Global continuation of monotone wavefronts

by

Adrian Gomez

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Mathematics
in the Universidad de Talca
2012

Thesis adviser:

Professor Dr. Sergei Trofimchuk
Universidad de Talca

Doctoral Committee:

Professor Dr. Manuel Elgueta
Pontificia Universidad Catolica de Chile
Professor Dr. Mokthar Hassaine
Universidad de Talca
Professor Dra. Salomé Martínez
Universidad de Chile

Este Trabajo fue financiado por CONICYT, Beca de doctorado nacional n^0 21110135 y Beca de Pasantía doctoral n^0 75120084.

TABLE OF CONTENTS

CHAPTER I

Introduction

In this thesis are studied Reaction diffusion equations with a single delay h that have the form

$$(1.1) \qquad u_t(t,x) = \Delta u(t,x) + f(u(t,x), u(t-h,x))$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $h \geq 0$ is the delay. Function f satisfies the *monostablity condition*: If $g(x) := f(x,x)$, then

(i) $g(0) = 0$, $g(\kappa) = 0$ for some $\kappa > 0$,

(ii) $g(x) > 0$ for $x \in (0,\kappa)$,

(iii) $g'(0) > 0$, $g'(\kappa) < 0$

Also, some differentiability conditions are imposed to f, precisely $f \in C^{1,\alpha}$, $\alpha \in (0,1]$, that is, the partial derivatives f_i, $i = 1,2$ satisfies the *α-Hölder condition*

$$|f_i(x_1,y_1) - f_i(x_2,y_2)| \leq C\|(x_1 - x_2, y_1 - y_2)\|^\alpha.$$

This type of equations are used as models in population dynamics and commonly in its applications, quantity u is the size of population in some time and place, then it is considered a positive quantity.

Definition 1. A solution $u(t,x)$ of (1.1) is called *travelling wave solution* if $u(t,x) = \phi(x \cdot \nu + ct)$, where $\phi : \mathbb{R} \to \mathbb{R}$ satisfies $\phi(-\infty) = 0$, $\phi(+\infty) = \kappa$. The constant $c \in \mathbb{R}_+$ is called *propagation speed of the wave.* The function ϕ is called *the profile of the wave* and if a travelling wave u has a monotone profile, u is a monotone travelling wavefront solution.

The study of travelling wave solution for reaction diffusion equations was developed first at the work of Kolmogorov et al.[37] and in Fischer [23], but is a extensively studied topic in many partial differential equations (see for example [25]).

In the present thesis is studied the existence and in some cases the uniqueness of monotone travelling waves solutions for equations as (1.1) but with delay, this is a great additional complication in the theory.

Differential equations with delay are also a extensively studied topic. They are more realistic models than differential equations without delay, many examples of application can be founded in [15] or [17], where delay symbolizing lifetime of member in a specie or, the time spending some substance for mixed or, the time of response of some mechanism faced with some stimulus, etc. Delay differential equations can model phenomena as oscillations better than without delayed equations, one interesting example of this can be found in ([17], pp. 51) where Nicholson's equation

$$\frac{dN}{dt} = ry(t-\tau)e^{-N(t-\tau)/K} - mN(t)$$

was used as model for a blowflies specie whose population show an oscillatory behavior.

However, mathematical theory of delayed differential equations is more complicated than theory for without delay equations, some difference can be illustrated with a simple first order equations

$$x'(t) = ax(t-\tau), \tag{1.2}$$

where $a \in \mathbb{R}$ and $\tau \geq 0$. If $\tau = \pi/2,\ a \neq 0$, equation (1.2) have a solution of the form $x(t) = c_1 \cos(at) + c_2 \sin(at)$. That is impossible for a linear first order equations without delay.

Another important difference is that if you want to write an initial value problem associated to equation 1.2, then you will need a space of initial conditions $C([-\tau, 0], \mathbb{R})$, that is, the initial conditions space has infinity dimension, for this reason, tools of ODE as phase space analysis do not work in a ODE with delay.

Study the problem of existence and uniqueness of monotone travelling waves solutions $u(t,x) = \phi(x \cdot \nu + ct)$ for equation (1.1) is equivalent to study the problem of existence and uniqueness(up to translation) of heteroclinic solutions $\phi(s)$, with $\phi(-\infty) = 0,\ \phi(+\infty) = \kappa$ for the second order ordinary differential equation with delay

$$x''(t) - cx'(t) + f(x(t), x(t-ch)) = 0. \tag{1.3}$$

There are essentially two approach to solve this problem, the *super and sub solution method*, proposed in [61] and *Lyapunov-Schmidt Reduction* proposed in [20].

In the second chapter of this thesis is used the super and subsolutions for studing the KPP-Fischer equation with delay

$$u_t(t,x) = \Delta u(t,x) + u(t,x)\big(1 - u(t-h,x)\big), \tag{1.4}$$

but with a different operator that in [61], (see equations 2.5 and 2.6 bellow). Is noteworthy that after the work [61] there are many works using super and subsolution method but, all with similar integral operator (3.23). In our approach, one of the keys is to use a different integral operator. Also, the study of the asymptotic behavior of a monotone heteroclinic of (1.3) in $\pm\infty$ gives the key for constructing sub and super solutions gluing properly eigenfunctions of autonomous linearizing equations around the equilibria.

Also, this work gives a explicit method for approximate a monotone travelling wave solution using a computer, because construction of super and subsolution are give completely in terms of real roots of eigenvalues of linearizing equation. This is an important aspect if we observe that equation (1.4) is a model in biomathematics (see, for example [47]).

In the third chapter, is studied the most general case (1.1). This generalization includes, in addition to KPP-Fischer equation, many important and extensively studied equations, such that the Mackey-Glass equation and the diffusive Nicholson's blowflies equation (See section 2 of third chapter).

In this case we use the Lyapunov-Schmidt reduction method. Is known that, in the case $h = 0$, equation (1.1) has monotone travelling waves solution for all $c \geq c_*$ and also are knowledge its asymptotic formulae in $\pm\infty$ (see [55]), then the idea is to extend the existence of waves to a set $\overline{\mathfrak{D}}_{\mathfrak{N}}$ of parameters $(h, c) \in \mathbb{R}_+ \times \mathbb{R}_+$ (delay and speed, see fig.3.2). For this, you need studying the linearizing equation around a heteroclinic solution, this equation is an asymptotically autonomous equation that, over an appropriated weighted space is a Fredholm operator, surjective and with one-dimensional Kernel. This is one of the difficult steps to conclude our result, where is used theory of adjoint operators and exponential dichotomies for functional differential equations developed by Hale and Lin in [32] and the theory of discrete Lyapunov functions developed by J. Mallet-Paret and G. Sell in [45].

The monotonicity of the waves is an important key of our program because it allows to enclose the set of parameters where monotone waves exist by take limit (see proof of Lemma 26).
The main result of third chapter (theorem III.1) gives a criterion for the existence of monotone wavefront in some important cases as sub tangential nonlinearity (see lemma 19), in particular, the problem of existence of monotone travelling wavefronts for Nicholson's equation is completely answered. Is remarkable that before this work, the existence of monotone fronts for Nicholson's equations was known only for the case $p/\delta \in (1, e]$ (see [53]).
Theorem III.1 gives another proof of existence of monotone travelling wavefronts for the KPP- Fischer equation studied in second chapter.

The second chapter was published at the year 2011 [26], and the third chapter was recently submitted and can be found in www.arxiv.org.[27].

CHAPTER II

Monotone traveling wavefronts of the KPP-Fischer delayed equation

2.1 Introduction and main results

It is well known that the traveling waves theory was initiated in 1937 by Kolmogorov, Petrovskii, Piskunov [37] and Fisher [23] who studied the wavefront solutions of the diffusive logistic equation

$$u_t(t,x) = \Delta u(t,x) + u(t,x)(1-u(t,x)),\ u \geq 0,\ x \in \mathbb{R}^m. \tag{2.1}$$

We recall that the classical solution $u(x,t) = \phi(\nu \cdot x + ct)$, $\|\nu\| = 1$, is a wavefront (or a traveling front) for (2.1), if the profile function ϕ is positive and satisfies $\phi(-\infty) = 0$, $\phi(+\infty) = 1$.

The existence of the wavefronts in (2.1) is equivalent to the presence of positive heteroclinic connections in an associated second order non-linear differential equation. The phase plane analysis is the natural geometric way to study these heteroclinics. The method is conclusive enough to demonstrate that (a) for every $c \geq 2$, the KPP-Fisher equation has exactly one traveling front $u(x,t) = \phi(\nu \cdot x + ct)$; (b) Eq. (2.1) does not have any traveling front propagating at the velocity $c < 2$; (c) the profile ϕ is necessarily strictly increasing function.

The stability of traveling fronts in (2.1) represents another important aspect of the topic: however, we do not discuss it here. Further reading and relevant information can be found in [9, 39, 50, 62].

Eq. (2.1) can be viewed as a natural extension of the ordinary logistic equation $u'(t) = u(t)(1-u(t))$. An important improvement of this growth model was proposed by Hutchinson [35] in 1948 who incorporated the maturation delay $h > 0$ in the following way:

$$u'(t) = u(t)(1 - u(t-h)),\ \ u \geq 0. \tag{2.2}$$

This model is now commonly known as the Hutchinson' s equation. Since then, the delayed KPP-Fisher equation or the diffusive Hutchinson's equation

$$u_t(t,x) = \Delta u(t,x) + u(t,x)(1 - u(t-h,x)),\ \ u \geq 0,\ \ x \in \mathbb{R}^m, \tag{2.3}$$

is considered as a natural prototype of delayed reaction-diffusion equations. It has attracted the attention of many authors, see [4, 5, 20, 24, 28, 30, 41, 58, 61, 63]. In particular, the existence of traveling fronts connecting the trivial and positive steady states in (2.3) (and its non-local generalizations) was studied in [4, 5, 12, 20, 29, 49, 58, 61]. Observe that the biological meaning of u is the size of an adult population, therefore *only* non-negative solutions of (2.3) are of interest. It is worth to mention that there is another delayed version of Eq. (2.1) derived by Kobayashi [36] from a branching process:

$$u_t(t,x) = \Delta u(t,x) + u(t-h,x)(1 - u(t,x)),\ \ u \geq 0,\ \ x \in \mathbb{R}^m.$$

However, since the right-hand side of this equation is monotone increasing with respect to the delayed term, the theory of this equation is fairly different (and seems to be simpler) from the theory of (2.3), see [52, 61, 64].

This paper deals with the problem of existence and uniqueness of monotone wavefronts for Eq. (2.3). The phase plane analysis does not work now because of the infinite dimension of phase spaces associated to delay equations. Recently, the existence problem was considered by using two different approaches. The first method, which was proposed in [61], uses the positivity and monotonicity properties of the integral operator

$$(2.4)\qquad (A\phi)(t) = \frac{1}{\epsilon'}\left\{\int_{-\infty}^{t} e^{r_1(t-s)}(\mathcal{H}\phi)(s)ds + \int_{t}^{+\infty} e^{r_2(t-s)}(\mathcal{H}\phi)(s)ds\right\},$$

where $(\mathcal{H}\phi)(s) = \phi(s)(\beta+1-\phi(s-h))$ for some appropriate $\beta > 1$, and $\epsilon' = \epsilon(r_2-r_1)$ with $r_1 < 0 < r_2$ satisfying $\epsilon z^2 - z - \beta = 0$, and $\epsilon^{-1/2} = c > 0$ is the front velocity. A direct verification shows that the profiles $\phi \in C(\mathbb{R}, \mathbb{R}_+)$ of traveling waves are completely determined by the integral equation $A\phi = \phi$. Wu and Zou have found a subtle combination of the usual and the Smith and Thieme nonstandard orderings on an appropriate profile set $\Gamma^* \subset C(\mathbb{R}, (0,1))$ which allowed them (under specific quasimonotonicity conditions) to indicate a pair of upper and lower solutions $\phi^{\pm}$ such that $\phi^- \leq A^{j+1}\phi^+ \leq A^j\phi^+,\ j = 0, 1, \ldots$ Then the required traveling front profile is given by $\phi = \lim A^j\phi^+$. More precisely, in [61, Theorem 5.1.5], Wu and Zou established the following

Proposition 1. *For any $c > 2$, there exists $h^*(c) > 0$ such that if $h \leq h^*(c)$, then Eq. (2.3) has a monotone traveling front with wave speed c.*

The above result was complemented in [58, Remark 5.15] and [49], where it was shown that Proposition 1 remains valid if $c = 2$. It should be observed that Wang *et al.* [58] have also used the method of upper and lower solutions, however their lower solution is different from that in [61]. Recently, Ou and Wu [48] showed that Proposition 1 can be proved by means of a perturbation argument (considering $h > 0$ as a small parameter).

The second method was proposed in [20]. It essentially relies on the fact that, in a 'good' Banach space, the Frechet derivative of $\lim_{\epsilon\to 0} A$ along a heteroclinic solution ψ of the limit delay differential equation (2.2) is a surjective Fredholm operator. In consequence, the Lyapunov-Schmidt reduction was used to prove the existence of a smooth family of wave solutions in some neighborhood of ψ. The following result was proved in [20, Corollary 6.6.]:

Proposition 2. *There exists $c^* > 0$ such that if $0 < h < 1/e$ then for any $c > c^*$, Eq. (2.3) has a wave solution $u(x,t) = \phi(\nu \cdot x + ct)$, $|\nu| = 1$, satisfying $\phi(-\infty) = 0$, $\phi(+\infty) = 1$.*

We remark that the positivity of this wave was not proved in [20] and the value of $c^* > 0$ was not given explicitly. Nevertheless, as it was shown in [21] for the case of the Mackey-Glass type equations, the method of [20] may be refined to establish the existence of positive wavefronts as well. Moreover, it follows from [21] that Proposition 2 is still valid for $h \in (0, 3/2)$. The recent work [3] suggests that the approach of [20] can be also used to prove the uniqueness (up to shifts) of the positive traveling solution of (2.3) for sufficiently fast speeds.

In this paper, motivated by ideas in [14, 61], we give a criterion for the existence of positive monotone wavefronts in (2.3) and prove their uniqueness (modulo translation). In order to do this, instead of using operator (3.23) as it was done in all previous works, we work with different integral operators, namely:

$$(\mathcal{A}\varphi)(t) = \frac{1}{\epsilon(\mu-\lambda)} \int_t^{+\infty} (e^{\lambda(t-s)} - e^{\mu(t-s)})\varphi(s)\varphi(s-h)ds, \tag{2.5}$$

where $\epsilon \in (0, 0.25)$ and $0 < \lambda < \mu$ are the roots of $\epsilon z^2 - z + 1 = 0$, and with

$$(\mathcal{B}\varphi)(t) = 4 \int_t^{+\infty} (s-t)e^{2(t-s)}\varphi(s)\varphi(s-h)ds \tag{2.6}$$

which can be considered as the limit of $\mathcal{A}$ when $\epsilon \to 0.25$. Remarkably, all monotone wavefronts (in particular, the wavefronts propagating with the minimal speed $c = 2$)

can be found via a monotone iterative algorithm which uses $\mathcal{A}, \mathcal{B}$ and converges uniformly on $\mathbb{R}$.

Before stating our main results, let us introduce the critical delay $h_1 = 0.560771160\ldots$ This value coincides with the positive root of the equation

$$2h^2 \exp(1 + \sqrt{1+4h^2} - 2h) = 1 + \sqrt{1+4h^2}$$

and plays a key role in the following result (which is proved in Section 2):

Lemma 1. *Let* $\epsilon \in (0, 0.25]$, $h > 0$. *Then the characteristic function* $\psi(z, \epsilon) := \epsilon z^2 - z - \exp(-zh)$ *has exactly two (counting multiplicity) negative zeros* $\lambda_1 \le \lambda_2 < 0$ *if and only if one of the following conditions holds*

1. $0 < h \le 1/e$,

2. $\epsilon \ge \epsilon^*(h)$ *and* $1/e < h \le h_1$.

Here the continuous $\epsilon^*(h)$ is defined in parametric form by

$$\epsilon^*(h(t)) = th(t),\ h(t) = (2t + \sqrt{4t^2+1}) \exp(-1 - \frac{2t}{1+\sqrt{4t^2+1}}),\ t \in [0, 0.445\ldots].$$

Let us state now the main results of this paper.

Theorem II.1. *Eq. (2.3) has a positive monotone wavefront* $u = \varphi(\nu \cdot x + ct)$, $|\nu| = 1$, *connecting* 0 *with* 1 *if and only if one of the following conditions holds*

1. $0 \le h \le 1/e = 0.367879441...$ *and* $2 \le c < c^*(h) := +\infty$;

2. $1/e < h \le h_1 = 0.560771160\ldots$ *and* $2 \le c \le c^*(h) := 1/\sqrt{\epsilon^*(h)}$.

Furthermore, set $\phi(s) := \varphi(cs)$. *Then for some appropriate* ϕ_- *(given below explicitly), we have that* $\phi = \lim_{j\to+\infty} \mathcal{A}^j \phi_-$ *(if* $c > 2$*), and* $\phi = \lim_{j\to+\infty} \mathcal{B}^j \phi_-$ *(if* $c = 2$*), where the convergence is monotone and uniform on* $\mathbb{R}$. *For each fixed* $c \ne c^*(h)$, $\phi(t)$

is the only possible monotone profile (modulo translation) and ϕ, ϕ_- *have the same asymptotic representation* $1 - e^{\lambda_2 t}(1 + o(1))$ *at* $+\infty$.

Corollary 1. *If* $h > h_1 = 0.560771160\ldots$ *then the delayed KPP-Fisher equation does not have any positive monotone traveling wavefront.*

Next, let us define the continuous function $\epsilon^{\#}(h)$ parametrically by

$$\epsilon^{\#}(h(t)) = \frac{t + 2 + \sqrt{2t+4}}{t^2}, h(t) = -\frac{\ln(2 + \sqrt{2t+4})}{t}, \ t \in (-2, -1.806\ldots] \tag{2.7}$$

Set $h_0 := 0.5336619208\ldots$ (see also Lema 2 for its complete definition) and

$$c^{\#}(h) := \begin{cases} +\infty, & \text{when } h \in (0, 0.5\ln 2], \\ 1/\sqrt{\epsilon^{\#}(h)}, & \text{when } h \in (0.5\ln 2, h_0], \\ 2, & \text{when } h > h_0. \end{cases}$$

□

Figure 2.1: Schematic presentation of the critical speeds and delays.

Theorem II.2. *Let* $u = \varphi(\nu \cdot x + ct)$, $|\nu| = 1$, *be a positive monotone traveling front of Eq. (2.3). Set* $\phi(s) := \varphi(cs)$. *Then, for some appropriate* t_0, *positive* K_j *and every small positive* σ, *we have at* $t = -\infty$

$$\phi(t+t_0) = \begin{cases} -K_2te^{\lambda t} + O(e^{(2\lambda-\sigma)t}), & \textit{when } c = 2, \\ e^{\lambda t} - K_1 e^{\mu t} + O(e^{(2\lambda-\sigma)t}), & \textit{when } 2 < c < 1.5\sqrt{2}, \\ e^{\lambda t} + O(e^{(2\lambda-\sigma)t}), & \textit{when } c \geq 1.5\sqrt{2} = 2.121\ldots \end{cases}$$

Similarly, at $t = +\infty$

$$\phi(t+t_0) = \begin{cases} 1 - e^{\lambda_2 t} + O(e^{(2\lambda_2+\sigma)t}), & \textit{when} \quad h \leq h_0, \ c \in [2, c^{\#}(h)] \cap \mathbb{R}, \\ 1 - e^{\lambda_2 t} + K_3 e^{\lambda_1 t} + & \textit{when} \quad h \in (0.5\ln 2, h_1] \\ \quad +O(e^{(\lambda_1-\sigma)t}), & \textit{and } c \in (c^{\#}(h), c^*(h)), \\ 1 - K_4 t e^{\lambda_2 t} + O(e^{(\lambda_2-\sigma)t}), & \textit{when } c = c^*(h) \textit{ and } h \in (1/e, h_1]. \end{cases}$$

Theorem II.2 suggests the way of approximating the traveling front profile: e.g., for $c \neq 2, c^*(h)$, we can take functions $a_-(t) := c_1 e^{-\lambda t}$ and $a_+(t) := 1 - e^{\lambda_2 t}$ and glue them together at some point τ. The point τ and $c_1 > 0$ have to be chosen to assure maximal smoothness of the approximation at τ. As we will see in Section 3, this idea allows to construct reasonable lower approximations to the exact traveling wave. See also Figure 2 below.

Remark 1. As it was showed by Ablowitz and Zeppetella [1], equation (2.1) has the explicit exact wavefront solution $u = \varphi_\star(\nu\cdot x + ct)$, $|\nu| = 1$, with $c = 5/\sqrt{6} = 2.041\ldots$ and the (scaled) profile

$$\phi_\star(s) = \left(\frac{1}{2} + \frac{1}{2}\tanh(\frac{5s}{12} + s_0)\right)^2, \ \phi_\star(s) := \varphi_\star(cs).$$

If we select $s_0 = 0.5\ln 2$, then

$$\phi_\star(s) = 1 - 2e^{-5s/6-2s_0} + O(e^{-5s/4}) = 1 - e^{-5s/6} + O(e^{-5s/4}), \ s \to +\infty,$$

so that $\phi_\star = \lim_{j\to+\infty} \mathcal{A}^j \phi_-$ in view of Theorem II.1 and the uniqueness (up to translations) of the traveling front for the non-delayed KPP-Fisher equation. Figure

2 (on the left) shows five approximations $\mathcal{A}^j\phi_-, j = 0, 1, 2, 3, 4$, and the exact solution $\phi_\star$, the graphs are ordered as $\phi_- < \mathcal{A}\phi_- < \mathcal{A}^2\phi_- < \mathcal{A}^3\phi_- < \phi_\star$. On the right, the four first approximations $\mathcal{B}^j\phi_-, j = 0, 1, 2, 3$, of ϕ are plotted when $c = 2,\ h = 0.56$. It should be noted that the limit function ϕ and the initial approximation ϕ_- have the same first two terms $(1 - \exp(\lambda_2 t))$ of their asymptotic expansions at $+\infty$. See Theorem II.1 and Sections 3,4. However, as the analysis of the Ablowitz-Zeppetella solution shows, these ϕ and ϕ_- may have different first terms of their expansions at $-\infty$. This partially explains a better agreement between the exact solution and their approximations for $t \geq \tau = 0.487\ldots$ on the left picture (the value of τ is given in Section 3).

□ □

Figure 2.2: On the left: increasing sequence of approximated waves $\mathcal{A}^j\phi_-, j = 0, 1, 2, 3, 4$, and the Ablowitz-Zeppetella exact solution $\phi_\star$ ($\epsilon = 0.24$ and $h = 0$). On the right: approximations $\mathcal{B}^j\phi_-, j = 0, 1, 2, 3$ ($\epsilon = 0.25$ and $h = 0.56$).

