hline 1\\\hline 0\\\hline 12\\\hline 11\\\hline {\color{red}10}\\\hline 9\\\hline 8\\\hline 7\\\hline\end{array}, \begin{array}{|c|}\hline 6\\\hline 5\\\hline 4\\\hline 3\\\hline 2\\\hline 1\\\hline 0\\\hline {\color{red}12}\\\hline 11\\\hline 10\\\hline 9\\\hline\end{array}, \begin{array}{|c|}\hline 8\\\hline\end{array}, \begin{array}{|c|}\hline 10\\\hline 9\\\hline\end{array} \right). \tag{6.0.1}$$

The numbers appearing in $\mathrm{Snake}(\gamma)$ of $\mathfrak{g}$ have been colored red. We are supposing that $\mathfrak{t} \lhd \mathfrak{g}$. Consider the case where $\mathfrak{g}(k_0) = (8, 1, 2)$, that is $k_0 = 35$. Then for $\mathfrak{t}$ to be standard we must have either $\mathfrak{t}(35) = \mathfrak{g}(40)$ or $\mathfrak{t}(35) = \mathfrak{g}(41)$. But $\mathfrak{t}(35)$ is of residue 8 whereas neither $\mathfrak{g}(40)$ nor $\mathfrak{g}(41)$ is of residue 8, and so we get the desired contradiction in this case. The other cases for $\mathfrak{g}(k_0)$ are treated similarly.

For completeness, we now give a tableau $\mathfrak{t}$ in $[\mathfrak{g}]$. One checks easily that $\mathfrak{t} \rhd \mathfrak{g}$.

$$\mathfrak{t} = \left(\begin{array}{|c|}\hline 1\\\hline 7\\\hline 12\\\hline 16\\\hline 20\\\hline 24\\\hline 28\\\hline 32\\\hline {\color{red}33}\\\hline 40\\\hline 44\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 8\\\hline 13\\\hline 17\\\hline 21\\\hline 25\\\hline 29\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 9\\\hline 14\\\hline 18\\\hline 22\\\hline 26\\\hline 30\\\hline {\color{red}36}\\\hline 38\\\hline 42\\\hline 46\\\hline\end{array}, \begin{array}{|c|}\hline 4\\\hline 10\\\hline 15\\\hline 19\\\hline 23\\\hline 27\\\hline 31\\\hline {\color{red}37}\\\hline 39\\\hline 43\\\hline 47\\\hline\end{array}, \begin{array}{|c|}\hline 5\\\hline {\color{red}34}\\\hline 41\\\hline 45\\\hline\end{array}, \begin{array}{|c|}\hline 6\\\hline 11\\\hline {\color{red}35}\\\hline\end{array} \right). \tag{6.0.2}$$

□

We can now generalize the first statement of Lemma 4.0.3.

Lemma 6.0.3. *For $\boldsymbol{\lambda}$ any one-column multipartition and any k we have that*

$$y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = e(\boldsymbol{i}^{\boldsymbol{\lambda}}) y_k = \sum_{\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}), \boldsymbol{\mu} \rhd \boldsymbol{\lambda}} c_{\mathfrak{st}} m_{\mathfrak{st}} \tag{6.0.3}$$

where the sum runs over one-column multipartitions $\boldsymbol{\mu}$ of n and $c_{\mathfrak{st}} \in \mathbb{F}$.

Proof. We first note that by construction of the $m_{\mathfrak{st}}$'s we have that

$$e(\boldsymbol{i}) m_{\mathfrak{st}} = \begin{cases} m_{\mathfrak{st}} & \text{if } \boldsymbol{i} = \boldsymbol{i}^{\mathfrak{s}} \\ 0 & \text{otherwise.} \end{cases} \tag{6.0.4}$$

Let us now consider the expansion of $y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}})$ in the basis $\mathcal{C}_n$:

$$y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = \sum_{\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathrm{Par}_n^1} c_{\mathfrak{st}} m_{\mathfrak{st}} \tag{6.0.5}$$

where $c_{\mathfrak{st}} \in \mathbb{F}$. We have that

$$\sum_{\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathrm{Par}_n^1} c_{\mathfrak{st}} m_{\mathfrak{st}} = y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = e(\boldsymbol{i}^{\boldsymbol{\lambda}}) y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = \sum_{\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathrm{Par}_n^1} c_{\mathfrak{st}} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) m_{\mathfrak{st}} \tag{6.0.6}$$

and hence we get via (6.0.4) that $\mathfrak{t} \in [\boldsymbol{i}^{\boldsymbol{\lambda}}]$ whenever $c_{\mathfrak{st}} \neq 0$ and so also $\mathrm{shape}(\mathfrak{s}) \rhd \boldsymbol{\lambda}$, via Lemma 6.0.3. The Lemma is proved. □

We can also generalize the third statement (4.0.32) of Lemma 4.0.3 in the relevant case of a Garnir tableau $\mathfrak{g}$.

Lemma 6.0.4. *Let $\mathfrak{g}$ be a Garnir tableau for the multipartition $\boldsymbol{\lambda}$. Then we have an expansion of the form*

$$e(\boldsymbol{i}^{\mathfrak{g}}) = \sum_{\mathfrak{s},\mathfrak{t}\in \mathrm{Std}(\boldsymbol{\mu}), \boldsymbol{\mu}\rhd\boldsymbol{\lambda}} c_{\mathfrak{st}} m_{\mathfrak{st}} \tag{6.0.7}$$

where $c_{\mathfrak{st}} \in \mathbb{F}$.

Proof. From the Lemmas 4.0.3 and 4.0.5 we have the expansion

$$e(\boldsymbol{i}^{\mathfrak{g}}) = \sum_{\mathfrak{s},\mathfrak{t}\in \mathrm{Std}(\boldsymbol{\mu}), \boldsymbol{\mu}>\boldsymbol{\lambda}} c_{\mathfrak{st}} m_{\mathfrak{st}} \tag{6.0.8}$$

with unique coefficients $c_{\mathfrak{st}} \in \mathbb{F}$ since the $m_{\mathfrak{st}}$'s are a basis. Thus arguing as in the previous Lemma 6.0.3 we get that $\mathfrak{s} \in [\mathfrak{g}]$ and so shape$(\mathfrak{s}) \rhd \boldsymbol{\lambda}$ by Lemma 6.0.2. □

The following Lemma generalizes Lemma 4.0.11, replacing $<$ by $\lhd$.

Lemma 6.0.5. *Suppose that $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$ and that $\mathfrak{s},\mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda})$. If $\mathfrak{t} \in \mathrm{NStd}(\boldsymbol{\lambda})$ then there is an expansion*

$$m_{\mathfrak{st}} = \sum_{\mathfrak{t}_1\in \mathrm{Std}(\boldsymbol{\lambda}), \mathfrak{t}_1\rhd\mathfrak{t},} c_{\mathfrak{st}_1} m_{\mathfrak{st}_1} + \sum_{\boldsymbol{\mu}\rhd\boldsymbol{\lambda}, \mathfrak{s}_2,\mathfrak{t}_2\in \mathrm{Std}(\boldsymbol{\mu})} c_{\mathfrak{s}_2\mathfrak{t}_2} m_{\mathfrak{s}_2\mathfrak{t}_2} \tag{6.0.9}$$

where $c_{\mathfrak{st}_1}, c_{\mathfrak{s}_2\mathfrak{t}_2} \in \mathbb{F}$. A similar statement holds for $\mathfrak{s}$.

Proof. We go through the proof of Lemma 6.0.5, checking that each occurrence of $>$ can be replaced by $\rhd$. There are two types of occurrences of $>$. The first ones are in reference to (4.0.30) of Lemma 4.0.3. But here Lemma 6.0.3 allows us to replace $>$ by $\rhd$. The second ones are the use of Garnir tableaux in (4.0.67) and (4.0.70). But in view of Lemma 6.0.4 we can also here replace $>$ by $\rhd$. □

The following Lemma corresponds to the JM-property of the y_k's, that we shall consider in more detail later on.

Lemma 6.0.6. *Suppose that $m_{\mathfrak{st}}$ is an element of $\mathcal{C}_n$. Then we have that*

$$y_k m_{\mathfrak{st}} = \sum_{\mathfrak{s}_1\rhd\mathfrak{s}} c_{\mathfrak{s}_1\mathfrak{t}} m_{\mathfrak{s}_1\mathfrak{t}} + \textit{ higher terms} \tag{6.0.10}$$

where $c_{\mathfrak{s}_1\mathfrak{t}} \in \mathbb{F}$ and where 'higher terms' means a linear combination of $m_{\mathfrak{s}_2\mathfrak{t}_2}$ where shape$(\mathfrak{s}_2) \rhd$ shape$(\mathfrak{s})$. A similar formula holds for y_k acting on the right of $m_{\mathfrak{st}}$.

Proof. We have that $m^*_{\mathfrak{st}} = m_{\mathfrak{ts}}$ and so we get the formula for $m_{\mathfrak{st}} y_k$ by applying $*$ to the formula for $y_k m_{\mathfrak{st}}$. Suppose that $d(\mathfrak{s}) = s_{i_1}\cdots s_{i_{N-1}} s_{i_N}$ is the official reduced expression for $d(\mathfrak{s})$ so that we have $\psi_{d(\mathfrak{s})} = \psi_{i_1}\cdots\psi_{i_{N-1}}\psi_{i_N}$. We now have from relations (3.0.9), (3.0.10), (3.0.11) and (3.0.12) that

$$y_k m_{\mathfrak{st}} = y_k \psi^*_{d(\mathfrak{s})} e(\boldsymbol{i}^{\boldsymbol{\lambda}})\psi_{d(\mathfrak{t})} = \begin{cases} \psi_{i_N} y_k \psi_{i_{N-1}}\cdots\psi_{i_1} e(\boldsymbol{i}^{\boldsymbol{\lambda}})\psi_{d(\mathfrak{t})} & \text{if } i \neq i_N, i_N+1 \\ \psi_{i_N} y_{k\pm1}\psi_{i_{N-1}}\cdots\psi_{i_1} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) + \delta\psi_{i_{N-1}}\cdots\psi_{i_1} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) & \text{if } i = i_N, i_N+1 \end{cases} \tag{6.0.11}$$

where $\delta = 0, \pm1$. Using relations (3.0.9), (3.0.10), (3.0.11) and (3.0.12) once again, we continue commuting the appearing $y_{k\pm1}$'s to the right as far as possible, until they meet $e(\boldsymbol{i}^{\boldsymbol{\lambda}})$. This gives rise to a linear combination of terms of the form

$$\pm\psi_{j_K}\psi_{j_{K-1}}\cdots\psi_{j_1} e(\boldsymbol{i}^{\boldsymbol{\lambda}})\psi_{d(\mathfrak{t})} \tag{6.0.12}$$

where $s_{j_1}\cdots s_{j_{K-1}} s_{i_K}$ is a strict subexpression of $s_{i_1}\cdots s_{i_{N-1}} s_{i_N}$, together with $\psi^*_{d(\mathfrak{s})} y_j e(\boldsymbol{i}^{\boldsymbol{\lambda}})\psi_{d(\mathfrak{t})}$ for some j, corresponding to y_k commuted all the way through $\psi^*_{d(\mathfrak{s})}$, But this last term belongs to the 'higher terms', by the previous Lemma 6.0.3. The other terms that arise are linear combinations of $m_{\mathfrak{s}_1\mathfrak{t}}$'s where $\mathfrak{s}_1 \rhd \mathfrak{s}$ by the proof of Theorem 4.0.11. This proves the Lemma. □

We can now prove the promised cellularity of $\mathcal{C}_n$.

Theorem 6.0.7. *The pair $(\mathcal{C}_n, \mathrm{Par}_n^1)$ is a graded cellular basis for $\mathbb{B}_n$ with respect to $\lhd$, in the sense of Definition 1.1.1.*

Proof. Condition (**i**) of Definition1.1.1 is easily verified so let us concentrate on the multiplication Condition (**ii**). It is enough to check it for a any of the generators $e(\boldsymbol{i})$, y_i and ψ_i. Here the case $a = e(\boldsymbol{i})$ is easy and the case $a = y_i$ is given by Lemma 6.0.6, so we are left with the case $a = \psi_i$. We here consider right multiplication on $m_{\mathfrak{st}}$ with ψ_i. We first write $\psi_{d(\mathfrak{t})}\psi_i$ as a linear combination of the elements $\mathcal{S} = \{e(\boldsymbol{i})\, y^{\underline{k}}\, \psi_w \mid \boldsymbol{i} \in I_e^n, \underline{k} \in \mathbb{N}^n, w \in \mathfrak{S}_n\}$ from (4.0.51). Upon right multiplication we get that $m_{\mathfrak{st}}\psi_i$ is a linear combination of $\psi^*_{d(\mathfrak{s})}e(\boldsymbol{i}^{\boldsymbol{\lambda}})\psi_w$ modulo higher terms. For each appearing w we consider $\mathfrak{t}_1 := \mathfrak{t}^{\boldsymbol{\lambda}}w$ and get that $\psi^*_{d(\mathfrak{s})}e(\boldsymbol{i}^{\boldsymbol{\lambda}})\psi_w = m_{\mathfrak{st}_1}$. If $\mathfrak{t}_1$ is standard we have that $m_{\mathfrak{st}_1} \in \mathcal{C}_n$. Otherwise, we use Lemma 6.0.5 to rewrite $m_{\mathfrak{st}_1}$ in terms of elements of $\mathcal{C}_n$, modulo higher terms. Hence Condition (**ii**) has been verified and since $\mathcal{C}_n$ consists of homogeneous elements we are done. □

We remark that $\mathbb{B}_n$ even satisfies the stronger property of being a *quasi-hereditary algebra*. This follows from Remark 3.10 of [14].

The following definition appears for the first time in [32]. It formalizes important properties of Jucys-Murphy elements. These properties go back to Murphy's work on the symmetric group and the Hecke algebra of finite type A_n, see [33], [34] and [36].

Definition 6.0.8. *Let A be an $\mathbb{F}$-algebra which is cellular with respect to $\mathcal{C} = \{c_{\mathfrak{st}} \mid \lambda \in \Lambda, \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$. Suppose also that each set $T(\lambda)$ is endowed with a poset structure with order relation $\rhd_\lambda$. Then we say that a commuting subset $\mathcal{L} = \{L_1, \ldots, L_M\} \subseteq A$ is a family of JM-elements for A with respect to $\mathcal{C}$ if it satisfies that $L_i^* = L_i$ for all i and if there exists a set of scalars $\{c_{\mathfrak{t}}(i) \mid \mathfrak{t} \in T(\lambda),\ 1 \le i \le M\}$, denoted the content functions for λ, such that for all $\lambda \in \Lambda$ and $\mathfrak{t} \in T(\lambda)$ we have that*

$$c_{\mathfrak{st}}L_i = c_{\mathfrak{t}}(i)c_{\mathfrak{st}} + \sum_{\substack{\mathfrak{v}\in T(\lambda)\\ \mathfrak{v}\rhd_\lambda \mathfrak{t}}} r_{\mathfrak{sv}}c_{\mathfrak{sv}} \mod A^{\lambda} \tag{6.0.13}$$

for some $r_{\mathfrak{sv}} \in \mathbb{F}$.

We can now prove the following main Theorem of our thesis, proving that the Jucys-Murphy elements introduced in (5.0.4) give rise to JM-elements in the sense of the previous Lemma.

Theorem 6.0.9. *Let $L_i \in \mathcal{H}_n(q,\kappa)$ be the Jucys-Murphy element introduced in (5.0.4) and define $\mathcal{L}_i := f^{-1}(L_i) \in \mathcal{R}_n$. Then the set $\{\mathcal{L}_i \mid i = 1, \ldots, n\}$ is a family of JM-elements for $\mathbb{B}_n$ with respect to the cellular basis $\mathcal{C}_n$. The corresponding content function is the one introduced in (5.0.7):*

$$c_{\mathfrak{s}}(i) = q^{res(\mathfrak{s}(i))}. \tag{6.0.14}$$

Proof. By Theorem 1.1 of Brundan and Kleshchev's work, [7], we have that

$$\mathcal{L}_k = \sum_{\boldsymbol{i}\in I_e^n} q^{i_k}(1 - y_k)e(\boldsymbol{i}) \tag{6.0.15}$$

from which we get

$$\mathcal{L}_k e(\boldsymbol{i}^{\mathfrak{s}}) = (c_{\mathfrak{s}}(k) - y_k)e(\boldsymbol{i}^{\mathfrak{s}}) \tag{6.0.16}$$

for any standard tableau $\mathfrak{s}$. The Theorem now follows from Lemma 6.0.6. □

Chapter 7

Comparison with the original definition of $\mathbb{B}_n$.

In this chapter we show that $\mathbb{B}_n$ is isomorphic to the original generalized blob algebra, introduced by Martin and Woodcock in [28]. For the original blob algebra the coincidence of these two definitions was proved in [38]. Our proof is an extension of an argument presented in [38].

Let $\mathcal{H}_2$ be the cyclotomic Hecke algebra for $n = 2$, as introduced in Definition 5.0.1. It follows from strong adjacency-freeness of $\hat{\kappa}$ that $\mathcal{H}_2$ is a semisimple $\mathbb{F}$-algebra. Following [28], for $j = 1, \ldots, n$ we let e_2^j be the primitive, central idempotents associated with the one-dimensional module given by the multipartition $\boldsymbol{\lambda}_2^j := (\emptyset, \ldots, (2), \ldots, \emptyset)$ of 2, that has the partition (2) positioned in the j'th position. Since $\mathcal{H}_2 \subseteq \mathcal{H}_n$ we may consider e_2^j as an element of $\mathcal{H}_n$ and so we may consider $\mathcal{I}_n \subseteq \mathcal{H}_n$, the two-sided ideal generated by e_2^j for $j = 1, \ldots, n$. The generalized blob algebra $\mathbb{B}_n'$ introduced in [28] was now defined via

$$\mathbb{B}_n' := \mathcal{H}_n / \mathcal{I}_n. \tag{7.0.1}$$

In [28], concrete formulas for e_2^j were found. For $l = 2$ these formulas gave rise to an isomorphism between $\mathbb{B}_n'$ and the usual blob algebra. The following Lemma gives another description of e_2^j.

Lemma 7.0.1. *Let $F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}} \in \mathcal{H}_2^{\mathcal{K}}$ be the idempotent defined in (5.0.9). Then $F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}} \in \mathcal{H}_2^{\mathcal{O}}$ and $e_2^j = F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}} \otimes_{\mathcal{O}} \mathbb{F}$.*

Proof. It follows from strong adjacency-freeness of $\hat{\kappa}$ that the only standard tableau in the class $[\mathbf{t}^{\boldsymbol{\lambda}_2^j}]$ is $\mathbf{t}^{\boldsymbol{\lambda}_2^j}$ itself and so

$$E_{[\mathbf{t}^{\boldsymbol{\lambda}_2^j}]} = \sum_{\mathbf{t} \in [\mathbf{t}^{\boldsymbol{\lambda}_2^j}] \cap \mathrm{Std}(n)} F_{\mathbf{t}} = F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}}. \tag{7.0.2}$$

Since $E_{[\mathbf{t}^{\boldsymbol{\lambda}_2^j}]} \in \mathcal{H}_2^{\mathcal{O}}$ this shows that $F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}} \in \mathcal{H}_2^{\mathcal{O}}$. On the other hand, we have by (5.0.10) that

$$L_i F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}} = c_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}}(i) F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}} = \begin{cases} q^{\kappa_j} F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}} & \text{if } i = 1 \\ q^{\kappa_j + 1} F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}} & \text{if } i = 2 \end{cases} \tag{7.0.3}$$

and moreover, using (5.0.2) and (5.0.20), we have that

$$T_1 F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}} = q F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}}. \tag{7.0.4}$$

The two conditions (7.0.3) and (7.0.4) characterize e_2^j uniquely and so the Lemma is proved. $\square$

We can now prove the promised isomorphism between the two definitions of the generalized blob algebra.

Theorem 7.0.2. *Viewing $F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}}$ as elements of $\mathcal{H}_n$ we have the following equality in $\mathcal{R}_n$*

$$f^{-1}(F_{\mathbf{t}^{\boldsymbol{\lambda}_2^j}}) = \sum_{\substack{\boldsymbol{i} \in I_e^n \\ i_1 = \kappa_j, i_2 = \kappa_j + 1}} e(\boldsymbol{i}) \tag{7.0.5}$$

corresponding to relation (3.0.3) of $\mathbb{B}_n$. In particular, $\mathbb{B}_n' = \mathbb{B}_n$.

Proof. We have that $1 = \sum_{\boldsymbol{i} \in I_e^n} e(\boldsymbol{i}) = \sum_{\mathfrak{s} \in \mathrm{Std}(n)} f^{-1}(E_{\mathfrak{s}})$. On the other hand we have that

$$F_{\mathfrak{t}^{\lambda_2^j}} E_{\mathfrak{s}} = \sum_{\mathfrak{t} \in \mathrm{Std}(n)} F_{\mathfrak{t}^{\lambda_2^j}} F_{\mathfrak{t}} = \begin{cases} E_{\mathfrak{s}} & \text{if } i_1 = \kappa_j, i_2 = \kappa_j + 1 \\ 0 & \text{otherwise} \end{cases} \tag{7.0.6}$$

and so the Theorem follows. □

Part III

The Nil-blob algebra: An incarnation of type $\tilde{A}_1$ Soergel calculus and of the truncated blob algebra

Chapter 8

The nil-blob algebra

For the rest of this thesis, we fix a field $\mathbb{F}$ with $\text{char}(\mathbb{F}) \neq 2$. All our algebras are associative and unital $\mathbb{F}$-algebras. We also shall denote by $\mathbb{B}_n$ the blob algebra (the generalized blob algebra of level 2).

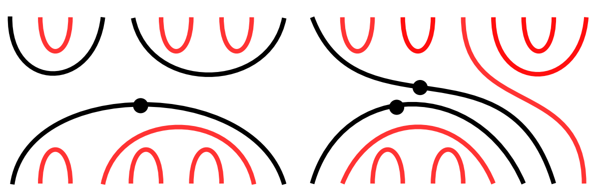
In this chapter we introduce and study the basic properties of the nil-blob algebra. Let us first recall the definition of the classical blob algebra $\mathbb{B}_n$. It was introduced by Martin and Saleur in [27]. We fix $q \in \mathbb{F}^\times$ and define for any $k \in \mathbb{Z}$ the usual Gaussian integer

$$[k] := q^{k-1} + q^{k-3} + \ldots + q^{-k+3} + q^{-k+1}. \tag{8.0.1}$$

Definition 8.0.1. *Let $m \in \mathbb{Z}$ with $[m] \neq 0$. The blob algebra $\mathbb{B}_n(m) = \mathbb{B}_n$ is the algebra generated by $\mathbb{V}_0, \mathbb{V}_1, \ldots, \mathbb{V}_{n-1}$ subject to the relations*

$$\mathbb{V}_i^2 = -[2]\mathbb{V}_i, \qquad \text{if } 1 \leq i < n; \tag{8.0.2}$$
$$\mathbb{V}_i\mathbb{V}_j\mathbb{V}_i = \mathbb{V}_i, \qquad \text{if } |i-j| = 1 \text{ and } i, j > 0; \tag{8.0.3}$$
$$\mathbb{V}_i\mathbb{V}_j = \mathbb{V}_j\mathbb{V}_i, \qquad \text{if } |i-j| > 1; \tag{8.0.4}$$
$$\mathbb{V}_1\mathbb{V}_0\mathbb{V}_1 = [m-1]\mathbb{V}_1, \tag{8.0.5}$$
$$\mathbb{V}_0^2 = -[m]\mathbb{V}_0. \tag{8.0.6}$$

An important feature of $\mathbb{B}_n$ is the fact that it is a diagram algebra. The diagram basis consists of blobbed (marked) Temperley-Lieb diagrams on n points where only arcs exposed to the left side of the diagram may be marked and at most once. The multiplication D_1D_2 of two diagrams D_1 and D_2 is given by concatenation of them, with D_1 on top of D_2. This concatenation process may give rise to internal marked or unmarked loops, as well as arcs with more than one mark. The internal unmarked loops are removed from a diagram by multiplying it by $-[2]$, whereas the internal marked loops are removed from a diagram by multiplying it by $-[m-1]/[m]$. Finally, any diagram with $r > 1$ marks on an arc is set equal to the same diagram with the $(r-1)$ extra marks removed. These marked Temperley-Lieb diagrams are called blob diagrams. Here is an example with $n = 20$.



(8.0.7)

The color red is here only used to indicate those arcs that are not exposed to the left side of the diagram and therefore cannot not be marked. For any of the black arcs the blob is optional.

Motivated in part by $\mathbb{B}_n$ we now define the nil-blob algebra $\mathbb{NB}_n$ and its extended version $\widetilde{\mathbb{NB}}_n$. They are the main objects of study of this part of the thesis.

Definition 8.0.2. *The nil-blob algebra* $\mathbb{NB}_n$ *is the algebra on the generators* $\mathbb{U}_0, \mathbb{U}_1, \ldots, \mathbb{U}_{n-1}$ *subject to the relations*

$$\mathbb{U}_i^2 = -2\mathbb{U}_i, \qquad \text{if } 1 \leq i < n; \tag{8.0.8}$$

$$\mathbb{U}_i\mathbb{U}_j\mathbb{U}_i = \mathbb{U}_i, \qquad \text{if } |i-j| = 1 \text{ and } i, j > 0; \tag{8.0.9}$$

$$\mathbb{U}_i\mathbb{U}_j = \mathbb{U}_j\mathbb{U}_i, \qquad \text{if } |i-j| > 1; \tag{8.0.10}$$

$$\mathbb{U}_1\mathbb{U}_0\mathbb{U}_1 = 0, \tag{8.0.11}$$

$$\mathbb{U}_0^2 = 0. \tag{8.0.12}$$

The extended nil-blob algebra $\widetilde{\mathbb{NB}}_n$ *is the algebra obtained from* $\mathbb{NB}_n$ *by adding an extra generator* $\mathbb{J}_n$ *which is central and satisfies* $\mathbb{J}_n^2 = 0$.

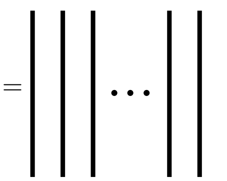
Remark 8.0.3. Note that the sign in (8.0.8) is unimportant. Indeed, replacing $\mathbb{U}_i$ with $-\mathbb{U}_i$ we get a presentation as in Definition 8.0.2 but with the sign in (8.0.8) positive.

It is known from [38] that $\mathbb{B}_n$ is a $\mathbb{Z}$-graded algebra. This is also the case for $\mathbb{NB}_n$ and $\widetilde{\mathbb{NB}}_n$ but is actually much easier to prove.

Lemma 8.0.4. *The rules* $\deg(\mathbb{U}_i) = 0$ *for* $i > 0$ *and* $\deg(\mathbb{U}_0) = \deg(\mathbb{J}_n) = 2$ *define (positive)* $\mathbb{Z}$*-gradings on* $\mathbb{NB}_n$ *and* $\widetilde{\mathbb{NB}}_n$.

