Black holes with source

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Why three dimensions?

Studying gravity in three dimensions

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- Many problems simplify tremendously in three dimensions, yet the geometry is still rich enough to provide interesting results. Techniques may then be applied to 4D.
- Particularly quantum gravity simplifies drastically in 3D which makes it more interesting to study.
- 3D gravity provides a nice laboratory to study the AdS/CFT correspondence.

Scalar-Tensor theories in general

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- We know that General Relativity does not solve everything, neither is it quantizable. In fact, while being a beautiful theory, there are still plenty of open questions.

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- In physics it is common to modify theories, using the correspondence principle, to explain new phenomena. Hence it is reasonable to study modified theories of gravity.
- Scalar tensor theories constitute one of the simplest extensions/modifications of General Relativity.

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A brief history

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- His student, Gregory Horndeski, then determined in 1974 the most general such action constructed from the metric tensor and a scalar field in $D = 4$.
- The requirement of having at most second order field equations is to avoid so-called Ostrogradsky instabilities (ghosts), which are extra degrees of freedom with negative energy.

Degenerate higher order scalar tensor theories

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- These theories are a generalization of Horndeski theories: While the equations of motion of Horndeski theories contain only two derivatives, DHOST theories contain equations of motion of higher order.
- Despite being of higher order, DHOST theories do not generate ghosts.

Degenerate higher order scalar tensor theories

Action

$$
\begin{aligned}
S &= \int d^4x\sqrt{-g}\Big[Z(X,\phi) + G(X,\phi)R \\
&+ A_1(X,\phi)\phi_{\mu\nu}\phi^{\mu\nu} + A_2(X,\phi)(\Box\phi)^2 \\
&+ A_3(X,\phi)\Box\phi\,\phi^{\mu}\phi_{\mu\nu}\phi^{\nu} + F_1(X,\phi)\Box\phi + F_2(X,\phi)G^{\mu\nu}\phi_{\mu\nu} \\
&+ A_4(X,\phi)\phi^{\mu}\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho} + A_5(X,\phi)\left(\phi^{\mu}\phi_{\mu\nu}\phi^{\nu}\right)^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
\phi_\mu &= \nabla_\mu\phi \\
\phi_{\mu\nu} &= \nabla_\mu\nabla_\nu\phi \\
X &= \phi_\mu\,\phi^\mu
\end{aligned}
$$

The model

Action

$$
\begin{aligned}
S &= \int d^3x\sqrt{-g}\Big[Z(X) + G(X)R \\
&+ A_2(X)\left((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right) \\
&+ A_3(X)\Box\phi\,\phi^{\mu}\phi_{\mu\nu}\phi^{\nu} \\
&+ A_4(X)\phi^{\mu}\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho} + A_5(X)\left(\phi^{\mu}\phi_{\mu\nu}\phi^{\nu}\right)^2\Big]
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Symmetries of the action

Scalar field transformations

- Shift symmetry: $\phi \to \phi + \text{const.} \to$ Noether current
- Discrete symmetry: $\phi \to -\phi$

Symmetries of the action

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- Shift symmetry: $\phi \to \phi + \text{const.} \to$ Noether current
- Discrete symmetry: $\phi \to -\phi$

Disformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} + K(X)\phi_\mu\phi_\nu$$

Transforms one DHOST theory into another by mixing the coupling functions in the action. Possibly can be used to encounter solutions of one theory by transforming those of another (work in progress).

Symmetries of the action

Kerr-Schild transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g^{(0)}_{\mu\nu} - a(x) l_\mu l_\nu,$$

with l_μ being a null and geodesic w.r.t. both metrics:

$$g^{\mu\nu} l_\mu l_\nu = g^{(0)\mu\nu} l_\mu l_\nu = 0, \qquad l^\mu \nabla_\mu l_\nu = l^\mu \nabla^{(0)}_\mu l_\nu = 0.$$

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Invariance of the action

- Action is quasi-invariant under a KS transformation given that $a(r)$ satisfies a first order differential equation.
- If X is constant, the solution to said equation is $a(r) = M$, where M is a constant (mass term of the metric in 3D).