The structure of the remainder of this paper is as follows. In Section 2, the characteristic function of the variational equation at the positive steady state is analyzed. In the third [the fourth] section, we present a lower [an upper] solution. Section 2.5 contains some comments on the smoothness of upper and lower solutions. Theorems II.1 and II.2 are proved in Sections 2.6 and 2.7, respectively.

Remark 2. After this article had been submitted for publication, the problem of existence and nonexistence of monotone traveling fronts to Eq. (2.3) has been recently considered by Kwong and Ou in [38], where a different approach based on a shooting technique was developed. By presenting a constructive approximation algorithm, indicating asymptotic formulas and proving the uniqueness of monotone fronts, our work complements the interesting investigation in [38]. Next, the existence of fast positive traveling fronts to Eq. (2.3) in the case $h \in [0, 3/2]$ has been recently established in [22] by Faria and one of the authors (cf. the paragraph after Proposition 2).

2.2 Characteristic equation at the positive steady state

In this section, we study the zeros of $\psi(z,\epsilon) := \epsilon z^2 - z - \exp(-zh),\ \epsilon, h > 0$. It is straightforward to see that ψ always has a unique positive simple zero. Since $\psi'''(z,\epsilon)$ is positive, ψ can have at most three (counting multiplicities) real zeros, one of them positive and the other two (when they exist) negative. Lemma 1 in the introduction provides a criterion for the existence of two negative zeros $\lambda_1 \le \lambda_2 < 0$. We start by proving this result:

Proof Lemma 1. Consider the equation $-z = \exp(-zh)$. An easy analysis shows that (i) this equation has exactly two real simple solutions $z_1 < z_2 < 0$, $z_2 > -e$, if $h \in (0, 1/e)$, (ii) it has one double real root $z_1 = z_2 = -e$ if $h = 1/e$, and (iii) it does not have any real root if $h > 1/e$. As a consequence,

$$\epsilon z^2 - z = \exp(-zh) \tag{2.8}$$

has two negative simple solutions if $\epsilon > 0$ and $h \in (0, 1/e]$.

A similar argument shows that for every $h > 1/e$ there exists $\epsilon^*(h) > 0$ such that Eq. (2.8) (a) has two negative simple roots if $\epsilon > \epsilon^*(h)$, (b) has one negative double root if $\epsilon = \epsilon^*(h)$, (c) does not have any solution if $\epsilon < \epsilon^*(h)$. In particular, $\epsilon = \epsilon^*(h), z = \lambda_1(h) = \lambda_2(h)$, solve the system

$$\epsilon z^2 - z = \exp(-zh), \quad 2\epsilon z - 1 = -h\exp(-zh),$$

which yields the parametric representation for $\epsilon^*(h)$ given in the introduction.

Finally, a direct graphical analysis of (2.8) shows that $\epsilon^*(h)$ is increasing with respect to h. Hence, since $\epsilon^*(h) \le 0.25$, we conclude that $h \le (\epsilon^*)^{-1}(0.25) =: h_1 =$

0.560771 . . . □

Lemma 2. *Let $\lambda_1 \le \lambda_2 < 0$ be two negative zeros of $\psi(z,\epsilon)$ and $\epsilon \in (0, 0.25]$ be fixed. Then $\lambda_1 \le 2\lambda_2$ if and only if one of the following conditions holds*

1. $0 < h \le 0.5 \ln 2 = 0.347\ldots$;
2. $\epsilon \ge \epsilon^{\#}(h)$ *and* $0.5 \ln 2 < h \le \ h_0 := 0.5336619208\ldots$.

Proof. This lemma can be proved analogously to the previous one, we briefly outline the main arguments. First, for each fixed positive $\epsilon^{\#}$ we may find $h(\epsilon^{\#}) > 0$ such that $\lambda_1 < 2\lambda_2$ if $h \in (0, h(\epsilon^{\#}))$ and $\lambda_1 = 2\lambda_2$ if $h = h(\epsilon^{\#})$. In this way,

$$\epsilon^{\#}\lambda_2^2 - \lambda_2 = \exp(-\lambda_2 h(\epsilon^{\#})), \quad 4\epsilon^{\#}\lambda_2^2 - 2\lambda_2 = \exp(-2\lambda_2 h(\epsilon^{\#})),$$

which yields representation (2.7). Now, we complete the proof by noting that $h(\epsilon)$ is continuous and strictly increasing on $(0, +\infty)$ and $h(0+) = 0.5 \ln 2,\ h_0 = h(0.25)$. □

Lemma 3. *Let $\lambda_1 \le \lambda_2 < 0$ be two negative zeros of $\psi(z,\epsilon)$ and $\epsilon \in (0, 0.25]$ be fixed. Then $\Re\lambda_j < \lambda_1$ for every complex root of $\psi(z,\epsilon) = 0$.*

Proof. Set $\alpha := (1 + 2\epsilon - \sqrt{1 + 4\epsilon^2})/(2\epsilon)$, $a := -e^{-\alpha h}/(\sqrt{1 + 4\epsilon^2} - 2\epsilon)$, $k := \epsilon/(\sqrt{1 + 4\epsilon^2} - 2\epsilon)$. Then $\alpha, k > 0, a < 0$, and

$$\psi(z + \alpha) = (\sqrt{1 + 4\epsilon^2} - 2\epsilon)(kz^2 - z - 1 + ae^{-zh}).$$

It is easy to see that $p(z) := kz^2 - z - 1 + ae^{-zh}$ also has two negative and one positive root. Since the translation $z \to z + \alpha$ of the complex plain does not change the mutual position of zeros of ψ, the statement of Lemma 3 follows now from [57, Remarks 19,20]. □

2.3 A lower solution when $\lambda_1 < \lambda_2$

In this section, we assume either condition (1) or condition (2) of Theorem II.1 holds. In addition, let $c \in [2, c^*(h))$ so that $\lambda_1 < \lambda_2$ (where $\lambda_1 := -\infty$ if $h = 0$) and $\lambda \le \mu$. Set

$$\tau = \frac{1}{\lambda_2} \ln \frac{\lambda}{\lambda - \lambda_2} > 0, \quad \phi_-(t) = \begin{cases} \frac{-\lambda_2}{\lambda - \lambda_2} e^{\lambda(t-\tau)}, & \text{if } t \le \tau, \\ \\ 1 - e^{\lambda_2 t} & \text{if } t \ge \tau. \end{cases}$$

It is easy to see that $\phi_- \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{\tau\})$ with $\phi_-'(t) > 0,\ t \in \mathbb{R}$, and

$$\epsilon \phi_-''(t) - \phi_-'(t) + \phi_-(t)(1 - \phi_-(t-h)) < 0, \quad t \in \mathbb{R} \setminus (\tau, \tau + h]. \tag{2.9}$$

Lemma 4. *Inequality (2.9) holds for all $t \in \mathbb{R}$.*

Proof. The case $h = 0$ is obvious, so let $h > 0$. It suffices to consider $t \in (\tau, \tau + h]$. If we take $t \in (\tau, \tau + h]$, then

$$\epsilon \phi_-''(t) - \phi_-'(t) + \phi_-(t)(1 - \phi_-(t-h)) = -\epsilon \lambda_2^2 e^{\lambda_2 t} + \lambda_2 e^{\lambda_2 t} +$$

$$(1 - e^{\lambda_2 t})(1 + \frac{\lambda_2}{\lambda - \lambda_2} e^{\lambda(t-\tau-h)}) = -e^{\lambda_2(t-h)} + (1 - e^{\lambda_2 t})(1 + \frac{\lambda_2}{\lambda - \lambda_2} e^{\lambda(t-\tau-h)}) =$$

$$1 - e^{\lambda_2(t-h)} + \frac{\lambda_2}{\lambda - \lambda_2} e^{\lambda(t-\tau-h)} - e^{\lambda_2 t} - e^{\lambda_2 t} \frac{\lambda_2}{\lambda - \lambda_2} e^{\lambda(t-\tau-h)} =$$

$$1 + \frac{\lambda_2}{\lambda - \lambda_2} e^{\lambda s} - \frac{\lambda}{\lambda - \lambda_2} e^{\lambda_2 s} - \frac{\lambda}{\lambda - \lambda_2} e^{\lambda_2(s+h)} - \frac{\lambda \lambda_2}{(\lambda - \lambda_2)^2} e^{\lambda_2(s+h)} e^{\lambda s} =: \rho(s)$$

where $s = t - \tau - h \in (-h, 0]$. The direct differentiation shows that

$$\rho'(s) = \frac{-\lambda_2 \lambda}{\lambda - \lambda_2} \left[-e^{\lambda s} + e^{\lambda_2 s} + e^{\lambda_2(s+h)} (1 + \frac{\lambda + \lambda_2}{\lambda - \lambda_2} e^{\lambda s}) \right] > 0,$$

since $e^{\lambda s} \le 1,\ e^{\lambda_2 s} \ge 1,$ and $(1 + \frac{\lambda + \lambda_2}{\lambda - \lambda_2} e^{\lambda s}) > 1,$ if $\lambda + \lambda_2 \ge 0,$

$$(1 + \frac{\lambda + \lambda_2}{\lambda - \lambda_2} e^{\lambda s}) \ge 1 + \frac{\lambda + \lambda_2}{\lambda - \lambda_2} = \frac{2\lambda}{\lambda - \lambda_2} > 0, \text{if } \lambda + \lambda_2 < 0.$$

Finally, we have that $\rho(s) < 0$ for all $s \in [-h, 0]$ since $\rho'(s) > 0$ and

$$\rho(0) = -\lambda^2 (\lambda - \lambda_2)^{-2} e^{\lambda_2 h} < 0.$$

□

Remark 3 (A lower solution when $\lambda_1 = \lambda_2$**).** We can not use ϕ_- as a lower solution when $c = c^*(h),\ 1/e < h \le\ h_1$. Indeed, by Theorem II.2, in this case ϕ_- converges to the positive steady state faster than the heteroclinic solutions. In Section 2.5, we will present an adequate lower solution for this situation. However, it will not be C^1-smooth.

2.4 An upper solution when $\lambda_1 < \lambda_2$

Suppose that $\lambda_1 < \lambda_2$ and set $\phi_2(t) := 1 - e^{\lambda_2 t} + e^{rt}$ for some $r \in (\lambda_1, \lambda_2)$. Recall that $\lambda_1 := -\infty$ if $h = 0$. Obviously, $\psi(r, \epsilon) > 0$ and $\phi_2(t) \in (0, 1)$ for $t > 0$. Next, it is immediate to check that $\phi_2 : \mathbb{R} \to \mathbb{R}$ has a unique critical point (absolute minimum) $t_0 = t_0(r) > 0$:

$$t_0(r) = \frac{\ln(-r) - \ln(-\lambda_2)}{\lambda_2 - r}, \quad \lambda_2 e^{\lambda_2 t_0} = r e^{r t_0}.$$

Observe that if $h \in (0, 1/e)$, then we can assume that $t_0(r) \ge h$ since

$$\lim_{r \to \lambda_2 -} t_0(r) = -1/\lambda_2 > 1/e > h,$$

where the last inequalities were established in the proof of Lemma 1. It is clear that the function

$$\phi_+(t) = \begin{cases} \phi_2(t), & \text{if } t \ge t_0(r), \\ \phi_2(t_0(r)), & \text{if } t \le t_0(r) \end{cases}$$

is C^1-continuous and increasing on $\mathbb{R}$. Moreover, $\phi_+(t) \in C^2(\mathbb{R} \setminus \{t_0(r)\})$.

Lemma 5. *For all $r < \lambda_2$ sufficiently close to λ_2, ϕ_+ satisfies the inequality*

$$\epsilon \phi''(t) - \phi'(t) + \phi(t)(1 - \phi(t - h)) \ge 0, \quad t \in \mathbb{R}.$$

Proof. Step I. First we prove that, for all $t \ge t_0$, the following inequality holds:

$$(\mathfrak{N}\phi_2)(t) := \epsilon \phi_2''(t) - \phi_2'(t) + \phi_2(t)(1 - \phi_2(t - h)) \ge 0.$$

In particular, this implies that $(\mathfrak{N}\phi_+)(t) \geq 0$ if $t \geq t_0 + h$. For $t = t_0 + s$, we have that

$$(\mathfrak{N}\phi_2)(t) = \psi(r,\epsilon)e^{rt} - \psi(\lambda_2,\epsilon)e^{\lambda_2 t} + (-e^{\lambda_2 t} + e^{rt})(e^{\lambda_2(t-h)} - e^{r(t-h)}) =$$

$$\psi(r,\epsilon)e^{rt} + (-e^{\lambda_2 t} + e^{rt})(e^{\lambda_2(t-h)} - e^{r(t-h)}) =$$

$$e^{rt_0}\left[\psi(r,\epsilon)e^{rs} + (-\frac{r}{\lambda_2}e^{\lambda_2 s} + e^{rs})e^{rt_0}(\frac{r}{\lambda_2}e^{\lambda_2(s-h)} - e^{r(s-h)})\right] =$$

$$e^{r(t_0+s)}\left[\psi(r,\epsilon) + (-\frac{r}{\lambda_2}e^{(\lambda_2-0.5r)s} + e^{0.5rs})e^{rt_0}(\frac{r}{\lambda_2}e^{-\lambda_2 h}e^{(\lambda_2-0.5r)s} - e^{-rh}e^{0.5rs})\right] =$$

$$e^{rt}\left[\psi(r,\epsilon) + A_1(s)e^{rt_0}A_2(s)\right].$$

It is easy to see that $A_j(+\infty) = 0$ and that A_j has a unique critical point s_j, with

$$\lim_{r\to\lambda_2-} s_1(r) = -1/\lambda_2, \lim_{r\to\lambda_2-} s_2(r) = h - 1/\lambda_2.$$

Therefore, for some small $\delta > 0$ and for all r close to λ_2, the function $A_1(s)e^{rt_0}A_2(s)$ is strictly increasing to 0 on the interval $[h-1/\lambda_2+\delta, +\infty)$ and it is strictly decreasing on $[0, -1/\lambda_2-\delta]$. This means that if $(\mathfrak{N}\phi_2)(t) \geq 0$ for all $t \in [t_0-1/\lambda_2-\delta, t_0+h-1/\lambda_2+\delta]$ then $(\mathfrak{N}\phi_2)(t) \geq 0$ for $t \geq t_0$. In order to prove the former, consider the expression

$$\frac{e^{-rt_0}}{r-\lambda_2}\left(\epsilon\phi_2''(t) - \phi_2'(t) + \phi_2(t)(1-\phi_2(t-h))\right) =$$

$$\frac{\psi(r,\epsilon)e^{rs} + (-\frac{r}{\lambda_2}e^{\lambda_2 s} + e^{rs})e^{rt_0}(\frac{r}{\lambda_2}e^{\lambda_2(s-h)} - e^{r(s-h)})}{r-\lambda_2} := \Gamma_\epsilon(r,s).$$

Since $\Gamma_\epsilon(r,s)$ is analytical on some open neighborhood $\Omega \subset \mathbb{R}^2$ of the compact segment $\{\lambda_2\} \times [-1/\lambda_2 - \delta, h - 1/\lambda_2 + \delta] \subset \mathbb{R}^2$, we find that, for every fixed $\epsilon > 0$,

$$\lim_{r\to\lambda_2-} \Gamma_\epsilon(r,s) = \psi'(\lambda_2,\epsilon)e^{\lambda_2 s} < 0$$

uniformly on $[-1/\lambda_2 - \delta, h - 1/\lambda_2 + \delta]$. As a consequence, we obtain that

$$\epsilon\phi_2''(t) - \phi_2'(t) + \phi_2(t)(1-\phi_2(t-h)) > 0, \quad t \in [t_0 - 1/\lambda_2 - \delta, t_0 + h - 1/\lambda_2 + \delta].$$

Step II. Now, we are ready to prove that $(\mathfrak{N}\phi_+)(t) \geq 0,\ t \in [t_0, t_0+h]$. Indeed, since $\phi_2(t_0) \leq \phi_2(t-h)$ for $t \in [t_0, t_0+h]$, we have that

$$(\mathfrak{N}\phi_+)(t) = \epsilon\phi_2''(t) - \phi_2'(t) + \phi_2(t)(1-\phi_2(t_0)) \geq$$

$$\phi_2''(t) - \phi_2'(t) + \phi_2(t)(1-\phi_2(t-h)) \geq 0, \quad t \in [t_0, t_0+h].$$

Finally, since the inequality $(\mathfrak{N}\phi_+)(t) > 0,\ t \leq t_0$, is obvious, the proof of the lemma is completed. □

Remark 4 (An upper solution when $\lambda_1 = \lambda_2$)**.** We can not use ϕ_+ as an upper solution when $c = c^*(h),\ 1/e < h \leq\ h_1$. Moreover, in this case it is not difficult to show that ϕ_+ satisfies inequality (2.9) for all $r < \lambda_2$ sufficiently close to λ_2 and for large positive t.

2.5 Some comments on upper and lower solutions

2.5.1 Non-smooth solutions

The problem of smoothness of the lower (upper) solutions is an interesting and important aspect of the topic, see [8, 42]. As we have seen in the previous sections, C^1-smoothness condition can be rather restrictive even when a simple nonlinearity (the birth function) is considered. The above mentioned works [42] show that continuous and piece-wise C^1-continuous lower (upper) solutions $\phi_\pm$ still can be used if some sign conditions are fulfilled at the points of discontinuity of $\phi_\pm'$. Moreover, as we prove it below even discontinuous functions $\phi_\pm$ can be also used. We start with a simple result of the theory of impulsive systems [51] which can be viewed as a version of the Perron theorem for piece-wise continuous solutions, cf. [8].

Lemma 6. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a bounded classical solution of the second order* *impulsive equation*

$$\psi'' + a\psi' + b\psi = f(t), \quad \Delta\psi|_{t_j} = \alpha_j, \quad \Delta\psi'|_{t_j} = \beta_j,$$

where $\{t_j\}$ *is a finite increasing sequence,* $f : \mathbb{R} \to \mathbb{R}$ *is bounded and continuous at every* $t \neq t_j$ *and the operator* Δ *is defined by* $\Delta w|_{t_j} := w(t_j+) - w(t_j-)$*. Assume that* $z^2 + az + b = 0$ *has two positive roots* $0 < \lambda \le \mu$*. Then*

$$\text{if } \lambda < \mu \text{ we have that } \psi(t) = \frac{1}{\mu-\lambda}\int_t^{+\infty} \left(e^{\lambda(t-s)} - e^{\mu(t-s)}\right) f(s)ds$$
$$+\frac{1}{\mu-\lambda}\sum_{t<t_j}\left[\left(\lambda e^{\mu(t-t_j)} - \mu e^{\lambda(t-t_j)}\right)\alpha_j + \left(e^{\lambda(t-t_j)} - e^{\mu(t-t_j)}\right)\beta_j\right], \quad t \neq t_j; \tag{2.10}$$
$$\text{if } \lambda = \mu = -0.5a \text{ we have that } \psi(t) = \int_t^{+\infty}(s-t)e^{-0.5a(t-s)}f(s)ds +$$
$$\sum_{t<t_j} e^{-0.5a(t-t_j)}\left[(t_j - t)(\beta_j + 0.5a\alpha_j) - \alpha_j\right], \quad t \neq t_j.$$

Proof. See [51, Theorem 87]. □

Next, the corollary below shows that our lower solution is an upper solution in the sense of Wu and Zou [61]:

Corollary 2. *Assume that* $\psi : \mathbb{R} \to \mathbb{R}$ *is bounded and such that the derivatives* $\psi', \psi'' : \mathbb{R} \setminus \{t_j\} \to \mathbb{R}$ *exist and are bounded. Suppose also that* ψ *is a classical solution of the impulsive inequality*

$$\psi'' + a\psi' + b\psi \le f(t), \quad \Delta\psi|_{t_j} = \alpha_j, \quad \Delta\psi'|_{t_j} = \beta_j.$$

If $\alpha_j \ge 0, \ \beta_j \le 0$*, then*

$$\psi(t) \le \frac{1}{\mu-\lambda}\int_t^{+\infty}\left(e^{\lambda(t-s)} - e^{\mu(t-s)}\right) f(s)ds, \quad \text{when } \lambda < \mu,$$
$$\psi(t) \le \int_t^{+\infty}(s-t)e^{-0.5a(t-s)}f(s)ds, \quad \text{when } \lambda = \mu = -0.5a.$$

Proof. Suppose that $\lambda < \mu$, the case $\lambda = \mu$ is similar. Clearly, $q(t) := f(t) - (\psi''(t) + a\psi'(t) + b\psi(t)) \ge 0$ and $\lambda e^{\mu(t-t_j)} < \mu e^{\lambda(t-t_j)}, \ e^{\lambda(t-t_j)} > e^{\mu(t-t_j)}$ for $t < t_j$. Thus the desired inequality follows from (2.10). □

2.5.2 A lower solution when $\lambda_1 = \lambda_2,\ h \in (1/e, h_1]$

In Section 2.3, a lower C^1- solution was presented for the case when $\lambda_1 < \lambda_2$. However, to apply our iterative procedure in the critical case $\lambda_1 = \lambda_2$, we also need to construct a lower solution for the corresponding range of parameters. It is worth to mention that our approach does not require any upper solution once a lower solution is found and the existence of the heteroclinic is proved, see Corollary 4. Here, we provide a continuous and piece-wise analytic lower solution $\phi_-(t)$ if $\lambda_1 = \lambda_2$. Our solution has a unique singular point τ' where $\Delta\phi_-|_{\tau'} = 0, \quad \Delta\phi'_-|_{\tau'} > 0$. This shows that, in general, the sign conditions of Corollary 2 need not to be satisfied.