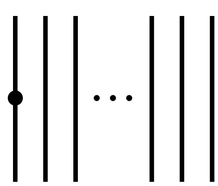
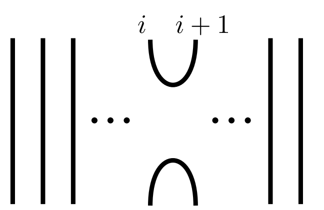
Proof. One checks easily that the relations are homogeneous with respect to deg. □

Our first goal is to show that $\mathbb{NB}_n$ is a diagram algebra with the same diagram basis as for $\mathbb{B}_n$, but with a slightly different multiplication rule. Indeed, in $\mathbb{NB}_n$ internal unmarked loops are removed from a diagram by multiplying it with -2, whereas diagrams in $\mathbb{NB}_n$ with a marked loop are set to zero. Moreover, in $\mathbb{NB}_n$ diagrams with a multiple marked arc are also set equal to zero. This defines an associative multiplication with identity element

$1 =$  (8.0.13)

That $\mathbb{NB}_n$ has this diagram realization follows from the results presented in the Appendix of [9], but for the reader's convenience we here present a different more self-contained proof of this fact, avoiding the theory of projection algebras. Let us denote by $\mathbb{NB}_n^{diag}$ the diagram algebra indicated above, with basis given by blob diagrams and multiplication rule as explained in the previous paragraph. We then prove the following Theorem:

Theorem 8.0.5. *There is an isomorphism between* $\mathbb{NB}_n$ *and* $\mathbb{NB}_n^{diag}$ *induced by*

$\mathbb{U}_0 \mapsto$  , $\mathbb{U}_i \mapsto$ 

(8.0.14)

In particular, $\mathbb{NB}_n$ *has the same dimension as* $\mathbb{B}_n$*, in other words*

$$\dim_{\mathbb{F}}(\mathbb{NB}_n) = \binom{2n}{n}. \tag{8.0.15}$$

Proof. One easily checks that the diagrams in (8.0.14) satisfy the relations for the $\mathbb{U}_i$'s in Definition 8.0.2 and so at least (8.0.14) induces an algebra homomorphism $\varphi : \mathbb{NB}_n \to \mathbb{NB}_n^{diag}$.

Although it is not possible to determine the dimension of $\mathbb{NB}_n$ directly, we can still get an upper bound for it using normal forms as follows. For $0 \leq j \leq i \leq n-1$ we define

$$\mathbb{U}_{ij} := \mathbb{U}_i\mathbb{U}_{i-1}\cdots\mathbb{U}_{j+1}\mathbb{U}_j \in \mathbb{NB}_n. \tag{8.0.16}$$

We consider ordered pairs (I, J) formed by sequences of numbers in $\{0, 1, 2, \dots, n-1\}$ of the same length k such that $I = (i_1, i_2, \dots, i_k)$ is strictly increasing, such that $J = (j_1, j_2, \dots, j_k)$ is strictly increasing too, except that there may be repetitions of 0, and such that $j_s \le i_s$ for all $1 \le s \le k$. For such pairs we define

$$\mathbb{U}_{IJ} := \mathbb{U}_{i_1 j_1} \mathbb{U}_{i_2 j_2} \cdots \mathbb{U}_{i_k j_k}. \tag{8.0.17}$$

A monomial of this form is called *normal*. We denote by $\mathcal{NM}_n$ the set formed by all normal monomials in $\mathbb{NB}_n$ together with 1. For $n = 2$ we have

$$\mathcal{NM}_1 = \{1, \mathbb{U}_0, \mathbb{U}_1, \mathbb{U}_1\mathbb{U}_0, \mathbb{U}_0\mathbb{U}_1, \mathbb{U}_0\mathbb{U}_1\mathbb{U}_0\}, \tag{8.0.18}$$

whereas for $n = 3$

$$\begin{aligned} \mathcal{NM}_2 = \{&1, \mathbb{U}_0, \mathbb{U}_1\mathbb{U}_0, \mathbb{U}_1, \mathbb{U}_2\mathbb{U}_1\mathbb{U}_0, \mathbb{U}_2\mathbb{U}_1, \mathbb{U}_2, \mathbb{U}_0\mathbb{U}_1\mathbb{U}_0, \mathbb{U}_0\mathbb{U}_1, \mathbb{U}_0\mathbb{U}_2\mathbb{U}_1\mathbb{U}_0, \mathbb{U}_0\mathbb{U}_2\mathbb{U}_1, \mathbb{U}_0\mathbb{U}_2, \mathbb{U}_1\mathbb{U}_0\mathbb{U}_2\mathbb{U}_1\mathbb{U}_0, \\ &\mathbb{U}_1\mathbb{U}_0\mathbb{U}_2\mathbb{U}_1, \mathbb{U}_1\mathbb{U}_0\mathbb{U}_2, \mathbb{U}_1\mathbb{U}_2, \mathbb{U}_0\mathbb{U}_1\mathbb{U}_0\mathbb{U}_2\mathbb{U}_1\mathbb{U}_0, \mathbb{U}_0\mathbb{U}_1\mathbb{U}_0\mathbb{U}_2\mathbb{U}_1, \mathbb{U}_0\mathbb{U}_1\mathbb{U}_0\mathbb{U}_2, \mathbb{U}_0\mathbb{U}_1\mathbb{U}_2\}. \end{aligned} \tag{8.0.19}$$

In general, using the relations given in Definition 8.0.2 one easily checks that $\mathcal{NM}_n$ spans $\mathbb{NB}_n$. Indeed, we have that $\{\mathbb{U}_0, \mathbb{U}_1, \dots, \mathbb{U}_{n-1}\} \subseteq \mathcal{NM}_n$ and that any product of the form $\mathbb{U}_i\mathbb{U}_{IJ}$ can be written as a linear combination of elements of $\mathcal{NM}_n$. On the other hand, the set $\mathcal{NM}_n$ is in bijection with the set of positive fully commutative elements of the Coxeter group of type B_n. In particular, the cardinality of $\mathcal{NM}_n$ is known to be $\binom{2n}{n}$, see for example [1]. Hence we deduce that

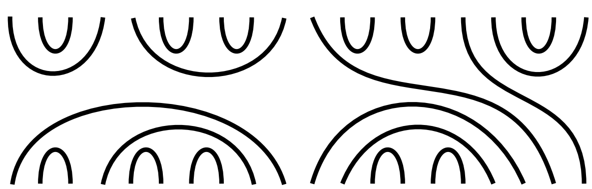
$$\dim \mathbb{NB}_n \le \dim \mathbb{NB}_n^{diag} \tag{8.0.20}$$

since $\dim \mathbb{NB}_n^{diag} = \dim \mathbb{B}_n = \binom{2n}{n}$. Thus, in order to show the Theorem we must check that φ is surjective, or equivalently that the diagrams in (8.0.14) generate $\mathbb{NB}_n^{diag}$.

Let us first focus on the 'Temperley-Lieb part' of $\mathbb{NB}_n^{diag}$, that is the subalgebra of $\mathbb{NB}_n^{diag}$ consisting of the linear combinations of Temperley-Lieb diagrams, the unmarked diagrams from $\mathbb{NB}_n^{diag}$. There is a concrete algorithm for obtaining any Temperley-Lieb diagram as a product of the $\varphi(\mathbb{U}_i)$'s, where $i > 0$, and so these diagrams generate the subalgebra. Although it is well known, we still explain how it works since we need a small variation of it.

In the following, whenever $\mathbb{U} \in \mathbb{NB}_n$ we shall often write $\mathbb{U} \in \mathbb{NB}_n^{diag}$ for $\varphi(\mathbb{U})$. This should not cause confusion.

Let D be a Temperley-Lieb diagram on n points with l through lines and let $k = (n-l)/2$. We associate with D two standard tableaux *top*(D) and *bot*(D) of shape $\lambda = (1^{l+k}, 1^k)$ as follows. For *top*(D) we go through the upper points of D, placing 1 in position $(1,1)$ of *top*(D), then 2 in position $(1,2)$ if 2 is the right end point of a horizontal arc, otherwise in position $(2,1)$, and so on recursively. Thus, having placed $1, 2 \dots, i-1$ in *top*(D) we place i in the first vacant position of the second column if i is the right end point of a horizontal arc, otherwise in the first vacant position of the first column. The standard tableau *bot*(D) is constructed the same way, using the bottom points of D. For example for the following diagram

$D =$  (8.0.21)

we have that

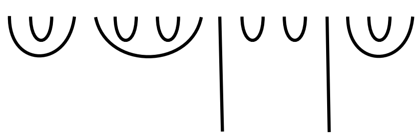
$$top(D) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 7 \\ \hline 6 & 9 \\ \hline 8 & 10 \\ \hline 11 & 13 \\ \hline 12 & 15 \\ \hline 14 & 19 \\ \hline 16 & 20 \\ \hline 17 & \\ \cline{1-1} 18 & \\ \cline{1-1} \end{array}, \qquad bot(D) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 & 8 \\ \hline 5 & 9 \\ \hline 7 & 10 \\ \hline 11 & 14 \\ \hline 12 & 16 \\ \hline 13 & 17 \\ \hline 15 & 18 \\ \hline 19 & \\ \cline{1-1} 20 & \\ \cline{1-1} \end{array} \tag{8.0.22}$$

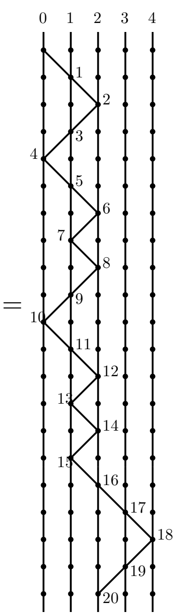
It is well known, and easy to see, that the map $D \mapsto (top(D), bot(D))$ is a bijection between Temperley-Lieb diagrams and pairs of two column standard tableaux of the same shape.

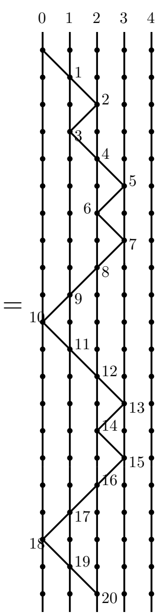
For $\mathbf{t}$ any Young tableau and $1 \leq k \leq n$ we define $\mathbf{t}|_k$ as the restriction of $\mathbf{t}$ to the set $\{1, 2, \ldots, k\}$. We may then consider a two-column standard tableaux $\mathbf{t}$ as a sequence of pairs $(i, \mathrm{diff}(\mathbf{t}|_i))$ for $i = 0, 1, 2 \ldots, n$, where $\mathrm{diff}(\mathbf{t}|_i)$ is the difference between the lengths of the first and the second column of the underlying shape of $\mathbf{t}|_i$ (here $i = 0$ corresponds to the pair $(0, 0)$). We then plot these pairs in a coordinate system, using matrix convention for the coordinates.

This may be viewed as a walk in this coordinate system, where at level i we step once to the left if $i + 1$ is in the second column of $\mathbf{t}$ and otherwise once to the right. In (8.0.24) we have indicated the corresponding walks for $top(D)$ and $bot(D)$ where D is as above in (8.0.21).

A Temperley-Lieb diagram D is given uniquely by $(top(D), bot(D))$ and so we introduce the corresponding half-diagrams. For example the top and bottom half-diagrams for D in (8.0.21) are as follows

$T :=$  $B :=$  (8.0.23)

$top(D) =$ 

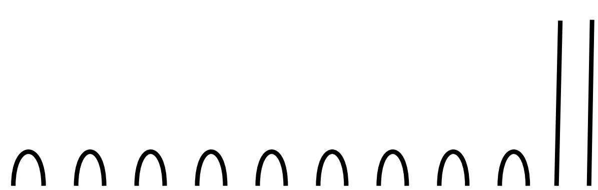
$bot(D) =$ 

(8.0.24)

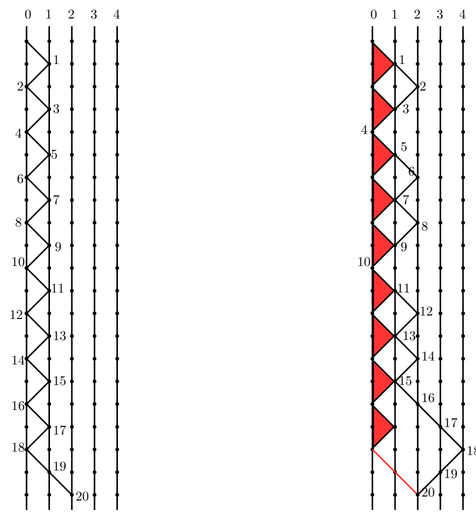
Recall that for any two column partition λ there is unique maximal λ-tableau $\mathbf{t}^\lambda$ under the dominance order. It is constructed as the row reading of λ. For example, for $\lambda = (1^{11}, 1^9)$ we have $\mathbf{t}^\lambda$ and its corresponding bottom half-diagram as follows

$\mathbf{t}^\lambda =$

1	2
3	4
5	6
7	8
9	10
11	12
13	14
15	16
17	18
19	
20	

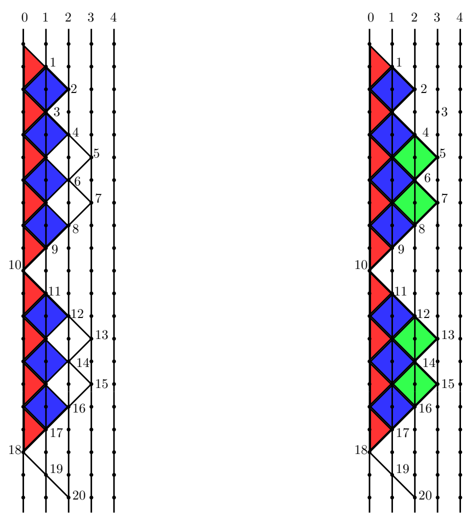
$B^\lambda :=$  (8.0.25)

The walk corresponding to $\mathbf{t}^\lambda$ is as follows where in the second and third figures we have colored it red and have combined it with the walks for $top(D)$ and $bot(D)$ from (8.0.24).



(8.0.26)

The algorithm for generating the Temperley-Lieb diagrams consists now in filling in the area between the walks for $\mathbf{t}^\lambda$ and $bot(D)$ (resp. $top(D)$) one column at the time, and then multiplying with the corresponding $\mathbb{U}_i$'s. For example, using the below figure (8.0.27),



(8.0.27)

we find that to obtain $bot(D)$ from the walk for $\mathbf{t}^\lambda$ we should first multiply by $\mathbb{U}_2\mathbb{U}_4\mathbb{U}_6\mathbb{U}_8\mathbb{U}_{12}\mathbb{U}_{14}\mathbb{U}_{16}$ corresponding to the blue area, and then with $\mathbb{U}_5\mathbb{U}_7\mathbb{U}_{13}\mathbb{U}_{15}$, corresponding to the green area, that is we have that

$$B = B^\lambda(\mathbb{U}_2\mathbb{U}_4\mathbb{U}_6\mathbb{U}_8\mathbb{U}_{12}\mathbb{U}_{14}\mathbb{U}_{16})(\mathbb{U}_5\mathbb{U}_7\mathbb{U}_{13}\mathbb{U}_{15}) \tag{8.0.28}$$

where B is the half-diagram in (8.0.23) and B^λ is the diagram defined in (8.0.25). Similarly, we have that

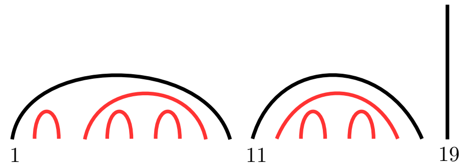
$$T = \mathbb{U}_{18}(\mathbb{U}_{17}\mathbb{U}_{19})(\mathbb{U}_2\mathbb{U}_6\mathbb{U}_8\mathbb{U}_{12}\mathbb{U}_{14}\mathbb{U}_{16}\mathbb{U}_{18})T^\lambda \tag{8.0.29}$$

where T is the half-diagram in (8.0.23) and T^λ is the reflection through a horizontal axis of B^λ. Since $T^\lambda B^\lambda = \mathbb{U}_1\mathbb{U}_3\mathbb{U}_5\mathbb{U}_7\mathbb{U}_9\mathbb{U}_{11}\mathbb{U}_{13}\mathbb{U}_{15}\mathbb{U}_{17}$ we get now D as a product of $\mathbb{U}_i$'s:

$$D = TB = \mathbb{U}_{18}(\mathbb{U}_{17}\mathbb{U}_{19})(\mathbb{U}_2\mathbb{U}_6\mathbb{U}_8\mathbb{U}_{12}\mathbb{U}_{14}\mathbb{U}_{16}\mathbb{U}_{18})T^\lambda B^\lambda(\mathbb{U}_2\mathbb{U}_4\mathbb{U}_6\mathbb{U}_8\mathbb{U}_{12}\mathbb{U}_{14}\mathbb{U}_{16})(\mathbb{U}_5\mathbb{U}_7\mathbb{U}_{13}\mathbb{U}_{15}). \tag{8.0.30}$$

Summing up, we have shown that any unmarked blob diagram can be obtained as a product of the generators $\mathbb{U}_i$'s, for $i > 0$.

We now explain how to obtain the marks on the arcs. In the case of B as before there are three arcs that may carry a mark, namely the black arcs below

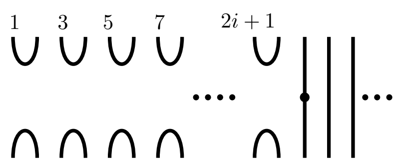


(8.0.31)

A main general observation for what follows is that these arcs are in correspondence with the 'contacts' between the associated walk and the vertical 0-line. To be precise for $i = 0, 1, \ldots, n-1$ we have that $(i, 0)$ belongs to the walk for B if and only if $i+1$ is the leftmost point of an arc that may be marked. For instance, using the walk in (8.0.27) for the above B we see that these points are $1, 11$ and 19, as one indeed observes in (8.0.31).

These contacts points induce a partition of the indices $1 \leq i \leq n$ and we call the corresponding classes for blocks. Thus in the above example (8.0.27), the first block consists of the indices $1 \leq i \leq 10$, the second of $11 \leq i \leq 18$ and the third of 19 and 20. We stress that the smallest number in each block is odd. On the other hand, under the above process of filling in the areas, the $\mathbb{U}_i$'s, where i corresponds to the rightmost index of some block, are not needed. But from this we deduce that the indices corresponding to distinct blocks give rise to commuting $\mathbb{U}_i$'s and hence we can in fact fill in one block at the time. We choose to do so going through the blocks of each walk from bottom to the top.

Our second observation is that any diagram of the form

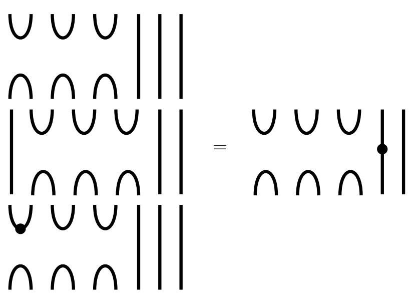


(8.0.32)

can be generated by the $\mathbb{U}_i$'s since indeed it is equal to

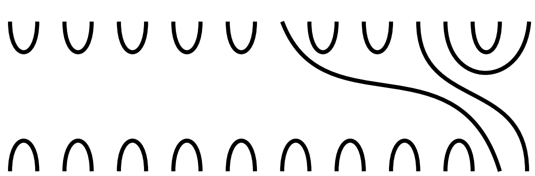
$$(\mathbb{U}_1\mathbb{U}_3\mathbb{U}_5 \cdots \mathbb{U}_{2i+1})\mathbb{U}_0(\mathbb{U}_2\mathbb{U}_4\mathbb{U}_6 \cdots \mathbb{U}_{2i+2})(\mathbb{U}_1\mathbb{U}_3\mathbb{U}_5 \cdots \mathbb{U}_{2i+1}). \tag{8.0.33}$$

Here is for example the case $i = 2$ and $n = 9$



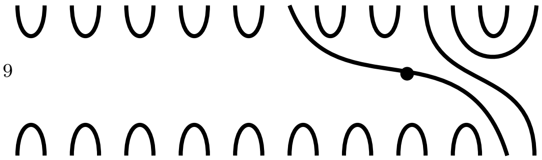
(8.0.34)

The algorithm for obtaining any marked diagram now consists in filling in by blocks, from bottom to top, and multiplying by a diagram of the form given in (8.0.32), for each block that requires a mark. Let us illustrate a few step of it on the blob diagram given in (8.0.7). Its bottom and top halves are given in (8.0.23). Both of them have three blocks. The third block is $\{11, 12, \ldots, 20\}$ for the top diagram and, as we have already seen, $\{19, 20\}$ for the bottom diagram. Multiplying with the corresponding $\mathbb{U}_i$'s on $T^\lambda B^\lambda$ we get the diagram



(8.0.35)

Suppose now that we want to produce the blob diagram from (8.0.7). Then we need a mark on the first through line and thus we multiply below with a diagram of the form (8.0.32) with $i = 8$ which gives us

$(-2)^9$  (8.0.36)

settling the third block, at least up to a unit in $\mathbb{F}$. The algorithm now goes on with the second block, etc. The Theorem is proved. □

In view of the Theorem 8.0.5 we shall write $\mathbb{NB}_n = \mathbb{NB}_n^{diag}$. Similarly we shall in general write $\mathbb{U}$ for $\varphi(\mathbb{U})$.

The next two corollaries are an immediate consequence of Theorem 8.0.5.

Corollary 8.0.6. *The set $\mathcal{NM}_n$ is a basis for $\mathbb{NB}_n$. Similarly, the set*

$$\widetilde{\mathcal{NM}}_n := \{X\mathbb{J}_n^i \mid X \in \mathcal{NM}_n,\ i \in \{0,1\}\} \tag{8.0.37}$$

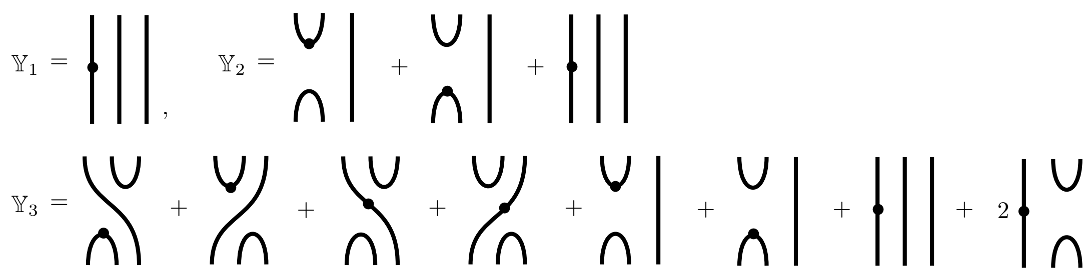
is a basis for $\widetilde{\mathbb{NB}}_n$. Consequently, $\dim \widetilde{\mathbb{NB}}_n = 2\binom{2n}{n}$.
We refer to the set $\mathcal{NM}_n$ (resp. $\widetilde{\mathcal{NM}}_n$) as the normal basis of $\mathbb{NB}_n$ (resp. $\widetilde{\mathbb{NB}}_n$).

Corollary 8.0.7. *$\mathbb{NB}_n$ is a cellular algebra in the sense of Graham and Lehrer, see [14], with the same cellular datum as for $\mathbb{B}_n$, see for example [38] for this cellular structure.*

Definition 8.0.8. *We define the JM-elements $\mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_n$ of $\mathbb{NB}_n$ via $\mathbb{Y}_1 = \mathbb{U}_0$ and recursively*

$$\mathbb{Y}_{i+1} = (\mathbb{U}_i + 1)\mathbb{Y}_i(\mathbb{U}_i + 1),\ i \geq 1. \tag{8.0.38}$$

Here are the JM-elements for $n = 3$.



(8.0.39)

Lemma 8.0.9. *The $\mathbb{Y}_i$'s have the following properties.*

a) $\mathbb{Y}_i\mathbb{Y}_j = \mathbb{Y}_j\mathbb{Y}_i$ *for all i, j.*

b) $\mathbb{Y}_i^2 = 0$ *for all i.*

Proof. We give the proof in Remark 11.0.13. □

The $\mathbb{Y}_i$'s are (nilpotent) JM-elements for $\mathbb{NB}_n$ in the sense of Mathas, see [32], with respect to the cellular structure on $\mathbb{NB}_n$ given in Corollary 8.0.7, On the other hand, in the next chapter we shall show that there is a completely different cellular structure on $\mathbb{NB}_n$, given by Soergel calculus. That cellular structure is also endowed with a family of JM-elements, that we define now.

Definition 8.0.10. *We define the JM-elements $\mathbb{L}_1, \mathbb{L}_2, \ldots, \mathbb{L}_n$ of $\mathbb{NB}_n$ via $\mathbb{L}_1 = \mathbb{U}_0$ and recursively*

$$\mathbb{L}_{i+1} = \mathbb{U}_i\mathbb{L}_i + \mathbb{L}_i\mathbb{U}_i - 2\mathbb{U}_i\sum_{j=1}^{i-1}\mathbb{L}_j,\ i \geq 1. \tag{8.0.40}$$

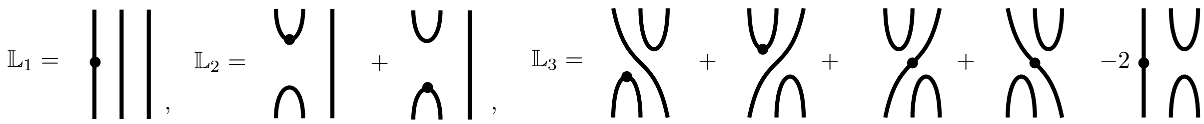
Lemma 8.0.11. *The $\mathbb{L}_i$'s have the following properties.*

a) $\mathbb{L}_i\mathbb{L}_j = \mathbb{L}_j\mathbb{L}_i$ *for all* i, j.

b) $\mathbb{L}_1^2 = 0$ *and that* $\mathbb{L}_i^2 = -2\mathbb{L}_i \sum_{j=1}^{i-1} \mathbb{L}_j$ *for all* $1 < i \leq n$.

Proof. We shall give the proof in Remark 9.0.10. □

Here are these JM-elements for $n = 3$.



(8.0.41)

Chapter 9

Soergel calculus for $\tilde{A}_1$.

In this chapter, we start out by briefly recalling the diagrammatical Soergel category $\mathcal{D}$ associated with the affine Weyl group W of type $\tilde{A}_1$. This category $\mathcal{D}$ was introduced in [11], in the complete generality of any Coxeter system (W, S). The objects of $\mathcal{D}$ are expressions $\underline{w}$ over S and hence for any such $\underline{w}$ we can introduce an algebra $\tilde{A}_w := \mathrm{End}_{\mathcal{D}}(\underline{w})$. In the main result of this chapter we show that $\tilde{A}_w$ and a natural subalgebra $A_w \subset \tilde{A}_w$ of it are isomorphic to the nil-blob algebras $\widetilde{\mathbb{NB}}_n$ and $\mathbb{NB}_n$ from the previous section.