Symmetries of the action

KS transformation: The Kerr metric as an example

Writing the seed metric of flat space in ellipsodial coordinates:

$$ds_0^2 = -dt^2 + \frac{\Sigma}{r^2+\omega^2}dr^2 + \Sigma d\theta^2 + \left(r^2+\omega^2\right)\sin^2\theta d\varphi^2$$

with $\Sigma = r^2 + \omega^2\cos^2\theta$. Then the Kerr metric can be generated in the following way:

$$\begin{aligned} ds^2 &= ds_0^2 + \frac{M}{\Sigma} l \otimes l \\ l &= dt + \frac{\Sigma}{r^2+\omega^2}dr - \omega\sin^2\theta d\varphi \end{aligned}$$

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Now we can look at the solutions!

The road towards a solution

Stationary, axialsymmetric ansatz

$$
\begin{aligned}
ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + H^2(r)\left[d\theta - k(r)dt\right]^2 , \\
\phi &= \phi(r).
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Additional condition

$$\mathcal{Z}_2^2 - 2\mathcal{Z}_1\mathcal{Z}_3 = 0$$

$$\begin{aligned} \mathcal{Z}_1 &= G + XA_2, \\ \mathcal{Z}_2 &= 2A_2 + XA_3 + 4G_X, \\ \mathcal{Z}_3 &= A_3 + A_4 + XA_5. \end{aligned}$$

The road towards a solution

Solution~~s~~

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2\left(d\theta + N^\theta(r)dt\right)^2,$$

$$F = \left(\frac{Z}{2\mathcal{Z}_1}r^2 - M + \frac{J^2}{4r^2}\right), \quad N^\theta = \frac{J}{2r^2}.$$

Where X has to satisfy $0 = (\mathcal{Z}_1 Z)_X - Z\mathcal{Z}_2$.
Note that the solution is completely determined by the previously defined combinations $\mathcal{Z}_1$ and $\mathcal{Z}_2$, hence different functions in the action can lead to the same solution with effective cosmological constant $\Lambda_{\text{eff}} = -Z/2\mathcal{Z}_1$.

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Solution~~s~~

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Other properties

- Equations remain solved by this metric without imposing the condition $\mathcal{Z}_2^2 - 2\mathcal{Z}_1\mathcal{Z}_3 = 0$.
- The ansatz $\phi = qt + \psi(r) + L\theta$ admits the same metric as a solution.

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Solution~~s~~

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2\left(d\theta + N^\theta(r)dt\right)^2,$$

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Open questions and ongoing work

Does a disformal transformation map to a differend DHOST theory violating the condition we imposed on the action functions? Can we use it (or find another sophisticated transformation) to encounter new solutions?

The road towards a solution

Solution~~s~~

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2\left(d\theta + N^\theta(r)dt\right)^2,$$

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Open questions and ongoing work

What is the meaning behind the condition we imposed on the functions? We found some hints relating it to the degeneracy conditions in different dimensions. Moreover it appears to be related to the Kerr-Schild invariance of the Noether current.

More work in 4D

Regular BH solutions

- Using similar techniques we are studying regular (no curvature singularity) black hole solutions of different topologies in four dimensions.

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- We have derived the equations of motion for the ansatz

$$ds^2 = -h(r)dt^2 + \frac{1}{f(r)}dr^2 + \frac{r^2}{1+\kappa\theta}d\theta^2 + r^2\theta^2 d\varphi^2,$$

with $\kappa \in \{-1, 0, 1\}$.

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  with $\kappa \in \{-1, 0, 1\}$.
- The equations admit a rich set of different solutions. The ongoing work at the moment is to study their properties.

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Black hole thermodynamics

- 1972 Hawking proved that the area of the event horizon can never decrease, just like entropy due to the 2nd law of thermodynamics.

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- 1972 Hawking proved that the area of the event horizon can never decrease, just like entropy due to the 2nd law of thermodynamics.
- Nowadays BH thermodynamics is a well established topic of research. Particularly theories of quantum gravity may want to explain the meaning of entropy in this context.
- There are different methods to calculate the entropy. We have applied the Euclidean method (Euclidean continuation of the action is used in order to add a boundary term that extremizes the action) and a generalized Cardy formula (the entropy can be computed in the corresponding CFT due to the AdS/CFT, provided that the theory admits a regular scalar soliton which is identified with its ground state).