Take some positive $A > (e^{-\lambda_2 h} - 1)/h$ and let τ' be the positive root of the equation $At + 1 = e^{-\lambda_2 t}$. It is easy to see that $\tau' > h$. Consider the piece-wise smooth function $\phi_- : \mathbb{R} \to [0, 1)$ defined by

$$\phi_-(t) = \begin{cases} 0, & \text{if } t \le \tau', \\ 1 - (At+1)e^{\lambda_2 t}, & \text{if } t \ge \tau'. \end{cases} \tag{2.11}$$

Proposition 3. *The inequality $(\mathcal{K}\phi_-)(t) > \phi_-(t)$ holds for all $t \in \mathbb{R}$.*

Proof. Below, we are assuming that $h \neq h_1$ so that $\lambda < \mu$ and $\mathcal{K} = \mathcal{A}$; however, a similar argument works also in the case $h = h_1$ (when $\mathcal{K} = \mathcal{B}$). It suffices to prove that $(\mathcal{A}\phi_-)(t) > \phi_-(t)$ for $t \ge \tau'$. Let C^2- smooth function ψ be defined by

$$\psi(t) = \begin{cases} 1 - (At+1)e^{\lambda_2 t}, & \text{if } t \ge \tau' - h, \\ B(t), & \text{if } 0 \le t \le \tau' - h, \\ 0, & \text{if } t \le 0, \end{cases}$$

for some appropriate continuous decreasing $B(t)$. Set

$$\zeta(t) := \epsilon\psi''(t) - \psi'(t) + \psi(t)(1 - \psi(t-h)).$$

It is easy to check that $\zeta \in C(\mathbb{R},\mathbb{R})$ is bounded on $\mathbb{R}$ and $\zeta(t) < 0$ for all $t > \tau'$. But then, for all $t > \tau'$, we have that

$$\phi_-(t) = \psi(t) = (\mathcal{A}\psi)(t) + \frac{1}{\epsilon(\mu-\lambda)} \int_t^{+\infty} \left(e^{\lambda(t-s)} - e^{\mu(t-s)}\right) \zeta(s)ds < (\mathcal{A}\psi)(t) \leq$$

$$= \frac{1}{\epsilon(\mu-\lambda)} \int_t^{+\infty} (e^{\lambda(t-s)} - e^{\mu(t-s)})\phi_-(s)\phi_-(s-h)ds = (\mathcal{A}\phi_-)(t).$$

□

2.5.3 Ordering the upper and lower solutions

Finally, we show that the condition of the correct ordering $\phi_- \leq \phi_+$ is not at all restrictive provided that solutions $\phi_\pm$ are monotone and satisfy some natural asymptotic relations.

Lemma 7. *Assume that functions* $\phi_\pm : \mathbb{R} \to [0,1)$, $j = 1,2$, *are increasing and, for some fixed* $k \in \{0,1\}$, *the following holds*

$$\lim_{t\to-\infty} \phi_\pm(t)e^{-\lambda t} = \alpha_\pm, \ \lim_{t\to+\infty} (1-\phi_\pm(t))t^{-k}e^{-\lambda_2 t} = \beta_{\pm,k},$$

where $\beta_{\pm,k} > 0$, *and* $\alpha_- \in [0,+\infty)$, $\alpha_+ \in (0,+\infty]$. *Then there exists a real number* σ *such that* $\phi_-(t) < \phi_+(t+\sigma)$ *for all* $t \in \mathbb{R}$.

Proof. It is clear that $\phi_-(-\infty) = 0$ and $\phi_\pm(+\infty) = 1$. Let σ_0 be sufficiently large to satisfy $\beta_{+,k}e^{\lambda_2\sigma_0} < \beta_{-,k}, \ \alpha_-e^{-\lambda\sigma_0} < \alpha_+$. Then there exist t_1, t_2 such that $t_1 < t_2$ and $\phi_-(t-\sigma_0) < \phi_+(t), \ t \in \mathcal{I} := (-\infty, t_1] \cup [t_2, +\infty)$. Now, set $\sigma = \sigma_0 + (t_2 - t_1)$. Since both functions are increasing, we have

$$\phi_-(t-\sigma) \leq \phi_-(t-\sigma_0) < \phi_+(t), \quad t \in \mathcal{I},$$

$$\phi_-(t-\sigma) < \phi_+(t-(t_2-t_1)) \leq \phi_+(t), \quad t \in [t_1, t_2].$$

□

2.6 Proof of Theorem II.1

6.1. Necessity. Let $u(t,x) = \zeta(ct + \nu \cdot x)$ be a positive bounded monotone solution of the delayed KPP-Fisher equation. Then $\varphi(t) = \zeta(ct)$ satisfies

$$\epsilon\varphi''(t) - \varphi'(t) + \varphi(t)(1 - \varphi(t-h)) = 0, \quad t \in \mathbb{R}, \tag{2.12}$$
$$\epsilon\varphi'(t) = \epsilon\varphi'(0) - \varphi(0) + \varphi(t) + \int_0^t \varphi(s)(1 - \varphi(s-h))ds.$$

The latter relation implies that $\varphi(\pm\infty) \in \{0, 1\}$ since otherwise $\varphi'(\pm\infty) = \infty$. Hence $\varphi : \mathbb{R} \to (0,1)$. Let $\phi \in C^2(\mathbb{R}, (0,1))$ be an arbitrary solution of (2.12). Suppose for a moment that $\phi'(t_0) = 0$. Then necessarily $\phi''(t_0) < 0$ so that t_0 is the unique critical point (absolute maximum) of ϕ. But then $\phi'(s) < 0$ for $s > t_0$, so that $\phi''(s) < 0,\ s \geq t_0$, which yields the contradiction $\phi(+\infty) = -\infty$. In consequence, either $\phi'(s) > 0$ or $\phi'(s) < 0$ for all $s \in \mathbb{R}$. But as we have seen, $\phi'(s) < 0$ implies $\phi(+\infty) = -\infty$, a contradiction. Hence, any solution $\phi \in C^2(\mathbb{R}, (0,1))$ of (2.12) satisfies $\phi'(t) > 0,\ \phi(-\infty) = 0,\ \phi(+\infty) = 1$.

Lemma 8. *If $\phi \in C^2(\mathbb{R}, (0,1))$ satisfies (2.12), then $\epsilon \in (0, 0.25]$.*

Proof. Suppose for a moment that $\epsilon > 0.25$. Then the characteristic equation $\epsilon\lambda^2 - \lambda + 1 = 0$ associated with the trivial steady state of (2.12) has two simple complex conjugate roots $\omega_\pm = (2\epsilon)^{-1}(1 \pm i\sqrt{4\epsilon - 1})$.

Since $\phi \in C^2(\mathbb{R}, (0,1))$ is a solution of (2.12), it holds that $\phi'(t) > 0,\ t \in \mathbb{R}$, $\phi(-\infty) = 0$. Set $z(t) = (\phi(t), \phi'(t))^T$, it is easy to check that $z(t)$ satisfies the following asymptotically autonomous linear differential equation

$$z'(t) = (A + R(t))z(t),\ t \in \mathbb{R}, \quad A = \begin{pmatrix} 0 & 1 \\ -1/\epsilon & 1/\epsilon \end{pmatrix},\ R(t) = \begin{pmatrix} 0 & 0 \\ \phi(t-h)/\epsilon & 0 \end{pmatrix}.$$

Since $R(-\infty) = 0,\ \int_{-\infty}^0 |R'(t)|dt = \phi(-h)$ and the eigenvalues $\omega_\pm$ of A are complex conjugate, we can apply the Levinson theorem [16, Theorem 1.8.3] to obtain the following asymptotic formulas at $t = -\infty$:

$$\phi(t) = (a + o(1))e^{t/(2\epsilon)} \cos(t\sqrt{4\epsilon - 1}(1 + o(1)) + b + o(1)),$$

$$\phi'(t) = (c + o(1))e^{t/(2\epsilon)} \sin(t\sqrt{4\epsilon - 1}(1 + o(1)) + d + o(1)),$$

where $a^2 + c^2 \neq 0$. But this means that either $\phi(t)$ or $\phi'(t)$ is oscillating around zero, a contradiction. □

Lemma 9. *If $h > h_1$ or $h \in (1/e, h_1]$ and $c > c^*(h)$ then Eq. (2.12) does not have any solution $\phi \in C^2(\mathbb{R}, (0, 1))$.*

Proof. On the contrary, let us assume that Eq. (2.12) has a solution $\phi \in C^2(\mathbb{R}, (0, 1))$. Then Lemma 8 implies that $\epsilon \in (0, 0.25]$ and therefore the assumptions of this lemma imply that $\psi(z, \epsilon)$ does not have negative zeros. Following the approach in [54], we will show that this will force $\phi(t)$ to oscillate about the positive equilibrium. For the convenience of the reader, the proof is divided in several steps.

Claim I: $y(t) := 1 - \phi(t) > 0$ has at least exponential decay as $t \to +\infty$. First, observe that

$$\epsilon y''(t) - y'(t) = \phi(t)y(t - h), \quad t \in \mathbb{R}. \tag{2.13}$$

Therefore, with $\gamma := \phi(t_0)$, which is close to 1, and $g(t) := \phi(t)y(t - h) - \phi(t_0)y(t)$, we obtain that

$$\epsilon y''(t) - y'(t) - \gamma y(t) - g(t) = 0, \ t \in \mathbb{R}.$$

Note that $g(t) > 0$ for all sufficiently large t. Since $y(t), g(t)$ are bounded on $\mathbb{R}$, it holds that

$$y(t) = -\frac{1}{\epsilon(m - l)} \left(\int_{-\infty}^{t} e^{l(t-s)} g(s)ds + \int_{t}^{+\infty} e^{m(t-s)} g(s)ds \right),$$

where $l < 0$ and $0 < m$ are roots of $\epsilon z^2 - z - \gamma = 0$. The latter representation of $y(t)$ implies that there exists T_0 such that

$$y'(t) - ly(t) = -\frac{1}{\epsilon}\int_t^{+\infty} e^{m(t-s)}g(s)ds < 0,\ t \geq T_0. \tag{2.14}$$

Hence, $(y(t)\exp(-lt))' < 0,\ t \geq T_0$, and therefore

$$y(t) \leq y(s)e^{l(t-s)}, \quad t \geq s \geq T_0, \quad g(t) = O(e^{lt}),\ t \to +\infty. \tag{2.15}$$

It is easy to see that these estimates are valid for every negative $l > (2\epsilon)^{-1}(1 - \sqrt{1+4\epsilon})$. Finally, (3.25), (3.26) imply that $y'(t) = O(e^{lt}),\ t \to +\infty$.

Claim II: $y(t) := 1 - \phi(t) > 0$ *is not superexponentially small as* $t \to +\infty$.

We already have proved that $y(t)$ is strictly decreasing and positive on $\mathbb{R}$. Since the right hand side of Eq. (3.24) is positive and integrable on $\mathbb{R}_+$, and since $y(t)$ is a bounded solution of (3.24) satisfying $y(+\infty) = 0$, we find that

$$y(t) = \int_t^{+\infty}(1 - e^{(t-s)/\epsilon})\phi(s)y(s-h)ds. \tag{2.16}$$

As a consequence, there exists T_1 such that

$$y(t) \geq 0.5(1 - e^{-0.5h/\epsilon})\int_{t-0.5h}^{t} y(s)ds := \xi\int_{t-0.5h}^{t} y(s)ds, \quad t \geq T_1 - h.$$

Now, since $y(t) > 0$ for all t, we can find positive C, ρ such that $y(s) > Ce^{-\rho s}$ for all $s \in [T_1 - h, T_1]$. We can assume that ρ is large enough to satisfy the inequality $\xi(e^{0.5\rho h} - 1) > \rho$. Then we claim that $y(s) > Ce^{-\rho s}$ for all $s \geq T_1 - h$. Conversely, suppose that $t' > T_1$ is the leftmost point where $y(t') = Ce^{-\rho t'}$. Then we get a contradiction:

$$y(t') \geq \xi\int_{t'-0.5h}^{t'} y(s)ds > C\xi\int_{t'-0.5h}^{t'} e^{-\rho s}ds = Ce^{-\rho t'}\xi\frac{e^{0.5\rho h} - 1}{\rho} > Ce^{-\rho t'}.$$

Claim III: $y(t) > 0$ *can not hold when* $\psi(z,\epsilon)$ *does not have any zero in* $(-\infty, 0)$. Observe that $y(t) = 1 - \phi(t)$ satisfies

$$\epsilon y''(t) - y'(t) - (1 - y(t))y(t-h) = 0,\ t \in \mathbb{R},$$

where in virtue of Claim I, it holds that $(y(t), y'(t)) = O(lt)$ at $t = +\infty$. Then [44, Proposition 7.2] implies that there exists $\gamma < l$ such that $y(t) = v(t) + O(\exp(\gamma t))$,

$t \to +\infty$, where v is a *non empty* (due to Claim II) finite sum of eigensolutions of the limiting equation

$$\epsilon y''(t) - y'(t) - y(t-h) = 0, \ t \in \mathbb{R},$$

associated to the eigenvalues $\lambda_j \in F = \{\gamma < \Re\lambda_j \le l\}$. Now, since the set F does not contain any real eigenvalue by our assumption, we conclude that $y(t)$ should be oscillating on $\mathbb{R}_+$, a contradiction. □

6.2. Sufficiency. Suppose that $\epsilon \in (0, 0.25]$ and let $0 < \lambda \le \mu$ be the roots of the equation $\epsilon z^2 - z + 1 = 0$. In Lemmas 10-13 below, $\mathcal{K}$ stands either for $\mathcal{A}$ or $\mathcal{B}$ (defined by (2.5), (2.6)).

Lemma 10. *If $\phi, \psi \in C(\mathbb{R}, (0,1))$ and $\phi(t) \le \psi(t)$ for all $t \in \mathbb{R}$, then $\mathcal{K}\phi, \mathcal{K}\psi \in C(\mathbb{R}, (0,1))$ and $(\mathcal{K}\phi)(t) \le (\mathcal{K}\psi)(t), \ t \in \mathbb{R}$. Moreover, if ϕ is increasing then $\mathcal{K}\phi$ is also increasing.*

Proof. The proof is straightforward. □

Lemma 11. *Let $\epsilon \in (0, 0.25]$. If $\phi_+ \in C^1(\mathbb{R}, (0,1))$ satisfies the inequality*

$$\epsilon\phi''(t) - \phi'(t) + \phi(t)(1 - \phi(t-h)) \ge 0$$

for all $t \in \mathbb{R}' := \mathbb{R} \backslash \{T_1, \dots, T_m\}$ and $\phi_+''(t), \phi_+'(t)$ are bounded on $\mathbb{R}'$, then $(\mathcal{K}\phi_+)(t) \le \phi_+(t)$ for all $t \in \mathbb{R}$.

Proof. If $\omega(T_i) := 0$ and

$$\omega(t) := \epsilon\phi_+''(t) - \phi_+'(t) + \phi_+(t)(1 - \phi_+(t-h)), \quad t \in \mathbb{R}' = \mathbb{R} \setminus \{T_1, \dots, T_m\}$$

then $\omega(t) \ge 0$ for all $t \in \mathbb{R}'$, $\omega(t)$ is bounded on $\mathbb{R}'$ and

$$\epsilon\phi_+''(t) - \phi_+'(t) + \phi_+(t) = \omega_1(t), \quad t \in \mathbb{R}',$$

where $\omega_1(t) := \omega(t) + \phi_+(t)\phi_+(t-h)$ is bounded on $\mathbb{R}'$. Let now $\epsilon \in (0, 0.25)$. By Lemma 6, we obtain that

$$\begin{aligned}\phi_+(t) &= \frac{1}{\epsilon(\mu-\lambda)}\int_t^{+\infty}\left(e^{\lambda(t-s)} - e^{\mu(t-s)}\right)\omega_1(s)ds = \\ &(\mathcal{A}\phi_+)(t) + \frac{1}{\epsilon(\mu-\lambda)}\int_t^{+\infty}\left(e^{\lambda(t-s)} - e^{\mu(t-s)}\right)\omega(s)ds \geq (\mathcal{A}\phi_+)(t).\end{aligned}$$

The case $\epsilon = 0.25$ (which corresponds to $\mathcal{K} = \mathcal{B}$) is completely analogous to the previous one. □

The proof of the next lemma is similar to that of Lemma 11:

Lemma 12. *Let $\epsilon \in (0, 0.25]$. If $\phi_- \in C^1(\mathbb{R}, (0,1))$ satisfies the inequality*

$$\epsilon\phi''(t) - \phi'(t) + \phi(t)(1 - \phi(t-h)) \leq 0$$

for all $t \in \mathbb{R} \setminus \{T_1, \dots, T_m\}$ and $\phi''_-(t), \phi'_-(t)$ are bounded on $\mathbb{R} \setminus \{T_1, \dots, T_m\}$, then $(\mathcal{K}\phi_-)(t) \geq \phi_-(t)$ for all $t \in \mathbb{R}$.

Set $\phi^{\pm}_{j+1} := (\mathcal{K}\phi^{\pm}_j),\ j \geq 0,\ \phi^{\pm}_0 := \phi_\pm$, and let the increasing functions $\phi_- \leq \phi_+$ be as in Lemmas 11, 12. Then

$$\phi_- \leq \phi_1^- \leq \dots \leq \Phi_- \leq \Phi_+ \leq \dots \phi_j^- \dots \leq \dots \phi_1^+ \leq \phi_+,$$

where $\Phi_\pm(t) = \lim_{j\to\infty}\phi^\pm_j(t)$ pointwise and $\phi^\pm_j$ are increasing (by Lemma 10).

Lemma 13. *$\Phi_\pm$ are wavefronts and $\Phi_\pm(t) = \lim_{j\to\infty}\phi^\pm_j(t)$ uniformly on $\mathbb{R}$.*

Proof. Applying the Lebesgue's dominated convergence theorem to $\phi^-_{j+1} := \mathcal{K}\phi^-_j$, we obtain that $\Phi_-(t) = (\mathcal{K}\Phi_-)(t)$. Differentiating this equation twice with respect to t, we deduce that $\Phi_- : \mathbb{R} \to (0,1)$ is a C^2-solution of (2.12) (and thus $\Phi'_-(t) > 0$). As a consequence of the Dini's theorem, we have that $\Phi_-(t) = \lim_{j\to\infty}\phi^-_j(t)$ uniformly on compact sets. Since Φ_-, ϕ^-_j are asymptotically constant and increasing, this convergence is uniform on $\mathbb{R}$. The proof for Φ_+ is similar. □

Corollary 3. *Eq. (2.3) has a monotone wavefront* $u(x,t) = \zeta(x\cdot\nu + ct)$, $|\nu| = 1$, *connecting* 0 *with* 1 *if one of the following conditions holds*

1. $0 \le h \le 1/e$ *and* $2 \le c$;

2. $1/e < h < \ h_1$ *and* $2 \le c < c^*(h)$.

Proof. It is an immediate consequence of Lemmas 4, 5, 7, 11-13. □

If $c = c^*(h)$, the reasoning of the last proof does not apply because of the lack of explicit upper solutions. Below, we follow an idea from [54, Section 6]:

Lemma 14. *Eq. (2.3) has a positive monotone wavefront* $u(x,t) = \zeta(x\cdot\nu + ct)$, $|\nu| = 1$, *connecting* 0 *with* 1 *if* $1/e < h \le \ h_1$ *and* $c = c^*(h)$.

Proof. Case I. Fix some $h \in (1/e, h_1)$ and $\epsilon = \epsilon^*(h)$. Then there exists a decreasing sequence $\epsilon_j \downarrow \epsilon^*(h)$ such that Eq. (2.12) has at least one monotone positive heteroclinic solution $\phi_j(t)$ normalized by $\phi_j(0) = 0.5$. It is clear that $\phi_j(t) = (\mathcal{A}\phi_j)(t)$. Moreover, each $y_j(t) := 1 - \phi_j(t) > 0$ solves (2.16) so that

$$|\phi_j'(t)| = |\frac{1}{\epsilon}\int_t^{+\infty} e^{(t-s)/\epsilon}\phi(s)(1-\phi_j(s-h))ds| \le 1,\ t \in \mathbb{R}.$$

Thus, by the Ascoli-Arzelà theorem combined with the diagonal method, $\{\phi_j\}$ has a subsequence $\{\phi_{j_k}\}$ converging (uniformly on compact subsets of $\mathbb{R}$) to some continuous non-decreasing non-negative function ϕ_*, $\phi_*(0) = 0.5$. Applying the Lebesgue's dominated convergence theorem to $\phi_{j_k}(t) = (\mathcal{A}\phi_{j_k})(t)$, we find that ϕ_* is also a fixed point of $\mathcal{A}$. Hence, $\phi_* : \mathbb{R} \to [0,1]$ is a monotone solution of Eq. (2.12) considered with $\epsilon = \epsilon^*(h)$. Since $\phi_*(0) = 0.5$, $\phi_* : \mathbb{R} \to (0,1)$ is actually a monotone wavefront.

Case II. Finally, let $\epsilon = 0.25$ and $h = h_1$. This case can be handled exactly in the same way as Case I if we keep $\epsilon = 0.25$ fixed, replace $\mathcal{A}$ with $\mathcal{B}$, and take some increasing sequence $h_j \uparrow h_1$ instead of $\epsilon_j \downarrow \epsilon^*(h)$. □

Corollary 4. *Assume that $c = c^*(h)$, $1/e < h \le h_1$, and let ϕ_- be as in (2.11). If A is sufficiently large, then*

$$\phi_- \le \phi_1^- \le \dots \le \phi_j^- \dots \le \Phi = \mathcal{K}\Phi,$$

where Φ is a wavefront and $\Phi(t) = \lim_{j\to\infty} \phi_j^-(t)$ uniformly on $\mathbb{R}$.

Proof. If $c = c^*(h)$ we will take the heteroclinic solution Φ_- whose existence was established in Lemma 14 as an upper solution. Due to (2.8), we can assume that

$$\beta_{+,1} := \lim_{t\to+\infty} (1 - \Phi_-(t))t^{-1}e^{-\lambda_2 t} > 0.$$

Next, let ϕ_- be defined by (2.11). Since $\alpha_- := \lim_{t\to-\infty} \phi_-(t)e^{-\lambda t} = 0$ and

$$\beta_{-,1} := \lim_{t\to+\infty} (1 - \phi_-(t - \frac{1}{A}))t^{-1}e^{-\lambda_2 t} = Ae^{-\lambda_2/A} > \beta_{+,1},$$

for sufficiently large A, Lemma 7 implies that $\phi_-(t) < \Phi_-(t + \sigma)$, $t \in \mathbb{R}$, for some σ. Finally, it suffices to take $\phi_+(t) := \Phi_-(t + \sigma)$ and repeat the proof of Lemma 13. □

6.3. Uniqueness. The uniqueness of wavefronts is an important and interesting topic. Among the most influential contributions to it, we would like to mention the seminal papers [14] and [11], e.g. see [12, 49, 59]. In particular, our method of proof follows a nice idea due to Diekmann and Kaper, see [14, Theorem 6.4]. Suppose that $c \ne c^*(h)$ and let ϕ_1, ϕ_2 be two different (modulo translation) profiles of wavefronts propagating at the same speed c. Due to Theorem II.2, we may assume that ϕ_1, ϕ_2 have the same asymptotic representation $\phi_j(t) = 1 - e^{\lambda_2 t}(1 + o(1))$ at $+\infty$. Moreover, $\phi_j = \mathcal{K}\phi_j$, where $\mathcal{K} = \mathcal{A}$ if $c > 2$ and $\mathcal{K} = \mathcal{B}$ if $c = 2$. Set $\omega(t) := |\phi_2(t) - \phi_1(t)|e^{-\lambda_2 t}$. Then $\omega(\pm\infty) = 0$, $\omega(t) \ge 0$, $t \in \mathbb{R}$, and $\omega(\tau) = \max_{s\in\mathbb{R}} \omega(s) := |\omega|_0 > 0$ for some τ. From the identity $\phi_2 - \phi_1 = \mathcal{K}\phi_2 - \mathcal{K}\phi_1$, we deduce that

$$\omega(\tau) < \frac{e^{-\lambda_2\tau}}{\epsilon(\mu - \lambda)} \int_\tau^{+\infty} (e^{\lambda(\tau-s)} - e^{\mu(\tau-s)})(\omega(s)e^{\lambda_2 s} + \omega(s-h)e^{\lambda_2(s-h)})ds <$$

$$\frac{|\omega|_0 e^{-\lambda_2 \tau}}{\epsilon(\mu-\lambda)} \int_\tau^{+\infty} (e^{\lambda(\tau-s)} - e^{\mu(\tau-s)})(e^{\lambda_2 s} + e^{\lambda_2(s-h)})ds = |\omega|_0 = \omega(\tau), \text{ if } c > 2;$$

$$\omega(\tau) < 4e^{-\lambda_2 \tau} \int_\tau^{+\infty} (s-\tau)e^{\lambda(\tau-s)}(\omega(s)e^{\lambda_2 s} + \omega(s-h)e^{\lambda_2(s-h)})ds <$$

$$4|\omega|_0 e^{-\lambda_2 \tau} \int_\tau^{+\infty} (s-\tau)e^{\lambda(\tau-s)}(e^{\lambda_2 s} + e^{\lambda_2(s-h)})ds = |\omega|_0 = \omega(\tau), \quad \text{if } c = 2,$$

which is impossible. Hence, $|\omega|_0 = 0$ and the proof is complete.