Let $S := \{s, t\}$ and let W be the Coxeter group on S defined by

$$W := \langle s, t \mid s^2 = t^2 = e \rangle. \tag{9.0.1}$$

Thus W is the infinite dihedral group or the affine Weyl group of type $\tilde{A}_1$. Given a non-negative integer n, we let

$$n_s := \underbrace{sts\ldots}_{n\text{-times}} \qquad n_t := \underbrace{tst\ldots}_{n\text{-times}} \tag{9.0.2}$$

with the conventions that $0_s := 0_t := e$. It is easy to see from (9.0.1) that n_s and n_s are reduced expressions and that each element in W is of the form n_s or n_t for a unique choice of n and s or t. Note that the elements of W are *rigid*, that is they have a unique reduced expression.

The construction of $\mathcal{D}$ depends on the choice of a *realization* $\mathfrak{h}$ of (W, S), which by definition is a representation $\mathfrak{h}$ of W, with associated *roots* and *coroots*, see [11, Section 3.1] for the precise definition.

In this thesis, our $\mathfrak{h}$ will be the *geometric representation* of W defined over $\mathbb{F}$, see [16, Section 5.3]. The coroots are the basis of $\mathfrak{h}$, that is $\mathfrak{h} = \mathbb{F}\alpha_s^\vee \oplus \mathbb{F}\alpha_t^\vee$ and in terms of this basis the representation $\mathfrak{h}$ of W is given by

$$s \to \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad t \to \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}. \tag{9.0.3}$$

The roots $\alpha_s, \alpha_t \in \mathfrak{h}^*$ are now given by

$$\alpha_s(\alpha_s^\vee) = 2, \quad \alpha_t(\alpha_s^\vee) = -2, \quad \alpha_s(\alpha_t^\vee) = -2, \quad \alpha_t(\alpha_t^\vee) = 2 \tag{9.0.4}$$

and so the Cartan matrix is

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{9.0.5}$$

Note that we have

$$\alpha_s = -\alpha_t. \tag{9.0.6}$$

Let $R := S(\mathfrak{h}^*) = \oplus_{i \geq 0} S^i(\mathfrak{h}^*)$ be the symmetric algebra of $\mathfrak{h}^*$, or in view of (9.0.6)

$$R = \mathbb{F}[\alpha_s] = \mathbb{F}[\alpha_t]. \tag{9.0.7}$$

In other words, this is a just the usual one variable polynomial algebra. We consider it a $\mathbb{Z}$-graded algebra by setting the degree of α_s equal to 2. Since W acts on $\mathfrak{h}$ it also acts on $\mathfrak{h}^*$ and this action extends in a canonical way to R. We now introduce the *Demazure operators* $\partial_s, \partial_t : R \to R(-2)$ via

$$\partial_s(f) = \frac{f - sf}{\alpha_s}, \qquad \partial_t(f) = \frac{f - tf}{\alpha_t}. \tag{9.0.8}$$

We have that

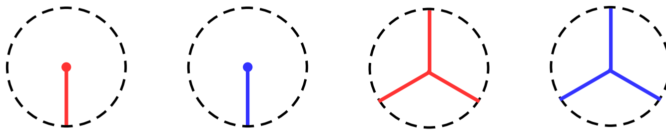
$$s\alpha_s = \alpha_t, \;\; t\alpha_t = \alpha_s \tag{9.0.9}$$

and so we get

$$\partial_s(\alpha_s) = \partial_t(\alpha_t) = 2, \;\; \partial_s(\alpha_t) = \partial_t(\alpha_s) = -2. \tag{9.0.10}$$

We now come to the diagrammatical ingredients of $\mathcal{D}$.

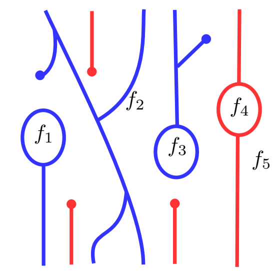
Definition 9.0.1. *A Soergel graph for (W, S) is a finite and decorated graph embedded in the planar strip $\mathbb{R} \times [0, 1]$. The arcs of a Soergel graph are colored by s and t. The vertices of a Soergel graph are of two types as indicated below, univalent vertices (dots) and trivalent vertices where all three incident arcs are of the same color.*



(9.0.11)

A Soergel graph may have its regions, that is the connected components of the complement of the graph in $\mathbb{R} \times [0, 1]$, decorated by elements of R.

Here is an example of a Soergel graph



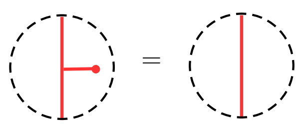
(9.0.12)

where the f_i's belong to R. Shortly we shall give many more examples. We define

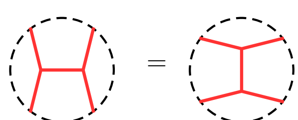
$$\mathbf{exp} := \{\underline{w} = (s_1, s_2, \ldots, s_k) \mid s_i \in S, \, k = 1, 2, \ldots\} \cup \emptyset. \tag{9.0.13}$$

as the set of expressions over S, that is words over the alphabet S. The points where an arc of a Soergel graph intersects the boundary of the strip $\mathbb{R} \times [0, 1]$ are called *boundary points*. The boundary points provide two elements of **exp** called the *bottom boundary* and *top boundary*, respectively. In the above example the bottom boundary is (t, s, t, t, s, s) and the top boundary is (t, s, t, t, s).

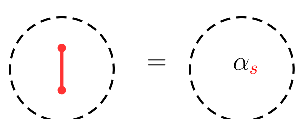
Definition 9.0.2. *The diagrammatical Soergel category $\mathcal{D}$ is defined to be the monoidal category whose objects are the elements of* **exp** *and whose homomorphisms $Hom_{\mathcal{D}}(\underline{x}, \underline{y})$ are the $\mathbb{F}$-vector space generated by all Soergel graphs with bottom boundary $\underline{x}$ and top boundary $\underline{y}$, modulo isotopy and modulo the following local relations*



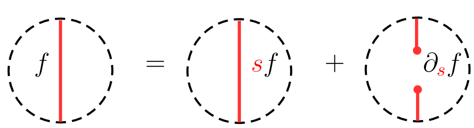
(9.0.14)



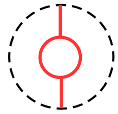
(9.0.15)



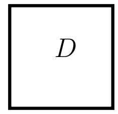
(9.0.16)



(9.0.17)

 $= 0$ (9.0.18)

There is a final relation saying that any Soergel graph D which is decorated in its leftmost region by an $f \in (\alpha_s)$, that is a polynomial with no constant term, is set equal to zero. We depict it as follows

α_s  $= 0$ (9.0.19)

The relations (9.0.14)–(9.0.19) *also hold if red is replaced by blue, of course.*

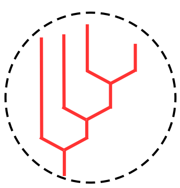
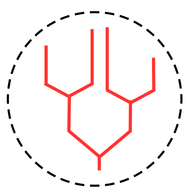
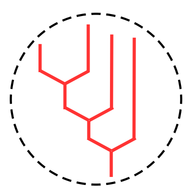
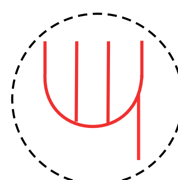
For $\lambda \in \mathbb{F}$ and D a Soergel diagram, the scalar product λD is identified with the multiplication by λ in any region of D. The multiplication $D_1 D_2$ of diagrams D_1 and D_2 is given by vertical concatenation with D_1 on top of D_2 and the monoidal structure by horizontal concatenation. There is natural $\mathbb{Z}$-grading on $\mathcal{D}$, extending the grading on R, in which the dots, that is the first two diagrams in (9.0.11) have degree 1, *and the trivalents, that is the last two diagrams in (9.0.11), have degree* -1.

Remark 9.0.3. Strictly speaking the category defined in Definition 9.0.2 is not the diagrammatic Soergel category introduced in [11]. To recover the category from [11] the relation (9.0.19) should be omitted.

Let us comment on the isotopy relation in Definition 9.0.2. It follows from it that the arcs of a Soergel graph may be assumed to be piecewise linear. It also follows from it together with (9.0.15) that the following relation holds

 $=$  (9.0.20)

In other words the two trees on three downwards leaves are equal. We also have equality for other trees. Here is the case with four upwards leaves. Note the last diagram which represents the way we shall often depict trees.

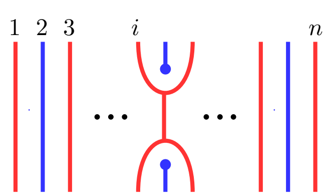
 $=$  $=$  $=$  (9.0.21)

Let now n be a fixed positive integer and fix $\underline{w} := n_s \in \mathbf{exp}$ as in (9.0.2). We then define

$$\tilde{A}_w := \mathrm{End}_{\mathcal{D}}(\underline{w}). \tag{9.0.22}$$

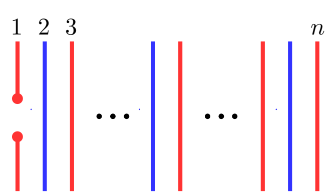
As mentioned above, w is a rigid element of W and therefore we use the notation $\tilde{A}_w$ instead of $\tilde{A}_{\underline{w}}$.

By construction, $\tilde{A}_w$ is an $\mathbb{F}$-algebra with multiplication given by concatenation and the goal of this chapter is to study the properties of this algebra. First, for $i = 1, \ldots, n-2$ we define the following element of $\tilde{A}_w$

$U_i :=$ 

(9.0.23)

and similarly

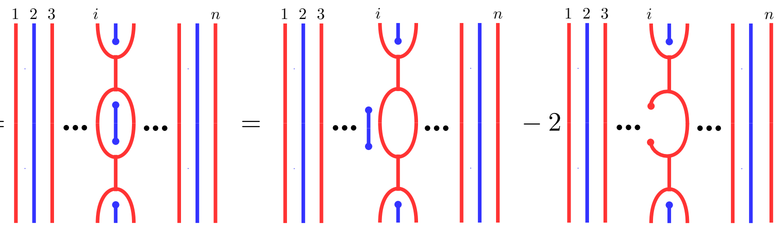
$U_0 :=$ 

(9.0.24)

The following Theorem is fundamental for what follows.

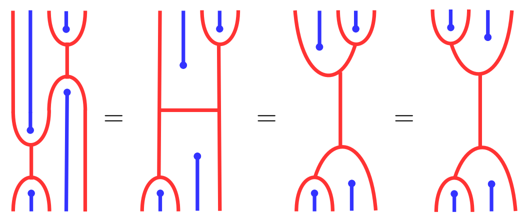
Theorem 9.0.4. *There is a homomorphism of* $\mathbb{F}$*-algebras* $\varphi : \mathbb{NB}_{n-1} \to \tilde{A}_w$ *given by* $\mathbb{U}_i \mapsto U_i$ *for* $i = 0, 1, \ldots, n-2$.

Proof: We must check that $U_0, U_1, \ldots, U_{n-2}$ satisfy the relations given by the $\mathbb{U}_i$'s in Definition 8.0.2. In order to show the quadratic relation (8.0.8) we argue as follows

$U_i^2 =$  $= -2U_i$ (9.0.25)

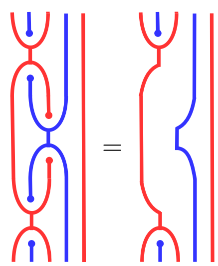
where we used (9.0.14), (9.0.16), (9.0.17) and (9.0.18).

We next show that (8.0.10) holds. If $|i-j| > 2$ then (8.0.10) clearly holds, that is $U_iU_j = U_jU_i$, but for $|i-j| = 2$ it is not completely clear that it holds. We shall only show it in the case $n = 5$, $i = 1$ and $j = 3$: the general case is proved the same way. We have that

$U_3U_1 =$  (9.0.26)

where we used the 'H'-relation (9.0.15) for the third equality and (9.0.20) for the last equality. But U_1U_3 is obtained from U_3U_1 by reflecting along a horizontal axis, and since the last diagram of (9.0.26) is symmetric along this axis, we conclude that $U_1U_3 = U_3U_1$ as claimed.

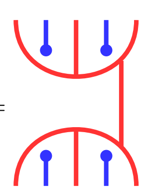
The relation (8.0.9), in the case $n = 4, i = 1$ and $j = 2$, is shown as follows.

$U_1U_2U_1 =$  $= U_1$ (9.0.27)

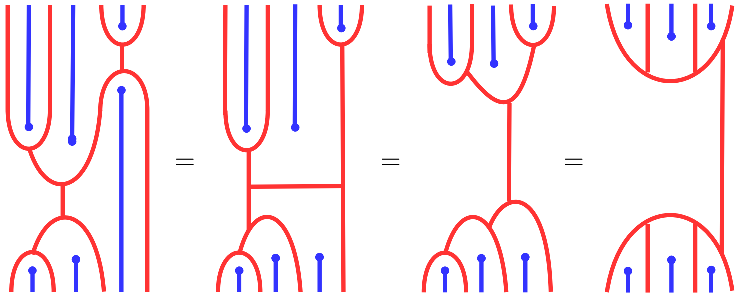
The general case is treated the same way. We finally notice that (8.0.11) and (8.0.12) are a direct consequence of (9.0.19). The Theorem is proved. □

For a general Coxeter system (W, S), Elias and Williamson found in [11] a recursive procedure for constructing an $\mathbb{F}$-basis for the homomorphism space $\text{Hom}_{\mathcal{D}}(\underline{x}, \underline{y})$, for any $\underline{x}, \underline{y} \in \textbf{exp}$. It is a diagrammatical version of Libedinsky's *double light leaves basis* for Soergel bimodules and the basis elements are also called double light leaves in this case. On the other hand we have fixed W as the infinite dihedral group, and in this particular case there is a non-recursive description of the double light leaves basis that we shall use.

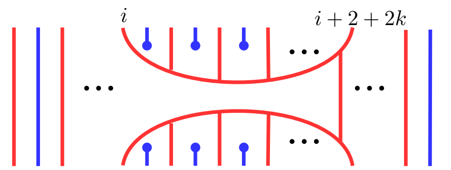
In order to describe it we first introduce some diagram conventions. First, in view of our tree conventions given in (9.0.21) we shall represent the diagram from (9.0.26) as follows

$U_1U_3 =$  (9.0.28)

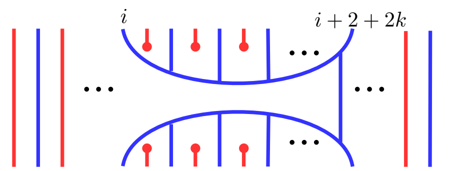
This can be generalized: for example using the last diagram in (9.0.26) we get that

$U_1U_3U_5$ =  (9.0.29)

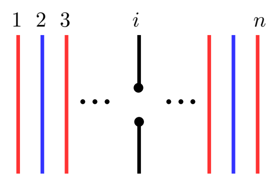
Even more generally, we have that

$U_iU_{i+2}\cdots U_{i+2k} =$  (9.0.30)

if i is odd and

$U_iU_{i+2}\cdots U_{i+2k} =$  (9.0.31)

if i is even. We now introduce a different kind of elements in $\tilde{A}_w$, namely the *JM-elements* L_i of $\tilde{A}_w$, via

$L_i :=$  (9.0.32)

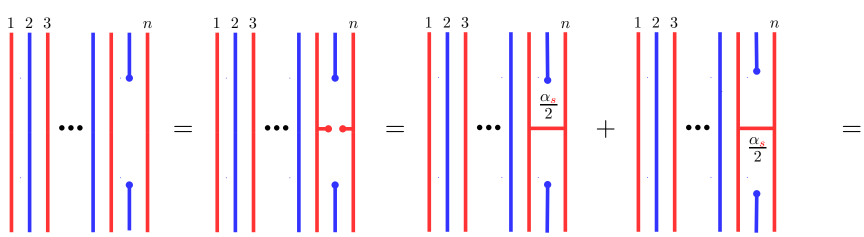
where black means red if i is odd and blue if i is even. Note that $L_1 = U_0$. (The name JM-element is motivated by the thesis [40] where it is shown that L_i indeed is a JM-element in the sense of Mathas [32], for any Coxeter system).

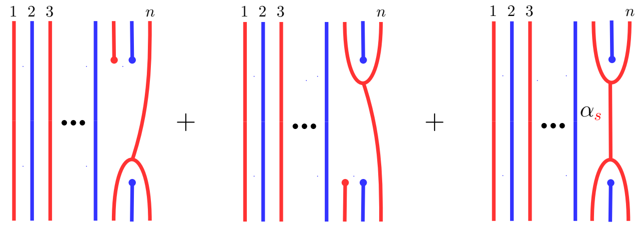
Lemma 9.0.5. *Let $1 < i < n$. Then we have the following formula in $\tilde{A}_w$*

$$L_i = U_{i-1}L_{i-1} + L_{i-1}U_{i-1} - 2U_{i-1}\sum_{j=1}^{i-2} L_j. \tag{9.0.33}$$

Consequently, for all $1 < i < n$ we have that L_i belongs to the subalgebra of $\tilde{A}_w$ generated by the elements $L_1, U_1, \ldots, U_{n-2}$.

Proof: Let us show the formula (9.0.33) in the case $i = n-1$ and i odd. The general case of the formula, that is the case where i is any number strictly smaller than n, is shown the same way. We have that

$L_i =$  (9.0.34)

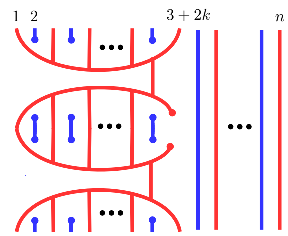


(9.0.35)

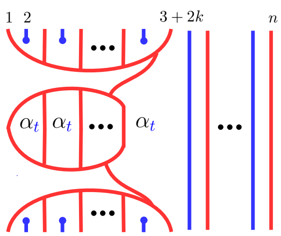
The first two diagrams of (9.0.35) are $U_{i-1}L_{i-1}$ and $L_{i-1}U_{i-1}$ and so we only have to check that the last diagram of (9.0.35) is equal to $-2U_{i-1}\sum_{j=1}^{i-2} L_j$. But this follows via repeated applications of the polynomial relation (9.0.17), moving $\alpha_s = -\alpha_t$ all the way to the left. □

The L_i's are important since they allow us to generate variations of (9.0.30) and (9.0.31) with no 'connecting' arcs, as follows

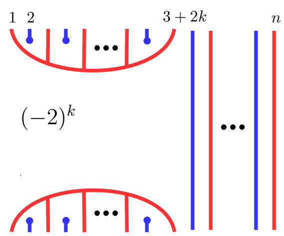
$$(U_1U_3U_5\cdots U_{2k+1})L_{2k+3}(U_1U_3U_5\cdots U_{2k+1}) \quad =$$



$$=$$

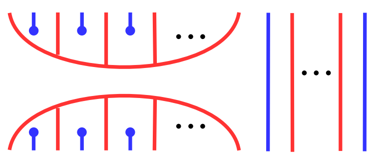


$$=$$



(9.0.36)

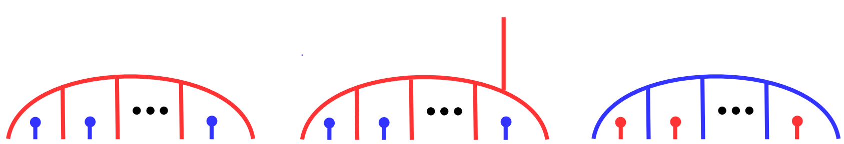
where we for the last equality used the polynomial relation (9.0.17) as well as (9.0.19). Thus any diagram of the form (9.0.37) belongs to the subalgebra of $\tilde{A}_w$ generated by the L_i's and the U_i's. Note on the other hand that in order for this argument to work, the diagram in question must be left-adjusted, that is without any through arcs on the left as in (9.0.37).



(9.0.37)

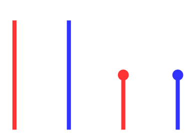
The diagrams corresponding to double light basis elements of $\tilde{A}_w$ are built up of top and bottom 'half-diagrams', similarly to the Temperley-Lieb diagrams and the blob diagrams considered in the previous chapter. These half-diagrams are called light leaves.

We now introduce the following bottom half-diagrams, called *full birdcages* by Libedinsky in [22].



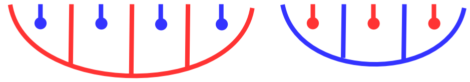
(9.0.38)

We say that the first and the last of these half-diagrams are *non-hanging full birdcages*, whereas the middle one is *hanging*. We also say that the first two full birdcages are *red*, and the third one is *blue*. We define the *length* of a full birdcage to be the number of dots contained in it. We view the half-diagrams



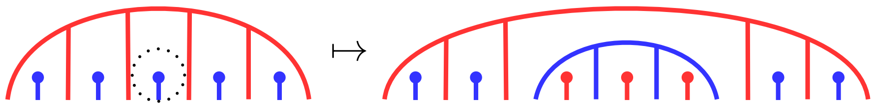
(9.0.39)

as *degenerate* full birdcages of lengths 0. A full birdcage which is not degenerate is called *non-degenerate*. We shall also consider *top full birdcages*, that are obtained from bottom full birdcages, by a reflection through a horizontal axis. Here are two examples of lengths four and three.



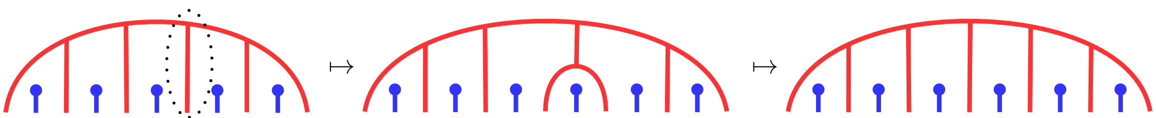
(9.0.40)

Light leaves are built up of full birdcages in a suitable sense that we shall now explain. We first consider the operation of *replacing a degenerate non-hanging full birdcage by a non-hanging non-degenerate full birdcage of the same color*. Here is an example



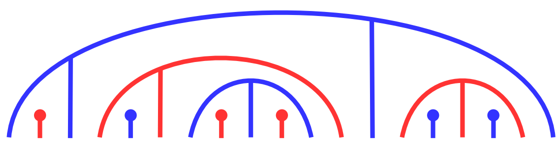
(9.0.41)

The reason why we only consider the application of this operation to non-hanging birdcages is that applying it to a degenerate hanging birdcage only gives a new, larger full birdcage; in other words nothing new. Here is an example



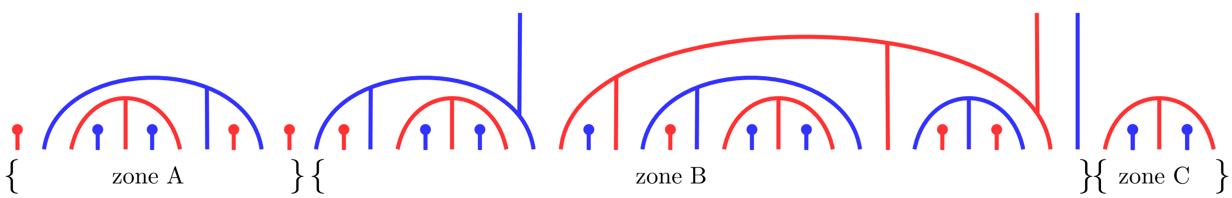
(9.0.42)

Following Libedinsky, we now define a *birdcagecage* to be any diagram that can be obtained from a degenerate non-hanging birdcage by performing the above operation recursively a finite number of times on the degenerate birdcages that appear at each step. Here is an example of a birdcagecage.



(9.0.43)

Now, according to [22], any light leaf is built up of birdcagecages as indicated below in (9.0.44). Here in (9.0.44) the number of bottom boundary points is n. Zone A consists of a number of non-hanging birdcagecages whereas zone B consists of a number of hanging birdcagecages. On the other hand zone C consists of at most one non-hanging birdcagecage.

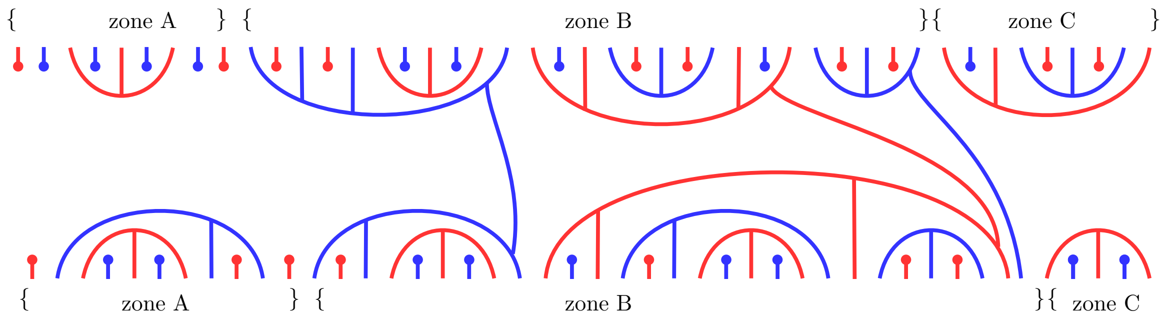


(9.0.44)

Note that each of the three zones may be empty, but they cannot all be empty since $n > 0$. In the case where zone B is empty, we define zone C to be the last birdcagecage. In other words, if zone B is empty then zone C is always nonempty, whereas zone A may be empty.

The hanging birdcagcages of zone B define an element $v \in W$. It satisfies $v \leq w$ where $\leq$ denotes the Bruhat order on W. In the above example we have $v = tst$. The *double leaves basis of* $\tilde{A}_w$ is now obtained by running over all $v \leq w$ and over all pairs of light leaves that are associated with that v. For each such pair (D_1, D_2) the second component D_2 is reflected through a horizontal axis, and finally the two components are glued together.

The resulting diagram is a double leaf. Here is an example



(9.0.45)

Note that although the total number of top and bottom boundary points of each double leaf is the same, the number of boundary points in each of the three zones need not coincide, although the parities do coincide. In the above example, there are for instance nine top boundary points in zone C but only five bottom boundary points in zone C. Note also that the number of top and bottom birdcagecages in zone B always is the same, three in the above example. This is of course also the case in zone C but not necessarily in zone A, although the parities must coincide. In the above example, we have five top birdcagecages in zone A but only three bottom birdcagecages in zone A. Moreover, there are nine top boundary points in zone A but eleven bottom boundary points in zone A.

For future reference we formulate the Theorem already alluded to several times.

Theorem 9.0.6. *The double leaves form an $\mathbb{F}$-basis for $\tilde{\mathcal{A}}_w$.*

Proof: This is mentioned in [22]. It is a consequence of the recursive construction of the light leaves. □

Definition 9.0.7. *Let $\mathcal{A}_w$ be the subspace of $\tilde{\mathcal{A}}_w$ spanned by the double leaves with empty zone C .*

With these notions and definitions at hand, we can now formulate and prove the following Theorem.