Black hole thermodynamics

Thermodynamic parameters

$$
\begin{aligned}
\mathcal{S} &= 8\mathcal{Z}_1\pi^2 r_h, \\
\mathcal{M} &= 2\pi\mathcal{Z}_1 M = 2\pi\mathcal{Z}_1\left(\frac{r_h^2}{L^2}+\frac{J^2}{4r_h^2}\right), \\
\mathcal{J} &= -2\pi\mathcal{Z}_1 J, \\
T &= \frac{1}{4\pi}\left(\frac{2r_h}{L^2}-\frac{J^2}{2r_h^3}\right).
\end{aligned}
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With $L^2 = 2\mathcal{Z}_1(X)/Z(X)$, so imposing $\mathcal{Z}_1 > 0$ and $Z > 0$ ensures positive mass and entropy solutions.

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With $L^2 = 2\mathcal{Z}_1(X)/Z(X)$, so imposing $\mathcal{Z}_1 > 0$ and $Z > 0$ ensures positive mass and entropy solutions.
First law of thermodynamics holds: $d\mathcal{M} = Td\mathcal{S} + \Omega d\mathcal{J}$!

Phase transition

Gibbs free energy (static case)

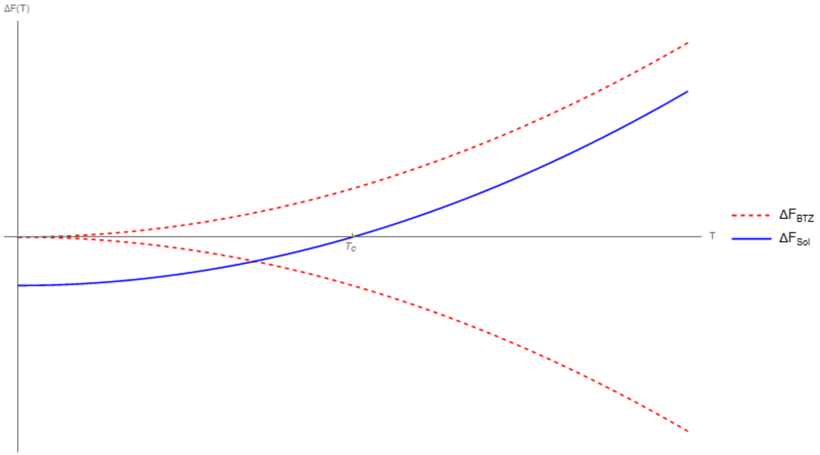
$$\Delta\mathcal{F}_{\text{BTZ}} = \mathcal{F}_{\text{BTZ}} - \mathcal{F} = 16\pi^3 T^2 \left[\frac{\mathcal{Z}_1^2(X)}{\mathcal{Z}(X)} - \frac{\mathcal{Z}_1^2(0)}{\mathcal{Z}(0)}\right]$$

$$\Delta\mathcal{F}_{\text{Sol}} = \mathcal{F}_{\text{Sol}} - \mathcal{F} = 16\pi^3 T^2 \frac{\mathcal{Z}_1^2(X)}{\mathcal{Z}(X)} - 2\pi\mathcal{Z}_1(X)$$

For the soliton there is a Hawking-Page phase transition at

$$T_c = \frac{\sqrt{2}}{4\pi}\sqrt{\frac{\mathcal{Z}(X)}{\mathcal{Z}_1(X)}}$$

Phase transition



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Conclusions and outlook

Summary and future work

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- Can we understand the condition imposed on the coupling functions better?
- The thermodynamic properties of the solution correspond exactly to what one would expect from a BTZ-like metric.
- Extend/continue work in four dimensions. How do the different types of black holes change their properties?

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Modifying gravity by adding torsion

- Another way to modify gravity is making use of a different geometry.
- Going from special to general relativity, one considers a non-zero curvature. Hence another simple extension would be non-zero torsion.
- While bosons do not "feel" the torsion, the geodesics of fermions are affected due to their non-commutative nature, hence it becomes particularly interesting in the study of quantum effects.
- We "source" torsion through the addition of a scalar field to the action. Hence we effectively modify the action again by means of a scalar field (like before).

Why torsion?

The tale of Einstein and Cartan

- General Relativity is formulated using pseudo-Riemannian geometry with symmetric connection (torsion free). All the geometric information is contained in the metric.

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- In 1922 Élie Cartan proposed reformulating the theory into what is now called the first-order formulation of gravity (using Cartan geometry).

Why not torsion?

The tale of Einstein and Cartan

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- In 1922 Élie Cartan proposed reformulating the theory into what is now called the first-order formulation of gravity (using Cartan geometry).
- This allows for a new interpretation of the geometry and reveals the vanishing of torsion to be an additional constraint on the field equations.