2.7 Proof of Theorem II.2

First, using the bilateral Laplace transform $(\mathcal{L}y)(z) := \int_{\mathbb{R}} e^{-sz} y(s)ds$ (see e.g. [60]), we extend [44, Proposition 7.1] (see also [3, Lemma 4.1] and [54, Lemma 22]) for the case $J = \mathbb{R}$.

Lemma 15. *Set* $\chi(z) := z^2 + \alpha z + \beta + pe^{-zh}$ *and let* $y \in C^2(\mathbb{R}, \mathbb{R})$ *satisfy*

$$y''(t) + \alpha y'(t) + \beta y(t) + py(t-h) = f(t), \ t \in \mathbb{R}, \tag{2.17}$$

where $\alpha, \beta, p, h \in \mathbb{R}$ *and*

$$y(t) = \begin{cases} O(e^{-Bt}), & \text{as } t \to +\infty, \\ O(e^{bt}), & \text{as } t \to -\infty; \end{cases} \quad f(t) = \begin{cases} O(e^{-Ct}), & \text{as } t \to +\infty, \\ O(e^{ct}), & \text{as } t \to -\infty, \end{cases} \tag{2.18}$$

for some non-negative $b < c, B < C, b + B > 0$. *Then, for each sufficiently small* $\sigma > 0$, *it holds that*

$$y(t) = \begin{cases} w_+(t) + e^{-(C-\sigma)t}o(1), & \text{as } t \to +\infty, \\ w_-(t) + e^{(c-\sigma)t}o(1), & \text{as } t \to -\infty, \end{cases}$$

where

$$w_\pm(t) = \pm \sum_{\lambda_j \in F_\pm} \mathrm{Res}_{z=\lambda_j} \left[\frac{e^{zt}}{\chi(z)} \int_{\mathbb{R}} e^{-zs} f(s)ds \right]$$

is a finite sum of eigensolutions of equation (2.17) associated to the eigenvalues $\lambda_j \in F_+ = \{-C + \sigma < \Re\lambda_i \le -B\}$ *and* $\lambda_j \in F_- = \{b \le \Re\lambda_i < c - \sigma\}$.

Proof. We will divide our proof into several parts.

Step I. We claim that there exist non-negative B', b' such that $B' \le B, b' \le b$, $B' + b' > 0$ and

$$y'(t), y''(t) = \begin{cases} O(te^{-B't}), & \text{as } t \to +\infty, \\ O(te^{b't}), & \text{as } t \to -\infty. \end{cases} \tag{2.19}$$

We will distinguish two cases:

Case A. Suppose that $\alpha = 0$. Then clearly $y''(t) = O(e^{-Bt})$ at $t = -\infty$, is bounded on $\mathbb{R}$ and therefore $y'(t)$ is uniformly continuous on $\mathbb{R}$. Since $B + b > 0$ then either $y(+\infty) = 0$, $\limsup_{s\to-\infty} |y(s)| < \infty$ or $y(-\infty) = 0$, $\limsup_{s\to+\infty} |y(s)| < \infty$. Suppose, for example that $B > 0$ (hence $y(+\infty) = 0$), the other case being similar. Then, applying the Barbalat lemma, see e.g. [61], we find that $y'(+\infty) = 0$. This implies that $y'(t) = -\int_t^{+\infty} y''(s)ds = O(e^{-Bt})$ at $t = +\infty$. Thus we may set $B' = B$. Now, $y'(t) = y'(0) + \int_0^t y''(s)ds = O(t)$ at $t = -\infty$ so that we can choose $b' = 0$.

Case B. Let now $\alpha \neq 0$. For example, suppose that $\alpha > 0$ (the case $\alpha < 0$ is similar). Then, for some ξ,

$$y'(t) = \xi e^{-\alpha t} + \int_{-\infty}^{t} e^{-\alpha(t-s)}\{f(s) - \beta y(s) - py(s-h)\}ds.$$

In fact, since the second term of the above formula is bounded on $\mathbb{R}$ and we can not have $y'(-\infty) = \pm\infty$ (due to the boundedness of $y(t)$), we obtain that $\xi = 0$. But then $y'(t) = O(e^{bt})$, $t \to -\infty$ and $y'(t) = O(te^{-\min\{\alpha,B\}t})$, $t \to +\infty$. Note that $b' + B' = \min\{\alpha + b, B + b\} > 0$. Finally, (2.17) assures that (2.19) is also valid for $y''(t)$.

Step II. Applying the bilateral Laplace transform $\mathcal{L}$ to (2.17), we obtain that $\chi(z)\tilde{y}(z) = \tilde{f}(z)$, where $\tilde{y} = \mathcal{L}y$, $\tilde{f} = \mathcal{L}f$ and $-B' < \Re z < b'$. Moreover, from the growth restrictions (2.18), we conclude that $\tilde{y}$ is analytic in $-B < \Re z < b$ while $\tilde{f}$ is analytic in $-C < \Re z < c$. As a consequence, $H(z) = \tilde{f}(z)/\chi(z)$ is analytic in $-B < \Re z < b$ and meromorphic in $-C < \Re z < c$. Observe that $H(z) = O(z^{-2}),\ z \to \infty$, for each fixed strip $\Pi(s_1, s_2) = \{s_1 \le \Re z \le s_2\}$, $-C < s_1 < s_2 < c$. Now, let $\sigma > 0$ be such that the vertical strips $c - 2\sigma < \Re z < c$ and $-C < \Re z < -C + 2\sigma$ do not contain any zero of $\chi(z)$. By the inversion formula [60, Theorem 5a], for each $\delta \in (-B, b)$, we obtain that

$$y(t) = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} e^{zt}\tilde{y}(z)dz = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} e^{zt}H(z)dz = w_\pm(t) + u_\pm(t),\ t \in \mathbb{R},$$

$$\text{where } w_\pm(t) = \pm\sum_{\lambda_j \in F_\pm} \mathrm{Res}_{z=\lambda_j}\frac{e^{zt}\tilde{f}(z)}{\chi(z)},\ \ u_\pm(t) = \frac{1}{2\pi i}\int_{\mp(c-\sigma)-i\infty}^{\mp(c-\sigma)+i\infty} e^{zt}H(z)dz.$$

The above sum is finite, since $\chi(z)$ has a finite set of the zeros in $F_\pm$. Now, for $a(s) = H(\mp(c-\sigma) + is)$, we obtain that

$$u_\pm(t) = \frac{e^{\mp(c-\sigma)t}}{2\pi}\left\{\int_{\mathbb{R}} e^{ist}a(s)ds\right\},\ t \in \mathbb{R}.$$

Next, since $a \in L_1(\mathbb{R})$, we have, by the Riemann-Lebesgue lemma, that

$$\lim_{t\to\infty}\int_{\mathbb{R}} e^{ist}a_1(s)ds = 0.$$

Thus we get $u_\pm(t) = e^{\mp(c-\sigma)t}o(1)$ at $t = \infty$, and the proof is completed. □

Now we can prove Theorem II.2:

Theorem II.2. <u>Case I: asymptotics at $t = +\infty$</u>. It follows from (3.26) that $y(t) = 1 - \phi(t)$ satisfies $y(t) = O(e^{lt}),\ t \to +\infty$, for every negative $l > (2\epsilon)^{-1}(1 - \sqrt{1 + 4\epsilon})$. Moreover, $f(t) := -y(t)y(t-h) = O(e^{2lt}), t \to +\infty,\ y(t) = O(1),\ t \to -\infty$ and

$$\epsilon y''(t) - y'(t) - y(t-h) = -y(t)y(t-h),\ t \in \mathbb{R}.$$

Therefore Lemma 15 implies that, for every small $\sigma > 0$,

$$y(t) = \sum_{2l+\sigma < \Re\lambda_j \le l} \mathrm{Res}_{z=\lambda_j}\frac{e^{zt}\tilde{f}(z)}{\chi(z)} + e^{(2l+\sigma)t}o(1),\ t \to +\infty.$$

Now, observe that $(2\epsilon)^{-1}(1-\sqrt{1+4\epsilon}) > \lambda_2$ so that either $\lambda_2 \in (2l+\sigma, l)$ or $\lambda_2 \leq 2l$. In the latter case, we obtain $y(t) = e^{(2l+\sigma)t}o(1),\ t \to +\infty$, which allows to repeat the above procedure till the inclusion $\lambda_2 \in (2^j l+\sigma, 2^{j-1}l)$ is reached for some integer j. In this way, assuming that $\lambda_1 < \lambda_2$, for each small $\sigma > 0$, we find that

$$y(t) = \eta e^{\lambda_2 t} + O(e^{(\lambda_2-\sigma)t}), \text{ where } \eta := \frac{\int_{\mathbb{R}} e^{-\lambda_2 s} y(s)y(s-h)ds}{-\chi'(\lambda_2)} > 0. \tag{2.20}$$

Now, if $c = c^*(h)$ (i.e. $\lambda_1 = \lambda_2$), we obtain analogously that

$$y(t+t_0) = \xi t e^{\lambda_2 t} + O(e^{(\lambda_2-\sigma)t}),\ t \to +\infty,$$

for some appropriate t_0 and $\xi > 0$.

Suppose now that $h \in (0, h_0],\ c \leq c^{\#}(h)$. Then Lemmas 2, 3 imply that $\Re\lambda_j < \lambda_1 \leq 2\lambda_2$. This means that formula (2.20) can be improved as follows:

$$y(t) = \eta e^{\lambda_2 t} + O(e^{(2\lambda_2+\sigma)t}),\ t \to +\infty.$$

Finally, if $h \in (0.5\ln 2, h_0]$ and $c \in (c^{\#}(h), c^*(h))$, it holds that $2\lambda_2 < \lambda_1 < \lambda_2$. Then

$$\begin{aligned} y(t) \quad &= \sum_{2\lambda_2+\sigma<\Re\lambda_j\leq\lambda_2} \mathrm{Res}_{z=\lambda_j} \frac{e^{zt}\tilde{f}(z)}{\chi(z)} + e^{(2\lambda_2+\sigma)t}o(1) = \\ &\eta e^{\lambda_2 t} + \theta e^{\lambda_1 t} + e^{(\lambda_1-\sigma)t}o(1), \text{ where } \theta := \frac{\int_{\mathbb{R}} e^{-\lambda_1 s} y(s)y(s-h)ds}{-\chi'(\lambda_1)} < 0. \end{aligned}$$

<u>Case II: asymptotics at $t = -\infty$.</u> This case is much easier to analyze since the characteristic polynomial $\epsilon z^2 - z + 1$ of the variational equation

$$\epsilon y''(t) - y'(t) + y(t) = 0,\ \epsilon \in (0, 0.25], \tag{2.21}$$

along the trivial equilibrium of (2.12) has only two real zeros $0 < \lambda \leq \mu$. It is easy to check that $2\lambda \leq \mu$ if and only if $c \geq 1.5\sqrt{2} = 2.121\dots$.

Since $\phi(-\infty) = 0$ and equation (2.21) is exponentially unstable on $\mathbb{R}_-$, we conclude that the perturbed equation

$$\epsilon y''(t) - y'(t) + y(t)(1 - \phi(t-h)) = 0$$

is also exponentially unstable on $\mathbb{R}_-$ (e.g. see [13]). As a consequence, $\phi(t) = O(e^{mt}),\ t \to -\infty$, for some $m > 0$. Now we can proceed as in Case I, since

$$\epsilon\phi''(t) - \phi'(t) + \phi(t) = f_1(t),$$

with $f_1(t) := \phi(t)\phi(t-h) = O(e^{2mt})$. The details are left to the reader. $\square$

CHAPTER III

Global continuation of monotone wavefronts

3.1 Introduction and main result

The aim of this paper is to obtain efficient criteria of existence of monotone travelling waves $u = \phi(\nu \cdot x + ct)$, $\phi(-\infty) = 0$, $\phi(+\infty) = \kappa > 0$, for the non-quasi-monotone functional reaction-diffusion equations

$$(3.1) \qquad u_t(t, x) = \Delta u(t, x) + f(u(t, x), u(t - h, x)),\ u \geq 0,\ x \in \mathbb{R}^m,$$

in that case when the function $g(x) := f(x, x)$ is of non-degenerate monostable type: $g(0) = g(\kappa) = 0$, $g'(0) > 0, g'(\kappa) < 0$, and $g(x) > 0$ for $x \in (0, \kappa)$. Here $\nu \in \mathbb{R}^m$ is a fixed unit vector, $c > 0$ is the propagation speed and $h \geq 0$ is the delay. Henceforth we will assume that f is $C^{1,\gamma}$-smooth function, $\gamma \in (0, 1]$.

There is a long list of studies that consider the wavefront existence for equation (3.1) either with or without delays, let us mention here only several of them: [5, 20, 25, 26, 33, 38, 40, 43, 55, 56, 61, 64]. The problem is quite well understood when $h = 0$. In particular, there exists $c_*^{\mathfrak{N}} > 0$ (called the minimal speed of propagation) such that, for every $c \geq c_*^{\mathfrak{N}}$, equation (3.1) has exactly one wavefront $u = \phi(\nu \cdot x + ct)$, see [25, Theorems 8.3(ii) and 8.7] or [40, 55]. In addition, (3.1) does not have any front propagating at the velocity $c < c_*^{\mathfrak{N}}$. There are several variational principles describing $c_*^{\mathfrak{N}}$ [6, 25]. If $g(x) \leq g'(0)x$, $x \geq 0$, then $c_*^{\mathfrak{N}} = 2\sqrt{g'(0)}$. In general, however, simple analytical formulas for $c_*^{\mathfrak{N}}$ are not available. The profile ϕ is necessarily strictly increasing [25, Theorem 2.39] and the following asymptotic formulae are valid [55] for $c > c_*^{\mathfrak{N}}$ and appropriate $s_j = s_j(c, \phi),\ \sigma > 0$:

$$\begin{aligned} (\phi, \phi')(t + s_0, c) &= e^{\lambda(c)t}(1, \lambda(c)) + O(e^{(\lambda(c)+\sigma)t}),\ t \to -\infty, \\ (\phi, \phi')(t + s_1, c) &= (\kappa, 0) - e^{\lambda_2(c)t}(1, \lambda_2(c)) + O(e^{(\lambda_2(c)-\sigma)t}),\ t \to +\infty. \end{aligned} \tag{3.2}$$

Here $\lambda(c)$ [respectively, $\lambda_2(c)$] is the closest to 0 positive [respectively, negative] zero of the characteristic polynomial $z^2 - cz + g'(0)$ [respectively, $z^2 - cz + g'(\kappa)$].

However, when $h > 0$, there are numerous gaps in our knowledge about the wavefronts of equation (3.1). As for now, neither of the questions concerning the existence, uniqueness, geometric shape of fronts has been completely answered even for such quite studied models as the Nicholson's blowflies diffusive equation [2, 43, 46, 56, 64] and the KPP-Fisher delayed equation [5, 7, 10, 19, 20, 26, 38]. An additional complication appearing in the delayed case is the possible non-monotonicity of wavefronts [5, 7, 56]. But even the existence of *monotone* fronts is usually proved only under the quasi-monotonicity assumption on $f(u, v)$. In particular, it is an open problem whether the minimal speed of propagation $c_*^{\mathfrak{N}} > 0$ for (3.1) can be well defined in the situation when $f(u, v)$ is not quasi-monotone and is not dominated by its linear part at $(0, 0)$ (cf. [55] and Lemma 19 below). In fact, even in the case of quasi-monotone nonlinearities, $c_*^{\mathfrak{N}} > 0$ was defined in full generality only very recently, in the fundamental contribution [40] by X. Liang and X.-Q. Zhao. Another example: due to the relatively 'bad' monotonicity properties of $f(u, v) = u(1 - v)$, an efficient criterion of existence of monotone wavefronts to the delayed KPP-Fisher equation was obtained just a few years ago [19, 26, 38] (in Section 3.2, we present a significant extension of this result). For the Nicholson's blowflies equation where

$f(u, v) = -u + pve^{-v}, p > e$, the similar question was not yet answered: in Section 3.2, we present a complete solution to the existence problem when $p \in (e, e^2]$ and we describe partially this solution when $p > e^2$.

Now, there are very few approaches which can be used to address the wavefront existence for equation (3.1). It should be noted that the profile ϕ of travelling front $u(t, x) = \phi(\nu \cdot x + ct)$, $\phi(-\infty) = 0$, $\phi(+\infty) = \kappa > 0$, defines a heteroclinic solution of the delay differential equation

$$\phi''(t) - c\phi'(t) + f(\phi(t), \phi(t - ch)) = 0, \; t \in \mathbb{R}. \tag{3.3}$$

Therefore the phase plane analysis, which is usually invoked in the non-delayed case, does not work when $h > 0$ because of the infinite dimension of phase spaces associated to equation (3.3). As a consequence, several alternative ideas were proposed, see e.g. [7, 20, 38, 61]. Between them, the upper-lower solution method [10, 26, 43, 61] and a perturbation approach based on the Lyapunov-Schmidt procedure [20, **?**, 32] are the most used by the researchers. The latter method relies essentially on the fact that delay differential equation (3.3) simplifies in the limit cases $c = +\infty$ and $h = 0$. For instance, the limit form (as $c \to +\infty$) of (3.3) is $\phi'(t) = f(\phi(t), \phi(t-h))$. Assume that this equation linearized along its heteroclinic solution ψ defines a surjective Fredholm operator in an appropriate Banach space. In consequence, the Lyapunov-Schmidt reduction can be used to prove the existence of a smooth family of fast (i.e. $c > c_*$ for some large c_*) wave solutions in some neighborhood of ψ. We remark that the value of $c_* > 0$ is at least very difficult to compute or estimate. Therefore, the existence results obtained by this technique so far have local nature (e.g., the existence is proved for velocities in some neighborhood of $c = +\infty$). This constitutes a serious drawback for the applications because of the special importance that the minimal fronts have for the description of propagation phenomena. Nevertheless, as we show in this paper, the described approach still can be extended to prove the existence of the global families of wavefronts for several important classes of equations. The key property of wavefronts which is needed for the mentioned extension is their monotonicity. It seems that our methodology does not apply to non-monotone travelling fronts.

Before stating the main theorem of this work, we need to discuss several properties of the spectra of the following linearizations of (3.3) along the equilibria $0, \kappa$:

$$v''(t) - cv'(t) + \alpha_j v(t) + \beta_j v(t - ch) = 0, \quad j \in \{0, \kappa\}. \tag{3.4}$$

Here $\alpha_0 := f_1(0,0)$, $\beta_0 := f_2(0,0)$, $\alpha_\kappa := f_1(\kappa,\kappa)$, $\beta_\kappa := f_2(\kappa,\kappa)$ and $f_j(x_1,x_2) := f_{x_j}(x_1,x_2)$. Recall that the monostable function $g(x) := f(x,x)$ satisfies

$$g'(0) = f_1(0,0) + f_2(0,0) = \alpha_0 + \beta_0 > 0, \; g'(\kappa) = f_1(\kappa,\kappa) + f_2(\kappa,\kappa) = \alpha_\kappa + \beta_\kappa < 0.$$

Additionally, in view of applications in population dynamics (see Section 3.2), we will assume that $\beta_0 = f_2(0,0) \geq 0$.

Lemma 16. *Given $\alpha_\kappa + \beta_\kappa < 0$, $\beta_\kappa < 0$, there exists $c_\kappa^{\mathfrak{L}} = c_\kappa^{\mathfrak{L}}(h) \in (0, +\infty]$ such that the characteristic equation*

$$\chi_\kappa(z) := z^2 - cz + \alpha_\kappa + \beta_\kappa e^{-chz} = 0, \; c > 0, \tag{3.5}$$

has three real roots $\lambda_1 \leq \lambda_2 < 0 < \lambda_3$ if and only if $c \leq c_\kappa^{\mathfrak{L}}$. If $c_\kappa^{\mathfrak{L}}$ is finite and $c = c_\kappa^{\mathfrak{L}}$, then equation (3.5) has a double root $\lambda_1 = \lambda_2 < 0$, while for $c > c_\kappa^{\mathfrak{L}}$ there does not exist any negative root to (3.5). Moreover, if $\lambda_j \in \mathbb{C}$ is a complex root of (3.5) for $c \in (0, c_\kappa^{\mathfrak{L}}]$ then $\Re\lambda_j < \lambda_2$.