Theorem 9.0.8. *Let $w \in W$ with $w = n_{s}$. Then, we have*

a) *As an algebra $\tilde{A}_w$ is generated by the elements $U_1, \ldots, U_{n-2}$ and $L_1, \ldots, L_n$.*

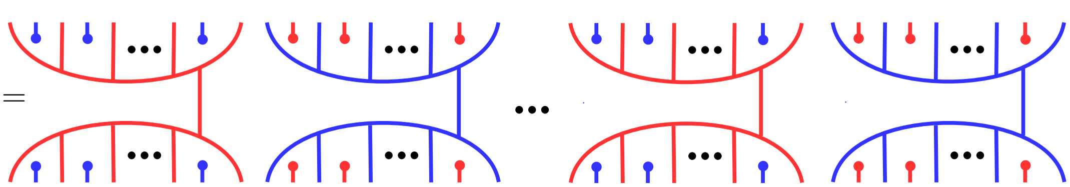
b) *A_w is a subalgebra of $\tilde{A}_w$. It is generated by $U_1, \ldots, U_{n-2}$ and $L_1 = U_0$.*

c) *The dimensions of A_w and $\tilde{A}_w$ are given by the formulas*

$$\dim_{\mathbb{F}}(A_w) = \binom{2n}{n} \qquad \text{and} \qquad \dim_{\mathbb{F}}(\tilde{A}_w) = 2\binom{2n}{n}. \tag{9.0.46}$$

Proof: We first prove *a*) of the Theorem. We define $\tilde{A}'_w$ as the subalgebra of $\tilde{A}_w$ generated by the U_i's and the L_i's. Thus, in order to show *a*) we must prove that $\tilde{A}'_w = \tilde{A}_w$. We shall do so by proving that $\tilde{A}'_w$ contains all the double leaves basis elements for $\tilde{A}_w$.

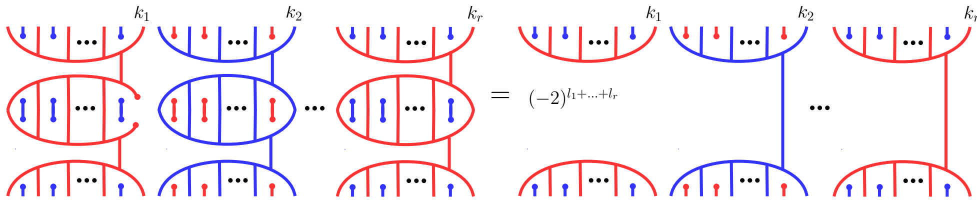
We first observe that the diagrams in (9.0.30) and (9.0.31) both belong to $\tilde{A}'_w$. In fact, multiplying them together we get that any diagram of the form

$D =$  (9.0.47)

belongs to $\tilde{A}'_w$. Here the length of each full birdcage on the bottom (which may be zero) is equal to the length of the corresponding full birdcage on top of it, that is the diagram in (9.0.47) is symmetric with respect to a horizontal axis. Note that the diagram D in (9.0.47) is a preidempotent; to be precise we have that

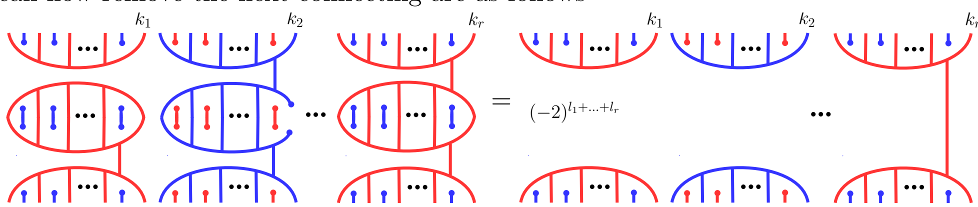
$$D^2 = (-2)^{l_1+\ldots+l_r} D, \tag{9.0.48}$$

where $l_1, l_2, \ldots, l_r$ are the lengths of the bottom full birdcages that appear in D. Now we can repeat the calculations from (9.0.36) and (9.0.37) in order to remove the connecting arc between the first bottom full birdcage of D and its top mirror image:

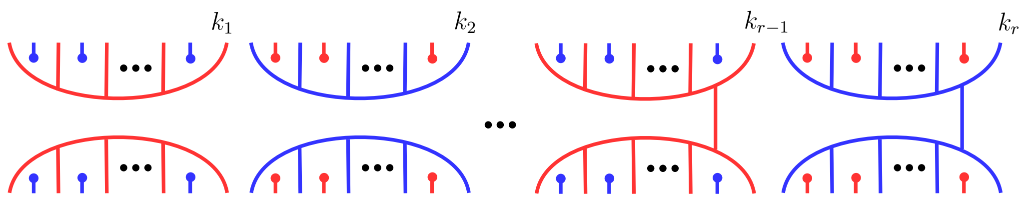
$DL_{k_1}D =$  (9.0.49)

In other words, we get that $D_1 := (-2)^{-(l_1+\ldots+l_r)}DL_{k_1}D$ is equal to D, but with the first connecting arc removed, and that D_1 belongs to $\tilde{A}'_w$.

From D_1 we can now remove the next connecting arc as follows

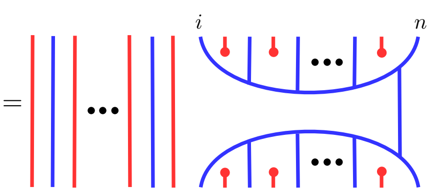
$D_1L_{k_2}D =$  (9.0.50)

Continuing this way we find that any diagram of the form

 (9.0.51)

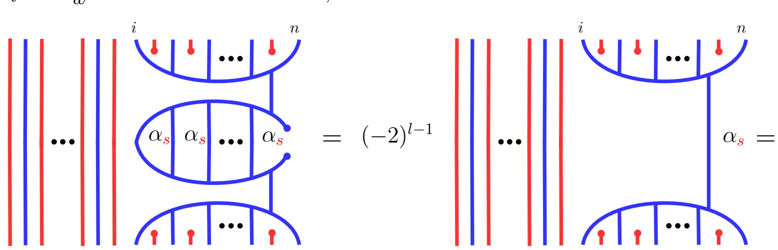
belongs to $\tilde{A}'_w$.

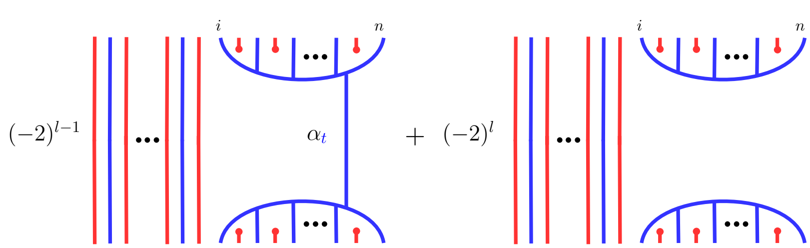
The diagrams in (9.0.51) consist of a number of non-hanging full birdcages followed by a number of hanging full birdcages. We shall now prove that the rightmost hanging full birdcage of (9.0.51) may be transformed into a non-hanging full birdcage and still give rise to an element of $\tilde{A}'_w$. Let $i < n$ be a positive integer of the same parity as n. We consider the diagram $F_i := U_iU_{i+3}\cdots U_{n-2}$:

$F_i =$  (9.0.52)

We notice that only the rightmost top and bottom full birdcages of F_i are non-degenerate, of length $l := (n-i)/2$.

Then we have that $F_iL_nF_i \in \tilde{A}'_w$. On the other hand, we also have that

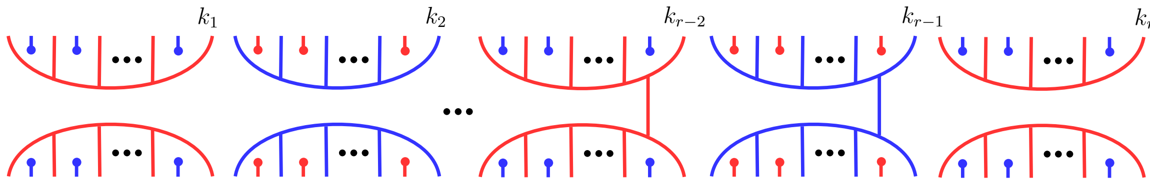
$F_iL_nF_i =$  (9.0.53)

 (9.0.54)

We consider the first diagram X of the last sum. Moving α_t all the way to the left we get that

$$X = -2F_i \sum_{j=1}^{i-1} L_i \tag{9.0.55}$$

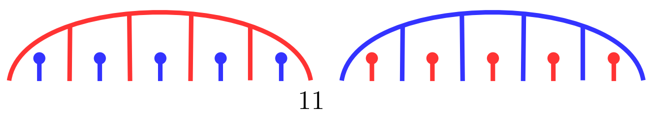
Therefore, X belongs to $\tilde{A}'_w$. But from this we conclude that also the second diagram of the sum belongs to $\tilde{A}'_w$. Finally, multiplying this diagram with diagrams from (9.0.51) we conclude that any diagram of the form



(9.0.56)

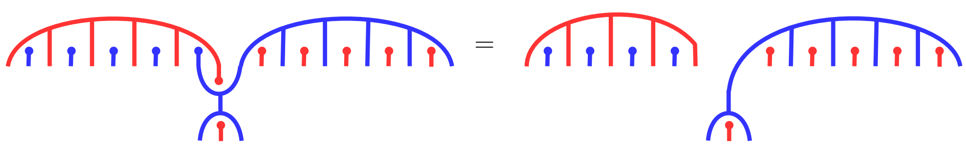
belongs to $\tilde{A}'_w$, proving the above claim. In other words, we have shown that any double leaves basis element of $\tilde{A}_w$, that is built up of full birdcages and is symmetric with respect to a horizontal axis, belongs to $\tilde{A}'_w$.

We next show that omitting the symmetry condition in the diagrams (9.0.56) still gives rise to an element of $\tilde{A}'_w$. Our first step for this is to produce a way of 'moving points' from a full birdcage to its neighboring full birdcage. We do this by multiplying by 'overlapping' U_i's. Consider the following example

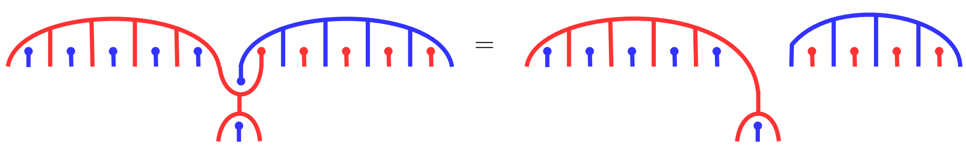
$D =$ 

(9.0.57)

consisting of two full birdcages, both of length 5. In this case the overlapping U_i's are U_{10} and U_{11}. Multiplying D below with U_{10} produces a diagram with two full birdcages as well, but this time of lengths 4 and 6, whereas multiplying D below by U_{11} produces a diagram with two full birdcages, of lengths 6 and 4:

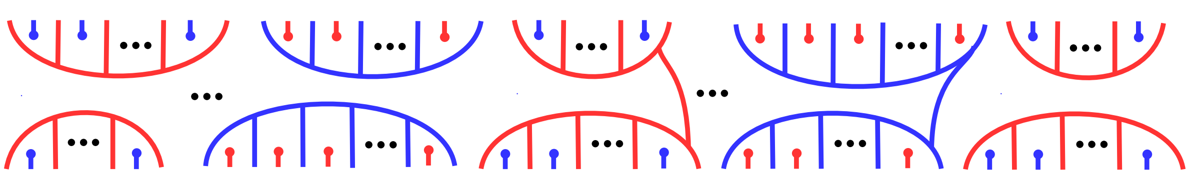
$DU_{10} =$ 

(9.0.58)

$DU_{11} =$ 

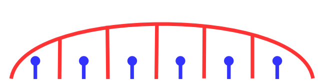
(9.0.59)

This gives us a method for moving points from one full birdcage to a neighboring full birdcage that works in general, for hanging as well as for non-hanging full birdcages, and so we get that any diagram of the form



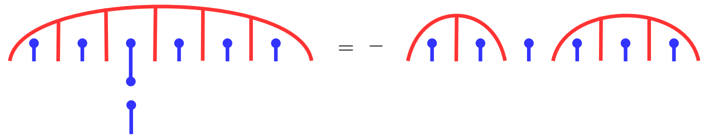
(9.0.60)

belongs to $\tilde{A}'_w$. These diagram are not horizontally symmetric anymore but still the total number of top full birdcages is equal to the total number of bottom full birdcages. Actually, by the description of the light leaves basis, this is expected in zones B and C, but not in zone A. However, multiplying a full birdcage in zone A with an JM-element L_i of the opposite color it breaks up in three smaller full birdcages, the middle one being degenerate. For example, for

$D :=$ 

(9.0.61)

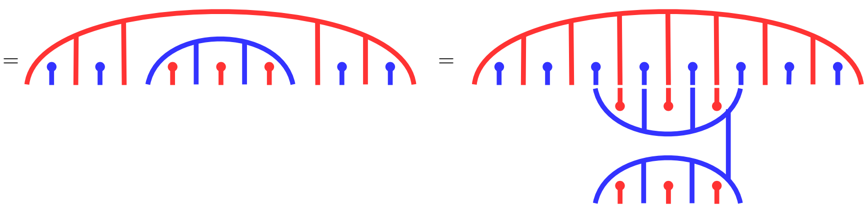
we have that

$DL_6 =$  (9.0.62)

Combining this with the procedure of moving points from a full birdcage to a neighboring full birdcage, we conclude that in the diagram (9.0.60) we may assume that the number of top full birdcages in zone A is different from the number of bottom full birdcages and still the diagram belongs to $\tilde{A}'_w$.

Thus, to finish the proof of a) we now only have to show that the full birdcages in the diagram (9.0.60) may be replaced by birdcagecages. It is here enough to consider a single bottom birdcage.

The replacing of a degenerate non-hanging birdcage by a non-degenerate full birdcage can be viewed as the insertion of a non-hanging birdcage in a full birdcage of the opposite color. But this can be achieved via multiplication with appropriate diagrams of the form (9.0.30) and (9.0.31). Consider for example the birdcagecage D in (9.0.41). It can be obtained as follows

$D =$  (9.0.63)

Repeating this process we can obtain any birdcagecage. This finishes the proof of a).

We next show c). For this we first note that there is a bijection between double leaves with empty zone C and double leaves with nonempty zone C, given by removing the connecting line between the last bottom and top birdcagecage. Hence we have that

$$\dim_{\mathbb{F}}(\tilde{A}_w) = 2\dim_{\mathbb{F}}(A_w). \tag{9.0.64}$$

On the other hand, from the vector space isomorphism given in Corollary 8.3 of [10] it follows that $\dim(\tilde{A}_w) = \dim \widetilde{\mathbb{NB}_n}$ and so c) follows from Theorem 8.0.5 and Corollary 8.0.6. (Note that in [10] the authors use the notation A_w for $\tilde{A}_w$).

We finally show b). Let A'_w be the subalgebra of $\tilde{A}_w$ generated by $U_1, \dots, U_{n-2}$ and $U_0 = L_1$. In view of Lemma 9.0.5 we first observe that A'_w is the same as the subalgebra of $\tilde{A}_w$ generated by $U_1, \dots, U_{n-2}$ and $L_1, \dots, L_{n-1}$. On the other hand, going through the proof of a) we see that the last JM-element L_n is only needed for the steps (9.0.52) and (9.0.53) where a hanging birdcage at the right end of the diagram is transformed into a non-hanging one, and so we have that $A_w \subseteq A'_w$. But from Theorem 9.0.4 we have that $\dim(A'_w) \leq \dim \mathbb{NB}_{n-1} = \dim(A_w)$ where we used c) for the last equality. Hence the inclusion $A_w \subseteq A'_w$ is an equality and b) is proved. □

Corollary 9.0.9. *Let $w \in W$ with $w = n_s$. Then, we have*

a) *The map φ defined in Theorem 9.0.4 induces an algebra isomorphism $\varphi : \mathbb{NB}_{n-1} \to A_w$.*

b) *Setting $J_n := L_1 + L_2 + \ldots + L_n$ we have that the extension of φ to $\widetilde{\mathbb{NB}}_{n-1}$ given by $\tilde{\varphi}(\mathbb{J}_{n-1}) = J_n$ induces an algebra isomorphism $\tilde{\varphi} : \widetilde{\mathbb{NB}}_{n-1} \to \tilde{A}_w$.*

Proof: Part a) was already proved in the previous Theorem so let us concentrate on part b). Here we have already checked all the relations that do not involve J_n and so we only have to check that $J_n^2 = 0$ and that J_n is central in $\tilde{A}_w$. Now by [10, Lemma 3.4] we know that $L_1^2 = 0$ and that

$$L_i^2 = -2L_i \sum_{j=1}^{i-1} L_j, \tag{9.0.65}$$

for all $2 \leq i \leq n$. Thus we obtain

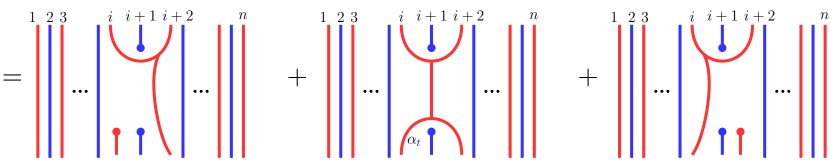
$$J_n^2 = (L_1 + L_2 + \ldots + L_n)^2 = \sum_{i=2}^{n} L_i^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} L_i L_j \tag{9.0.66}$$

$$= -2\sum_{i=2}^{n}\sum_{j=1}^{i-1} L_i L_j + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} L_i L_j = 0, \tag{9.0.67}$$

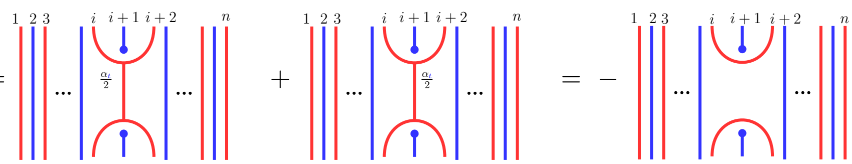
as claimed. Now let us show that J_n is central in $\tilde{A}_w$. It is enough to show that $[U_j, J_n] = 0$, for all $1 \leq j \leq n-2$, where $[\cdot,\cdot]$ denotes the usual commutator bracket. We notice that $[U_j, L_i] = 0$ if $i \neq j, j+1, j+2$. Then we are done if we are able to show that

$$[U_i, L_i + L_{i+1} + L_{i+2}] = 0. \tag{9.0.68}$$

But we have that

$U_i \cdot (L_i + L_{i+1} + L_{i+2}) =$ 

In the second diagram we first rewrite $\alpha_t = -\frac{\alpha_s}{2} - \frac{\alpha_s}{2}$ and next use the polynomial relation (9.0.17), to take the first $-\frac{\alpha_s}{2}$ out of the birdcage to the left and the second $-\frac{\alpha_s}{2}$ out of the birdcage to the right. This will give rise to a cancellation of the first and the third terms in the expression for $U_i \cdot (L_i + L_{i+1} + L_{i+2})$ and so we have that

$U_i \cdot (L_i + L_{i+1} + L_{i+2}) =$ 

This last diagram is symmetric with respect to a horizontal reflection and so

$$U_i \cdot (L_i + L_{i+1} + L_{i+2}) = (L_i + L_{i+1} + L_{i+2}) \cdot U_i \tag{9.0.69}$$

as claimed. The Corollary is proved. □

Remark 9.0.10. Combining the isomorphism $\mathbb{NB}_{n-1} \cong A_w$ with Lemma 9.0.5, we obtain a proof of Lemma 8.0.11.

Remark 9.0.11. All the results in this chapter consider the case $w = n_s$. Of course, they remain valid if we replace n_s by n_t.

Chapter 10

Idempotent truncations of $\mathbb{B}_n$ and related alcove geometry

10.1 Idempotent truncations of $\mathbb{B}_n$

From now on we shall study a certain subalgebra of $\mathbb{B}_n$ that arises from idempotent truncation of $\mathbb{B}_n$. This subalgebra has already appeared in the literature, for example in [10], [23].

Definition 10.1.1. Suppose that $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$. Then the subalgebra $\mathbb{B}_n(\boldsymbol{\lambda})$ of $\mathbb{B}_n$ is defined as

$$\mathbb{B}_n(\boldsymbol{\lambda}) := e(\boldsymbol{i}^{\boldsymbol{\lambda}})\mathbb{B}_n e(\boldsymbol{i}^{\boldsymbol{\lambda}}). \tag{10.1.1}$$

Let us mention the following Lemma without proof.

Lemma 10.1.2. *Let $\boldsymbol{\lambda} = (1^{\lambda_1}, 1^{\lambda_2}) \in \mathrm{Par}_n^1$. Set $\boldsymbol{\mu} := (1^{\lambda_2}, 1^{\lambda_1}) \in \mathrm{Par}_n^1$ and $\boldsymbol{\nu} = (1^{\lambda_1 - M}, 1^{\lambda_2 - M}) \in Par^1_{2,n-2M}$ where $M = \min\{\lambda_1, \lambda_2\}$. There is an isomorphism $\mathbb{B}_n(\boldsymbol{\lambda}) \cong \mathbb{B}_{n-2M}(\boldsymbol{\nu})$ of $\mathbb{F}$-algebras.*

We shall from now on fix $\boldsymbol{\lambda}$ of the form

$$\boldsymbol{\lambda} = (1^n, 1^0). \tag{10.1.2}$$

Remark 10.1.3. When defining $\mathbb{B}_n(\boldsymbol{\lambda})$ we could have taken more general $\boldsymbol{\lambda}$, but in view of the Lemma it is enough to consider $\boldsymbol{\lambda}$ either of the form $(1^n, 1^0)$ or $\boldsymbol{\mu} := (1^0, 1^n)$. Moreover, we have that

$$e(\boldsymbol{i}^{\boldsymbol{\mu}})\mathbb{B}_n e(\boldsymbol{i}^{\boldsymbol{\mu}}) \cong e(\boldsymbol{i}^{\boldsymbol{\lambda}})\mathbb{B}'_n(e-m)e(\boldsymbol{i}^{\boldsymbol{\lambda}}). \tag{10.1.3}$$

On the other hand, the methods and results for $\mathbb{B}_n(\boldsymbol{\lambda})$ that we shall develop during the rest of the thesis will have almost identical analogues for the right hand side of (10.1.3), as the reader will notice during the lecture, with the only difference that one-column bipartitions and tableaux are replaced by one-row bipartitions and tableaux. Thus, there is no loss of generality in assuming that $\boldsymbol{\lambda}$ is of the form given in (10.1.2).

One of the advantages of the choice of $\boldsymbol{\lambda}$ in (10.1.2) is that the residue sequence $\boldsymbol{i}^{\boldsymbol{\lambda}}$ is particularly simple since it decreases in steps by one. Let us state it for future reference

$$\boldsymbol{i}^{\boldsymbol{\lambda}} = (0, -1, -2, -3, \dots, -n+1) \in I_e^n. \tag{10.1.4}$$

In the main theorems of this chapter we shall find generators for $\mathbb{B}_n(\boldsymbol{\lambda})$, verifying the same relations as the generators $\mathbb{NB}_n$ or $\widetilde{\mathbb{NB}}_n$. The following series of definitions and recollections of known results from the literature are aimed at introducing these generators.

It follows from general principles that $\mathbb{B}_n(\boldsymbol{\lambda})$ is a graded cellular algebra with identity element $e(\boldsymbol{i}^{\boldsymbol{\lambda}})$. Let us describe the corresponding cellular basis. Set first $\mathrm{Std}(\mathrm{Par}_n^1) := \bigcup_{\boldsymbol{\mu} \in \mathrm{Par}_n^1} \mathrm{Std}(\boldsymbol{\mu})$ and define for $\boldsymbol{i} \in I_e^n$:

$$\mathrm{Std}(\boldsymbol{i}) := \{\mathbf{t} \in \mathrm{Std}(\mathrm{Par}_n^1) \mid \boldsymbol{i}^{\mathbf{t}} = \boldsymbol{i}\}. \tag{10.1.5}$$

Furthermore, for $\boldsymbol{\mu} \in \mathrm{Par}_n^1$ define

$$\mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) := \mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}}) \cap \mathrm{Std}(\boldsymbol{\mu}). \tag{10.1.6}$$

Then we have the following Lemma.

Lemma 10.1.4. a) *For* $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\mu})$ *we have that*

$$e(\boldsymbol{i}^{\boldsymbol{\mu}})\psi_{d(\mathfrak{t})} = \psi_{d(\mathfrak{t})} e(\boldsymbol{i}^{\mathfrak{t}}) \quad \textit{and} \quad \psi^*_{d(\mathfrak{s})} e(\boldsymbol{i}^{\boldsymbol{\mu}}) = e(\boldsymbol{i}^{\mathfrak{s}})\psi^*_{d(\mathfrak{s})}. \tag{10.1.7}$$

b) *The set* $\mathcal{C}_n(\boldsymbol{\lambda}) := \{m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}}), \boldsymbol{\mu} = \mathit{shape}(\mathfrak{s}) = \mathit{shape}(\mathfrak{t})\}$ *is a graded cellular basis for* $\mathbb{B}_n(\boldsymbol{\lambda})$*.*

Proof. From the multiplication rule in $\mathbb{B}_n$ we have that $\psi_k e(\boldsymbol{i}) = e(s_k \boldsymbol{i})\psi_k$ for any $k = 1, \ldots, n-1$ and $\boldsymbol{i} \in I_e^n$. Hence if $d(\mathfrak{t}) = s_{i_1} s_{i_2} \cdots s_{i_N}$ is a reduced expression we get that

$$e(\boldsymbol{i}^{\boldsymbol{\mu}})\psi_{d(\mathfrak{t})} = \psi_{d(\mathfrak{t})} e(s_{i_N} \cdots s_{i_2} s_{i_1} \boldsymbol{i}^{\boldsymbol{\mu}}) = \psi_{d(\mathfrak{t})} e(\boldsymbol{i}^{\mathfrak{t}}), \tag{10.1.8}$$

proving the first formula of a). The second formula of a) is proved the same way. On the other hand, by using a) and 3.0.1 we obtain

$$e(\boldsymbol{i}^{\boldsymbol{\lambda}}) m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = e(\boldsymbol{i}^{\boldsymbol{\lambda}})\psi^*_{d(\mathfrak{s})} e(\boldsymbol{i}^{\boldsymbol{\mu}})\psi_{d(\mathfrak{t})} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = e(\boldsymbol{i}^{\boldsymbol{\lambda}}) e(\boldsymbol{i}^{\mathfrak{s}})\psi^*_{d(\mathfrak{s})}\psi_{d(\mathfrak{t})} e(\boldsymbol{i}^{\mathfrak{t}}) e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = \delta_{\boldsymbol{i}^{\mathfrak{s}}, \boldsymbol{i}^{\boldsymbol{\lambda}}} \delta_{\boldsymbol{i}^{\mathfrak{t}}, \boldsymbol{i}^{\boldsymbol{\lambda}}} m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}} \tag{10.1.9}$$

and so b) follows. □

10.2 An explicit algorithm for the elements $d(\mathfrak{t})$

We now explain an algorithm for producing a reduced expression for the elements $d(\mathfrak{t})$ for $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. This algorithm has already been used in [38], [15], [10] and [23].