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Rewriting the theory

Introducing a tetrad

Instead of working in a coordinate basis, itis often more convenient to work in a local orthonormal frame, with basis one-forms $e^a(x) = e^a{}_\mu(x) dx^\mu$:

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x) dx^\mu dx^\nu = \eta_{ab} e^a(x) e^b(x) \\ g_{\mu\nu}(x) &= \eta_{ab}\, e^a{}_\mu(x)\, e^b{}_\nu(x), \\ \delta^a_b &= e^a{}_\mu(x) e_b{}^\mu, \quad \delta^\mu_\nu = e^a{}_\nu(x) e_a{}^\mu \end{aligned}$$

All the information of the metric is contained in the tetrad, yet it is not uniquely determined by the metric tensor! We always have the freedom of "rotating" the tetrad by means of a local Lorentz transformation!

Rewriting the theory

The spin connection

In order to obtain a consistent covariant derivative, we need to define the so-called "spin connection", $\omega^a{}_b(x) = \omega_\mu{}^a{}_b(x)dx^\mu$:

$$DV^a_b = dV^a_b + \omega^a{}_c \wedge V^c_b + \omega^c{}_b \wedge V^a_c$$

With this, the torsion two-form can be written as:

$$De^a = de^a + \omega^a{}_b \wedge e^b = T^a.$$

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$$De^a = de^a + \omega^a{}_b \wedge e^b = T^a.$$

The curvature form

With the spin-connection the curvature two-form is defined as:

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu$$

Rewriting the theory

The basic ingredients

$$
\begin{aligned}
e^a &= e^a{}_\mu dx^\mu \\
\omega^a{}_b &= \omega_\mu{}^a{}_b dx^\mu \\
R^a{}_b &= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \\
T^a &= De^a = de^a + \omega^a{}_b \wedge e^b
\end{aligned}
$$

Example: Einstein-Hilbert action

$$
\begin{aligned}
I_{\mathrm{EH}}[g] &= \kappa' \int d^4x \sqrt{-g} R \\
\rightarrow I_{\mathrm{EH}}[e, \omega] &= \kappa \int \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d
\end{aligned}
$$

Calculating the field equations

Equations of motion

$$\begin{aligned} \frac{1}{4\kappa} S_{ab} &= \epsilon_{abcd} e^c \wedge T^d \\ \frac{1}{2\kappa} \tau_a &= \epsilon_{abcd} e^b \wedge R^{cd} \end{aligned}$$

General action

$$\begin{aligned} I &= I_{\mathrm{EH}} + I_{\mathrm{M}} \\ &= \kappa \int \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d + I_{\mathrm{M}} \end{aligned}$$

Calculating the field equations

Equations of motion

$$\frac{1}{4\kappa}S_{ab} = \epsilon_{abcd}e^c \wedge T^d$$
$$\frac{1}{2\kappa}\tau_a = \epsilon_{abcd}e^b \wedge R^{cd}$$

Remarks

It can be shown that in order to have torsion the spin current S_{ab} must not be zero, while curvature can still "propagate" outside of matter! However, in order to avoid torsion one has to choose the matter Lagrangians in such a way that they do not depend on the connection!

Calculating the field equations

Equations of motion

$$
\begin{aligned}
0 &= \epsilon_{abcd} e^c \wedge T^d \\
0 &= \epsilon_{abcd} e^b \wedge R^{cd}
\end{aligned}
$$

Example: Schwarzschild

$$ds^2 = -f(r)^2 dt^2 + g(r)^2 dr^2 + r^2 d\Sigma_2^2$$

$$e^0 = f(r)dt, \quad e^1 = g(r)dr, \quad e^2 = rd\theta, \quad e^3 = r\sin(\theta)d\varphi$$

Calculating the field equations

Spin connection

Solve torsion equation $0 = de^a + \bar{\omega}^a{}_b \wedge e^b$:

$$\bar{\omega}^0{}_1 = \frac{f'}{g} dt \qquad \bar{\omega}^1{}_2 = -\frac{1}{g} d\theta$$

$$\bar{\omega}^2{}_3 = -\cos(\theta) d\varphi \qquad \bar{\omega}^1{}_3 = -\frac{\sin(\theta)}{g} d\varphi$$

Example: Schwarzschild

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Calculating the field equations

Equations (not independent!)