Furthermore, $c_\kappa^{\mathfrak{L}}(0) = +\infty$ and $c_\kappa^{\mathfrak{L}}(h)$ is strictly decreasing in its domain. In fact,

$$c_\kappa^{\mathfrak{L}}(h) = \frac{\theta(\alpha_\kappa, \beta_\kappa) + o(1)}{h}, \quad h \to +\infty, \quad \text{where } \theta(\alpha_\kappa, \beta_\kappa) := \sqrt{\frac{2\omega_\kappa}{\beta_\kappa}} e^{\omega_\kappa/2},$$

and ω_κ is the unique negative root of

$$-2\alpha_\kappa = \beta_\kappa e^{-\omega_\kappa}(2 + \omega_\kappa). \tag{3.6}$$

Lemma 17. *Given $\alpha_0 + \beta_0 > 0,\ \beta_0 \geq 0$, there exists $c_0^{\mathfrak{L}} = c_0^{\mathfrak{L}}(h) > 0$ such that the characteristic equation*

$$\chi_0(z) := z^2 - cz + \alpha_0 + \beta_0 e^{-chz} = 0,\ c > 0, \tag{3.7}$$

has exactly two simple real roots $\lambda = \lambda(c), \mu = \mu(c)$ if and only if $c > c_0^{\mathfrak{L}}$. These roots are positive so that we can suppose that $0 < \lambda < \mu$. Next, if $c > c_0^{\mathfrak{L}}$ and $\beta_0 > 0$, then all complex roots $\{\lambda_j\}_{j\geq 1}$ of (3.7) are simple and can be ordered in such a way that

$$\dots \leq \Re\lambda_3(c) \leq \Re\lambda_4(c) \leq \Re\lambda_2(c) = \Re\lambda_1(c) < \lambda < \mu. \tag{3.8}$$

If $c = c_0^{\mathfrak{L}}$, then the above equation has a double positive root $\lambda(c_0^{\mathfrak{L}}) = \mu(c_0^{\mathfrak{L}})$, while for $c < c_0^{\mathfrak{L}}$ there does not exist any real root to (3.7). Furthermore, each complex root $z_0 = x_0 + iy_0$ with $\Re z_0 = x_0 \leq \lambda(c)$ must have its imaginary part $|\Im z_0| > \pi/ch$. Finally, $c_0^{\mathfrak{L}} = c_0^{\mathfrak{L}}(h) > 0$ is a decreasing function, with $c_0^{\mathfrak{L}}(+\infty) = 0$ if $\alpha_0 \leq 0$ and $c_0^{\mathfrak{L}}(+\infty) = 2\sqrt{\alpha_0}$ if $\alpha_0 > 0$. In fact, for $\alpha_0 \leq 0$, we have

$$c_0^{\mathfrak{L}}(h) = \frac{\theta_1(\alpha_0, \beta_0) + o(1)}{h}, \quad h \to +\infty, \quad \textit{where} \quad \theta_1(\alpha_0, \beta_0) := \sqrt{\frac{2\omega_0}{\beta_0}} e^{\omega_0/2},$$

and ω_0 is the unique positive root of

$$-2\alpha_0 = \beta_0 e^{-\omega_0}(2 + \omega_0).$$

Lemma 18. *Assume that all conditions of Lemmas 16 and 17 are satisfied. Then equation $c_\kappa^{\mathfrak{L}}(h) = c_0^{\mathfrak{L}}(h)$ has exactly one non-negative solution h_0 if $\theta(\alpha_\kappa, \beta_\kappa) < \theta_1(\alpha_0, \beta_0)$ and does not have any non-negative solution otherwise.*

Corollary 5. *Set $\mathcal{D}_{\mathfrak{L}} = \{(h, c) : h \geq 0,\ c_0^{\mathfrak{L}}(h) \leq c \leq c_\kappa^{\mathfrak{L}}(h)\} \cap \mathbb{R}^2 \subset \mathbb{R}^2_+$. Then $\mathcal{D}_{\mathfrak{L}}$ is a connected closed domain containing $\{0\} \times [c_0^{\mathfrak{L}}(0), +\infty)$.*

Figure 2 below presents two possible forms of $\mathcal{D}_{\mathfrak{L}}$, in the second case $\theta_1(\alpha_0, \beta_0) < \theta(\alpha_\kappa, \beta_\kappa)$.

Next, let ϕ be a strictly monotone wavefront of (3.3). The characteristic exponents $\Lambda_\pm$ of ϕ are defined as $\Lambda_\pm(\phi) := \lim_{t\to\pm\infty}(1/t)\ln|\phi(\pm\infty) - \phi(t)|$.

Definition 2. Let $\mathcal{D}_{\mathfrak{N}}$ stand for the maximal connected open (in topology of $\mathbb{R}^2_+$) component of the set

$$\{(h,c) \in \mathcal{D}_{\mathfrak{L}} : \Lambda_-(\phi) = \lambda(c), \Lambda_+(\phi) = \lambda_2(c) \text{ for each monotone wavefront } \phi\}$$

which has a non-empty intersection (cf. Lemma 24) with the vertical line $h = 0$.

In general, description of $\mathcal{D}_{\mathfrak{N}}$ is a very difficult task, related to the determination of the minimal speed of propagation [55]. But when the nonlinearity f is dominated by its linearizations at the equilibria 0 and κ, this task can be easily accomplished:

Lemma 19. *$\overline{\mathcal{D}_{\mathfrak{N}}}$ coincides with $\mathcal{D}_{\mathfrak{L}}$ for each $f(x,y) \in C^{1,\gamma}$ satisfying*

$$f(x,y) \le \alpha_0 x + \beta_0 y, \quad f(x,y) \le \alpha_\kappa(x-\kappa) + \beta_k(y-\kappa), \ (x,y) \in [0,\kappa]^2.$$

In other cases, we can still indicate explicitly a substantial subset of $\mathcal{D}_{\mathfrak{N}}$, see Section 2.

In the sequel, we will consider the following sign/monotonicity assumptions:

(M) Each profile $\phi : \mathbb{R} \to (0,\kappa)$ of travelling front to (3.3) is a monotone function.

(MG) $\alpha_0 + \beta_0 > 0,\ \alpha_0 < 0,\ \beta_0 > 0,\ \alpha_\kappa < 0,\ \beta_\kappa < 0$, and for each strictly increasing $\zeta \in C^2(\mathbb{R})$, $\zeta(-\infty) = 0,\ \zeta(+\infty) = \kappa$, it holds that $f_1(\zeta(t), \zeta(t-ch)) \le 0$, $t \in \mathbb{R}$, while $f_2(\zeta(t), \zeta(t-ch))$ has a unique zero on $\mathbb{R}$.

(KPP) $\beta_0 = 0, \alpha_0 > 0, \alpha_\kappa = 0, \beta_\kappa < 0$, and for each strictly increasing C^2-function $\zeta = \zeta(t),\ \zeta(-\infty) = 0,\ \zeta(+\infty) = \kappa$, it holds that $\alpha_0 \ge f_1(\zeta(t), \zeta(t-ch)) \ge 0$, $t \in \mathbb{R}$, while $f_2(\zeta(t), \zeta(t-ch)) \le 0$ on $\mathbb{R}$.

Now we are in position to state the main result of this work:

Theorem III.1. *Assume that either hypotheses* (**M**)&(**MG**) *or* (**M**)&(**KPP**) *are satisfied. Then there is a global family* $\mathcal{F} = \{\phi(\cdot, h, c), (h, c) \in \overline{\mathcal{D}_{\mathfrak{N}}}\}$ *of monotone travelling fronts to (3.1). Moreover, if* $u = \phi(\nu \cdot x + ct)$ *is an eventually monotone front to (3.1), then* $(h, c) \in \mathcal{D}_{\mathfrak{L}}$.

Remark 1. a) In consequence, if $\overline{\mathcal{D}_{\mathfrak{N}}} = \mathcal{D}_{\mathfrak{L}}$ then Theorem III.1 provides a criterion of existence of monotone wavefronts. Moreover, what is quite important for applications, this criterion can be formulated explicitly (in terms of coefficients of the characteristic equations (3.7), (3.5), see Section 2). b) Theorem 1.4 in [55] suggests that $\overline{\mathcal{D}_{\mathfrak{N}}}$ might be the maximal domain of the monotone fronts existence even when $\overline{\mathcal{D}_{\mathfrak{N}}} \neq \mathcal{D}_{\mathfrak{L}}$. In particular, this would imply that $c_*^{\mathfrak{N}}(h') = \inf\{c : (h', c) \in \mathcal{D}_{\mathfrak{N}}\}$ and that Theorem III.1 yields an existence criterion even when $\overline{\mathcal{D}_{\mathfrak{N}}} \neq \mathcal{D}_{\mathfrak{L}}$. In any case, as we have already mentioned, the explicit determination of $c_*^{\mathfrak{N}}$ (and, in consequence, of $\overline{\mathcal{D}_{\mathfrak{N}}}$) is a very difficult problem even for non-delayed equations.
c) As we will show, the family of all monotone wavefronts has the following property of local continuity: if $(h', c') \in \mathcal{D}_{\mathfrak{N}}$ then there exists an open neighborhood $\mathcal{U} \subset \mathbb{R}_+^2$ of (h', c') and a local family of monotone fronts ϕ_U such that $\phi_U(\cdot, h, c)$ depends continuously on $(h, c) \in \mathcal{U}$ in the metric of weighted uniform convergence on $\mathbb{R}$.

Finally, a few words about the organization of the paper. Theorem III.1 is proved in Sections 3 and 4, while in the next section it is applied to two important families of delayed diffusion equations. Appendix to this paper contains the proofs of all four lemmas announced in the introduction.

3.2 Applications

3.2.1 The KPP type delayed equations

Recently, a criterion of existence of monotone fronts for the KPP-Fisher equation

$$u_t(t,x) = \Delta u(t,x) + u(t,x)(1 - u(t-h,x)) \tag{3.9}$$

was established in [38] by means of the shooting techniques and in [26] by using a constructive monotone iteration algorithm. In this section, we apply Theorem III.1 to a broad family of equations (3.1) which contains (3.9) as a particular case. It is worth to mention that the monotone wavefronts of the KPP-Fisher delayed equation (3.9) have an additional nice property: they are *absolutely* unique [18, 26, 33]. Thus the family $\mathcal{F} = \{\phi(\cdot, h, c), (h,c) \in \mathcal{D}_{\mathfrak{L}}\}$ of monotone wavefronts to (3.9) is actually globally continuous.

We will say that monostable nonlinearity $f(x,y)$ in (3.1) is of the KPP type, if

$$f \in C^{1,\gamma} \text{ for some } \gamma \in \ (0,1], \quad \alpha_0 > 0,\ \beta_0 = 0,\ \alpha_\kappa = 0,\ \beta_\kappa < 0, \quad f(0,y) \equiv 0,$$

$$\text{and, for all } x, y \in (0,\kappa), 0 < f_1(x,y) \le \alpha_0,\ \ f_2(x,y) \le 0,\ \ 0 < f(x,y) \le \beta_k(y-\kappa).$$

It is then easy to see that the set $\mathcal{D}_{\mathfrak{L}}$ has the form given on Fig. 1, cf. [26].

□

Figure 3.1: Domain $\mathcal{D}_{\mathfrak{L}}$ for the KPP type delayed equation

In fact, $c = c_\kappa(h)$ can be found from the equation

$$2 + \sqrt{c^4h^2+4} = -\beta_\kappa c^2 h^2 \exp\left(1 + \frac{2}{c^2h + \sqrt{c^4h^2+4}}\right).$$

This allows to calculate easily h_0 (defined in Lemma 18) and the asymptote $h = -1/(e\beta_\kappa)$.

Theorem III.2. *Let f be of the KPP type. Then there is a monotone front $u = \phi(\nu \cdot x + ct)$, $|\nu| = 1$, $c > 0$, to (3.9) if and only if $(h,c) \in \mathcal{D}_{\mathfrak{L}}$.*

Proof. Since, by Lemma 19, $\overline{\mathcal{D}_{\mathfrak{N}}} = \mathcal{D}_{\mathfrak{L}}$, we have only to check that the hypotheses (**M**) and (**KPP**) are satisfied. First, (**KPP**) clearly holds due to the above definition of the KPP type nonlinearity. Next, suppose for a moment that $\phi'(t_0) = 0$ at some $t_0 \in \mathbb{R}$. Since $\phi(t_0)$, $\phi(t_0 - ch) \in (0, \kappa)$, we have $\phi''(t_0) = -f(\phi(t_0), \phi(t_0 - ch)) < 0$ and therefore t_0 is the only critical point of ϕ (strict local maximum), in contradiction with the boundary conditions at $\pm\infty$. Thus we have $\phi'(t) > 0$ for all t. □

3.2.2 The Mackey-Glass type delayed diffusion equations

Consider the following monostable equation

$$u_t(t,x) = \Delta u(t,x) - \delta u(t,x) + g(u(t-h,x)), \tag{3.10}$$

where $C^{1,\gamma}-$continuous $g : \mathbb{R}_+ \to \mathbb{R}_+,\ g(0) = 0,\ g(\kappa) = \delta\kappa,\ g'(0) > \delta > 0$, has a unique critical point (a global maximum) on $(0, \kappa)$. Clearly, $\alpha_0 = \alpha_\kappa = -\delta,\ \beta_0 = g'(0) > \delta$ and $\beta_\kappa = g'(\kappa) < 0$.

Theorem III.3. *Let g satisfy the above conditions. Then there exists a family of monotone wavefronts $u := \phi(x \cdot \nu + ct, h, c),\ |\nu| = 1,\ c > 0$, parametrized by $(h, c) \in \overline{\mathfrak{D}}_{\mathfrak{N}}$.*

Proof. Observe that the monotonicity assumption (**M**) is satisfied in view of [56, Theorem 1.1]. In order to check (**MG**), suppose that $\zeta \in C^2(\mathbb{R}, (0, \kappa))$ is a strictly increasing function such that $\zeta(-\infty) = 0$, $\zeta(+\infty) = \kappa$. Then $f_1(\zeta(t), \zeta(t - ch)) = -\delta < 0,\ t \in \mathbb{R}$, while $f_2(\zeta(t), \zeta(t - ch)) = g'(\zeta(t - ch))$ clearly has a unique zero on $\mathbb{R}$. □

3.2.3 The diffusive Nicholson's equation

Equation (3.10) with $g(x) = pxe^{-x}$ is called the diffusive Nicholson's equation. It is monostable when $p/\delta > 1$, with steady state solutions $u_1 := 0$ and $u_2 := \ln(p/\delta)$.

If, in addition, $p/\delta \le e$, then $g(x)$ is monotone on $[u_1, u_2]$ and therefore there exists a unique monotone travelling front for each fixed $c \ge c_0^{\mathfrak{L}}$, cf. [2, 55, 64]. In fact, this front can be found as a limit of a converging monotone functional sequence [64]. The uniqueness can be deduced either by using the Diekmann-Kaper theory [2] or by applying the sliding method of Berestycki and Nirenberg [55]. Now, if $e < p/\delta \le e^2$, then travelling fronts exist for every fixed $h \ge 0$ and $c \ge c_0^{\mathfrak{L}}$ [43, 57]. However, they are not monotone for large c and h [57]. Finally, if $p/\delta > e^2$ then the wavefronts exist only for h from some bounded set (depending on p, δ) [57, 56]. If $p/\delta > e^2$ and h is large, then the Nicholson's equation possesses positive and bounded semi-wavefront solutions, i.e. solutions $u = \phi(\nu \cdot x + ct)$, $\phi(-\infty) = 0$, $\liminf_{t\to+\infty} \phi(t) > 0$. It was also proved in [56] that, for $p/\delta \in (e^2, 16.99..)$, these solutions have monotone leading edge and that they are either eventually monotone or slowly oscillating at $+\infty$, cf. Corollary 6 below. It is an open problem whether there can exist an eventually monotone and non-monotone front for some $p/\delta > e^2$. Hence, excepting the above mentioned result from [64], nothing was known about the existence of monotone fronts for the Nicholson's equation. Our first assertion here gives the complete solution to the considered problem for $p/\delta \le e^2$:

Theorem III.4. *Assume that $g(x) = pxe^{-x}$ and $p/\delta \in (e, e^2]$. Then equation (3.10) has a unique (up to a translation) travelling front for each $c \ge c_0^{\mathfrak{L}}$, $h \ge 0$. This front is monotone if and only if $(h, c) \in \mathcal{D}_{\mathfrak{L}}$. The domain $\mathcal{D}_{\mathfrak{L}}$ has two main geometric forms presented on Fig. 2, where $\nu_0 := 2.808\ldots$ and $\delta h_a e^{\delta h_a} = (e \ln(p/e\delta))^{-1}$.*

□□

Figure 3.2: $\nu_0 < p/\delta$ $\nu_0 \ge p/\delta$

Proof. The front existence for $c \ge c_0^{\mathfrak{L}}$ was proved in [43, 57]. The uniqueness statement follows from [2]. If $p/\delta \in (e, e^2]$ then $\min_{[u_1,u_2]} g'(x) = g'(u_2)$ and therefore Lemma 19 assures that $\overline{\mathcal{D}_{\mathfrak{N}}} = \mathcal{D}_{\mathfrak{L}}$. Hence, in order to prove our criterion for the existence of monotone fronts, it suffices to invoke Theorem III.3.

Next, $\alpha_0 = -\delta$, $\beta_0 = p$, $\alpha_\kappa = -\delta$ and $\beta_\kappa = \delta \ln(e\delta/p)$. As a consequence, functions $c = c_0^{\mathfrak{L}}(h)$ and $c = c_\kappa^{\mathfrak{L}}(h)$ are determined, respectively, by the equations

$$\frac{c^2+4\delta}{2+\sqrt{c^4h^2+4c^2h^2\delta+4}} = ep\exp\left(-\frac{\sqrt{c^4h^2+4c^2h^2\delta+4}+c^2h}{2}\right), \ \ h \geq 0;$$

$$\frac{2+\sqrt{c^4h^2+4c^2h^2\delta+4}}{ec^2h^2|\beta_\kappa|} = \exp\left(\frac{\sqrt{c^4h^2+4c^2h^2\delta+4}-c^2h}{2}\right), h > h_a, \tag{3.11}$$

where h_a is such that $e|\beta_\kappa| h_a \exp(\delta h_a) = 1$. A simple analysis shows that $c_\kappa^{\mathfrak{L}}(h) = +\infty$ if and only if $h \in [0, h_a]$. Next, $\theta_1(\alpha_0, \beta_0) = \sqrt{\frac{2w_0}{p}}e^{w_0/2}$ where w_0 is the positive root of $2\delta/p = e^{-w}(2+w)$ (see Lemma 17). Similarly, from Lemma 16 we infer that $\theta(\alpha_\kappa, \beta_\kappa) = \sqrt{\frac{2|w_0|}{\delta\ln(p/e\delta)}}e^{w_0/2}$, where w_0 is the negative root of $-2/\ln(p/e\delta) = e^{-w}(2+w)$. By Lemma 18, the value of $\nu_0 = p/\delta$ is determined by the condition $\theta(\alpha_\kappa, \beta_\kappa) = \theta_1(\alpha_0, \beta_0)$. It is easy to show that $\nu_0 = \frac{p}{\delta}(t_0) = t_0^{-1}(-1+\sqrt{1+2t_0})e^{-1+\sqrt{1+2t_0}}$ with t_0 being the positive root of

$$t_0^{-1}(-1+\sqrt{1+2t_0})\exp\left(-2+\sqrt{1+2t_0}\right) = \exp\left(t_0^{-1}(1+\sqrt{1+2t_0})e^{-1-\sqrt{1+2t_0}}\right).$$

Finally, we find $\nu_0 = 2.808\dots \in (e, e^2)$. □

Corollary 6. *Suppose that $p/\delta \in (2.718\dots, 2.808\dots]$, then each minimal wavefront is monotone (independently on h). If $p/\delta \in (2.808\dots, 16.99904\dots]$ and $h > h_0$, then every minimal wavefront is slowly oscillating at $+\infty$.*

Proof. If $p/\delta \leq 2.80\dots$, then the domain $\overline{\mathcal{D}_{\mathfrak{N}}} = \mathcal{D}_{\mathfrak{L}}$ is unbounded from the right (see Fig. 2) and the first statement follows. If $p/\delta \in (2.80\dots, 16.99\dots]$ then the positive feedback assumption of [56, Theorem 3] is satisfied and therefore each wavefront is either eventually monotone or slowly oscillating. However, if $h > h_0$, then none front solution can be eventually monotone due to Theorem III.1. $\square$

Let now $p/\delta > e^2$. Then $\beta_\kappa^- := \inf_{x\in(0,u_2)}(g(x) - g(u_2))/(x - u_2) < 0$ and $g(x) \leq \beta_\kappa^-(x - u_2) + g(u_2), \ x \in [0, u_2]$. We also will need function $c := c_\kappa^-(h)$ which is implicitly (and analogously to c_κ) defined by equation (3.11) where β_κ, h_a are replaced with β_κ^- and h_a^- (such that $e|\beta_\kappa^-|h_a^- \exp(\delta h_a^-) = 1$), respectively. In particular, $c_\kappa^-(h) := +\infty$ for $h \in [0, h_a^-]$. It is easy to see that $0 < h_a^- \leq h_a$ and that $c_\kappa^-(h) \leq c_\kappa(h), h \in [0, h_0^-]$. Here h_0^- satisfies $c_\kappa^-(h_0^-) = c_0^{\mathfrak{L}}(h_0^-)$.

Theorem III.5. *Suppose that $p/\delta > e^2$ and $c \in [c_0^{\mathfrak{L}}(h), c_\kappa^-(h)], \ h \leq h_0^-$. Then the Nicholson's equation has a unique (up to a translation) monotone wavefront.*

Proof. This result follows from Theorem III.3 if we observe that Int $\mathfrak{D}_{\mathfrak{L}}^- := \{(h, c) : c \in (c_0^{\mathfrak{L}}(h), c_\kappa^-(h)), \ h \in [0, h_0^-]\} \subset \overline{\mathfrak{D}}_{\mathfrak{N}}$ (this inclusion is justified in Appendix, Remark 2). The front uniqueness is due to the relation $g'(0) = \max_{s\geq 0}|g'(s)|$, e.g. see [2]. $\square$

3.3 Associated Fredholm operator

Let ϕ be a monotone solution of equation (3.3) connecting equilibria 0 and κ. The spectra of the linearization of (3.3) at $0, \kappa$ were analyzed in Lemmas 17, 16. In this section, we study the linear variational equation along the solution ϕ

$$v''(t) - cv'(t) + f_1(\phi(t), \phi(t - ch))v(t) + f_2(\phi(t), \phi(t - ch))v(t - ch) = 0.$$

With the notation $a(t) := f_1(\phi(t), \phi(t-ch)), \ b(t) := f_2(\phi(t), \phi(t-ch))$, this equation can be written as the system

$$v'(t) = w(t), \ w'(t) = -a(t)v(t) + cw(t) - b(t)v(t - ch), \tag{3.12}$$

or shortly as $\mathfrak{F}_c(v, w) = 0$, where

$$\mathfrak{F}_c(v, w)(t) = (v'(t) - w(t), w'(t) + a(t)v(t) - cw(t) + b(t)v(t - ch)).$$

For small $\delta > 0$ and fixed c, we define the following Banach spaces:

$$C_\delta = \{\psi \in C(\mathbb{R}, \mathbb{R}^2) : |\psi|_\delta := \sup_{s \le 0} e^{-(\lambda(c)-\delta)s}|\psi(s)| + \sup_{s \ge 0} e^{-(\lambda_2(c)+\delta)s}|\psi(s)| < \infty\},$$

$$C^1_\delta = \{\psi \in C_\delta : \psi, \psi' \in C_\delta,\ |\psi|_{1,\delta} := |\psi|_\delta + |\psi'|_\delta < +\infty\}.$$

We will consider $\mathfrak{F}_c$ as a linear operator defined on C^1_δ and taking its values in C_δ. The main result of this section follows:

Theorem III.6. *Let either (***MG***) or (***KPP***) hold with* $\zeta(t) = \phi(t)$. *If* $(h, c) \in \operatorname{Int} \mathcal{D}_{\mathfrak{L}}$ *then* $\mathfrak{F}_c : C^1_\delta \to C_\delta$ *is a surjective Fredholm operator, with* $\dim Ker\, \mathfrak{F}_c = 1$.