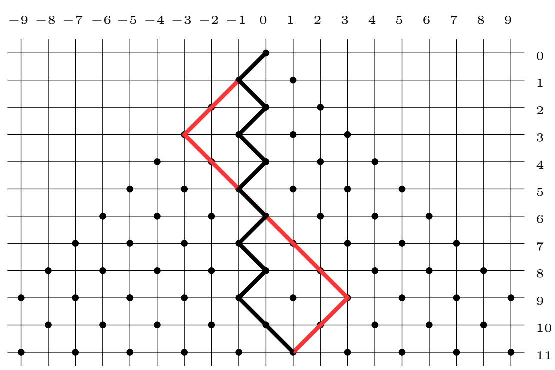
We first need to reinterpret standard tableaux as paths on the Pascal triangle.

Let $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then we define $p_{\mathfrak{t}} : \{0, 1, \ldots, n\} \to \mathbb{Z}$ as the function given recursively by $p_{\mathfrak{t}}(0) = 0$ and $p_{\mathfrak{t}}(k) = p_{\mathfrak{t}}(k-1) + 1$ (resp. $p_{\mathfrak{t}}(k) = p_{\mathfrak{t}}(k-1) - 1$) if k is located in the second (resp. first) column of $\mathfrak{t}$. Moreover, we define $P_{\mathfrak{t}} : [0, n] \to \mathbb{R}^2$ as the piecewise linear path such that $P_{\mathfrak{t}}(k) = (p_{\mathfrak{t}}(k), k)$ for $k = 0, 1, \ldots, n$ and such that $P_{\mathfrak{t}}|_{[k,k+1]}$ is a line segment for all $k = 0, 1, \ldots, n-1$.

We depict $P_{\mathfrak{t}}$ graphically inside the standard two-dimensional coordinate system, but reflected through the x-axis. For instance, if $\mathfrak{s}$ and $\mathfrak{t}$ are the standard tableaux in (10.2.1)

$$[\boldsymbol{\lambda}] = \left(\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \end{array}, \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \end{array} \right) \qquad \mathfrak{s} = \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 10 \\ \hline 11 \\ \hline \end{array}, \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline 6 \\ \hline 7 \\ \hline 8 \\ \hline 9 \\ \hline \end{array} \right) \qquad \mathfrak{t} = \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 5 \\ \hline 7 \\ \hline 9 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline 6 \\ \hline 8 \\ \hline 10 \\ \hline 11 \\ \hline \end{array} \right) \tag{10.2.1}$$

then $P_{\mathfrak{s}}$ and $P_{\mathfrak{t}}$ are depicted in (10.2.2), with $P_{\mathfrak{s}}$ in red and $P_{\mathfrak{t}}$ in black. In general, we denote by $P_{\boldsymbol{\lambda}}$ the path obtained from the tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$. Thus in (10.2.2) we have that $P_{\mathfrak{t}} = P_{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} = (1^5, 1^6)$.

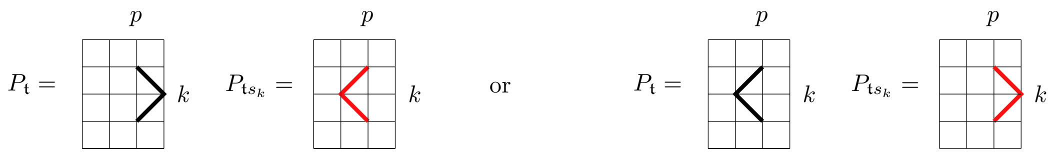


(10.2.2)

Note that in general the integral values of $P_{\mathfrak{t}}$ belong to the set $\{(p, k) \mid k \in \mathbb{Z}_{\geq 0},\ p = -k, -k+2, \ldots, k-2, k\}$. This set has a Pascal triangle structure which is why we say that standard tableaux correspond to paths on the Pascal triangle.

It is clear that the map $\mathfrak{t} \mapsto P_{\mathfrak{t}}$ defines a bijection between $\mathrm{Std}(\boldsymbol{\lambda})$ and the set of all such piecewise linear paths with final vertex $(\lambda_2 - \lambda_1, n)$. For this reason, we sometimes identify $\boldsymbol{\lambda}$ with the point $(\lambda_2 - \lambda_1, n)$.

Suppose now that both $\mathfrak{t}$ and $\mathfrak{t}s_k$ are standard tableaux for some $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$ and $s_k \in S$. Then k and $k+1$ are in different columns of $\mathfrak{t}$ and so we conclude that the functions $p_{\mathfrak{t}}$ and $p_{\mathfrak{t}s_k}$ are equal except that $p_{\mathfrak{t}}(k) = p_{\mathfrak{t}s_k}(k) \pm 2$, and hence also the paths $P_{\mathfrak{t}}$ and $P_{\mathfrak{t}s_k}$ are equal except in the interval $[k-1, k+1]$ where they are related in the following two possible ways



(10.2.3)

Conversely, if $\mathfrak{s}$ and $\mathfrak{t}$ are standard tableaux in $\mathrm{Std}(\boldsymbol{\lambda})$ such that $P_{\mathfrak{s}}$ and $P_{\mathfrak{t}}$ are equal except in the interval $[k-1, k+1]$ where they are related as in (10.2.3), then we have that $\mathfrak{s} = \mathfrak{t}s_k$. Let us now consider the following algorithm.

Algorithm 10.2.1. Let $\boldsymbol{\lambda} \in \mathrm{Par}_n^1$ and $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then we define a sequence $seq := (s_{i_1}, s_{i_2}, \ldots, s_{i_N})$ of elements of S_n as follows.

Step 1. Set $P_0 := P_{\boldsymbol{\lambda}}$. If $P_0 \neq P_{\mathfrak{t}}$ then choose i_1 any such that $\mathfrak{t}^{\boldsymbol{\lambda}} s_i \in \mathrm{Std}(\boldsymbol{\lambda})$ and such that the area bounded by $P_1 := P_{\mathfrak{t}^{\boldsymbol{\lambda}} s_i}$ and $P_{\mathfrak{t}}$ is strictly smaller than the area bounded by P_0 and $P_{\mathfrak{t}}$.

Step 2. If $P_1 = P_{\mathfrak{t}}$ then the algorithm stops with $seq := (s_{i_1})$. Otherwise choose any i_2 such that $\mathfrak{t}^{\boldsymbol{\lambda}} s_{i_1} s_{i_2} \in \mathrm{Std}(\boldsymbol{\lambda})$ and such that the area bounded by $P_2 := P_{\mathfrak{t}^{\boldsymbol{\lambda}} s_{i_1} s_{i_2}}$ and $P_{\mathfrak{t}}$ is strictly smaller than the area bounded by P_1 and $P_{\mathfrak{t}}$.

Step 3. If $P_2 = P_{\mathfrak{t}}$ then the algorithm stops with $seq := (s_{i_1}, s_{i_2})$. Otherwise choose any i_3 such that $\mathfrak{t}^{\boldsymbol{\lambda}} s_{i_1} s_{i_2} s_{i_3} \in \mathrm{Std}(\boldsymbol{\lambda})$ and such that the area bounded by $P_3 := P_{\mathfrak{t}^{\boldsymbol{\lambda}} s_{i_1} s_{i_2} s_{i_3}}$ and $P_{\mathfrak{t}}$ is strictly smaller than the area bounded by P_2 and $P_{\mathfrak{t}}$.

Step 4. Repeat until $P_N = P_{\mathfrak{t}}$. The resulting sequence $seq = (s_{i_1}, s_{i_2}, \ldots, s_{i_N})$ gives rise to a reduced expression for $d(\mathfrak{t})$ via $d(\mathfrak{t}) = s_{i_1} s_{i_2} \cdots s_{i_N}$.

Note that it follows from (10.2.3) that the i_k's in **Step 2** and **Step 3** do exist and so the Algorithm 10.2.1 makes sense. For example in the case of the tableau $\mathfrak{s}$ from (10.2.1) we get, using (10.2.2), that for example

$$d(\mathfrak{s}) = s_2 s_4 s_3 s_7 s_9 s_8 s_{10} s_9 \tag{10.2.4}$$

is a reduced expression for $d(\mathfrak{s})$. For completeness, we now present a proof of the correctness of the Algorithm.

Theorem 10.2.2. *Algorithm 10.2.1 computes a reduced expression for $d(\mathfrak{s})$.*

Proof: This is a statement about the symmetric group $\mathfrak{S}_n$ viewed as a Coxeter group. Let $\mathfrak{t}_k := \mathfrak{t}^{\boldsymbol{\lambda}} s_{i_1} s_{i_1} \cdots s_{i_k}$ be the tableau constructed after k steps of the algorithm. Then we have that $d(\mathfrak{t}_k) = s_{i_1} s_{i_1} \cdots s_{i_k}$ and we must show that $l(s_{i_1} s_{i_1} \cdots s_{i_k}) = k$ where $l(\cdot)$ is the length function for $\mathfrak{S}_n$. We therefore identify $d(\mathfrak{t}_k)$ with a permutation of $\{1, 2, \ldots, n\}$ via the row reading for $\mathfrak{t}_k$. To be precise, using the usual one line notation for permutations, we write

$d(\mathfrak{t}_k) =$

$\mathfrak{t}_k((\mathfrak{t}^{\boldsymbol{\lambda}})^{-1}(1))$	$\mathfrak{t}_k((\mathfrak{t}^{\boldsymbol{\lambda}})^{-1}(2))$	·····	$\mathfrak{t}_k((\mathfrak{t}^{\boldsymbol{\lambda}})^{-1}(n))$

(10.2.5)

We call this the *one line representation* for $d(\mathfrak{t}_k)$. If for example $\mathfrak{t}_k = \mathfrak{s}$ from (10.2.1) then we have the following one line representation for $d(\mathfrak{t}_k)$

$d(\mathfrak{s}) =$

1	4	2	5	3	6	10	7	11	8	9

(10.2.6)

whereas for $\mathfrak{t}_k = \mathfrak{t}^{\boldsymbol{\lambda}}$ from (10.2.1) we have the identity one line representation, that is

$d(\mathfrak{t}^{\boldsymbol{\lambda}}) =$

1	2	3	4	5	6	7	8	9	10	11

(10.2.7)

In general, by the Coxeter theory for $\mathfrak{S}_n$, we have that $l(d(\mathfrak{t}_k))$ is the number of *inversions* of the one line representation of $d(\mathfrak{t}_k)$ that is

$$l(d(\mathfrak{t}_k)) = inv(d(\mathfrak{t}_k)) := |\{(i,j) : i < j \text{ and } \mathfrak{t}_k((\mathfrak{t}^{\boldsymbol{\lambda}})^{-1}(i) > \mathfrak{t}_k((\mathfrak{t}^{\boldsymbol{\lambda}})^{-1}(j)\}| \tag{10.2.8}$$

To prove the Theorem we must now show that $inv(d(\mathfrak{t}_k)) = k$. We proceed by induction on k. For $k = 0$ we have that $inv(d(\mathfrak{t}_k)) = inv(d(\mathfrak{t}^{\boldsymbol{\lambda}})) = 0$, see (10.2.7), and so the induction basis is ok. We next assume that $inv(d(\mathfrak{t}_{k-1})) = k-1$ and must show that $inv(d(\mathfrak{t}_k)) = k$. At step k of Algorithm 10.2.1, we have that $\mathfrak{t}_{k-1}, \mathfrak{t}_k \in \mathrm{Std}(\boldsymbol{\lambda})$ and $\mathfrak{t}_{k-1} s_{i_k} = \mathfrak{t}_k$ and hence $\mathfrak{t}_{k-1}$ and $\mathfrak{t}_k$ are in one of the two situations described in 10.2.3. Let p be as in 10.2.3. Then, since $\mathfrak{t}_k$ is closer to $\mathfrak{t}$ than $\mathfrak{t}_{k-1}$, we have that $\mathfrak{t}_{k-1}$ and $\mathfrak{t}_k$ are in the first situation of 10.2.3 if $p \leq -1$ and in the second situation of 10.2.3 if $p \geq 0$. In other words, the first situation of 10.2.3 only takes places in the left half of the Pascal triangle (10.2.2) and the second situation of 10.2.3 only takes places in the right half of the Pascal triangle (10.2.2), with the vertical axis $p = 0$ is included.

These two situations translate into the following two possible relative positions for k and $k+1$ in $\mathfrak{t}_{k-1}$.

$$\left(\begin{array}{|c|} \hline * \\ \hline * \\ \hline * \\ \hline \vdots \\ \hline k+1 \\ \hline * \\ \hline \end{array} \, , \, \begin{array}{|c|} \hline * \\ \hline \vdots \\ \hline k \\ \hline * \\ \hline * \\ \hline * \\ \hline \end{array} \right) \qquad \left(\begin{array}{|c|} \hline * \\ \hline \vdots \\ \hline k \\ \hline * \\ \hline * \\ \hline * \\ \hline \end{array} \, , \, \begin{array}{|c|} \hline * \\ \hline * \\ \hline * \\ \hline \vdots \\ \hline k+1 \\ \hline * \\ \hline \end{array} \right) \tag{10.2.9}$$

Here, in both tableaux k and $k+1$ are in different columns, but in the first tableau, corresponding to $p < 0$, we have that $k+1$ is in a strictly lower row than k, whereas in the second tableau, corresponding to $p \geq 0$, we have that $k+1$ is in a lower or equal row than k.

On the other hand, in each of the two cases of (10.2.9) we have that k appears before $k+1$ in the one line representation for $\mathfrak{t}_{k-1}$ and so $inv(d(\mathfrak{t}_k)) = inv(d(\mathfrak{t}_{k-1})) + 1$. This proves the Theorem. □

Remark 10.2.3. We remark that the reduced expression for $d(\mathfrak{s})$ obtained via Algorithm 10.2.1 is by no means unique. In general, we have many choices for the i_k's and the reduced expression obtained depends on the choices we make. On the other hand, it is known that $d(\mathfrak{s})$ is fully commutative. In other words, any two reduced expressions for $d(\mathfrak{s})$ are related via the commuting braid relations.

10.3 Alcove geometry

We now introduce an $\tilde{A}_1$ alcove geometry on $\mathbb{R}^2$. For each $j \in \mathbb{Z}$ we introduce a *wall* M_j in $\mathbb{R}^2$ via

$$M_j := \{((j-1)e + m, a) \mid a \in \mathbb{R}\} \subset \mathbb{R}^2. \tag{10.3.1}$$

The connected components of $\mathbb{R}^2 \setminus \bigcup_j M_j$ are called *alcoves* and the alcove containing $(0,0)$ is denoted by $\mathcal{A}^0$ and is called the *fundamental alcove*. Recall that we have fixed W as the infinite dihedral group with generators s and t. We view W as the reflection group associated with this alcove geometry, where s and t are the reflections through the walls M_0 and M_1, respectively. This defines a right action of W on $\mathbb{R}^2$ and on the set of alcoves. For $w \in W$, we write $\mathcal{A}^w := \mathcal{A}^0 \cdot w$.

Let $P : [0, n] \to \mathbb{R}^2$ be a path on the Pascal triangle and suppose that $P(k) \in M_j$ for some integers k and j. Let r_j be the reflection through the wall M_j. We then define a new path $P^{(k,j)}$ by applying r_j to the part of P that comes after $P(k)$, that is

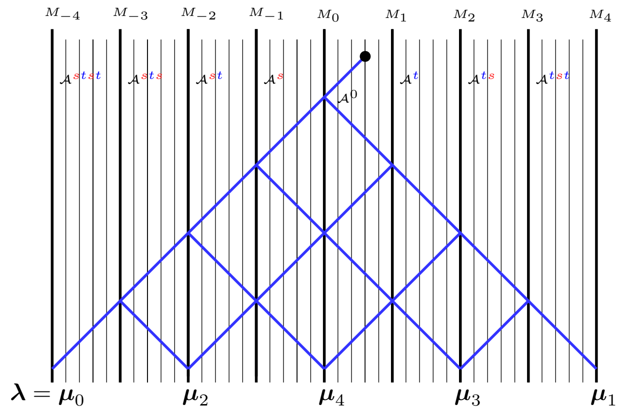
$$P^{(k,j)}(t) := \begin{cases} P(t), & \text{if } 0 \leq t \leq k; \\ P(t) r_j, & \text{if } k \leq t \leq n. \end{cases} \tag{10.3.2}$$

For two paths on the Pascal triangle we write $P \overset{(k,j)}{\sim} Q$ if $Q = P^{(k,j)}$ and denote by $\sim$ the equivalence relation on the paths on the Pascal triangle induced by the $\overset{(k,j)}{\sim}$'s. Then we have the following Lemma which is a straightforward consequence of the definitions.

Lemma 10.3.1. *Suppose that $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\mathrm{Par}_n^1)$. Then $\boldsymbol{i}^{\mathfrak{s}} = \boldsymbol{i}^{\mathfrak{t}}$ if and only if $P_{\mathfrak{s}} \sim P_{\mathfrak{t}}$.*

We can now provide an alcove geometrical description of $\mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$. It is a direct consequence of Lemma 10.3.1.

Lemma 10.3.2. *Let $[P_{\boldsymbol{\lambda}}]$ be the equivalence class of $P_{\boldsymbol{\lambda}}$ under the equivalence relation $\sim$. Then,* $\mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}}) = [P_{\boldsymbol{\lambda}}]$.



(10.3.3)

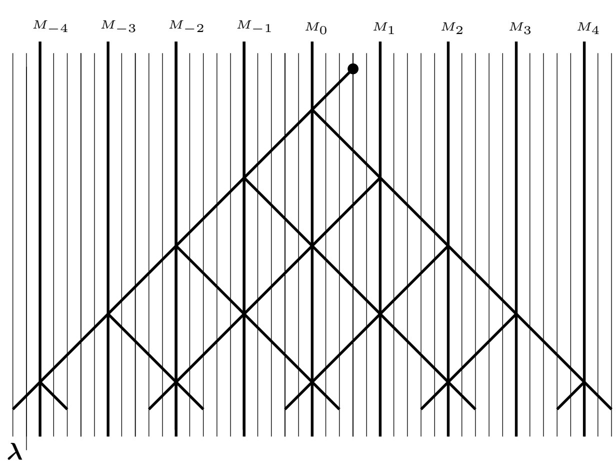
In (10.3.3) we indicate for $m = 2, e = 5$ and $n = 23$ the paths corresponding to elements in $\mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$, according to Lemma 10.3.4. The path $P_{\boldsymbol{\lambda}}$ is the one to the extreme left. The endpoints of the paths are enumerated according to the order relation $\lhd$ on Par_n^1, with $\boldsymbol{\mu}_0 = \boldsymbol{\lambda}$, $\boldsymbol{\mu}_1$ the rightmost path, and so on.

To illustrate the connection between paths and tableaux, we present in (10.3.5) the six elements of $\mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu}_4)$ for (10.3.3) as tableaux. We have colored the entries of each tableau by *blocks*. The *zero'th block* corresponds to the path segment from the origin $(0,0)$ to the first wall M_0 and its entries have been colored red. The first *full block* corresponds to the path segment from M_0 to the next wall which may be either M_{-1} or M_1 depending on the tableau and the corresponding elements have been colored blue, and so on. We shall give the precise definition of full blocks shortly.

In (10.3.5) we have also given the *residue tableau* $\mathrm{res}\,\boldsymbol{\mu}_4$ for $\boldsymbol{\mu}_4$. By definition, it is obtained from $[\boldsymbol{\mu}_4]$ by decorating each node A with its residue $\mathrm{res}(A)$. Using it, one checks that for each $\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu}_4)$ the corresponding residue sequence is $\boldsymbol{i}^{\boldsymbol{\lambda}}$, as it should be:

$$\boldsymbol{i}^{\boldsymbol{\lambda}} = \boldsymbol{i}^{\mathfrak{t}} = (0,4,3,2,1,0,4,3,2,1,0,4,3,2,1,0,4,3,2,1,0,4,3,2,1) \tag{10.3.4}$$

$$\mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu}_4) = \left(\begin{array}{cc} 1 & 14 \\ 2 & 15 \\ 3 & 16 \\ 4 & 17 \\ 5 & 18 \\ 6 & 19 \\ 7 & 20 \\ 8 & 21 \\ 9 & 22 \\ 10 & 23 \\ 11 & \\ 12 & \\ 13 & \end{array}, \begin{array}{cc} 1 & 9 \\ 2 & 10 \\ 3 & 11 \\ 4 & 12 \\ 5 & 13 \\ 6 & 19 \\ 7 & 20 \\ 8 & 21 \\ 14 & 22 \\ 15 & 23 \\ 16 & \\ 17 & \\ 18 & \end{array}, \begin{array}{cc} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 9 & 7 \\ 10 & 8 \\ 11 & 19 \\ 12 & 20 \\ 13 & 21 \\ 14 & 22 \\ 15 & 23 \\ 16 & \\ 17 & \\ 18 & \end{array}, \begin{array}{cc} 1 & 9 \\ 2 & 10 \\ 3 & 11 \\ 4 & 12 \\ 5 & 13 \\ 6 & 14 \\ 7 & 15 \\ 8 & 16 \\ 19 & 17 \\ 20 & 18 \\ 21 & \\ 22 & \\ 23 & \end{array}, \begin{array}{cc} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 9 & 7 \\ 10 & 8 \\ 11 & 14 \\ 12 & 15 \\ 13 & 16 \\ 19 & 17 \\ 20 & 18 \\ 21 & \\ 22 & \\ 23 & \end{array}, \begin{array}{cc} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 14 & 7 \\ 15 & 8 \\ 16 & 9 \\ 17 & 10 \\ 18 & 11 \\ 19 & 12 \\ 20 & 13 \\ 21 & \\ 22 & \\ 23 & \end{array} \right), \ [\mathrm{res}\,\boldsymbol{\mu}_4] = \left(\begin{array}{cc} 0 & 2 \\ 4 & 1 \\ 3 & 0 \\ 2 & 4 \\ 1 & 3 \\ 0 & 2 \\ 4 & 1 \\ 3 & 0 \\ 2 & 4 \\ 1 & 3 \\ 0 & \\ 4 & \\ 3 & \end{array} \right) \tag{10.3.5}$$



(10.3.6)

The structure of $\mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$ depends on whether $\boldsymbol{\lambda}$ is *singular or regular*:

Definition 10.3.3. *Let the integers $K_{n,m} = K$ and $0 \leq R_{n,m} = R < e$ be defined via integer division $n-(e-m) = Ke+R$. Then we say that $\boldsymbol{\lambda}$ is singular if $R = 0$ and otherwise we say $\boldsymbol{\lambda}$ that is regular. Graphically, $\boldsymbol{\lambda}$ is singular if it is located on a wall, otherwise it is regular.*

The paths in (10.3.3) represent a singular situation whereas the paths in (10.3.6) represent a regular situation. In both cases, regular or singular, the cardinality $|\mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})|$ is given by binomial coefficients and so we have the following Lemma.

Lemma 10.3.4. a) *Let $[P_{\boldsymbol{\lambda}}]$ be the equivalence class of $P_{\boldsymbol{\lambda}}$ under the equivalence relation $\sim$. Then,* $\mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}}) = [P_{\boldsymbol{\lambda}}]$.

b) *Suppose that $\boldsymbol{\lambda}$ is singular. Then* $\sum_{\boldsymbol{\mu}\in[P_{\boldsymbol{\lambda}}](n)} |\mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})|^2 = \binom{2K}{K}$.

c) *Suppose that $\boldsymbol{\lambda}$ is regular. Then* $\sum_{\boldsymbol{\mu}\in[P_{\boldsymbol{\lambda}}](n)} |\mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})|^2 = 2\binom{2K}{K}$.

We now define the integer valued function

$$f_{n,m}(j) = f(j) := -m + je \text{ for } j \in \mathbb{Z}_+. \tag{10.3.7}$$

Then for $\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ we have that $k = f(1), f(2), \ldots, f(K)$ are the values of k such that $P_{\mathfrak{t}}(k)$ belongs to a wall M_j and we then define for $i = 1, 2, \ldots, K$ the i'th *full block for* $\boldsymbol{\lambda}$ as the set

$$B_i := [f(i)+1, f(i)+2, \ldots, f(i)+e]. \tag{10.3.8}$$

For example, in the situations (10.3.3) and (10.3.6) we have the following full blocks

$$B_1 = [4,5,6,7,8],\ B_2 = [9,10,11,12,13],\ B_3 = [14,15,16,17,18],\ B_4 = [19,20,21,22,23]. \tag{10.3.9}$$

For $1 \leq i < K$ we next define $U_i \in \mathfrak{S}_n$ as the order preserving permutation that interchanges the blocks B_i and B_{i+1} that is

$$U_i := (f(i)+1, f(i+1)+1)\,(f(i)+2, f(i+1)+2)\cdots(f(i)+e, f(i+1)+e). \tag{10.3.10}$$

For example, in the situation (10.3.9) we have

$$U_1 = (4,9)(5,10)(6,11)(7,12)(8,13) \tag{10.3.11}$$

written as a product of non-simple transpositions. We need a reduced expression for (10.3.10) and therefore for $i \leq j$ of the same parity we introduce the following element of $\mathfrak{S}_n$

$$s_{[i,j]} := s_i s_{i+2}\cdots s_{j-2}s_j. \tag{10.3.12}$$

Then we have

$$U_i = s_{[a,a]}s_{[a-1,a+1]}\cdots s_{[a-e+1,a+e-1]}\cdots s_{[a-1,a+1]}s_{[a,a]} \tag{10.3.13}$$

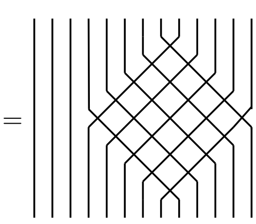
where $a = f(i+1)$ which upon expanding out the $s_{[i,j]}$'s becomes a reduced expression for U_i. We can now recall the following important definition from [23].

Definition 10.3.5. For $1 \leq i < K$ we define the *diamond* of $\boldsymbol{\lambda}$ at position $f(i)$ by

$$U_i^{\boldsymbol{\lambda}} := \psi_{U_i}e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = \psi_{[a,a]}\psi_{[a-1,a+1]}\cdots\psi_{[a-e+1,a+e-1]}\cdots\psi_{[a-1,a+1]}\psi_{[a,a]}e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \tag{10.3.14}$$

where $a = f(i+1)$ and $\psi_{[i,j]} := \psi_i\psi_{i+2}\cdots\psi_{j-2}\psi_j$.