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0 &= -gf' + rf'g' - rgf'' + fg' \\
0 &= 2rg' + g^3 - g \\
0 &= 2rf' + f - fg^2
\end{aligned}
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$$ds^2 = -f(r)^2 dt^2 + g(r)^2 dr^2 + r^2 d\Sigma_2^2$$

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Calculating the field equations

Equations (not independent!)

$$\begin{aligned} 0 &= -gf' + rf'g' - rgf'' + fg' \\ 0 &= 2rg' + g^3 - g \\ 0 &= 2rf' + f - fg^2 \\ \Rightarrow f(r)^2 &= g(r)^{-2} = 1 - \frac{2M}{r} \end{aligned}$$

Example: Schwarzschild

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Generating torsion

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- This can make the theory arbitrarily complex, in particular since it is necessary to construct a matter Lagrangian that depends on the spin connection.
- However, if we want the matter to directly interact with gravity, a natural way is to include a coupling of the field to the curvature form.
- One elegant approach is making use of the Gauß-Bonnet term, which is usually a topological invariant.

The action

Adding the Gauß-Bonnet term

$$
\begin{aligned}
I &= I_{EH} + I_C, \\
I_{EH} &= \kappa \int \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d, \\
I_C &= \lambda \int \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d.
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I_C &= \lambda \int \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d, \\
I_{GB} &= \int \epsilon_{abcd}\, \phi\, R^{ab} \wedge R^{cd}.
\end{aligned}
$$

Field equations

Variation of ω

$$0 = \epsilon_{abcd} \left(\kappa e^c \wedge T^d - d\phi \wedge R^{cd} \right)$$

Variation of e

$$0 = \epsilon_{abcd} \left(\kappa e^b \wedge R^{cd} + 2\lambda e^b \wedge e^c \wedge e^d \right)$$

Variation of ϕ (locally exact)

$$\begin{aligned} 0 &= \epsilon_{abcd} R^{ab} \wedge R^{cd} \\ &= d \left[\epsilon_{abcd} \left(\omega^{ab} \wedge \omega^{cd} + \frac{2}{3} \omega^{ab} \wedge \omega^{c}{}_{e} \wedge \omega^{ed} \right) \right] \end{aligned}$$

Solving the equations

Static, spherically symmetric ansatz

Solving the equations turns out to be a complex task. The first obvious choice is to consider a static and spherically symmetric ansatz:

$$ds^2 = -f(r)^2 dt^2 + g(r)^2 dr^2 + r^2 d\Sigma_2^2$$
$$e^0 = f(r)dt, \quad e^1 = g(r)dr, \quad e^2 = rd\theta, \quad e^3 = r\sin(\theta)d\varphi$$

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- Define contorsion: $\kappa^a{}_b = \omega^a{}_b - \bar{\omega}^a{}_b$.
- Then the torsion form can be written as $T^a = \kappa^a{}_b \wedge e^b$.
- We know the torsion free connection $\bar{\omega}^a{}_b$ from solving the zero torsion equation $0 = de^a + \bar{\omega}^a{}_b \wedge e^b$.

Solving the equations

Applying Killing symmetries to the torsion

A physically reasonable assumption would be to require the torsion to possess the same symmetries as the metric. Therefore one can apply the Lie derivative with respect to the Killing vector fields on the torsion tensor to find the most general form with these symmetries:

$$
\begin{aligned}
T^0 &= C_0 e^0 \wedge e^1 + D_0 e^2 \wedge e^3, \\
T^1 &= C_1 e^0 \wedge e^1 + D_1 e^2 \wedge e^3, \\
T^2 &= D_2 e^0 \wedge e^2 + A_2 e^0 \wedge e^3 + D_3 e^1 \wedge e^2 + B_2 e^1 \wedge e^3, \\
T^3 &= D_2 e^0 \wedge e^3 - A_2 e^0 \wedge e^2 + D_3 e^1 \wedge e^3 - B_2 e^1 \wedge e^2.
\end{aligned}
$$

Where the functions depend on r only.

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- Up to now we were not even able to show whether a solution can exist or not. At least we could not find an obvious contradiction to asymptotic AdS-like behaviour.
- Extensions may be necessary (non-static, axial symmetry, dynamical scalar field, $\cdots$).

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- At the same time the system seems quite restrictive, so a relaxation of the symmetries or other modifications in the ansatz may be needed to study the problem.
- It would be interesting to see the effects of torsion and compare it to a known system without torsion.

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That's all folks!

Thank you very much for your attention!