We will prove this theorem by using Hale and Lin analysis [32, Lemmas 4.5-4.6] of the linear functional differential equations

$$y'(t) = L(t)y_t, \quad y_t(s) := y(t+s), \quad L(t) : C([-ch, 0], \mathbb{R}^n) \to \mathbb{R}^n, \tag{3.13}$$

where linear bounded operators $L(t)$ depend continuously on $t \in \mathbb{R}$ in the operator norm and are uniformly bounded on $\mathbb{R}$. Let $Y(t, s)$ denote the evolution (solution) operator for (3.13). Then the equation is said [32] to have a shifted exponential dichotomy on a half-line J with the exponents $\alpha < \beta$ and projection $P_u(s), s \in J$, if

$$|Y(t, s)(I - P_u(s))| \le Ke^{\alpha(t-s)}, \quad |Y(t, s)P_u(s)| \le Ke^{\beta(t-s)}, \quad t \ge s \in J.$$

Take some $\nu \in (\alpha, \beta)$ and consider the change of variables $y(t) = x(t)e^{\nu t}$ which transforms (3.13) into $x'(t) = M(t)x_t$ with $M(t)\phi(\cdot) = L(t)(e^{\nu\cdot}\phi(\cdot)) - \nu\phi(0)$ and the evolution operator $X(t, s) = e^{-\nu(t-s)}e^{-\nu\cdot}Y(t, s)e^{\nu\cdot}$. It is clear that the transformed equation has a usual exponential dichotomy with the exponents $\alpha - \nu < 0 < \beta - \nu$, and projection $e^{-\nu\cdot}P_u(s)e^{\nu\cdot}, s \in J$, if and only if the original equation (3.13) has a shifted exponential dichotomy with the exponents $\alpha < \beta$ and projection $P_u(s), s \in J$.

For convenience of the reader, in Proposition 1 below we summarize the content of the mentioned lemmas from [32] for the special case of system (3.12) whose formal adjoint equation [31] is given by

$$(3.14) \qquad y_1'(t) = a(t)y_2(t) + b(t+ch)y_2(t+ch), \; y_2'(t) = -y_1(t) - cy_2(t).$$

Particular solutions $y = (y_1, y_2)$ of (3.14) which are defined on $\mathbb{R}$ and satisfy

$$(3.15) \qquad |y(t)| \le Ke^{-\beta_2 t}, \; t \ge 0, \quad |y(t)| \le Ke^{-\alpha_1 t}, \; t \le 0,$$

for some K, α_1, β_2 (specified below) will be of special importance:

Proposition 1. *Suppose that continuous functions $a, b : \mathbb{R} \to \mathbb{R}$ are bounded and, for some $\tau > 0$, system (3.12) has shifted dichotomies in $(-\infty, -\tau]$ and $[\tau, +\infty)$ with exponents $\alpha_1 = \lambda(c) - \delta < \beta_1$, $\alpha_2 < \lambda_2(c) + \delta < \beta_2$ and projections $P_u^-(t), P_u^+(t)$, respectively. Then $\mathfrak{F}_c : C_\delta^1 \to C_\delta$ is Fredholm of index $i(\mathfrak{F}_c) = \dim \mathcal{R}P_u^-(-\tau) - \dim \mathcal{R}P_u^+(\tau)$, and with the range*

$$\mathcal{R}(\mathfrak{F}_c) = \{h \in C_\delta : \int_{\mathbb{R}} y(s)h(s)ds = 0 \textit{ for all solutions } y(t) \textit{ of } (3.14) \textit{ satisfying } (3.15)\}.$$

Now, since system (3.12) is asymptotically autonomous and the eigenvalues $\lambda(c), \lambda_2(c)$ of the limit systems for (3.12) at $\pm\infty$ are real and isolated, the roughness property of the exponential dichotomy (cf. [32, Lemma 4.3]) implies the following. For sufficiently large $\tau > 0$, system (3.12) has shifted dichotomies in $(-\infty, \tau]$ and $[\tau, +\infty)$ with exponents, respectively,

$$\alpha_1 = \lambda(c) - \delta > 0, \; \beta_1 = \lambda(c) - \delta/2, \; \alpha_2 := \lambda_2(c) + \delta/2 < 0, \; \beta_2 := \delta,$$

$$\text{and } \dim \mathcal{R}P_u^-(-\tau) = 2, \; \dim \mathcal{R}P_u^+(\tau) = 1, \text{ so that } \; i(\mathfrak{F}) = 1.$$

Let $(h, c) \in \operatorname{Int} \mathcal{D}_{\mathfrak{L}}$, then, for each wavefront ϕ, we have $(\phi, \phi') \in C_\delta^1$ (cf. Remark 3) and $\mathfrak{F}_c(\phi, \phi')(t) = 0$. As a consequence, dim $\operatorname{Ker}(\mathfrak{F}_c) \ge 1$. Theorem III.6 claims that actually dim Ker$\mathfrak{F}_c = 1$ because of codim $\mathcal{R}\mathfrak{F}_c = 0$. In order to prove that $\mathcal{R}(\mathfrak{F}_c) = C_\delta$ it suffices to show that none nontrivial solution of (3.14) can satisfy (3.15). We establish this fact in the next lemmas.

At this stage, it is worth rewriting (3.14) and (3.15) in a more familiar way. First, we observe that (3.14) reduces to the second order equation

$$y''(t) = -cy'(t) - a(t)y(t) - b(t+ch)y(t+ch).$$

Next, after the change of variables $v(t) = y(-t),\ t \in \mathbb{R}$, we obtain that

$$v''(t) - cv'(t) + a(-t)v(t) + b(-t+ch)v(t-ch) = 0,$$

while inequalities (3.15) take the form

$$|v(t)| + |v'(t)| \le Ke^{\delta t},\ t \le 0, \quad |v(t)| + |v'(t)| \le Ke^{(\lambda(c)-\delta)t},\ t \ge 0. \tag{3.16}$$

Set $A(t) := a(-t),\ B(t) := b(-t+ch)$. It is clear that A, B are continuous with

$$A(-\infty) = \alpha_\kappa,\ B(-\infty) = \beta_\kappa,\ A(+\infty) = \alpha_0,\ B(+\infty) = \beta_0.$$

Lemma 20. *Let $(h, c) \in \operatorname{Int} \mathcal{D}_{\mathfrak{L}}$. Then there exists a unique (modulo a constant factor) nontrivial solution $v(t)$ of equation*

$$v''(t) - cv'(t) + A(t)v(t) + B(t)v(t-ch) = 0, \tag{3.17}$$

such that $v(t), v'(t) \to 0$ as $t \to -\infty$. Moreover, we can suppose that $v(t) > 0$, $v'(t) > 0$ for all sufficiently large negative t while $\lim_{t\to-\infty} v'(t)/v(t) = \lambda_3$.

Proof. Setting $C_2 := C([-ch, 0], \mathbb{R}^2)$, we can present (3.17) as the system

$$v'(t) = w(t),\ w'(t) = cw(t) - A(t)v(t) - B(t)v(t-ch). \tag{3.18}$$

Since $(h, c) \in \operatorname{Int} \mathcal{D}_{\mathfrak{L}}$, the limit system of (3.18) at $-\infty$ is exponentially dichotomic with some projection P. In fact, it possesses one-dimensional unstable invariant submanifold of C_2 generated by the element $(v, w)(s) = (e^{\lambda_3 s}, \lambda_3 e^{\lambda_3 s})$, $s \in [-ch, 0]$. Thus

$P(v,w) = (v,w)$. Using the roughness property [32, Lemma 4.3] of the exponential dichotomy, we obtain that the perturbed system (3.18) is also dichotomic on some interval $(-\infty, -\tau] \subset \mathbb{R}_-$ with the projection $P(t)$ such that $P(t) \to P,\ t \to -\infty$. Set $(v_t, w_t) = P(t)(v,w)$, then $P(t)(v_t, w_t) = (v_t, w_t)$ and

$$|(v,w) - (v_t, w_t)|_{C_2} = |(P(t) - P)(v,w)|_{C_2} \to 0,\ t \to -\infty.$$

As a consequence, $v_t(s) > 0, w_t(s) > 0,\ s \in [-ch, 0]$, for all sufficiently large negative $t \le -\tau_1 \le -\tau$. Next, it is clear that every bounded on $\mathbb{R}_-$ solution $(v(t), v'(t))$ of (3.18) can be written as

$$(v(t+s), v'(t+s)) = \lambda(t)(v_t(s), w_t(s)),\ t \le -\tau_1,\ s \in [-ch, 0],$$

for some continuous scalar function $\lambda : (-\infty, -\tau_1] \to \mathbb{R}$. It is easy to see from (3.18) that $\lambda(t_0) = 0$ for some $t_0 \le -\tau_1$ if and only if $\lambda(t) = 0, t \le -\tau_1$. Therefore components of each bounded solution $(v(t), v'(t))$ of (3.18) keep their sign on $(-\infty, -\tau_1]$. Finally, we have that $\lim_{t\to-\infty} v'(t)/v(t) = \lim_{t\to-\infty} w_t(0)/v_t(0) = w(0)/v(0) = \lambda_3$. □

Lemma 21. *Assume that either hypothesis* (**MG**) *or* (**KPP**) *is satisfied. Let* $(h,c) \in \operatorname{Int} \mathcal{D}_{\mathfrak{L}}$ *and* $A(t) = \alpha_0 + O(e^{-\gamma t}),\ B(t) = \beta_0 + O(e^{-\gamma t}),\ t \to +\infty$, *for some* $\gamma > 0$. *Then only the trivial solution* $v(t) \equiv 0$ *of equation (3.17) can satisfy inequalities (3.16).*

Proof. Assume, on the contrary, that there is a nontrivial $v(t)$ satisfying (3.16), (3.17). By Lemma 20, we can suppose that $v(t), v'(t) > 0$ on some maximal open interval $(-\infty, \sigma)$ and $v'(\sigma) = 0$ (whenever σ is finite).

In the first part of the proof, we will assume additionally that hypothesis **(MG)** is satisfied. Then the open set $Z_v := \{t \in \mathbb{R} : v(t) \ne 0\}$ is dense in $\mathbb{R}$. Indeed, otherwise $v(t) \equiv 0$ on some non-degenerate interval $[r_1, r_2]$ so that, in virtue of equation (3.17),

$v(t) \equiv 0$ for $t \in [r_1 - chj, r_2 - chj]$, $j \in \mathbb{N}$. This, however, contradicts to the inequality $v(t) > 0$, $t \leq \sigma$. Now, if $v(t)$ is not a small solution (the latter means that $\lim_{t\to+\infty} v(t)e^{st} = 0$ for every $s \in \mathbb{R}$), we obtain from [44, Proposition 7.2] and Lemma 17 that

$$v(t) = Ce^{x_j t}(\cos(y_j t + \varphi_j) + o(1)), \quad t \to +\infty, \tag{3.19}$$

for some $C > 0$, $\varphi_j \in \mathbb{R}$, and complex $\lambda_j := x_j + iy_j$, $|y_j| > \pi/ch$, $x_j < \lambda(c)$, satisfying (3.7). Therefore $v(t)$ oscillates on $\mathbb{R}_+$ and σ is finite. Let t_* denote the unique zero of $B(t)$ on $\mathbb{R}$. Since $v''(\sigma) \leq 0$, $v'(\sigma) = 0, v(\sigma) > 0, v(\sigma - ch) > 0$, we obtain that $\sigma \geq t_*$ because of

$$0 = v''(\sigma) - cv'(\sigma) + A(\sigma)v(\sigma) + B(\sigma)v(\sigma - ch) \leq B(\sigma)v(\sigma - ch).$$

Hence $B(t) > 0$, $A(t) \leq 0$ on $(\sigma, +\infty)$ and therefore the nonlinearity

$$(N_0, N_1) := (w(t), cw(t) - A(t)v(t) - B(t)v(t - ch))$$

satisfies the following feedback inequalities (with $\delta^* = -1$, see [45]) for $t \geq \sigma$:

$$\begin{cases} N_0(t, 0, w) = w \geq 0 & \text{if and only if } w \geq 0, \\ N_1(t, v, 0, v_t) = -A(t)v - B(t)v_t \geq 0 & \text{if } v \geq 0 \text{ and } \delta^* v_t \geq 0, \\ N_1(t, v, 0, v_t) = -A(t)v - B(t)v_t \leq 0 & \text{if } v \leq 0 \text{ and } \delta^* v_t \leq 0. \end{cases} \tag{3.20}$$

In the next stage of the proof, we make use of the discrete Lyapunov functional $V^-(\phi)$ introduced by J. Mallet-Paret and G. Sell in [45]. For the convenience of the reader, below we adopt to our situation the definition of V^- and a key result from [45] describing the monotonicity properties of $V^-(v_t)$, $t \geq \sigma$. Let us introduce a new notation: $\mathbb{K} = [-h, 0] \cup \{1\}$.

Definition 3. For any $v \in C(\mathbb{K}) \setminus \{0\}$ we define the number of sign changes by

$$\mathrm{sc}(v) = \sup\{k \geq 1 : \textit{there are } t_0 < \cdots < t_k,\ t_j \in \mathbb{K},\ \textit{such that } v(t_{i-1})v(t_i) < 0 \text{ for } i \geq 1\}.$$

We set $\mathrm{sc}(v) = 0$ if $v(s) \geq 0$ or $v(s) \leq 0$ for $s \in \mathbb{K}$. If $\varphi \in C^1[-ch, 0]$ is not identically zero, we write $(\bar{\varphi})(s) = \varphi(s)$ if $s \in [-ch, 0]$, and $(\bar{\varphi})(1) = \varphi'(0)$. Then the Lyapunov functional $V^- : C^1[-ch, 0] \setminus \{0\} \to \{1, 3, 5, \dots\}$ is defined by the relations: $V^-(\phi) = \mathrm{sc}(\bar{\phi})$ if $\mathrm{sc}(\bar{\phi})$ is odd or infinite; $V^-(\phi) = \mathrm{sc}(\bar{\phi}) + 1$ if $\mathrm{sc}(\bar{\phi})$ is even.

Proposition 2. *(By [45, Theorem 2.1]). Assume that the feedback inequalities (3.20) hold for $t \geq \sigma$. Let $v : [\sigma - ch, +\infty) \to \mathbb{R}$ be a nontrivial C^1-solution of equation (3.17), and set $v_t(s) := v(t+s)$, $s \in [-ch, 0]$. Then the discrete Lyapunov functional $V^-(v_t)$ is a nonincreasing function of $t \geq \sigma$ as long as v_t is not the zero function.*

Since $V^-(v_\sigma) = 1$, Proposition 2 assures that $V^-(v_t) = 1$ for $t \geq \sigma$. On the other hand, in view of $|y_j| > \pi/ch$ and representation (3.19), we find that $V^-(v_t) \geq 3$ for all large positive t. This contradiction shows that $v(t)$ must be a small solution. We will analyze the following two alternative cases:

i) $v(t) \geq 0$ for all t from some maximal subinterval $[\hat{t}, \infty) \subseteq [\sigma, \infty)$. Since

$$-A(t) = |A(t)| \leq \mathfrak{b}_0 := \max_{t \geq \sigma} |A(t)|, \quad -B(t) \leq \mathfrak{b}_1 = 0, \ t \geq \sigma,$$

we can apply [**?**, Lemma 3.1.1], under Assumption 3.1.2 with $\gamma = -1$, to conclude that $v \equiv 0$ on some interval $[t_\#, \infty) \subset \mathbb{R} \setminus Z_v$, a contradiction.

ii) $v(t)$ is oscillating on $[\sigma, \infty)$. Since we know that $V^-(v_t) = 1$ for $t \geq \sigma$, the number of sign changes of v_t on $[t-ch, t]$ is less than 1. This implies the existence of an infinite sequence $\{t_j\}_{j \geq 0}$, $t_{j+1} - t_j \geq ch$, such that $v(t_j) = 0$ and $v(t) > 0$ [respectively, $v(t) < 0$] almost everywhere on each (t_{2j}, t_{2j+1}) [respectively, (t_{2j+1}, t_{2j+2})]. Next, the property $V^-(v_t) = 1$, $t \geq \sigma$, yields additionally that $v'(t) \geq 0$ a.e. on $(t_{2j}, t_{2j} + ch)$. In consequence,

$$v''(t) = cv'(t) + |A(t)|v(t) + B(t)|v(t - ch)| \geq 0 \text{ a.e. on } [t_{2j}, t_{2j} + ch].$$

Therefore $v'(t), v(t) > 0$ for all $t \in (t_{2j}, t_{2j}+ch]$. This shows that, in fact, $t_{2j+1}-t_{2j} > ch$ and there is a rightmost $s_j \in (t_{2j}+ch, t_{2j+1})$ such that $v(s_j) = \max_{u\in[t_{2j},t_{2j+1}]} v(u)$. Since $v(+\infty) = 0$, without restricting the generality, in the sequel we can assume that t_{2j}, s_j are choosen in such a way that $0 < v(s_j) \geq |v(t)|,\ t \geq t_{2j}$ (otherwise, it suffices to consider $-v(t)$).

Hence, $\max_{u\geq s_j-ch} |v(u)| \leq v(s_j)$, and for every fixed $T \geq 0$ and $t \in [s_j-ch, s_j+T]$, it holds

$$|v'(t)| \leq v'(t_{2j}+ch) + \max_{u\in[q_j,s_j+T]} |v'(u)| \leq |\int_{t_{2j}+ch}^{s_j} e^{c(t_{2j}+ch-s)}(A(s)v(s)+B(s)v(s-ch))ds| +$$
$$\max_{t\in[q_j,s_j+T]} |\int_t^{s_j} e^{c(t-s)}(A(s)v(s)+B(s)v(s-ch))ds| < 4\frac{|\alpha_0|+\beta_0}{c}e^{cT}v(s_j),$$

where $q_j := \max\{s_j-ch, t_{2j}+ch\}$. Therefore, if we set $w_j(t) := v(t+s_j-ch)/v(s_j)$, we have that $|w_j(t)| \leq 1,\ t \geq 0,\ w_j(ch) = 1, w_j'(ch) = 0$, and, for every fixed $T > 0$,

$$|w_j'(s)| \leq 4\frac{|\alpha_0|+\beta_0}{c}e^{cT},\ s \in [0,T].$$

As a consequence, after an application of the Arzela-Ascoli theorem, we obtain that w_j has a subsequence (we will use the same notation w_j for it) such that $w_j'(ch) = 0,\ \lim w_j(t) = w_*(t),\ t \in \mathbb{R}_+$, where the convergence is uniform on compact subsets of $\mathbb{R}_+$. It is clear that continuous w_* is bounded: $1 = \max_{t\geq 0} w_*(t) = w_*(ch)$. Note that $w_j(t)$ satisfies

$$w''(t) - cw'(t) + A_j(t)w(t) + B_j(t)w(t-ch) = 0,\ t \in \mathbb{R},$$

where $A_j(t) := A(t+s_j-ch) \to \alpha_0,\ B_j(t) := B(t+s_j-ch) \to \beta_0$ uniformly on $\mathbb{R}_+$. Thus

$$w_j'(t) = w_j'(ch) + c(w_j(t) - w_j(ch)) - \int_{ch}^{t}(A_j(s)w_j(s) + B_j(s)w_j(s-ch))ds$$

converges (uniformly on compact subsets of $[ch, +\infty)$) to $w_*'(t)$ and

$$w_*'(t) = c(w_*(t) - w_*(ch)) - \int_{ch}^{t} (\alpha_0 w_*(s) + \beta_0 w_*(s - ch))ds, \ t \geq ch.$$

Thus $w_*(t)$ is a bounded solution of the linear delay differential equation (3.4, $j = 0$) considered for $t \geq ch$, with non-negative initial value $w_*(s)$, $s \in [0, ch]$, and $w_*'(ch) = 0$, $w_*(ch) = 1$. In view of (3.4, $j = 0$), this implies that $w_*(t) \not\equiv 0$ on every subinterval $[p, +\infty)$, $p \geq ch$. By [31, Theorem 3.1, p. 76] the latter assures that $w_*(t)$ is not a small solution of (3.4, $j = 0$). Moreover, since (3.4, $j = 0$) satisfies the feedback assumptions similar to (3.20) and $V^-(w_{*ch}) = 1$, Proposition 2 implies $V^-(w_{*t}) = 1$ for $t \geq ch$. However, invoking again representation (3.19), we find that $V^-(w_{*t}) \geq 3$ for all large positive t, a contradiction.

Assume now condition (**KPP**). By Lemma 20, without restricting the generality, we can suppose that $0 \in (-\infty, \sigma)$ and $v'(0)/v(0) \approx \lambda_3$. Let $(-\infty, \sigma_*)$ denote the maximal open interval where $v(t) > 0$ (it is clear that $\sigma_* \geq \sigma$). Observe that

$$v''(t) - cv'(t) + \alpha_0 v(t) = D(t), \text{ where } D(t) := (\alpha_0 - A(t))v(t) - B(t)v(t - ch) \geq 0,$$

$t < \sigma_*$. Integrating the latter equation, we find that

$$v(t) = C_1 e^{\lambda t} + C_2 e^{\mu t} + \frac{1}{\mu - \lambda} \int_0^t \left(e^{\mu(t-s)} - e^{\lambda(t-s)}\right) D(s)ds,$$

$$\text{where} \quad C_1 := v(0)\frac{\mu - v'(0)/v(0)}{\mu - \lambda} < 0, \quad C_2 := v(0)\frac{v'(0)/v(0) - \lambda}{\mu - \lambda} > 0,$$

and $0 < \lambda < \mu$ satisfy $z^2 - cz + \alpha_0 = 0$. We note here that a direct comparation of the latter equation with $z^2 - cz + \beta_\kappa e^{-zch} = 0$ shows that $\lambda < \mu < \lambda_3$. This also implies that $c(t) := C_1 e^{\lambda t} + C_2 e^{\mu t} > 0$ for $t > 0$. Indeed, $c(t)$ is positive for sufficiently large t and if $C_1 e^{\lambda T} + C_2 e^{\mu T} = 0$ for the rightmost T, then

$$e^{(\mu-\lambda)T} = \frac{v'(0) - \mu v(0)}{v'(0) - \lambda v(0)} \approx \frac{\lambda_3 - \mu}{\lambda_3 - \lambda} < 1 \quad \text{so that } T < 0.$$

All the above imply that $\sigma_* = +\infty$ and $v(t) > 0.5C_2e^{\mu t}$ for sufficiently large t, contradicting to the second inequality of (3.16). $\square$

3.4 Global continuation of wavefronts

This section contains the proof of Theorem III.1. It is divided into three parts.