The name 'diamond' comes from the diagrammatical realization of $\mathbb{B}_n(\boldsymbol{\lambda})$. Here is for example the $n = 13, m = 2, e = 5$ and $i = 1$ case

$U_1^{\boldsymbol{\lambda}} =$  (10.3.15)

Chapter 11

A presentation for $\mathbb{B}_n(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda}$ singular

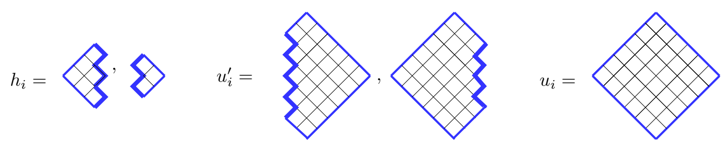
In this chapter we consider the case where $\boldsymbol{\lambda}$ is singular. Our aim is to show that $\mathbb{B}_n(\boldsymbol{\lambda})$ and $\mathbb{NB}_K$ are isomorphic $\mathbb{F}$-algebras. The first step towards this goal is to prove that the following subset of $\mathbb{B}_n(\boldsymbol{\lambda})$

$$G(\boldsymbol{\lambda}) := \{U_j^{\boldsymbol{\lambda}} \mid 1 \le j < K\} \cup \{y_i e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \mid 1 \le i \le n\} \tag{11.0.1}$$

is a generating set for $\mathbb{B}_n(\boldsymbol{\lambda})$. To be precise, letting $\mathbb{B}'_n(\boldsymbol{\lambda})$ be the subalgebra of $\mathbb{B}_n(\boldsymbol{\lambda})$ generated by $G(\boldsymbol{\lambda})$ we shall show that each element $m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}}$ of the cellular basis $\mathcal{C}_n(\boldsymbol{\lambda})$ for $\mathbb{B}_n(\boldsymbol{\lambda})$, given in Lemma 10.1.4, belongs to $\mathbb{B}'_n(\boldsymbol{\lambda})$. The proof of this will take up the next few pages.

We shall rely on a systematic way of applying Algorithm 10.2.1 to get reduced expressions for the elements $d(\mathfrak{t})$, $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$. Let us now explain it.

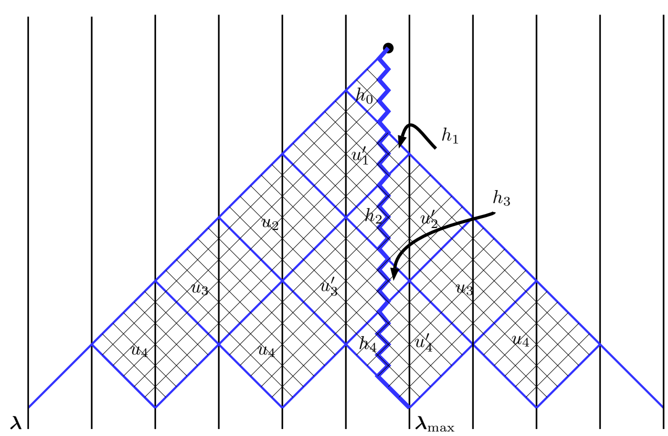
Let $\boldsymbol{\lambda}_{\max} \in \mathrm{Par}^1_n$ be the maximal element in the W-orbit of $\boldsymbol{\lambda}$ with respect to the order $\lhd$. Clearly, $\boldsymbol{\lambda}_{\max}$ is located on one of the two walls of the fundamental alcove. Recall that $P_{\boldsymbol{\lambda}_{\max}}$ is the path associated with the tableau $\mathfrak{t}^{\boldsymbol{\lambda}_{\max}}$; it zigzags along the vertical central axis of the Pascal triangle as long as possible, and finally goes linearly off to $\boldsymbol{\lambda}_{\max}$. The set of paths $P_{\mathfrak{t}}$ for $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$ together with $P_{\boldsymbol{\lambda}_{\max}}$, which does *not* belong to $\mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$, determine three kind of bounded regions that we denote by h_i, u_i and u'_i:



(11.0.2)

See also (11.0.3). In (11.0.2) as well as (11.0.3) we have indicated $P_{\boldsymbol{\lambda}_{\max}}$ with bold blue.

In general the h_i's are completely embedded in $\mathcal{A}^0$, whereas the 'diamond' regions u_i's have empty intersection with $\mathcal{A}^0$. The 'cut diamond' regions u'_i's have non-empty intersection with $\mathcal{A}^0$ but also with one of the alcoves $\mathcal{A}^{s}$ or $\mathcal{A}^{t}$. Note that the union of h_i and u'_i forms a diamond shape. We enumerate the regions from top to bottom as in (11.0.3), with the h_i's starting with $i = 0$ and the u'_i and u_i's with $i = 1$. Note that there are repetitions of the u_i's.



(11.0.3)

For each of the three kinds of regions h_i, u_i, u_i' we now introduce an element $H_i, U_i, U_i' \in \mathfrak{S}_n$ in the following way. For $R = h_i, u_i, u_i'$ we let $\partial(R)$ be the boundary of R with respect to the usual metric topology. Then for any $R = h_i, u_i, u_i'$ we have that $\partial(R)$ is a union of line segments and we define the outer boundary, $\partial_{out}(R)$, as the union of the two line segments that are the furthest away from $P_{\boldsymbol{\lambda}_{\max}}$. Moreover we define the *inner boundary* as $\partial_{in}(R) = \overline{\partial(R) \setminus \partial_{out}(R)}$, where the overline means closure with respect to the metric topology.

Suppose now that $R = h_i$ (resp. $R = u_i$ and $R = u_i'$). We then choose any tableau $\mathfrak{b} \in \mathrm{Std}(\mathrm{Par}_n^1)$ such that $\partial_{in}(R) \subseteq P_{\mathfrak{b}}$. Let $P_{\mathfrak{b}}'$ be the path obtained from $P_{\mathfrak{b}}$ by replacing $\partial_{in}(R)$ by $\partial_{out}(R)$. Then we define $H_i \in S_n$ (resp. $U_i \in \mathfrak{S}_n$ or $U_i' \in \mathfrak{S}_n$) by the equation

$$P_{\mathfrak{b}}' = P_{\mathfrak{b}H_i} \text{ (resp. } P_{\mathfrak{b}}' = P_{\mathfrak{b}U_i} \text{ and } P_{\mathfrak{b}}' = P_{\mathfrak{b}U_i'}). \tag{11.0.4}$$

In other words, H_i (resp. U_i and U_i') is simply the element of $\mathfrak{S}_n$ that is used to fill in the region h_i (resp. u_i and u_i') in the sense of Algorithm 10.2.1, where each s_i appearing in H_i (resp. U_i and U_i') corresponds to the filling in of one of the little squares of h_i (resp. u_i and u_i'). For example, in the situation (11.0.3) we have that

$$H_0 = s_2 s_4 s_6 s_3 s_5 s_4, \;\; H_1 = s_9 s_{11} s_{10}, \;\; U_1' = s_{[8,12]} s_{[7,13]} s_{[6,14]} s_{[5,15]} s_{[6,14]} s_{[7,13]} s_{[8,12]} s_{[9,11]} s_{[10,10]} \tag{11.0.5}$$

where we used the notation from (10.3.12) for the formula for U_1'. Note that the U_i's coincide with the U_i's defined in (10.3.10). It is also possible to give formulas for the H_i's and the U_i''s, in the spirit of (10.3.10), but we do not need them.

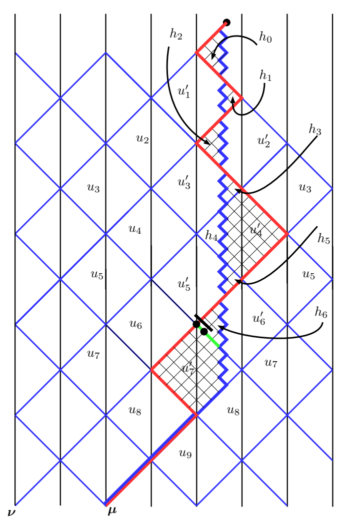
For any $\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ we now introduce a reduced expression for $d(\mathfrak{t})$ by applying Algorithm 10.2.1 in a way compatible with the regions. To be precise, starting with $P_{\boldsymbol{\lambda}_{\max}}$ we first choose those regions h_i that give rise to a path closer to $P_{\mathfrak{t}}$ than $P_{\boldsymbol{\lambda}_{\max}}$, by replacing the inner boundaries with the outer boundaries. Having adjusted $P_{\boldsymbol{\lambda}_{\max}}$ for those h_i's we next choose those regions u_i' that the same way give rise to a path even closer to $P_{\mathfrak{t}}$ and finally we repeat the process with the regions u_i. It may be necessary to repeat the last step more than once. The product of the corresponding symmetric group elements is now a reduced expression for $d(\mathfrak{t})$: this is *our favorite reduced expression* for $d(\mathfrak{t})$ that we shall henceforth use.

In (11.0.7) we give two examples with $e = 6$ and $m = 2$.

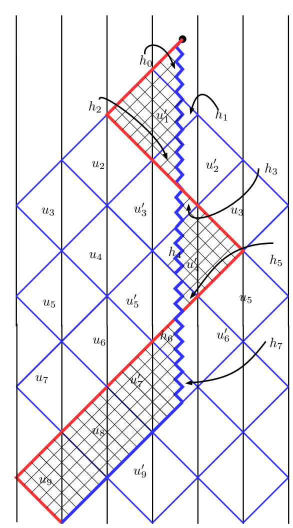
We let ψ_{H_i} (resp. ψ_{U_i} and $\psi_{U_i'}$) be the element of $\mathbb{B}_n$ obtained by replacing each $s_i \in \mathfrak{S}_n$ in H_i (resp. U_i and U_i') with the corresponding ψ_i. We then get an expression for $\psi_{d(\mathfrak{t})}$ by replacing each occurring H_i (resp. U_i and U_i') in the above expansion for $d(\mathfrak{t})$ by ψ_{H_i} (resp. ψ_{U_i} and $\psi_{U_i'}$). Note that $\psi_{U_i} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = U_i^{\boldsymbol{\lambda}} \in G(\boldsymbol{\lambda})$ from (11.0.1). For example, in the cases (11.0.7) we have

$$\psi_{d(\mathfrak{s})} = \psi_{H_0}\psi_{H_1}\psi_{H_2}\psi_{H_3}\psi_{H_5}\psi_{H_6}\psi_{U_4'}\psi_{U_7'} \qquad \text{and} \qquad \psi_{d(\mathfrak{t})} = \psi_{H_0}\psi_{H_2}\psi_{H_3}\psi_{H_5}\psi_{H_6}\psi_{U_1'}\psi_{U_4'}\psi_{U_7'}\psi_{U_8}\psi_{U_9}. \tag{11.0.6}$$

$P_{\mathfrak{s}} =$



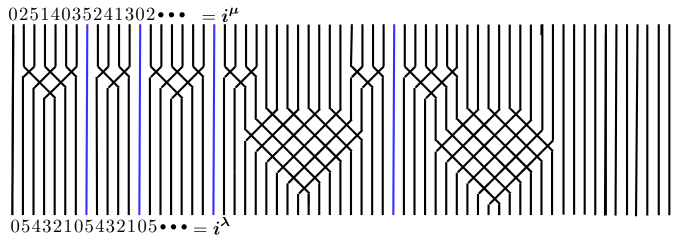
$P_{\mathfrak{t}} =$



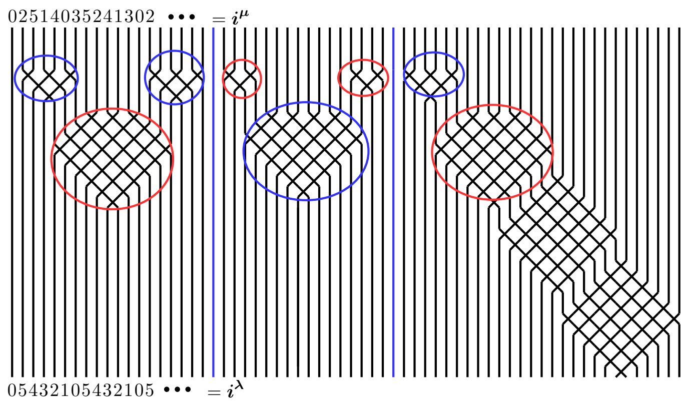
(11.0.7)

With the same $\mathfrak{s}$ and $\mathfrak{t}$ we have in terms of KLR-diagrams

$$e(\boldsymbol{i}^{\boldsymbol{\mu}})\psi_{d(\mathfrak{s})} = \tag{11.0.8}$$



$$e(\boldsymbol{i}^{\boldsymbol{\mu}})\psi_{d(\mathfrak{t})} = \tag{11.0.9}$$



Let us give some comments related to the combinatorial structure of (11.0.8) and (11.0.9); these hold in general. Note first that only the lower residue sequence of (11.0.8) and (11.0.9) is $\boldsymbol{i}^{\boldsymbol{\lambda}}$ and so $e(\boldsymbol{i}^{\boldsymbol{\mu}})\psi_{d(\mathfrak{s})}$ and $e(\boldsymbol{i}^{\boldsymbol{\mu}})\psi_{d(\mathfrak{t})}$ actually do not belong to $\mathbb{B}_n(\boldsymbol{\lambda})$, only to $\mathbb{B}_n$. Secondly, note that the KLR-diagrams for the ψ_{H_i}'s are located in the 'top lines' of (11.0.8) and (11.0.9), whereas the diagrams for the $\psi_{U_i'}$'s and the ψ_{U_i}'s are situated in 'the middle and the bottom lines' of (11.0.8) and (11.0.9), respectively. For each i only one of the diagrams ψ_{H_i} or $\psi_{U_i'}$ appears. The appearing ψ_{H_i}'s and $\psi_{U_i'}$'s are ordered from the left to the right, with ψ_{H_0}, that always appears, to the extreme left and so on. On the other hand, in general the ψ_{U_i}'s do not appear ordered.

Next, we observe that the shapes of ψ_{H_i}'s and the $\psi_{U_i'}$'s depend on their parity. In other words, if i and j have the same parity then ψ_{H_i} and ψ_{H_j} (resp. $\psi_{U_i'}$ and $\psi_{U_j'}$) have the same shape. In (11.0.9) we have encircled with blue the *even* diagrams ψ_{H_i} and $\psi_{U_i'}$ and with red the *odd* diagrams ψ_{H_i} and $\psi_{U_i'}$.

Our next observation is that the diagrams $\psi_{U_i'}$ always lie between two diagrams $\psi_{H_{i-1}}$ and $\psi_{H_{i+1}}$, except possibly for the rightmost $\psi_{U_i'}$. The rightmost $\psi_{U_i'}$ is always preceded by $\psi_{H_{i-1}}$ but it may be followed by $\psi_{U_{i+1}}$, as in (11.0.9), or by a number of through lines, as in (11.0.8).

In general, we have that the ψ_{H_i}'s are 'distant' apart and so pairwise commuting. This is not the case for the $\psi_{U_i'}$'s. However, we still have that $\psi_{U_i'}\psi_{U_j'} = \psi_{U_j'}\psi_{U_i'}$ if $|i-j| > 1$. By the previous paragraph we know that each occurrence of $\psi_{U_i'}$ is surrounded by $\psi_{H_{i-1}}$ and $\psi_{H_{i+1}}$. We conclude that if $\psi_{U_i'}$ and $\psi_{U_j'}$ occur in the diagram of some $\psi_{d(\mathfrak{t})}$ then $|i-j| > 1$, and therefore, they do commute. The relations between the ψ_{U_i}'s are known from [23], we shall return to them shortly. Between the different groups there is no commutativity in general, that is $\psi_{U_i'}$ does not commute with $\psi_{H_{i-1}}$ and $\psi_{H_{i+1}}$ and so on.

Finally, we observe that the all of the diagrams $\psi_{H_i}, \psi_{U_i'}$ and ψ_{U_i} are organized tightly. There are for example only two through lines in (11.0.9). In both (11.0.8) and (11.0.9) we have colored blue the through lines that correspond to the places where $P_{\mathfrak{s}}$ and $P_{\mathfrak{t}}$ change from the left to right half of the Pascal triangle, or reversely. In general these lines lie between two ψ_{H_i}'s. Thus the contours' of (11.0.8) and (11.0.9) are a mirror of the shapes of the paths (11.0.7), with the modification that the through blue lines indicate a change from left to right of reversely.

For $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$ we define $\theta(\mathfrak{t})$ as the element of $\mathfrak{S}_n$ obtained from the favorite reduced expression for $d(\mathfrak{t})$ by erasing all the U_i-factors and similarly we define $u(\mathfrak{t}) \in \mathfrak{S}_n$ by erasing both the H_i and the U_i'-factors. Then clearly

$$d(\mathfrak{t}) = \theta(\mathfrak{t})u(\mathfrak{t}). \tag{11.0.10}$$

We now have the following Lemma.

Lemma 11.0.1. *Suppose that $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ and let $P_{\mathfrak{s}_1}$ and $P_{\mathfrak{t}_1}$ be the paths obtained from $P_{\mathfrak{s}}$ and $P_{\mathfrak{t}}$ by replacing outer boundary with inner boundary for all the u_i-regions. Then we have that $\theta(\mathfrak{s}) = d(\mathfrak{s}_1)$ and $\theta(\mathfrak{t}) = d(\mathfrak{t}_1)$. Moreover*

$$m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}} = \psi^*_{u(\mathfrak{s})} m^{\boldsymbol{\mu}}_{\mathfrak{s}_1\mathfrak{t}_1} \psi_{u(\mathfrak{t})}. \tag{11.0.11}$$

Proof. The result is a direct consequence of the definitions. □

Our goal is to prove that $m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}}$ belongs to $\mathbb{B}'_n(\boldsymbol{\lambda})$. On the other hand, $\psi_{u(\mathfrak{s})}$ and $\psi_{u(\mathfrak{t})}$ in (11.0.11) are products of $U_i^{\boldsymbol{\lambda}}$'s and so it follows from Lemma 11.0.1 that to achieve this goal it is enough to consider the case where $\mathfrak{s} = \mathfrak{s}_1$ and $\mathfrak{t} = \mathfrak{t}_1$. Let us give the corresponding formal definition.

Definition 11.0.2. Let $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$. We say that $\mathfrak{t}$ is *central* if $u(\mathfrak{t})$ is the empty word. Equivalently, $\mathfrak{t}$ is *central* if $d(\mathfrak{t}) = \theta(\mathfrak{t})$.

Geometrically, $\mathfrak{t}$ is central if the path $P_{\mathfrak{t}}$ stays close to the central vertical axis of the Pascal triangle. In other words, $P_{\mathfrak{t}}$ does not cross the walls M_{-1} and M_2, except possible once in the final stage. For example, in (11.0.7) we have that $\mathfrak{s}$ is central but $\mathfrak{t}$ is not. In view of Lemma 11.0.1 we will from now on only consider central tableaux.

Suppose therefore that $\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu}_k)$ is central where $\boldsymbol{\mu}_k$ is as described in (10.3.3). Then one checks that the total number of ψ_{H_i}'s and $\psi_{U_i'}$'s appearing in $\psi_{d(\mathfrak{t})}$ is k. We now define a $(2 \times k)$-matrix $c(\mathfrak{t}) = (c_{ij})$ of symbols that completely determines $\psi_{d(\mathfrak{t})}$. It is given by the following rules.

1. If H_i appears in appears in $d(\mathfrak{t})$ then $c_{1,i+1} := H_i$ and $c_{2,i+1} := \emptyset$.
2. If U_i' appears in $d(\mathfrak{t})$ then $c_{2,i+1} = U_i{}'$ and $c_{1,i+1} := \emptyset$.

We view the matrix $c(\mathfrak{t})$ as a codification for $\psi_{d(\mathfrak{t})}$, where the first row of $c(\mathfrak{t})$ corresponds to the top line of $\psi_{d(\mathfrak{t})}$ and the second row of $c(\mathfrak{t})$ to the second line of $\psi_{d(\mathfrak{t})}$. The comments that were made on the structure of (11.0.8) and (11.0.9) carry over to the matrices $c(\mathfrak{t})$. In particular, exactly one of H_i or U_i' appears in $c(\mathfrak{t})$ for each i. Moreover, H_0 always appears and each U_i', except possibly U_{k-1}', is surrounded by H_{i-1} and H_{i+1}.

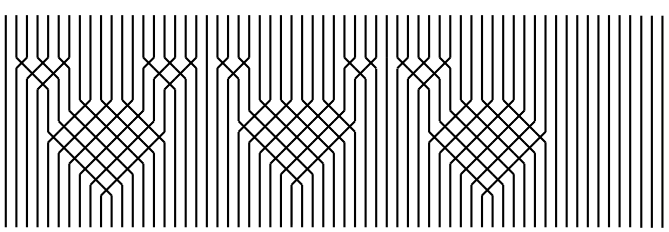
For example if $\psi_{d(\mathfrak{s})}$ is as in (11.0.8), then

$c(\mathfrak{s}) =$ (11.0.12)

H_0	H_1	H_2	H_3		H_5	H_6	
				U_4'			U_7'

Note that we leave the entries containing $\emptyset$ empty. Similarly, let $\mathfrak{t}$ be as in (11.0.7) but with the regions U_8 and U_9 eliminated. Then $\mathfrak{t}$ is central and $\psi_{d(\mathfrak{t})}$ is obtained by deleting ψ_{U_8} and ψ_{U_9} from (11.0.9) and we have

$\psi_{d(\mathfrak{t})} =$



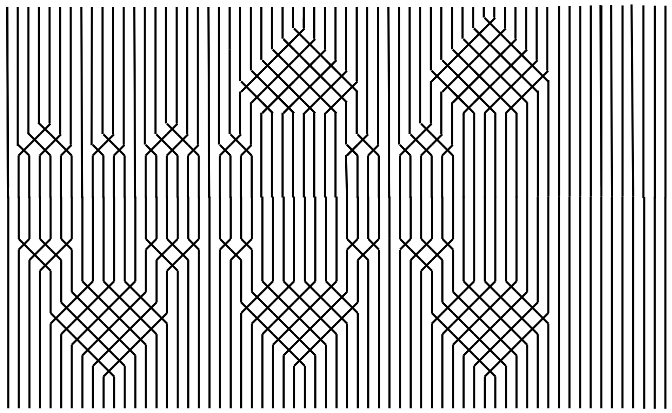
(11.0.13)

with corresponding matrix

$c(\mathfrak{t}) =$ (11.0.14)

H_0		H_2	H_3		H_5	H_6	
	U_1'			U_4'			U_7'

We are interested in the elements $m^{\boldsymbol{\mu}}_{\mathfrak{st}}$. In the above cases (11.0.8) and (11.0.13) it is as follows

$m^{\boldsymbol{\mu}}_{\mathfrak{st}} =$  $c(\mathfrak{s},\mathfrak{t}) =$ (11.0.15)

				$U_4'^*$			$U_7'^*$
H_0^*	H_1^*	H_2^*	H_3^*		H_5^*	H_6^*	
H_0		H_2	H_3		H_5	H_6	
	U_1'			U_4'			U_7'

In general, for $\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu}_k)$ central we define $c^*(\mathfrak{t})$ as the $(2 \times k)$-matrix (d_{ij}) where $d_{1j} = c^*_{2j}$ and $d_{2j} = c^*_{1j}$. Here we set $\emptyset^* := \emptyset$. Moreover, for $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu}_k)$ both central we define $c(\mathfrak{s},\mathfrak{t})$ as the $(4 \times k)$-matrix that has $c^*(\mathfrak{s})$ on top of $c(\mathfrak{t})$. Then $c(\mathfrak{s},\mathfrak{t})$ is our codification of $m^{\boldsymbol{\mu}}_{\mathfrak{st}}$. In (11.0.15) we have given $c(\mathfrak{s},\mathfrak{t})$ next to $m^{\boldsymbol{\mu}}_{\mathfrak{st}}$.

Our task is now to show that any diagram as in (11.0.15) can be written in terms of the elements from $G(\boldsymbol{\lambda})$. This requires calculations using the defining relations for $\mathbb{B}_n$. Let us first recall a couple of results from the literature.

Lemma 11.0.3. *The idempotent $e(\boldsymbol{i}) \in \mathbb{B}_n$ is nonzero only if $\boldsymbol{i} = \boldsymbol{i}^{\mathfrak{t}}$ for some $\mathfrak{t} \in \mathrm{Std}(\mathrm{Par}^1_n)$.*

Proof. This follows from Lemma 4.1(c) of [17], where it was proved for cyclotomic Hecke algebras in general, combined with the fact that $\mathbb{B}_n$ is a graded quotient of the cyclotomic Hecke algebra of type $G(2,1,n)$, see [38]. □

Lemma 11.0.4. *Let B_i be a full block for $\boldsymbol{\lambda}$ as introduced in (10.3.8) and suppose that $k, l \in B_i$. Then we have that*

$$y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = y_l e(\boldsymbol{i}^{\boldsymbol{\lambda}}). \tag{11.0.16}$$

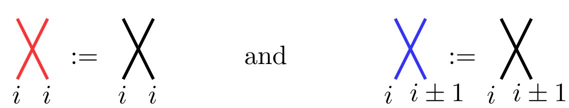
Proof. This follows from relation (3.0.21) and Lemma 11.0.3. □

Lemma 11.0.5. *Suppose that $\boldsymbol{\nu} \in \mathrm{Par}^1_n$ and that $\mathfrak{t} \in \mathrm{Std}(\mathrm{Par}^1_n)$. Suppose moreover that $P_{\mathfrak{t}}|_{[0,k]} = P_{\boldsymbol{\nu}}|_{[0,k]}$ for some integer $k \geq 0$. Then for all $1 \leq r \leq k$ we have in $\mathbb{B}_n$ that*

$$y_r e(\boldsymbol{i}^{\mathfrak{t}}) = 0. \tag{11.0.17}$$

Proof. Recall that $P_{\boldsymbol{\nu}}$ zigzags along the vertical central axis of the Pascal triangle and finally goes linearly off to $\boldsymbol{\nu}$. If r belongs to the zigzag part of $P_{\boldsymbol{\nu}}$, the result follows from the Lemmas 14 and 15 of [25], see also Theorem 6.4 of [10]. Otherwise, if r belongs to the linear part of $P_{\boldsymbol{\nu}}$, we argue as in the previous Lemma and get that $y_r e(\boldsymbol{i}^{\mathfrak{t}}) = y_{r-1} e(\boldsymbol{i}^{\mathfrak{t}})$. Continuing like this, we finally end up in the zigzag part of $P_{\boldsymbol{\nu}}$. □

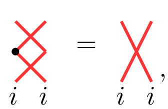
Henceforth, we color the intersections of our KLR-diagrams according to the difference of the relevant residues. More precisely, we shall use the following color scheme

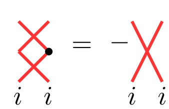


(11.0.18)

whereas for all other crossing we keep the usual black color. In this notation we now have the following Lemma which is a direct consequence of the relations (3.0.18) and 3.0.21.