3.4.1 Lyapunov-Schmidt reduction.

Take a fixed $(h_0, c_0) \in \text{Int}\ \mathcal{D}_{\mathfrak{L}}$ and suppose that there exists a monotone wavefront $u = \phi(\nu \cdot x + ct)$, $|\nu| = 1$, for equation (3.1) considered with $h = h_0$, and propagating at the velocity $c = c_0$. Then ϕ satisfies (3.3) or, equivalently, $(v, w) = (\phi(t), \phi'(t))$ is a solution of

$$v'(t) = w(t), \quad w'(t) = cw(t) - f(v(t), v(t-r)). \tag{3.21}$$

with $c = c_0, r = c_0h_0 =: r_0$. In what follows, the spaces C_δ, C^1_δ will be also considered with the fixed parameters $c = c_0, h = h_0$. The change of variables $z_1 + \phi(t) = v, z_2 + \phi'(t) = w$ transforms (3.21) into

$$\mathfrak{F}_{c_0}(z) = G(h, c, z),$$

where we use the notation $z(t) = (z_1(t), z_2(t))$, $z_{jr}(t) = z_j(t - r)$,

$$G(h, c, z) = (0, (c - c_0)\phi' + (c - c_0)z_2 + f(\phi, \phi_{r_0}) - f(z_1 + \phi, z_{1r} + \phi_r) + a(\cdot)z_1 + b(\cdot)z_{1r_0}).$$

By Theorem III.6, there exists a subspace $W \subset C^1_\delta$, $\text{codim}(W) = 1$, such that $C^1_\delta = \ker(\mathfrak{F}_{c_0}) \bigoplus W$. Clearly, the restriction

$$L := \mathfrak{F}_{c_0}\Big|_W : W \to C_\delta$$

is continuous one-to-one operator, hence L^{-1} exists and is bounded.

Set $W_\rho := W \cap \{z \in C^1_\delta : |z|_{1,\delta} < \rho\}$. We have the following

Lemma 22. *There exist* $\rho_1, \rho_2, K > 0$ *such that*

(i) $|G(h,c,z) - G(h,c,w)|_\delta \le K|z-w|_\delta$ *for all* $z, w \in \mathcal{U}_{\rho_1}(0) = \{z : |z|_\delta < \rho_1\}$ *and* $(h,c) \in \mathcal{U}_{\rho_2}(h_0,c_0) = \{(h,c) : |h-h_0| + |c-c_0| < \rho_2\}$.

(ii) $L^{-1}G(h,c,\cdot) : W_{\rho_1} \to W_{\rho_1}$ *is well defined and is a contraction uniformly in* $(h,c) \in \mathcal{U}_{\rho_2}(h_0,c_0)$.

Proof. (i) Set $R(s) := (\phi + sw_1 + (1-s)z_1, \phi_r + sw_{1r} + (1-s)z_{1r})$, where $z = (z_1, z_2),\ w = (w_1, w_2) \in C_\delta$. Then there exists $s_0 \in (0,1)$ such that

$$|G(h,c,z)(t) - G(h,c,w)(t)| \le |c-c_0||z_2-w_2| + |f_1(R(s_0)) - a(t)||w_1 - z_1| +$$

$$|f_2(R(s_0)) - b(t)||w_{1r} - z_{1r}| = |c-c_0||z_2-w_2| +$$

$$|f_1(R(s_0)) - f_1(\phi,\phi_{r0})||w_1 - z_1| + |f_2(R(s_0)) - f_2(\phi,\phi_{r0})||w_{1r} - z_{1r}|.$$

Now, since $f_j(x,y), j = 1,2,$ are continuous functions of real variables and $\phi(t)$ is bounded on $\mathbb{R}$, for each given $\sigma > 0$ there exists $\rho_0 > 0$ such that $\sup_{t\in\mathbb{R}, s\in[0,1]} |R(s)(t) - (\phi,\phi_{r0})(t)| \le \rho_0$ implies that $|f_j(R(s)) - f_j(\phi,\phi_{r0})| < \sigma$. Since

$$|R(s)(t) - (\phi,\phi_{r0})(t)| \le |\phi(t-r) - \phi(t-r_0)| + |w_1(t)| + |z_1(t)| + |w_1(t-r)| + |z_1(t-r)| \le$$

$$\sup_{s\in\mathbb{R}} \phi'(s)|r - r_0| + 2\sup_{s\in\mathbb{R}} |w_1(s)| + 2\sup_{s\in\mathbb{R}} |z_1(s)| \le |\phi'|_\delta |r-r_0| + 4\rho_1 < \rho_0$$

for sufficiently small ρ_1, ρ_2, we find that

$$|G(h,c,z)(t) - G(h,c,w)(t)| \le \sigma(|w(t) - z(t)| + |w_1(t-r) - z_1(t-r)|).$$

Therefore, for all $z, w \in \mathcal{U}_{\rho_1}(0)$ and $(h,c) \in \mathcal{U}_{\rho_2}(h_0,c_0)$, it holds that

$$|G(h,c,z) - G(h,c,w)|_\delta \le \sigma(|w-z|_\delta + |w_1(\cdot - r) - z_1(\cdot - r)|_\delta) \le 2\sigma\Theta|w-z|_\delta.$$

Here we use the continuity of the usual translation operator $T_r : C_\delta \to C_\delta,\ r = ch > 0$, defined by $T_r z(s) = z(s-r)$: $\|T_r\| \le \exp(-r\lambda_2(c_0)) \le \exp(-(h_0+\rho_2)(c_0 + \rho_2)\lambda_2(c_0)) =: \Theta$.

(ii) Take $\sigma < (2\Theta\|L^{-1}\|)^{-1}$ and observe that $\lim_{r\to 0}|T_r\phi - \phi|_\delta = 0$:

$$|T_r\phi - \phi|_\delta \leq \sup_{s\leq 0} e^{-(\lambda-\delta)s}|\phi(s)-\phi(s-r)| + \sup_{s\geq 0} e^{-(\lambda_2+\delta)s}|\phi(s)-\phi(s-r)| \leq$$

$$r\left(\sup_{s\leq 0} e^{-(\lambda-\delta)s}\phi'(\theta(s)) + \sup_{s\geq 0} e^{-(\lambda_2+\delta)s}\phi'(\omega(s))\right) = O(r).$$

Next, if $z \in W_{\rho_1}$ and $(h,c) \in \mathcal{U}_{\rho_2}(h_0,c_0)$, then

$$|G(h,c,z)|_\delta = |G(h,c,z) - G(h,c,0)|_\delta + |G(h,c,0)|_\delta < 2\sigma\Theta\rho_1 + |c-c_0||\phi'|_\delta +$$

$$|f(\phi,\phi_{r_0}) - f(\phi,\phi_r)|_\delta < 2\sigma\Theta\rho_1 + |c-c_0||\phi'|_\delta + \max_{[0,\kappa]\times[0,\kappa]}|f_2(x,y)||\phi_{r_0}-\phi_r|_\delta < \frac{\rho_1}{\|L^{-1}\|},$$

once ρ_1, ρ_2, σ are sufficiently small. Therefore, for the same c, h, z, we have

$$|L^{-1}G(h,c,z)|_{\delta,1} \leq \|L^{-1}\||G(h,c,z)|_\delta < \rho_1,$$

so that $L^{-1}G(h,c,\cdot): W_{\rho_1} \to W_{\rho_1}$ is well defined. Finally, for h, c, z as above,

$$|L^{-1}G(h,c,z) - L^{-1}G(h,c,w)|_{\delta,1} \leq \|L^{-1}\||G(h,c,z) - G(h,c,w)|_\delta \leq 2\sigma\Theta\|L^{-1}\||z-w|_\delta$$

which completes the proof of the lemma. □

Corollary 7. *Assume that either hypothesis* (**MG**) *or* (**KPP**) *holds. If* $\phi(h_0,c_0)(t)$ *is a monotone wavefront of (3.3) for some* $(h_0,c_0) \in \operatorname{Int}\mathcal{D}_{\mathfrak{L}}$ *then there exist* $\rho > 0$ *and continuous map* $\phi: \mathbb{R}^2_+ \cap \mathcal{U}_\rho(h_0,c_0) \to C^1_\delta$ *such that each* $\phi(h,c)(t)$ *is a travelling front of equation (3.3) considered with* $(h,c) \in \mathbb{R}^2_+ \cap \mathcal{U}_\rho(h_0,c_0)$.

Proof. Indeed, since $L^{-1}G(h,c,\cdot): W_{\rho_1} \to W_{\rho_1}$ is a uniform contraction, there exist a unique solution $z = z(h,c)$ of the equation $L^{-1}G(h,c,z) = z$. Moreover, the function $z: \mathcal{U}_{\rho_2}(h_0,c_0) \to W_{\rho_1}$ depends continuously on (h,c) (e.g. see [34, Section 1.2.6]) and $z(h_0,c_0) = 0$. As a consequence, $\mathfrak{F}_{c_0}(z(h,c)) = G(h,c,z(h,c))$ and therefore $\phi(h,c)(t) := \phi(t) + z_1(h,c)(t)$ is a travelling front of equation (3.3) considered with $(h,c) \in \mathbb{R}^2_+ \cap \mathcal{U}_\rho(h_0,c_0)$. □

3.4.2 Asymptotic analysis of $\phi(t,h,c) := \phi(h,c)(t)$.

Fix $(h_0, c_0) \in \mathcal{D}_{\mathfrak{N}}$ and suppose that there exists a monotone wavefront for equation (3.3) considered with $h = h_0$ and propagating with the velocity $c = c_0$. As we have proved, this implies the existence of an open neighborhood $\mathcal{O} \subset \mathbb{R}^2_+$ of (h_0, c_0) and a continuous family $\phi : \mathcal{O} \to C^1_\delta$ of wavefronts to (3.1). It should be observed that, at the present moment, we do not have any information either about the positivity or about the monotonicity properties of $\phi(h,c)$. In the next lemma, we analyze the main term of asymptotic expansions of each particular wavefront $\phi(h,c)$ at the infinity. Recall that $f \in C^{1,\gamma}$ for some $\gamma \in (0,1]$. Since δ can be taken arbitrarily small, there is no loss of generality in assuming that γ, δ satisfy $(\gamma+1)(\lambda(c_0)-\delta) > \lambda(c) + 2\sigma > \lambda(c_0) - \delta$ for some $\sigma > 0$ and all $(h,c) \in \mathcal{O}$.

Lemma 23. *Let $(h_0, c_0) \in \mathcal{D}_{\mathfrak{N}}$. Then there exist an open neighborhood $\mathcal{O}' \subset \mathcal{O}$ and continuous functions $K_1, K_2 : \mathcal{O}' \to (0, +\infty)$ such that, for some $\sigma > 0, M > 0$, independent of c, h, and for all $(h,c) \in \mathcal{O}'$, it holds that*

$$(\phi(t,h,c), \phi'(t,h,c)) = \begin{cases} K_1(h,c)e^{\lambda(c)t}(1, \lambda(c)) + R_1(t,h,c), & t \le 0, \\ (\kappa, 0) - K_2(h,c)e^{\lambda_2(c)t}(1, \lambda_2(c)) + R_2(t,h,c), & t \ge 0, \end{cases}$$

where $|R_1(t,h,c)| \le Me^{(\lambda(c)+\sigma)t},\ t \le 0, \quad |R_2(t,h,c)| \le Me^{(\lambda_2(c)-\sigma)t},\ t \ge 0.$

Proof. First, we will analyze the asymptotic behavior at $-\infty$. By Corollary 7, there exist a positive number $M_1 > 0$ and an open neighborhood $\mathcal{O}_1 \subset \mathcal{O}$ such that, for all $t \le 0,\ (h,c) \in \mathcal{O}_1$,

$$|\phi(t,h,c)| = |z(t,h,c) + \phi(t)| \le (|z(h,c,\cdot)|_\delta + |\phi|_\delta)e^{(\lambda(c_0)-\delta)t} \le M_1 e^{(\lambda(c_0)-\delta)t}.$$

Since

$$\phi''(t,h,c) - c\phi'(t,h,c) + \alpha_0\phi(t,h,c) + \beta_0\phi(t-ch,h,c) = F(t,h,c), \tag{3.22}$$

where $F(t,h,c) := \alpha_0\phi(t,h,c)+\beta_0\phi(t-ch,h,c)-f(\phi(t,h,c),\phi(t-ch,h,c))$ satisfies

$$|F(t,h,c)| \le |\alpha_0 - f_1(\theta(t)\phi(t),\theta(t)\phi(t-ch)||y(t)| + |\beta_0 - f_2(\theta(t)\phi(t),\theta(t)\phi(t-ch))||\phi(t-ch)|$$

$$\le C_1(|\theta(t)\phi(t)|+|\theta(t)\phi(t-ch)|)^\gamma(|\phi(t)|+|\phi(t-ch)|) \le C_2 e^{(\gamma+1)(\lambda(c_0)-\delta)t},$$

with C_j independent of $(h,c)\in\mathcal{O}_1$ and $\theta(t)\in(0,1)$ appearing due to an application of the mean value theorem. Thus $|F(t,h,c)|\le C_2e^{(\lambda(c)+2\sigma)t},\ t\le 0$, so that, by [26, Lemma 28], $\phi(t,h,c)=w_-(t)+u_-(t)$, where

$$w_-(t) = -\mathrm{Res}_{z=\lambda(c)}\left(\frac{e^{zt}}{\chi_0(z)}\int_{\mathbb{R}} e^{-zs}F(s,h,c)ds\right) = -e^{\lambda(c)t}\frac{\tilde F(\lambda(c),h,c)}{\chi_0'(\lambda(c))},$$

$$u_-(t) = \frac{e^{(\lambda(c)+\sigma)t}}{2\pi}\int_{\mathbb{R}} e^{ist}\frac{\tilde F(\lambda(c)+\sigma+is,h,c)}{\chi_0(\lambda(c)+\sigma+is,h,c)}ds,\ \ \tilde F(z,h,c) := \int_{\mathbb{R}} e^{-zs}F(s,h,c)ds,$$

whenever δ,σ are sufficiently small positive numbers. Set

$$K_1(h,c) := -\frac{\int_{\mathbb{R}} e^{-\lambda(c)s}F(s,h,c)ds}{\chi_0'(\lambda(c))}.$$

Since continuous $F(t,h,c)$ is uniformly bounded on $\mathbb{R}\times\mathcal{O}_1$ and, in addition, $e^{-\lambda(c)s}|F(s,h,c)|\le C_2e^{2\sigma s},\ s\le 0$, we conclude that $K_1(h,c)$ is also continuous on $\mathcal{O}_1$. We note that $\chi_0'(\lambda(c))<0$ for all $(h,c)\in\mathcal{O}_1\subset \mathrm{Int}\ \mathcal{D}_{\mathfrak{L}}$. Next, there exists an open subset $\mathcal{O}_2\subset\mathcal{O}_1$ such that, for $(h,c)\in\mathcal{O}_2$,

$$|u_-(t,h,c)| \le \frac{e^{(\lambda(c)+\sigma)t}}{2\pi}\int_{\mathbb{R}}\frac{1}{|\chi_0(\lambda(c)+\sigma+is,h,c)|}\int_{\mathbb{R}} e^{-t(\lambda(c)+\sigma)}|F(t,h,c)|dtds \le e^{(\lambda(c)+\sigma)t}C_3,$$

where C_3 is independent of (h,c). Indeed, as we have seen, the function $\int_{\mathbb{R}} e^{-t(\lambda(c)+\sigma)}|F(t,h,c)|dt$ is uniformly bounded on $\mathcal{O}_1$ and, on the other hand, for some open subset $\mathcal{O}_2\subset\mathcal{O}_1$ and positive C_4, C_5, it holds that $C_4+C_5s^2\le|\chi_0(\lambda(c)+\sigma+is,h,c)|,\ s\in\mathbb{R},\ (h,c)\in\mathcal{O}_2$.

In consequence, $K_1(h_0,c_0)\ne 0$, since otherwise $\Lambda_-(\phi)\ge\lambda(c_0)+\sigma>\lambda(c_0)$ (recall that $(h_0,c_0)\in\mathcal{D}_{\mathfrak{N}}$ and see Definition 2). Moreover, the positivity of ϕ implies that

$K_1(h_0, c_0) > 0$. Since $K_1(h, c)$ is continuous, there exists an open set $\mathcal{O}_3 \subset \mathcal{O}_2$ where $K_1(h, c)$ is positive.

Next, after integrating equation (3.22) on $(-\infty, t)$, we obtain

$$\phi'(t, h, c) = c\phi(t, h, c) + \int_{-\infty}^{t} (F(s, h, c) - \alpha_0\phi(s, h, c) - \beta_0\phi(s - ch, h, c))\, ds =$$
$$K_1(h, c)\lambda(c)e^{\lambda(c)t} + cu_-(t) + \int_{-\infty}^{t} (F(s, h, c) - \alpha_0 u_-(s) - \beta_0 u_-(s - ch))\, ds,$$

that proves the asymptotic formula of Lemma 23 at $-\infty$.

After applying the change of variables $y(t, h, c) = \kappa - \phi(t, h, c)$, the study of the asymptotic behavior of wavefronts at $+\infty$ becomes fully analogous to the first case and is left to the reader. □

3.4.3 The final part of the proof of Theorem III.1.

The proof of our main result is an easy consequence of the following three propositions.

Lemma 24. *Assume that either hypotheses* (**M**)&(**MG**) *or* (**M**)&(**KPP**) *are satisfied and* $(h_0, c_0) \in \mathcal{D}_{\mathfrak{N}}$. *Then in Corollary 7, we can choose* $\rho > 0$ *such that* $\phi(h, c)(t)$ *is a positive monotone wavefront of (3.3) for each* $(h, c) \in \mathbb{R}^2_+ \cap \mathcal{U}_\rho(h_0, c_0)$. *Hence, the non-empty set*

$$\mathcal{D}'_{\mathfrak{N}} := \{(h, c) \in \mathcal{D}_{\mathfrak{N}} : \textit{there is at least one monotone wavefront for (3.3)}\}$$

is open in topology of $\mathcal{D}_{\mathfrak{N}}$.

Proof. First, we observe that $\{0\} \times (c_*^{\mathfrak{N}}, +\infty) \subset \mathcal{D}'_{\mathfrak{N}} \neq \emptyset$ because of the existence results and asymptotic formulae (3.2) presented in the second paragraph of the introduction. Next, by Lemma 23, there exist $\rho' > 0$ and $T > 0$ independent of h, c, such that $\phi'(h, c)(t),\ \phi(h, c)(t) > 0$ for all $|t| \geq T,\ (h, c) \in \mathcal{U}_{\rho'}(h_0, c_0) \cap \mathbb{R}^2_+$. On the other hand, due to the continuity of application $\phi : \mathbb{R}^2_+ \cap \mathcal{U}_{\rho'}(h_0, c_0) \to C^1_\delta$, we find that, for an appropriate $\epsilon > 0$ and some $0 < \rho < \rho'$, it holds that

$$0 < \phi(h_0, c_0)(t) - \epsilon < \phi(h, c)(t) < \epsilon + \phi(h_0, c_0)(t) < \kappa, \ |t| \le T, \ (h, c) \in \mathcal{U}_\rho(h_0, c_0).$$

In consequence, $\phi(h, c)(t) \in (0, \kappa)$ for all $t \in \mathbb{R}$, $(h, c) \in \mathbb{R}^2_+ \cap \mathcal{U}_\rho(h_0, c_0)$. In addition, by assumption (**M**), each profile $\phi(h, c)(\cdot) : \mathbb{R} \to (0, \kappa)$ is a monotone function.

Finally, it is clear that $\Lambda_-(\phi(h, c)) = \lambda(c), \Lambda_+(\phi(h, c)) = \lambda_2(c)$ for each $(h, c) \in \mathcal{U}_\rho(h_0, c_0)$. This means that $\mathbb{R}^2_+ \cap \mathcal{U}_\rho(h_0, c_0) \subset \mathcal{D}'_{\mathfrak{N}}$. Since (h_0, c_0) was an arbitrary point from $\mathcal{D}'_{\mathfrak{N}}$, we conclude that $\mathcal{D}'_{\mathfrak{N}}$ is open in $\mathcal{D}_{\mathfrak{N}}$. □

Lemma 25. *For each $(h_0, c_0) \in \overline{\mathcal{D}'_{\mathfrak{N}}}$, equation (3.3) has at least one positive monotone front. Therefore $\mathcal{D}'_{\mathfrak{N}}$ is closed in topology of $\mathcal{D}_{\mathfrak{N}}$ so that $\mathcal{D}'_{\mathfrak{N}} = \mathcal{D}_{\mathfrak{N}}$.*

Proof. Suppose that a sequence of points $(h_n, c_n) \in \mathcal{D}'_{\mathfrak{N}}$ converges to (h_0, c_0). If we denote by $\phi_n(t)$ some associated sequence of monotone wavefronts normalized by $\phi_n(0) = \kappa/2$, a direct verification shows that

$$\phi_n(t) = \frac{1}{z_2 - z_1} \left\{ \int_{-\infty}^{t} e^{z_1(t-s)} (\mathcal{H}\phi_n)(s) ds + \int_{t}^{+\infty} e^{z_2(t-s)} (\mathcal{H}\phi_n)(s) ds \right\}, \tag{3.23}$$

where $(\mathcal{H}\phi)(s) = \phi(s) + f(\phi(t), \phi(t - ch))$ and $z_1 < 0 < z_2$ satisfy $z^2 - cz - 1 = 0$. It follows from (3.23) that $0 \le \phi'_n(t) \le \kappa + \max_{[0,\kappa]^2} |f(x, y)|$. Thus $\{\phi_n(t)\}$ has a subsequence (by abusing the notation, we will call it again $\{\phi_n(t)\}$) converging in the compact open topology of $C(\mathbb{R}, \mathbb{R})$. Let $\phi_0 = \lim \phi_n$, passing to the limit (as $n \to +\infty$) in (3.23), we find that $\phi_0(t)$ also satisfies (3.23). Therefore $\phi_0(t)$, $\phi_0(0) = \kappa/2$, $\phi_0(t) \le \kappa, 0 \le \phi'_0(t) \le \kappa + \max_{[0,\kappa]^2} |f(x, y)|$, is a monotone positive solution of (3.3). Since $\phi_0(\pm\infty)$ are finite and $\phi''_0(t)$ is bounded, we obtain that $\phi'_0(\pm\infty) = 0$. In consequence, taking into account that ϕ_0 is a bounded solution of equation (3.3), we find that $f(\phi_0(\pm\infty), \phi_0(\pm\infty)) = 0$. In this way, $\phi_0(-\infty) = 0, \phi_0(+\infty) = \kappa$.