Lemma 11.0.6. *We have the following relations in $\mathbb{B}_n$*



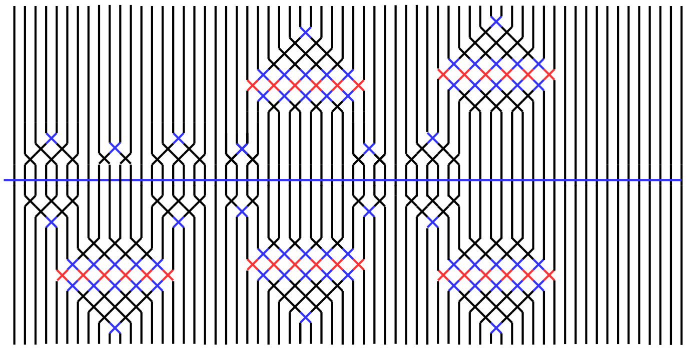


(11.0.19)

We can now finally prove the Theorem that was announced in the beginning of this chapter.

Theorem 11.0.7. *The set $G(\boldsymbol{\lambda})$ introduced in (11.0.1) generates $\mathbb{B}_n(\boldsymbol{\lambda})$.*

Proof. Using the coloring scheme introduced above, the diagram (11.0.15) looks as follows



				$U_4'^*$			$U_7'^*$
H_0^*	H_1^*	H_2^*	H_3^*		H_5^*	H_6^*	
H_0		H_2	H_3		H_5	H_6	
	U_1'			U_4'			U_7'

(11.0.20)

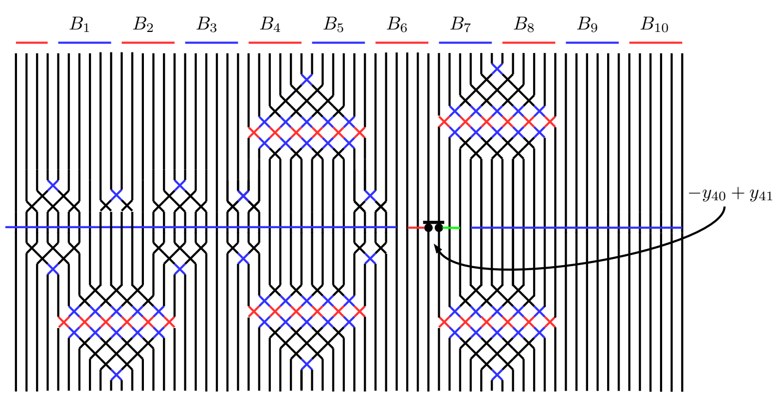
We must show that the elements $m^{\boldsymbol{\mu}}_{\mathfrak{st}}$ can be written in terms of the elements of $G(\boldsymbol{\lambda})$. We will do so by pairing the elements of the columns of the corresponding $c(\mathfrak{s},\mathfrak{t})$.

Note that the residue sequence for the middle blue horizontal of (11.0.20) is $\boldsymbol{i}^{\boldsymbol{\mu}}$. The idea is to apply Lemma 11.0.5 and therefore it is of importance to resolve the columns from the right to the left.

Let us first consider columns containing pairs $\{H_i^*, H_i\}$, starting with the rightmost of these columns. Thus in the above case we consider first $\{H_6^*, H_6\}$. We now use relation 3.0.21 to undo all the crossings in H_i^* and H_i, arriving at a diagram like (11.0.21). Here we use an overline on the two dots to denote that the result is a difference of two equal diagrams but each with *one* dot in the indicated place. Note that the residue sequence for the middle line has now changed, and correspondingly we have changed the color from blue to red and green around the two dots. In the above case, the new middle residue sequence is $\boldsymbol{i}^{\mathfrak{t}_1}$ where $\mathfrak{t}_1 = \mathfrak{t}^{\boldsymbol{\mu}} H_6$, that is $\mathfrak{t}_1$ is obtained from $\mathfrak{t}^{\boldsymbol{\mu}}$ by replacing $\partial_{in}(h_6)$ with $\partial_{out}(h_6)$. In the first figure of (11.0.7), we have indicated $P_{\mathfrak{t}_1}$, using the same colors red and green. On the leftmost dot, given by y_{40} in the above example, we can now apply Lemma 11.0.5, with $\mathfrak{t} = \mathfrak{t}_1$ and $\boldsymbol{\nu}$ as indicated in (11.0.7) We conclude from the Lemma that the corresponding diagram is zero.

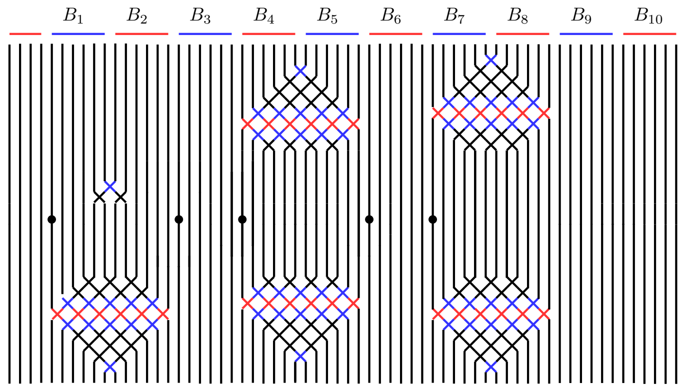
Thus in the above case (11.0.21) only the second term dot with y_{41} stays. We now repeat this process for all the other pairs of the form $\{H_i^*, H_i\}$, from the right to the left. For example in the case (11.0.21) we arrive at the diagram (11.0.22). We have indicated the blocks for $\boldsymbol{\lambda}$ on the top of the diagrams (11.0.21) and (11.0.22). Note that each H_i (resp. H_i^*, U_i' and $U_i'^*$) 'intersects' both of the blocks B_i and B_{i+1} and that the dots of (11.0.22) are all situated at the beginning of a block.

$m^{\boldsymbol{\mu}}_{\mathfrak{st}} =$



(11.0.21)

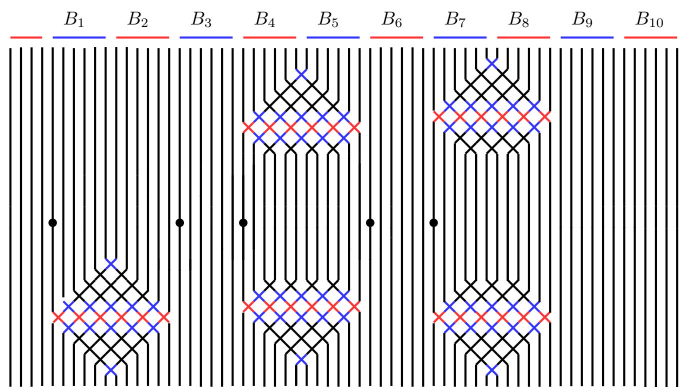
$m^{\boldsymbol{\mu}}_{\mathfrak{st}} =$



(11.0.22)

Next we treat the pairs of the form $\{U_i'^*, H_i\}$ or $\{H_i^*, U_i'\}$. By the combinatorial remarks made earlier, each appearing H_i-term (resp. H_i^*-term) fits perfectly with the corresponding $U_i'^*$-term (resp. U_i'-term) to form a diamond. We then move the H_i-term up (resp. the H_i^*-term down) to form this diamond. Note that this process does not involve any other terms since the H_i-terms (resp. the H_i^*-terms) are distant from the surrounding dots. In the above case (11.0.22) we get the following diagram.

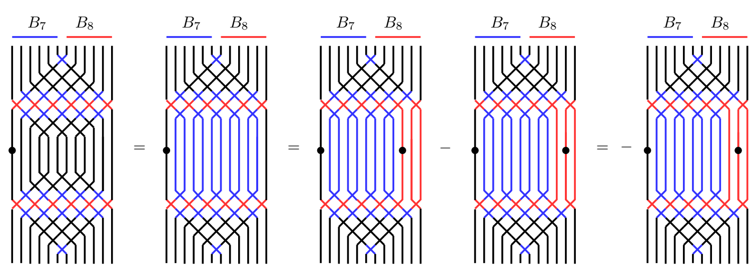
$m^{\boldsymbol{\mu}}_{\mathfrak{st}} =$



(11.0.23)

We are only left with columns containing pairs of the form $\{U_i'^*, U_i'\}$. By the previous step there is now a dot between the top $U_i'^*$ and the bottom U_i', at the left end of the 'line segment' between them, see (11.0.23). We show that this kind of configuration C_i is equal to diamond ψ_{U_i}. In fact, the arguments we employ for this have already appeared in the literature, see for example [23]. Let us give the details corresponding to $i = 7$ in (11.0.23); the general case is done the same way. Using relation (3.0.21) to undo the black double crosses, next relation (3.0.21) to undo the last blue cross and finally (3.0.21) on the red double cross, we have the following series of identities.

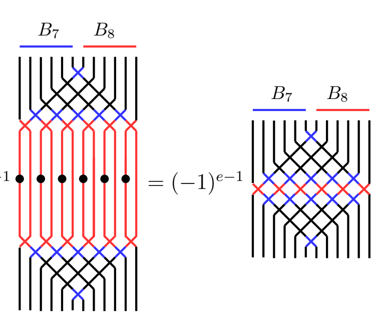
$C_7 =$



(11.0.24)

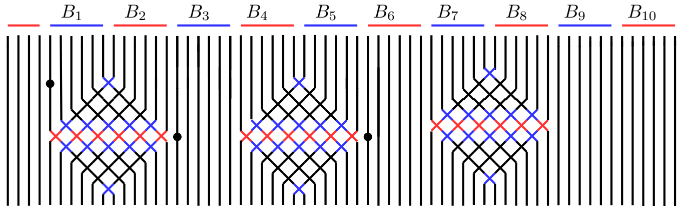
But this process can be repeated on all the blue double crosses and so we have via Lemma 11.0.6 that

$C_7 = (-1)^{e-1}$



$= (-1)^{e-1}\psi_{U_7}.$ (11.0.25)

The same procedure can be carried out for the other columns of the form $\{U_i'^*, U_i'\}$. In the above case there is only one such column, corresponding to $i = 4$ and so get finally that

$m^{\boldsymbol{\mu}}_{\mathfrak{st}} = \pm$  (11.0.26)

In other words, since multiplication in $\mathbb{B}_n$ is from top to bottom, we have that

$$m^{\boldsymbol{\mu}}_{\mathfrak{st}} = \pm\, y_5 U_1^{\boldsymbol{\lambda}} y_{17} U_4^{\boldsymbol{\lambda}} y_{35} U_7^{\boldsymbol{\lambda}}. \tag{11.0.27}$$

All appearing factors of $m^{\boldsymbol{\mu}}_{\mathfrak{st}}$ belong to $G(\boldsymbol{\lambda})$ and so we have proved the Theorem. □

Let us point out some remarks concerning Theorem 11.0.7 and its proof. First of all, we already saw that only a few of the y_i's are needed to generate $\mathbb{B}_n(\boldsymbol{\lambda})$. Let us make this more precise. Choose any k in the i'th block B_i. Then we define

$$\mathcal{Y}_i^{\boldsymbol{\lambda}} := y_k e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \in \mathbb{B}_n(\boldsymbol{\lambda}). \tag{11.0.28}$$

Note that by Lemma 11.0.4, we have that $\mathcal{Y}_i^{\boldsymbol{\lambda}}$ is independent of the choice of k. Moreover, it follows immediately from Theorem 11.0.7 that $\mathbb{B}_n(\boldsymbol{\lambda})$ is generated by the set

$$\{U_j^{\boldsymbol{\lambda}} \,|\, 1 \leq j < K\} \cup \{\mathcal{Y}_i^{\boldsymbol{\lambda}} \,|\, 1 \leq i \leq K\}. \tag{11.0.29}$$

Secondly we remark that the proof of Theorem 11.0.7 gives rise to an algorithm for writing the above $m^{\boldsymbol{\mu}}_{\mathfrak{st}}$ in terms of the generators in (11.0.29). Although the algorithm itself is not necessary for what follows, for the sake of completeness we prefer to establish it formally.

Algorithm 11.0.8. Let $\boldsymbol{\mu} \in \mathrm{Par}_n^1$ and let $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ be central tableaux. Let $c(\mathfrak{s}, \mathfrak{t})$ be the matrix associated with $m^{\boldsymbol{\mu}}_{\mathfrak{st}}$.

Step 0. Add an empty column to the right of $c(\mathfrak{s}, \mathfrak{t})$.

Step 1. For each column in $c(\mathfrak{s}, \mathfrak{t})$ containing $\{U_i'^*, H_i\}$ (resp. $\{H_i^*, U_i'\}$) we remove H_i (resp. H_i^*) from $c(\mathfrak{s}, \mathfrak{t})$ and replace $U_i'^*$ (resp. U_i') in $c(\mathfrak{s}, \mathfrak{t})$ by U_i.

Step 2. Working from the right two the left, for each column in $c(\mathfrak{s}, \mathfrak{t})$ containing $\{H_i^*, H_i\}$ we remove H_i^* and H_i from $c(\mathfrak{s}, \mathfrak{t})$ and write Y_{i+1} in one of the two middle boxes of the following column, one to the right.

Step 3. Each column in $c(\mathfrak{s}, \mathfrak{t})$ containing $\{U_i'^*, U_i'\}$ will now also contain Y_i. We replace these three ingredients of that column by one U_i which is placed in one of the two middle boxes of the column.

Step 4. Replacing each U_i by $U_i^{\boldsymbol{\lambda}}$ and each Y_i by $\mathcal{Y}_i^{\boldsymbol{\lambda}}$ we form the product of all appearing elements of $c(\mathfrak{s}, \mathfrak{t})$, starting with the top line, then the two middle lines and finally the bottom line. This product is $\pm m^{\boldsymbol{\mu}}_{\mathfrak{st}}$.

Let us give an example to illustrate how the algorithm works. Suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are central tableaux and that $c(\mathfrak{s}, \mathfrak{t})$ is as follows.

$c(\mathfrak{s}, \mathfrak{t}) =$ (11.0.30)

			$U_3'^*$			$U_6'^*$		
H_0^*	H_1^*	H_2^*		H_4^*	H_5^*		H_7^*	H_8^*
H_0		H_2		H_4	H_5	H_6	H_7	
	U_1'		U_3'					U_8'

Then going through the algorithm we get

$c(\mathfrak{s}, \mathfrak{t}) \rightarrow$

			$U_3'^*$			U_6			
	Y_1		Y_3		Y_5	Y_6		Y_8	
	U_1		U_3'					U_8	

$\rightarrow$ (11.0.31)

						U_6			
	Y_1		U_3		Y_5	Y_6		Y_8	
	U_1							U_8	

and so we conclude that

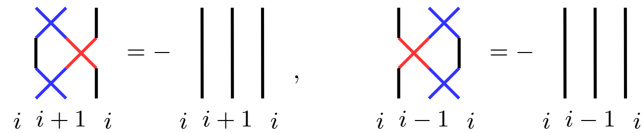
$$m^{\boldsymbol{\mu}}_{\mathfrak{st}} = \pm U^{\boldsymbol{\lambda}}_6 \mathcal{Y}^{\boldsymbol{\lambda}}_1 U^{\boldsymbol{\lambda}}_3 \mathcal{Y}^{\boldsymbol{\lambda}}_5 \mathcal{Y}^{\boldsymbol{\lambda}}_6 \mathcal{Y}^{\boldsymbol{\lambda}}_8 U^{\boldsymbol{\lambda}}_1 U^{\boldsymbol{\lambda}}_8. \tag{11.0.32}$$

Our next step is to show that actually only $\mathcal{Y}^{\boldsymbol{\lambda}}_1$ is needed in order to generate $\mathbb{B}_n(\boldsymbol{\lambda})$. Let us first prove the following result.

Lemma 11.0.9. *For all* $1 \leq i < K$ *we have*

$$\mathcal{Y}^{\boldsymbol{\lambda}}_{i+1} U^{\boldsymbol{\lambda}}_i = U^{\boldsymbol{\lambda}}_i \mathcal{Y}^{\boldsymbol{\lambda}}_i + (-1)^e (\mathcal{Y}^{\boldsymbol{\lambda}}_i - \mathcal{Y}^{\boldsymbol{\lambda}}_{i+1}). \tag{11.0.33}$$

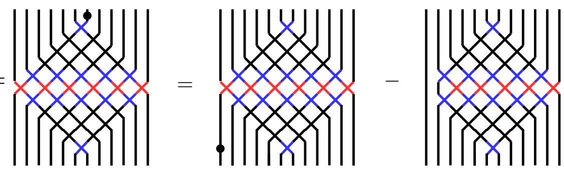
Proof. Let us first recall the following relations valid in $\mathbb{B}_n$, see Lemma 5.16 of [23].



(11.0.34)

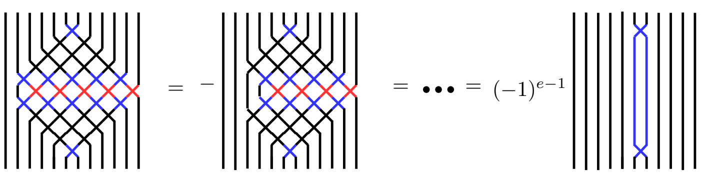
They are a consequence of the braid relation (3.0.20) together with Lemma 11.0.3.

Let us now show the Lemma for $i = 1$, since the general case is treated the same way. We take $e = 6$. Then we have that via repeated applications of relation (3.0.18) that

$\mathcal{Y}^{\boldsymbol{\lambda}}_2 U^{\boldsymbol{\lambda}}_1 =$ 

(11.0.35)

The first diagram is here $U^{\boldsymbol{\lambda}}_1 \mathcal{Y}^{\boldsymbol{\lambda}}_1$ so let us focus on the second diagram. Using the first relation in (11.0.34) repeatedly we get that it is equal to



$= (-1)^{e-1}(\mathcal{Y}^{\boldsymbol{\lambda}}_1 - \mathcal{Y}^{\boldsymbol{\lambda}}_2)$ (11.0.36)

where we used the quadratic relation (**??**) for the last step. Combining (11.0.35) and (11.0.36), we then get (11.0.33). ☐

Let us recall the commutation relations between the $U^{\boldsymbol{\lambda}}_i$'s, see Proposition 5.18 of [23].

Theorem 11.0.10. *The subset* $\{U^{\boldsymbol{\lambda}}_i \mid i = 1, \dots K-1\}$ *of* $\mathbb{B}_n(\boldsymbol{\lambda})$ *verifies the Temperley-Lieb relations, or to be more precise*

$$(U^{\boldsymbol{\lambda}}_i)^2 = (-1)^{e-1} 2 U^{\boldsymbol{\lambda}}_i, \qquad \text{if } 1 \leq i < K; \tag{11.0.37}$$

$$U^{\boldsymbol{\lambda}}_i U^{\boldsymbol{\lambda}}_j U^{\boldsymbol{\lambda}}_i = U^{\boldsymbol{\lambda}}_i, \qquad \text{if } |i-j| = 1; \tag{11.0.38}$$

$$U^{\boldsymbol{\lambda}}_i U^{\boldsymbol{\lambda}}_i = U^{\boldsymbol{\lambda}}_j U^{\boldsymbol{\lambda}}_i, \qquad \text{if } |i-j| > 1. \tag{11.0.39}$$

With this at our disposal we can now prove, as promised, that $\mathcal{Y}^{\boldsymbol{\lambda}}_1$ is the only $\mathcal{Y}^{\boldsymbol{\lambda}}_i$ which is needed in order to generate $\mathbb{B}_n(\boldsymbol{\lambda})$.

Theorem 11.0.11. *The set*

$$G_1(\boldsymbol{\lambda}) := \{U^{\boldsymbol{\lambda}}_i \,|\, 1 \leq i < K\} \cup \{\mathcal{Y}^{\boldsymbol{\lambda}}_1\} \tag{11.0.40}$$

generates $\mathbb{B}_n(\boldsymbol{\lambda})$ *as an* $\mathbb{F}$*-algebra.*

Proof. Recall that $e(\boldsymbol{i}^{\boldsymbol{\lambda}})$ is the identity element of $\mathbb{B}_n(\boldsymbol{\lambda})$, for simplicity we denote it by 1. Let us define

$$S_i^{\boldsymbol{\lambda}} := U_i^{\boldsymbol{\lambda}} + (-1)^e. \tag{11.0.41}$$

Then from Theorem 11.0.10 we get that

$$(S_i^{\boldsymbol{\lambda}})^2 = 1. \tag{11.0.42}$$

On the other hand, we notice that using the notation introduced above, the relation (11.0.33) becomes

$$\mathcal{Y}_{i+1}^{\boldsymbol{\lambda}} S_i^{\boldsymbol{\lambda}} = S_i^{\boldsymbol{\lambda}} \mathcal{Y}_i^{\boldsymbol{\lambda}}. \tag{11.0.43}$$

Finally, by combining (11.0.42) and (11.0.43) we obtain

$$\mathcal{Y}_{i+1}^{\boldsymbol{\lambda}} = S_i^{\boldsymbol{\lambda}} \mathcal{Y}_i^{\boldsymbol{\lambda}} S_i^{\boldsymbol{\lambda}} \tag{11.0.44}$$

and the result follows. □

We are now in position to prove the main result of this chapter.

Theorem 11.0.12. *There is an isomorphism* $f : \mathbb{NB}_K \to \mathbb{B}_n(\boldsymbol{\lambda})$ *given by*

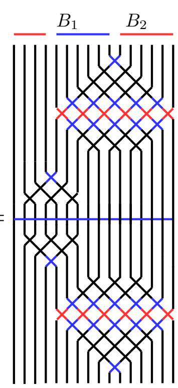
$$\mathbb{U}_0 \mapsto \mathcal{Y}_1^{\boldsymbol{\lambda}} \quad \text{and} \quad \mathbb{U}_i \mapsto (-1)^e U_i^{\boldsymbol{\lambda}} \text{ for } 1 \le i < K. \tag{11.0.45}$$

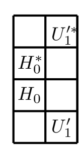
Proof. In view of Theorem 8.0.5 and the Pascal triangle description of the cellular basis for $\mathbb{B}_n(\boldsymbol{\lambda})$, the two algebras have the same dimension. Hence, we only have to show that f is well defined since, by Theorem 11.0.11, it will automatically be surjective.

Let us therefore check that $f(\mathbb{U}_0)$ and the $f(\mathbb{U}_i)$'s verify the relations for $\mathbb{NB}_K$. The Temperley-Lieb relations (8.0.8), (8.0.9) and (8.0.10) are clearly satisfied by Theorem 11.0.10 whereas the relation $(\mathcal{Y}_1^{\boldsymbol{\lambda}})^2 = 0$ follows from relation (3.0.17) and (**??**). Hence we are only left with checking relation (8.0.11). It corresponds to $U_1^{\boldsymbol{\lambda}} \mathcal{Y}_1^{\boldsymbol{\lambda}} U_1^{\boldsymbol{\lambda}} = 0$ which via Lemma 11.0.9 and (11.0.37) is equivalent to the relation

$$(\mathcal{Y}_1^{\boldsymbol{\lambda}} + \mathcal{Y}_2^{\boldsymbol{\lambda}}) U_1^{\boldsymbol{\lambda}} = 0. \tag{11.0.46}$$

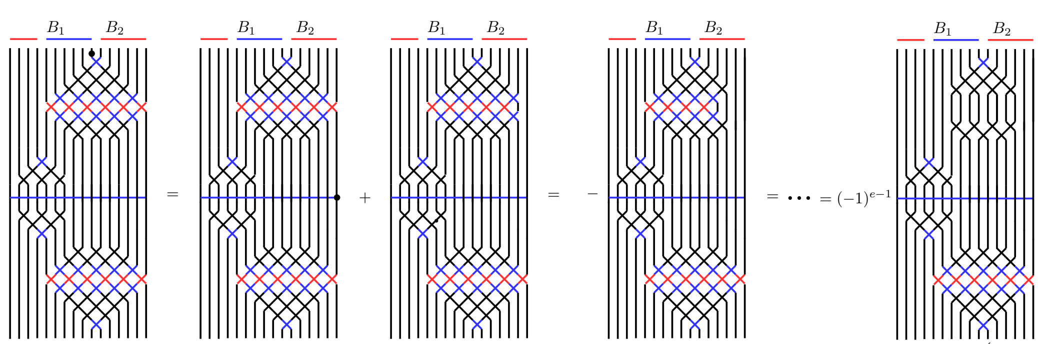
For this we first write $(-1)^{e-1} U_1^{\boldsymbol{\lambda}}$ in the following form

$(-1)^{e-1} U_1^{\boldsymbol{\lambda}} =$ 



(11.0.47)

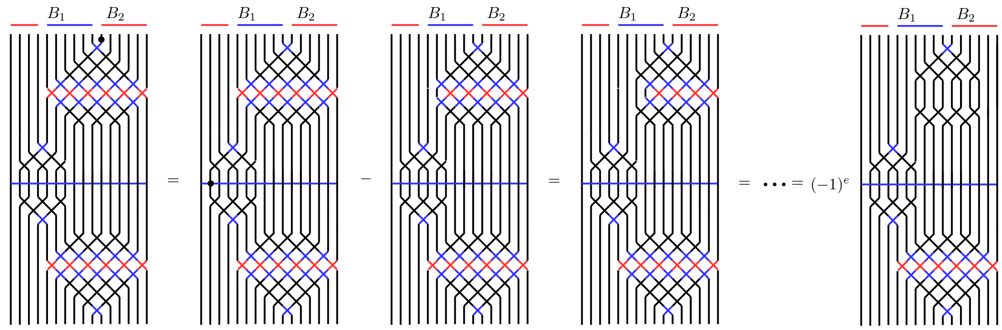
We have here used $e = 6$ as in the examples of the proof of Theorem 11.0.7. The middle blue horizontal line has the same meaning as in (11.0.21); its residue sequence is $\boldsymbol{i}^{\boldsymbol{\mu}}$ for the corresponding $\boldsymbol{\mu}$. Using this we get

$(-1)^{e-1} \mathcal{Y}_1^{\boldsymbol{\lambda}} U_1^{\boldsymbol{\lambda}} =$ 

(11.0.48)

where the first equality comes from relation (3.0.18), the second from Lemma 11.0.5 and the other equalities from (11.0.34). On the other hand, for $(-1)^{e-1}\mathcal{Y}_2^{\boldsymbol{\lambda}} U_1^{\boldsymbol{\lambda}}$ we have almost the same expansion with only a sign change coming from relation 3.0.18:

$(-1)^{e-1}\mathcal{Y}_2^{\boldsymbol{\lambda}} U_1^{\boldsymbol{\lambda}} =$



(11.0.49)

Comparing (11.0.48) and (11.0.49) we see that (11.0.46) holds. The Theorem is proved. □

Remark 11.0.13. Using (11.0.44) and (11.0.42) we extend $(\mathcal{Y}_1^{\boldsymbol{\lambda}})^2 = 0$ to $(\mathcal{Y}_i^{\boldsymbol{\lambda}})^2 = 0$ to all i. Thus the isomorphism $\varphi : \mathbb{NB}_n \cong A_w$ gives us a proof of Lemma 8.0.9. The recursive formula for the $\mathbb{Y}_i$'s is given by 11.0.44.

Chapter 12

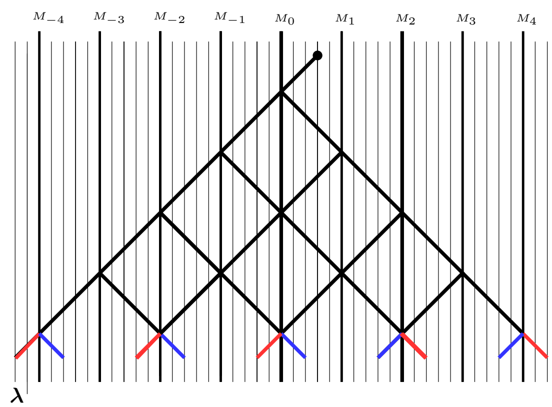
A presentation for $\mathbb{B}_n(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda}$ regular

In this chapter we consider the case where $\boldsymbol{\lambda}$ is regular, in other words we assume that $R > 0$, see Definition 10.3.3. We define $\mathbb{B}_n(\boldsymbol{\lambda}) := e(\boldsymbol{i}^{\boldsymbol{\lambda}})\mathbb{B}_n e(\boldsymbol{i}^{\boldsymbol{\lambda}})$ just as in the singular case but, as we shall see, the regular case is slightly more complicated than the singular case since we need an extra generator. Recall first the function $f = f_{n,m}$ from (10.3.7) which was used to define the full blocks in the singular case, see (10.3.8). Let K be as in Definition 10.3.3. Then in the regular case there is an extra *non-full* block B_{last} defined as follows

$$B_{last} := [f(K+1)+1, f(K+1)+2, \dots, f(K+1)+R] = [f(K+1)+1, f(K+1)+2, \dots, n]. \tag{12.0.1}$$

For example in the situation described in (10.3.6), we have $n = 25, e = 5, m = 2$ and so $K = 4, R = 2$ and therefore

$$B_1 = [4,5,6,7,8],\ B_2 = [9,10,11,12,13],\ B_3 = [14,15,16,17,18],\ B_4 = [19,20,21,22,23], B_{last} := [24,25]. \tag{12.0.2}$$



(12.0.3)

Let $\overline{n} := n - R$ and let $\overline{\boldsymbol{\lambda}} := (1^{\overline{n}}, 1^0) \in \mathrm{Par}^1_{\overline{n}}$. We notice that

$$\bar{n} = f(K+1). \tag{12.0.4}$$

It is clear from the definitions that $\overline{\boldsymbol{\lambda}}$ is singular. On the other hand, any $\bar{\mathfrak{s}} \in \mathrm{Std}(\boldsymbol{i}^{\bar{\boldsymbol{\lambda}}})$ gives rise to two tableaux $\bar{\mathfrak{s}}(I)$ and $\bar{\mathfrak{s}}(O)$, in $\mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$, as follows. The tableau $\bar{\mathfrak{s}}(I)$ (resp. $\bar{\mathfrak{s}}(O)$) is defined as the unique tableau $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$ whose path $P_{\mathfrak{t}}$ coincides with $P_{\bar{\mathfrak{s}}}$ on the restriction to $[1, 2, \dots, \overline{n}]$ and whose restriction to B_{last} is a straight line that moves $P_{\mathfrak{t}}$ closer to (resp. further away from) the central vertical axis of the Pascal triangle. We say that $\mathfrak{t}$ is an *inner tableau* (resp. an *outer tableau*) if it is of the form $\mathfrak{t} = \bar{\mathfrak{s}}(I)$ (resp. $\mathfrak{t} = \bar{\mathfrak{s}}(O)$) for some $\bar{\mathfrak{s}} \in \mathrm{Std}(\boldsymbol{i}^{\bar{\boldsymbol{\lambda}}})$. It is easy to see that any tableau $\mathfrak{t}$ in $\mathrm{Std}(\boldsymbol{i}^{\boldsymbol{\lambda}})$ is of the form $\mathfrak{t} = \bar{\mathfrak{s}}(I)$ or $\mathfrak{t} = \bar{\mathfrak{s}}(O)$ for a unique $\bar{\mathfrak{s}} \in \mathrm{Std}(\boldsymbol{i}^{\bar{\boldsymbol{\lambda}}})$.

In (12.0.3) we have indicated with blue the restriction to B_{last} of the paths corresponding to inner tableaux, and with red the restriction to B_{last} of the paths corresponding to outer tableaux. Note that $P_{\boldsymbol{\lambda}}$ is always the path of an outer tableau.

Let $\boldsymbol{i}^{last} \in I_e^R$ be the restriction to B_{last} of the residue sequence for $\boldsymbol{i}^{\boldsymbol{\lambda}}$ and let $e(\boldsymbol{i}^{last})$ be the corresponding idempotent diagram, consisting of R vertical lines with residue sequence $\boldsymbol{i}^{last}$. For $x \in \mathbb{B}_{\bar{n}}$ we define the element $\iota(x) := x \wedge e(\boldsymbol{i}^{last}) \in \mathbb{B}_n$ as the horizontal concatenation of x with $e(\boldsymbol{i}^{last})$ on the right. We notice that

$$\iota(xy) = xy \wedge e(\boldsymbol{i}^{last}) = (x \wedge e(\boldsymbol{i}^{last}))(y \wedge e(\boldsymbol{i}^{last})) = \iota(x)\iota(y), \tag{12.0.5}$$

for all $x, y \in \mathbb{B}_{\bar{n}}$. Furthermore,

$$\iota(e(\boldsymbol{i}^{\bar{\boldsymbol{\lambda}}})) = e(\boldsymbol{i}^{\bar{\boldsymbol{\lambda}}}) \wedge e(\boldsymbol{i}^{last}) = e(\boldsymbol{i}^{\boldsymbol{\lambda}}). \tag{12.0.6}$$

We shall shortly prove that $m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}} = \iota(m^{\overline{\boldsymbol{\mu}}}_{\bar{\mathfrak{s}}\bar{\mathfrak{t}}})$. Combining this with (12.0.5) and (12.0.6) we conclude that there is an algebra inclusion

$$\iota(\mathbb{B}_{\bar{n}}(\bar{\boldsymbol{\lambda}})) \subset \mathbb{B}_n(\boldsymbol{\lambda}). \tag{12.0.7}$$

We define $U^{\boldsymbol{\lambda}}_i := \iota(U^{\bar{\boldsymbol{\lambda}}}_i) \in \mathbb{B}_n(\boldsymbol{\lambda})$ and $\mathcal{Y}^{\boldsymbol{\lambda}}_j := \iota(\mathcal{Y}^{\bar{\boldsymbol{\lambda}}}_j) \in \mathbb{B}_n(\boldsymbol{\lambda})$, for $1 \le i < K$ and $1 \le j \le K$.

It turns out that the outer tableaux are easier to handle than the inner tableaux.

Lemma 12.0.1. *Let $\boldsymbol{\lambda}$ be regular and suppose that $\mathfrak{s} = \bar{\mathfrak{s}}(O)$ and $\mathfrak{t} = \bar{\mathfrak{t}}(O)$ are outer tableaux in $\mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$. Let $\overline{\boldsymbol{\mu}}$ be the shape of $\bar{\mathfrak{s}}$ and $\bar{\mathfrak{t}}$. Then we have that*

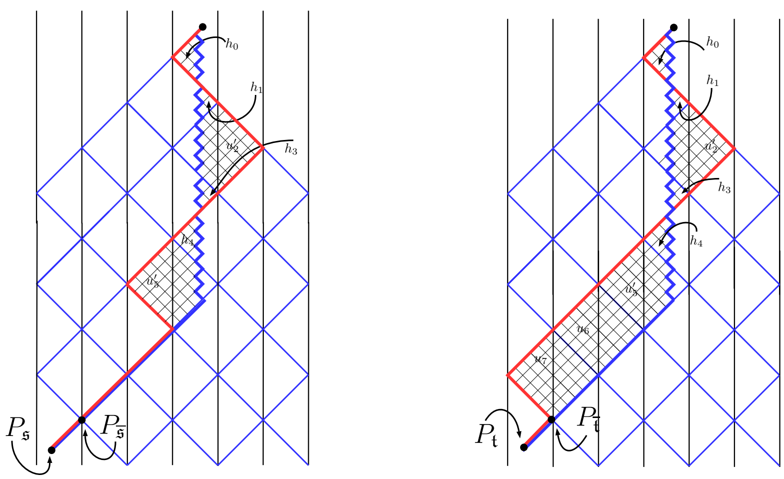
$$m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}} = \iota(m^{\overline{\boldsymbol{\mu}}}_{\bar{\mathfrak{s}}\bar{\mathfrak{t}}}). \tag{12.0.8}$$

Consequently, $m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}}$ belongs to the subalgebra of $\mathbb{B}_n(\boldsymbol{\lambda})$ generated by $\{U^{\boldsymbol{\lambda}}_i \mid 1 \le i < K\}$ and $\mathcal{Y}^{\boldsymbol{\lambda}}_1$.

Proof. Using Theorem 11.0.11 we see that the second statement follows from the first statement (12.0.8). In order to prove the first statement we note that since $\mathfrak{s}$ and $\mathfrak{t}$ are outer tableaux we have that

$$d(\mathfrak{s}) = d(\bar{\mathfrak{s}}) \ \text{ and } \ d(\mathfrak{t}) = d(\bar{\mathfrak{t}}). \tag{12.0.9}$$

Here are examples illustrating (12.0.9)



(12.0.10)

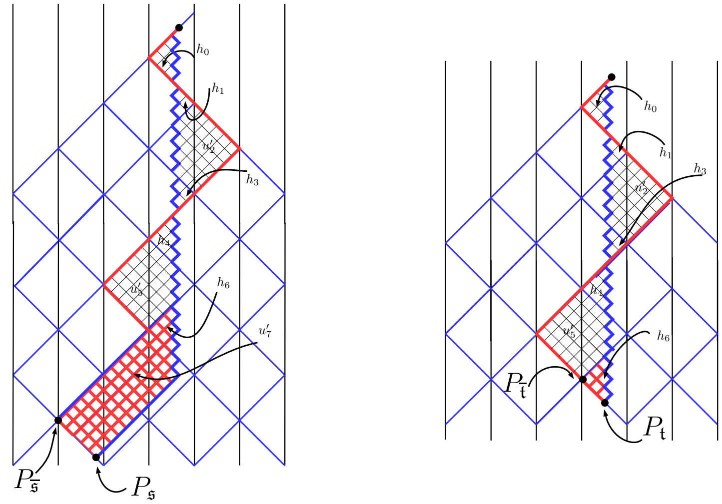
On the other hand we have that $e(\boldsymbol{i}^{\boldsymbol{\mu}}) = \iota(e(\boldsymbol{i}^{\overline{\boldsymbol{\mu}}}))$ and so we obtain

$$\iota(m^{\overline{\boldsymbol{\mu}}}_{\bar{\mathfrak{s}}\bar{\mathfrak{t}}}) = \iota(\psi^*_{d(\bar{\mathfrak{s}})} e(\boldsymbol{i}^{\bar{\boldsymbol{\mu}}}) \psi_{d(\bar{\mathfrak{t}})}) = \iota(\psi^*_{d(\bar{\mathfrak{s}})}) \iota(e(\boldsymbol{i}^{\bar{\boldsymbol{\mu}}})) \iota(\psi_{d(\bar{\mathfrak{t}})}) = \psi^*_{d(\mathfrak{s})} e(\boldsymbol{i}^{\boldsymbol{\mu}}) \psi_{d(\mathfrak{t})} = m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}}. \tag{12.0.11}$$

□

Suppose now that $\mathfrak{s} = \bar{\mathfrak{s}}(I) \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ is an inner tableau. Then $d(\mathfrak{s})$ and $d(\bar{\mathfrak{s}})$ are different but still closely related. Let $a_{\mathfrak{s}}$ be the region of the Pascal triangle bounded by $P_{\mathfrak{s}}$ and $P_{\boldsymbol{\mu}}$ and let $a_{\bar{\mathfrak{s}}}$ be the region bounded by $P_{\bar{\mathfrak{s}}}$ and $P_{\overline{\boldsymbol{\mu}}}$, where $\bar{\boldsymbol{\mu}}$ denotes the shape of $\bar{\mathfrak{s}}$. Then $a_{\mathfrak{s}} = a_{\bar{\mathfrak{s}}} \cup s_{\boldsymbol{\mu}}$ where $s_{\boldsymbol{\mu}}$ is the region bounded by $P_{\boldsymbol{\mu}}$ and $P_{\mathfrak{t}^{\bar{\mu}}(I)}$, see (12.0.13) for two examples in which we have indicated $s_{\boldsymbol{\mu}}$ with the color red. Note that $s_{\boldsymbol{\mu}}$ only depends on $\boldsymbol{\mu}$ and not on $\mathfrak{s}$, which is the reason for our notation. When applying Algorithm 10.2.1 there is an independence between the regions $a_{\bar{\mathfrak{s}}}$ and $s_{\boldsymbol{\mu}}$. Indeed, let $A_{\bar{\mathfrak{s}}} \in \mathfrak{S}_n$ be the element obtained by filling in $a_{\bar{\mathfrak{s}}}$ as in the algorithm, and let similarly $S_{\boldsymbol{\mu}} \in \mathfrak{S}_n$ be the element obtained by filling in $s_{\boldsymbol{\mu}}$. Then we have that

$$d(\mathfrak{s}) = S_{\boldsymbol{\mu}} A_{\bar{\mathfrak{s}}}. \tag{12.0.12}$$



(12.0.13)

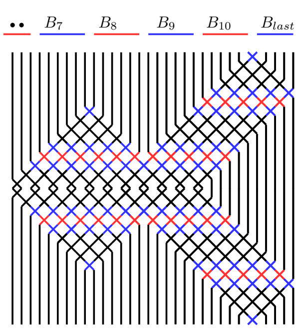
Definition 12.0.2. *Let $\mathfrak{s} = \bar{\mathfrak{s}}(I)$ be an inner tableau. We say that $\mathfrak{s}$ is central if $\bar{\mathfrak{s}}$ is central.*

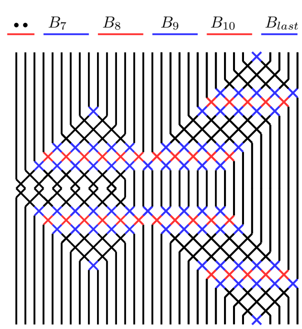
We can now prove the following Lemma.

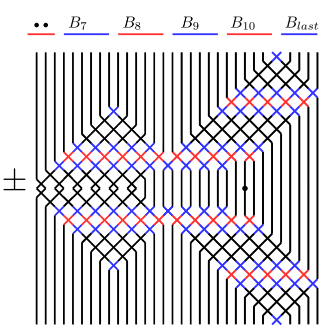
Lemma 12.0.3. *Let $\mathfrak{s} = \bar{\mathfrak{s}}(I)$ and $\mathfrak{t} = \bar{\mathfrak{t}}(I)$ be central inner tableaux in $\mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$. Let $\bar{\boldsymbol{\mu}}$ be the shape of $\bar{\mathfrak{s}}$ and $\bar{\mathfrak{t}}$. Then, we have*

$$m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}} = \pm \begin{cases} (y_{\bar{n}+1} - y_{\bar{n}})\iota(m^{\bar{\boldsymbol{\mu}}}_{\bar{\mathfrak{s}}\bar{\mathfrak{t}}}) = \iota(m^{\bar{\boldsymbol{\mu}}}_{\bar{\mathfrak{s}}\bar{\mathfrak{t}}})(y_{\bar{n}+1} - y_{\bar{n}}), & \textit{if } \boldsymbol{\mu} \notin \mathcal{A}^0; \\ y_{\bar{n}+1}\iota(m^{\bar{\boldsymbol{\mu}}}_{\bar{\mathfrak{s}}\bar{\mathfrak{t}}}) = \iota(m^{\bar{\boldsymbol{\mu}}}_{\bar{\mathfrak{s}}\bar{\mathfrak{t}}})y_{\bar{n}+1}, & \textit{if } \boldsymbol{\mu} \in \mathcal{A}^0. \end{cases} \tag{12.0.14}$$

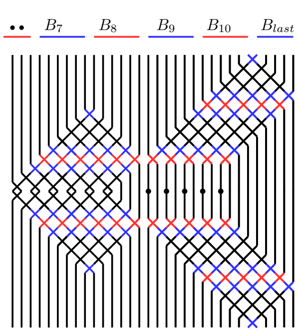
Proof. The proof is a calculation similar to the ones done in Lemma 11.0.9 and Theorem 11.0.12. Our general strategy is to first focus on the crosses that come from the region $s_{\boldsymbol{\mu}}$. Let us prove the first formula in (12.0.14). Thus we assume that we are in the case where $\boldsymbol{\mu}$ does not belong to the fundamental alcove. This case is a bit easier since, as we will see below, the crosses associated to the $s_{\boldsymbol{\mu}}$ region can be eliminated without altering the other parts of the diagram. We illustrate the computation in the case where $\mathfrak{s}$ is given by the first diagram of (12.0.13) and where $\mathfrak{t} = \mathfrak{s}$. For these choices we calculate as follows, using the defining relations in $\mathbb{B}_n$ together with (11.0.34).

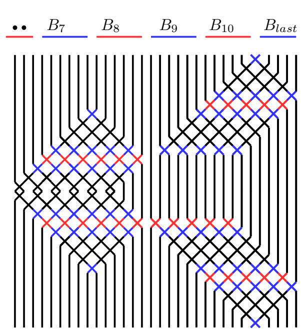
$m_{\mathfrak{s}\mathfrak{t}} =$ 

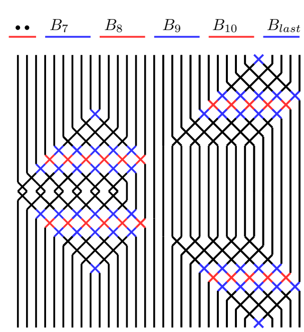
$=$ 

$= \pm$ 

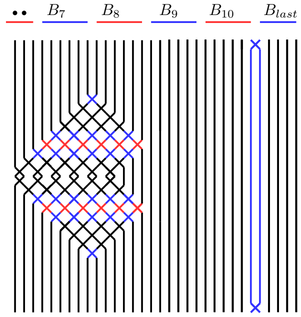
$=$ (12.0.15)

$\cdots\cdot = \pm$ 

$= \pm$ 

$= \pm$ 

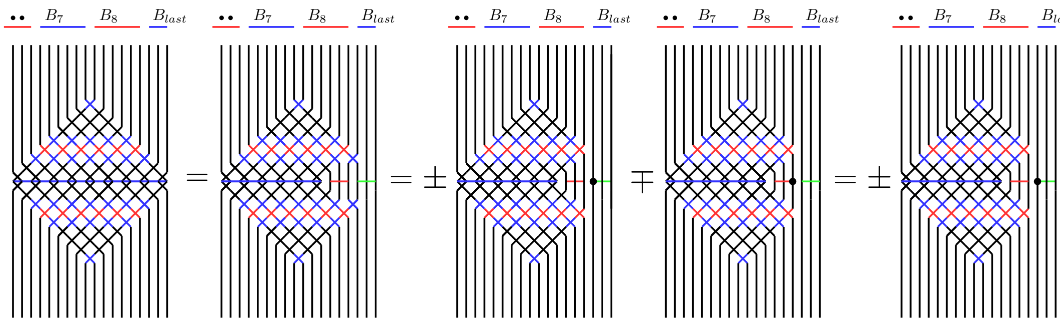
(12.0.16)



$$\ldots\ldots = \;\; = \iota(m^{\overline{\boldsymbol{\mu}}}_{\bar{\mathfrak{s}}\bar{\mathfrak{t}}})\psi^2_{\bar{n}} = (y_{\bar{n}+1} - y_{\bar{n}})\iota(m^{\overline{\boldsymbol{\mu}}}_{\bar{\mathfrak{s}}\bar{\mathfrak{t}}}) = \iota(m^{\overline{\boldsymbol{\mu}}}_{\bar{\mathfrak{s}}\bar{\mathfrak{t}}})(y_{\bar{n}+1} - y_{\bar{n}}) \tag{12.0.17}$$

as claimed. The general case is done the same way.

Let us now prove the second formula in (12.0.14), corresponding to the case where $\boldsymbol{\mu}$ belongs to the fundamental alcove. In this case $s_{\boldsymbol{\mu}}$ is as small as possible, as for example in the second diagram of (12.0.13). The proof is essentially the same as the proof of the first formula with the only difference being the vanishing of the factor $y_{\bar{n}}$ which is due to Lemma 11.0.5. Let us do the calculation in the case where $\mathfrak{s}$ is given by the second diagram of (12.0.13), and $\mathfrak{t} = \mathfrak{s}$. We have then



$$m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}} = \;\; = \;\; = \pm \;\; \mp \;\; = \pm \tag{12.0.18}$$

where the blue horizontal, red and green lines have the same meaning as in (11.0.21). The fact that the fourth diagram of (12.0.18) vanishes is shown using Lemma 11.0.5, arguing the same way as two paragraphs above 11.0.23, in the proof of Theorem 11.0.7. The proves the Lemma. □

Suppose that i in any element of B_{last}. Then we extend the definition in (11.0.28) by setting

$$\mathcal{Y}^{\boldsymbol{\lambda}}_{K+1} := y_i e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \in \mathbb{B}_n(\boldsymbol{\lambda}). \tag{12.0.19}$$

We get from Lemma 11.0.4 that $\mathcal{Y}^{\boldsymbol{\lambda}}_{K+1}$ is independent of the choice of i.

Corollary 12.0.4. *Let $G_1(\boldsymbol{\lambda})$ be as in Theorem 11.0.11. Then the set*

$$G_2(\boldsymbol{\lambda}) = G_1(\boldsymbol{\lambda}) \cup \{\mathcal{Y}^{\boldsymbol{\lambda}}_{K+1}\} \tag{12.0.20}$$

generates $\mathbb{B}_n(\boldsymbol{\lambda})$.

Proof. Let $\mathbb{B}_n(\boldsymbol{\lambda})'$ be the subalgebra of $\mathbb{B}_n(\boldsymbol{\lambda})$ generated by $G_2(\boldsymbol{\lambda})$. Let $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$. We need to show that $m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}} \in \mathbb{B}_n(\boldsymbol{\lambda})'$. If $\mathfrak{s}, \mathfrak{t}$ are outer tableaux then the result follows by a combination of Theorem 11.0.11 and Lemma 12.0.1. Suppose now that $\mathfrak{s}$ and $\mathfrak{t}$ are inner tableaux. If both tableaux are central then the result follows by combining Theorem 11.0.11 and Lemma 12.0.3. Otherwise, the same argument given in the proof of Lemma 11.0.1 allows us to conclude that there exist central standard tableaux $\mathfrak{s}_1, \mathfrak{t}_1 \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ and monomials $M_{\mathfrak{s}}$ and $M_{\mathfrak{t}}$ in the generators $\{U^{\boldsymbol{\lambda}}_1, \ldots, U^{\boldsymbol{\lambda}}_{K-1}\}$ such that

$$m^{\boldsymbol{\mu}}_{\mathfrak{s}\mathfrak{t}} = M_{\mathfrak{s}} m^{\boldsymbol{\mu}}_{\mathfrak{s}_1\mathfrak{t}_1} M_{\mathfrak{t}}, \tag{12.0.21}$$

and the result follows in this case as well. □

Corollary 12.0.5. *$\mathcal{Y}^{\boldsymbol{\lambda}}_{K+1}$ is a central element of $\mathbb{B}_n(\boldsymbol{\lambda})$.*

Proof. This follows from Corollary 12.0.4 once we notice that $\mathcal{Y}^{\boldsymbol{\lambda}}_{K+1}$ commutes with all the elements of $G_1(\boldsymbol{\lambda})$. □

Lemma 12.0.6. *We have that* $(\mathcal{Y}^{\boldsymbol{\lambda}}_{K+1})^2 = 0$.

Proof. For $i = 1, 2 \ldots, K+1$ we introduce the following elements of $\mathbb{B}_n(\boldsymbol{\lambda})$

$$\mathcal{L}^{\boldsymbol{\lambda}}_i := \mathcal{Y}^{\boldsymbol{\lambda}}_i - \mathcal{Y}^{\boldsymbol{\lambda}}_{i-1} \tag{12.0.22}$$

with the convention that $\mathcal{Y}^{\boldsymbol{\lambda}}_0 := 0$. Then in Theorem 6.9 of [10] it was shown that these elements $\mathcal{L}^{\boldsymbol{\lambda}}_i$ satisfy the JM-relations of Lemma 8.0.11. On the other hand we have that

$$\mathcal{Y}^{\boldsymbol{\lambda}}_{K+1} = \mathcal{L}^{\boldsymbol{\lambda}}_{K+1} + \mathcal{L}^{\boldsymbol{\lambda}}_{K} + \ldots + \mathcal{L}^{\boldsymbol{\lambda}}_{1} \tag{12.0.23}$$

and so the calculation done in (9.0.66) shows that $(\mathcal{Y}^{\boldsymbol{\lambda}}_{K+1})^2 = 0$, as claimed. The Lemma is proved. □

We can now establish the connection between the extended nil-blob algebra and $\mathbb{B}_n(\boldsymbol{\lambda})$.

Theorem 12.0.7. *Suppose that* $\boldsymbol{\lambda}$ *is regular. Then the assignment* $\mathbb{U}_0 \mapsto \mathcal{Y}^{\boldsymbol{\lambda}}_1$, $\mathbb{J}_K \mapsto \mathcal{Y}^{\boldsymbol{\lambda}}_{K+1}$ *and* $\mathbb{U}_i \mapsto (-1)^e U^{\boldsymbol{\lambda}}_i$ *for all* $1 \le i < K$, *induces an* $\mathbb{F}$*-algebra isomorphism between* $\widetilde{\mathbb{NB}}_K$ *and* $\mathbb{B}_n(\boldsymbol{\lambda})$.

Proof. Combining Theorem 11.0.12, Corollary 12.0.5 and Lemma 12.0.6 we get that the assignment of the Theorem defines an algebra homomorphism, which is surjective in view of Corollary 12.0.4. The two algebras have the same dimension $2\binom{2K}{K}$, and hence the Theorem is proved. □

The following is the main result of this part of this thesis. It establishes a connection between the algebras $\tilde{A}_w$ and $\mathbb{B}_n(\boldsymbol{\lambda})$, as predicted in [10] and [23].

Theorem 12.0.8. *Let* $\boldsymbol{\lambda}$ *be a regular bipartition. Suppose that* $\boldsymbol{\lambda}$ *is located in the alcove* $\mathcal{A}_w$. *Then,* $\tilde{A}_w \cong \mathbb{B}_n(\boldsymbol{\lambda})$ *as* $\mathbb{F}$*-algebras.*

Proof. This is an immediate consequence of Corollary 9.0.9 and Theorem 12.0.7. □

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