Since $\beta_\kappa < 0$, it follows from (3.3) that actually $0 < \phi_0(t) < \kappa,\ t \in \mathbb{R}$. Finally, since $\mathcal{D}'_{\mathfrak{N}}$ is simultaneously closed and open in connected space $\mathcal{D}_{\mathfrak{N}}$, we obtain that $\mathcal{D}'_{\mathfrak{N}} = \mathcal{D}_{\mathfrak{N}}$. □

Lemma 26. *If $(\bar{h}, \bar{c}) \in \mathbb{R}^2_+ \setminus \mathcal{D}_{\mathfrak{L}}$, then equation (3.3) does not have any positive eventually monotone front.*

Proof. Take some $(\bar{h}, \bar{c}) \in \mathbb{R}^2_+ \setminus \mathcal{D}_{\mathfrak{L}}$. Then either $\bar{c} < c_0^{\mathfrak{L}}(\bar{h})$ or $\bar{c} > c_\kappa^{\mathfrak{L}}(\bar{h})$. In the first case, the non-existence of positive fronts is a well known fact (cf. [54, Theorem 1]). Consequently, it suffices to consider the case $\bar{c} > c_\kappa^{\mathfrak{L}}(\bar{h})$. Then Lemma 16 implies that $\chi_\kappa(z)$ does not have negative zeros. Arguing by contradiction, suppose that, nevertheless, equation (3.3) has some positive eventually monotone front $\phi(t)$ for $h = \bar{h}, c = \bar{c}$. Then $\psi(t) := \pm(\kappa - \phi(t))$ is strictly positive on some interval $[T, +\infty)$ and satisfies

$$\psi''(t) - \bar{c}\psi'(t) \pm f(\kappa \pm \psi(t), \kappa \pm \psi(t - \bar{c}\bar{h})) = 0,\ \ \psi(+\infty) = 0,$$

where the sign "−" [respectively, "+"] corresponds to the case $\phi(t) < \kappa,\ t > T$ [to the case $\phi(t) > \kappa,\ t > T$, respectively]. Following the approach in [26], we will show that the inequality $\bar{c} > c_\kappa^{\mathfrak{L}}(\bar{h})$ will force $\psi(t)$ to oscillate about the zero. For the convenience of the reader, the proof is divided in several steps.

Claim I: $\psi(t)$ has at least exponential decay as $t \to +\infty$.

First, observe that

$$\psi''(t) - \bar{c}\psi'(t) = \Gamma\psi(t) - g(t),\ \ t \in \mathbb{R}, \tag{3.24}$$

where, with some $\mathbf{z}(t) := (\kappa \pm \theta(t)\psi(t), \kappa \pm \theta(t)\psi(t - \bar{c}\bar{h}))$, $\theta(t) \in (0, 1)$, $\Gamma > 0$, we set

$$g(t) := \Gamma\psi(t) \pm f(\kappa \pm \psi(t), \kappa \pm \psi(t - \bar{c}\bar{h})) = (\Gamma + f_1(\mathbf{z}(t)))\psi(t) + f_2(\mathbf{z}(t))\psi(t - \bar{c}\bar{h}).$$

Since $f_1(\mathbf{z}(+\infty)) + f_2(\mathbf{z}(+\infty)) = \alpha_\kappa + \beta_\kappa < 0$, $f_2(\mathbf{z}(+\infty)) = \beta_\kappa < 0$, and $\psi(t)$ is decreasing, we find that, for all sufficiently large t and some positive $0 < \Gamma < -\beta_\kappa - \alpha_\kappa$, it holds that

$$g(t) \leq (\Gamma + f_1(\mathbf{z}(t)) + f_2(\mathbf{z}(t)))\psi(t) < 0.$$

Since $\psi(t), g(t)$ are bounded on $\mathbb{R}$, we obtain that

$$\psi(t) = \frac{1}{m-l}\left(\int_{-\infty}^{t} e^{l(t-s)}g(s)ds + \int_{t}^{+\infty} e^{m(t-s)}g(s)ds\right),$$

where $l < 0$ and $0 < m$ are roots of $z^2 - \bar{c}z - \Gamma = 0$. The latter representation of $\psi(t)$ implies that there exists T_0 such that

$$\psi'(t) - l\psi(t) = \int_{t}^{+\infty} e^{m(t-s)}g(s)ds < 0,\ t \geq T_0. \tag{3.25}$$

Hence, $(\psi(t)\exp(-lt))' < 0,\ t \geq T_0$, and therefore

$$\psi(t) \leq \psi(s)e^{l(t-s)}, \quad t \geq s \geq T_0, \quad g(t) = O(e^{lt}),\ t \to +\infty. \tag{3.26}$$

Finally, (3.25), (3.26) imply that $\psi'(t) = O(e^{lt}),\ t \to +\infty$.

Claim II: $\psi(t) > 0$ is not superexponentially small as $t \to +\infty$.

Recall that $\psi(t)$ is decreasing and positive on $\mathbb{R}$. Since the right hand side of Eq. (3.24) is positive and integrable on $[T_0, +\infty)$, and since $\psi(t)$ is a bounded solution of (3.24) satisfying $\psi(+\infty) = 0$, we find that

$$\psi(t) = -\int_{t}^{+\infty}(1 - e^{\bar{c}(t-s)})(f_1(\mathbf{z}(s))\psi(s) + f_2(\mathbf{z}(s))\psi(s - \bar{c}\bar{h}))ds.$$

As a consequence, there exists T_1 such that

$$\psi(t) \geq 0.5|\beta_\kappa|(1 - e^{-0.5\bar{h}\bar{c}})\int_{t-0.5\bar{h}\bar{c}}^{t}\psi(s)ds := \xi\int_{t-0.5\bar{h}\bar{c}}^{t}\psi(s)ds, \quad t \geq T_1 - \bar{c}\bar{h}.$$

Now, since $\psi(t) > 0$ for all t, we can find positive C, ρ such that $\psi(s) > Ce^{-\rho s}$ for all $s \in [T_1 - \bar{c}\bar{h}, T_1]$. We can assume that ρ is large enough to satisfy the inequality

$\xi(e^{0.5\rho\bar{h}\bar{c}}-1)>\rho$. Then we claim that $\psi(s)>Ce^{-\rho s}$ for all $s\geq T_1-\bar{c}\bar{h}$. Conversely, suppose that $t'>T_1$ is the leftmost point where $\psi(t')=Ce^{-\rho t'}$. Then we get a contradiction:

$$\psi(t')\geq\xi\int_{t'-0.5\bar{h}\bar{c}}^{t'}\psi(s)ds>C\xi\int_{t'-0.5\bar{h}\bar{c}}^{t'}e^{-\rho s}ds=Ce^{-\rho t'}\xi\frac{e^{0.5\rho\bar{c}\bar{h}}-1}{\rho}>Ce^{-\rho t'}.$$

Claim III: $\psi(t)>0$ can not hold when $\chi_\kappa(z)$ does not have any zero in $(-\infty,0)$.

Observe that $\psi(t)$ satisfies

$$\psi''(t)-\bar{c}\psi'(t)+f_1(\mathbf{z}(t))\psi(t)+f_2(\mathbf{z}(t))\psi(t-\bar{c}\bar{h})=0,\ t\in\mathbb{R},$$

where in virtue of Claim I, it holds that $(\psi(t),\psi'(t))=O(e^{lt})$. Next, $f\in C^{1,\gamma}$ assures that $f_1(\mathbf{z}(t))=\alpha_k+O(\psi^\gamma(t)),\ f_2(\mathbf{z}(t))=\beta_k+O(\psi^\gamma(t))$ at $t=+\infty$. Then [44, Proposition 7.2] implies that there exists $q<l$ such that $\psi(t)=v(t)+O(e^{qt})$, $t\to+\infty$, where v is a *non empty* (due to Claim II) finite sum of eigensolutions of the limiting equation

$$y''(t)-\bar{c}y'(t)+\alpha_k y(t)+\beta_\kappa y(t-\bar{c}\bar{h})=0,\ t\in\mathbb{R},$$

associated to the eigenvalues $\lambda_j\in F=\{q<\Re\lambda_j\leq l\}$. Now, since the set F does not contain any real eigenvalue by our assumption, we conclude that $\psi(t)$ should be oscillating on $\mathbb{R}_+$, a contradiction. □

3.5 Appendix

3.5.1 Proof of Lemma 16

With $\lambda:=cz,\ \epsilon=c^{-2}$, equation (3.5) takes the form

$$F(\lambda):=\epsilon\lambda^2-\lambda+\alpha_\kappa+\beta_\kappa e^{-h\lambda}=0,\ \epsilon>0. \tag{3.27}$$

Since $F'''(x)>0,\ x\in\mathbb{R}$, equation (3.27) has at most three real roots. Since $F(0)<0,\ F(\pm\infty)=\pm\infty$, this equation has an even number (either 0 or 2) of negative roots (counting the multiplicity) and at least one positive root. A straightforward analysis of (3.27) shows that

(a) If this equation has a negative root for some $\epsilon_0 \geq 0$, it also has two negative roots for each $\epsilon > \epsilon_0$. We will denote the greatest negative root as λ_2. If $\epsilon_0 = 0$, we obtain $c^{\mathfrak{L}}_\kappa(h) = +\infty$.

(b) If equation (3.27) does not have any negative root for $\epsilon = 0$ (this happens when h is sufficiently large), there exists a unique $\epsilon_0 > 0$ such that (3.27) possesses two negative roots (counting the multiplicity) for $\epsilon \geq \epsilon_0$ and does not have a negative root for $\epsilon < \epsilon_0$. Thus $c^{\mathfrak{L}}_\kappa(h) = \epsilon_0^{-1/2}$ is finite for sufficiently large h and $\epsilon_0 = \epsilon_0(h)$ can be determined from the system

$$\epsilon\lambda^2 - \lambda + \alpha_\kappa = -\beta_\kappa e^{-h\lambda},\ 2\epsilon\lambda - 1 = h\beta_\kappa e^{-h\lambda}. \tag{3.28}$$

In particular, the double negative root $\lambda = \lambda(h)$ of (3.27) satisfies

$$-2\frac{\alpha_\kappa}{\beta_\kappa} + \frac{\omega}{\beta_\kappa h} = e^{-\omega}(2+\omega), \quad \omega := h\lambda(h), \tag{3.29}$$

while $c^{\mathfrak{L}}_\kappa(h)$ is strictly decreasing on some maximal open interval $(h_0, +\infty)$, $h_0 > 0$, because of $\epsilon_0'(h) = \beta_\kappa e^{-\lambda h}/\lambda > 0$. Observe that $\alpha_\kappa/|\beta_\kappa| < 1$ and the right-hand side of (3.29) has a unique inflection point at $\omega = 0$. This implies that $\omega(h) \to \omega_\kappa,\ h \to +\infty$, where $\omega_\kappa < 0$ satisfies (3.6).

It is clear that $c^{\mathfrak{L}}_\kappa(h) = +\infty$ for $h \in [0, h_0]$. From the second equation of (3.28), we also easily obtain that $\lim_{h\to+\infty} hc^{\mathfrak{L}}_\kappa(h) = \sqrt{\frac{2\omega_\kappa}{\beta_\kappa}}e^{\omega_\kappa/2}$, so that $c^{\mathfrak{L}}_\kappa(+\infty) = 0$.

(c) It is immediate to see that, for each fixed $c = 1/\sqrt{\epsilon} \in (0, c^{\mathfrak{L}}_\kappa]$, there exists $x_1 > 0$ (independent on h) such that $\Re\lambda_j < x_1$ for every λ_j satisfying (3.27). Furthermore, for every fixed $x_2 \in \mathbb{R}$ there is an increasing continuous function $y = y(h) > 0,\ h \geq 0$, such that all roots λ_j of (3.27) with $\Re\lambda_j \geq x_2$ are contained in the rectangle $\mathcal{R}(x_2, h) := [x_2, x_1] \times [-y(h), y(h)] \subset \mathbb{C}$. Next, observe that because of $\alpha_\kappa + \beta_\kappa < 0$

equation (3.27) with $h = 0$ has only two roots $\lambda_2 < 0 < \lambda_1$. By the Rouche's theorem, this implies that, for all small positive h, equation (3.27) does not have roots $\lambda_j = \lambda_j(h)$, $\Re\lambda_j \geq \lambda_2$, others than $\lambda_2(h), \lambda_1(h)$. Now, suppose for a moment that for some positive h_0 there exists complex $\lambda_j(h_0) \in \mathcal{R}(\lambda_2(h_0), h_0)$. Let h_0 be the minimal value with such a property, then the Rouche's theorem assures that $\Re\lambda_j(h_0) = \lambda_2(h_0)$. Moreover, $\Im\lambda_j(h_0) \neq 0$ since otherwise $\lambda_2(h_0)$ would have the multiplicity 3. Thus equation (3.27) with $h = h_0$ has at least three roots of the form $\lambda(y) := \lambda_2(h) + iy$ with $y \in \{-\theta, 0, \theta\}$ for some positive θ. Since $c \in (0, c_\kappa^{\mathfrak{L}}]$, the function $F(x)$ has exactly two critical points, one of them belongs to $[\lambda_3(h), \lambda_2(h)]$ and the second one is in $(\lambda_2(h), \lambda_1(h))$. In consequence,

$$F'(\lambda_2(h)) = 2\epsilon\lambda_2(h) - 1 + h|\beta_\kappa|e^{-h\lambda_2(h)} \leq 0.$$

However, this contradicts to the following relations: $F(\lambda(\theta)) = 0 = \Im F(\lambda(\theta)) =$

$$\theta\left(2\epsilon\lambda_2(h) - 1 + h|\beta_\kappa|e^{-h\lambda_2(h)}\frac{\sin(h\theta)}{h\theta}\right) < \theta\left(2\epsilon\lambda_2(h) - 1 + h|\beta_\kappa|e^{-h\lambda_2(h)}\right) \leq 0. \qquad \square$$

3.5.2 Proof of Lemma 17

The existence of the critical speed $c_0^{\mathfrak{L}}(h)$ which has properties mentioned in the lemma is a well known fact, and its proof is omitted. Clearly, it suffices to consider the case $\beta_0 > 0$. Next, if $c > c_0^{\mathfrak{L}}(h)$ then $0 < Q_0 := cq_0 - q_0^2 - \alpha_0 < \beta_0 e^{-chq_0}$ for some $q_0 = q_0(c) \in (0, \lambda)$, $c - 2q_0 > 0$. The change of variables $\omega := (z - q_0)(c - 2q_0)/Q_0$ transforms (3.7) into

$$\epsilon\omega^2 - \omega - 1 + \gamma e^{-\omega h'} = 0, \tag{3.30}$$

where

$$\epsilon := \frac{Q_0}{(c-2q_0)^2} > 0, \ \gamma := \beta_0 e^{-chq_0}/Q_0 > 1, \ h' := \frac{chQ_0}{c-2q_0} > 0.$$

Now, since inequalities (3.8) for equation (3.30) were established in [57, Lemma 2.3], we obtain that inequalities (3.8) hold also for equation (3.7) once $c > c_0^{\mathfrak{L}}(h)$.

Next, let $z_0 = x_0 + iy_0$ with $\Re z_0 = x_0 < \lambda$ be a complex root of (3.7). Then $0 > (2x_0 - c)|y_0| = \beta_0 e^{-chx_0}\sin(ch|y_0|)$ and therefore $ch|y_0| > \pi$.

Finally, the derivation of asymptotic representation and the proof of monotonicity of $c_0^{\mathfrak{L}}(h)$ repeat the arguments used in Subsection 3.5.1 (b) above and are omitted.

□

3.5.3 Proof of Lemma 18

Suppose that the graphs of the functions $c = c_0^{\mathfrak{L}}(h)$ and $c = c_\kappa^{\mathfrak{L}}(h)$ intersects at some $h = h_1$. Since $\lambda_2(h_1) < 0 < \lambda(h_1)$, after differentiating the first equation of (3.28) with respect to h, we obtain

$$\frac{d}{dh}c_\kappa^{\mathfrak{L}}(h)|_{h=h_1} = -\frac{c_\kappa^{\mathfrak{L}}(h_1)}{h_1} + \frac{(c_\kappa^{\mathfrak{L}}(h))^3}{2h_1\lambda_2(h_1)} < -\frac{c_0^{\mathfrak{L}}(h_1)}{h_1} + \frac{(c_0^{\mathfrak{L}}(h))^3}{2h_1\lambda(h_1)} = \frac{d}{dh}c_0^{\mathfrak{L}}(h)|_{h=h_1}.$$

This means that the above mentioned graphs have a unique transversal intersection on $\mathbb{R}_+$. As a consequence, if $\theta(\alpha_\kappa, \beta_\kappa) = \theta_1(\alpha_0, \beta_0)$ then $c_0^{\mathfrak{L}}(h) < c_\kappa^{\mathfrak{L}}(h)$ for all $h \geq 0$.

□

3.5.4 Proof of Lemma 19

It suffices to prove the inclusion Int $\mathcal{D}_{\mathfrak{L}} \subset \mathcal{D}_{\mathfrak{N}}$ where Int $\mathcal{D}_{\mathfrak{L}}$ denotes the interior of $\mathcal{D}_{\mathfrak{L}}$. So let us fix some $(h, c) \in$ Int $\mathcal{D}_{\mathfrak{L}}$. By the definition of $\mathcal{D}_{\mathfrak{N}}$, it holds automatically $(h, c) \in \mathcal{D}_{\mathfrak{N}}$ if there does not exist any monotone heteroclinic solution to equation (3.3) for the choosen pair (h, c). Therefore we can assume that (3.3) has a positive monotone front $\phi : \mathbb{R} \to (0, \kappa)$. Set $y(t) := \kappa - \phi(t)$ and $\mathbf{u}(t) = (\kappa - sy(t), \kappa - sy(t - ch))$, $s \in [0, 1]$. Then $y(t)$ satisfies the linear equation

$$x''(t) - cx'(t) + (\alpha_\kappa + N(t))x(t) + (\beta_\kappa + M(t))x(t - ch) = 0, \tag{3.31}$$

$$\text{where} \quad N(t) := \int_0^1 f_1(\mathbf{u}(t))ds - \alpha_\kappa, \ M(t) := \int_0^1 f_2(\mathbf{u}(t))ds - \beta_\kappa,$$

so that $N(+\infty) = M(+\infty) = 0$. Since the linear equation with constant coefficients

$$x''(t) - cx'(t) + \alpha_\kappa x(t) + \beta_\kappa x(t - ch) = 0 \tag{3.32}$$

is hyperbolic (i.e. it does not have eigenvalues on the imaginary axis) and $N(+\infty) = M(+\infty) = 0$, equation (3.31) possesses the property of exponential dichotomy on some infinite interval $[\tau, +\infty)$ (e.g. see [32, Lemma 4.3]). In particular, $y(+\infty) = y'(+\infty) = 0$ yield $y(t), y'(t) = O(e^{-\rho t}),\ t \to +\infty$, for some $\rho > 0$. Therefore, in view of $C^{1,\gamma}$-smoothness of f, we have that $M(t), N(t) = O(e^{-\rho\gamma t})$ at $t = +\infty$. Hence, invoking [44, Proposition 7.2] and Lemma 16, we obtain that $y(t) = ae^{\lambda_2 t} + o(e^{(\lambda_2 - \delta)t}),\ t \to +\infty$, for some a and $\delta > 0$. Note that $a \geq 0$ since we have $\phi(t) \in (0, \kappa)$ for all $t \in \mathbb{R}$. In fact, a can be found explicitly (e.g., see [26, Lemma 28]):

$$a = \mathrm{Res}_{z=\lambda_2} \frac{-1}{\chi_\kappa(z)} \int_{\mathbb{R}} e^{-zs} S(s)ds = \frac{-1}{\chi'_\kappa(\lambda_2)} \int_{\mathbb{R}} e^{-\lambda_2 s} S(s)ds > 0, \tag{3.33}$$

since

$$S(t) := N(t)y(t) + M(t)y(t-ch) = \alpha_\kappa(\phi(t) - \kappa) + \beta_\kappa(\phi(t-ch) - \kappa) - f(\phi(t), \phi(t-ch)) \geq 0,$$

is not identically zero. Indeed, if $S(t) \equiv 0$ then bounded and strictly decreasing $y(t)$ must satisfy (3.32). However, this is impossible due to the hyperbolicity of this equation. Thus $\Lambda_+(\phi(t)) = \lambda_2$. The proof of the relation $\Lambda_-(\phi(t)) = \lambda$ is completely similar and is left to the reader. □

Remark 2. The above argument needs a minor modification to imply the inclusion $\mathrm{Int}\ \mathfrak{D}^-_{\mathfrak{L}} := \{(h, c) : c \in (c_0^{\mathfrak{L}}(h), c_\kappa^-(h)),\ h \in [0, h_0^-]\} \subset \overline{\mathfrak{D}}_{\mathfrak{N}}$ stated in the proof of Theorem III.5. It suffices to show that a in (3.33) is positive for each $(h, c) \in \mathrm{Int}\ \mathfrak{D}^-_{\mathfrak{L}}$. Assuming, on the contrary, that $a = 0$, and again invoking [44, Proposition 7.2] and

Lemma 16, we find that $y(t) = be^{\lambda_1 t} + o(e^{(\lambda_1-\delta)t}),\ t \to +\infty$, for some $b \geq 0,\ \delta > 0$. Then $y(t)$ satisfies the equation

$$x''(t) - cx'(t) + \alpha_\kappa x(t) + \beta_\kappa^- x(t-ch) = Q(t),$$

where $Q(t) := \alpha_\kappa y(t) + \beta_\kappa^- y(t-ch) - f(\phi(t), \phi(t-ch)) \geq 0$. Furthermore, $Q(t) = -N(t)y(t) + (\beta_\kappa^- - \beta_\kappa - M(t))y(t-ch) = O(e^{\lambda_1 t}),\ t \to +\infty$, and we claim that $Q(t)$ is not identically zero. Indeed, by the proof of Lemma 16, we have that for $c \in (c_0^{\mathfrak{L}}(h), c_\kappa^-(h))$ the characteristic function $\chi_\kappa^-(z) := z^2 - cz + \alpha_\kappa + \beta_\kappa^- e^{-chz}$ has exactly three real zeros $\lambda_1^- < \lambda_2^- < 0 < \lambda_3^-$ and does not have any zero on $i\mathbb{R}$. Moreover, it is easy to see that $\lambda_1 \leq \lambda_1^- < \lambda_2^- \leq \lambda_2 < 0$. Therefore, if $Q(t) \equiv 0$ then bounded and strictly decreasing $y(t)$ must satisfy a hyperbolic equation with constant coefficient, a contradiction. Since $y(t) = O(e^{\lambda_2^- t}),\ t \to +\infty$, we also have that $y(t) = ce^{\lambda_2^- t} + o(e^{(\lambda_2^- - \delta_1)t}),\ t \to +\infty$, for some $c \geq 0,\ \delta_1 > 0$, where actually

$$c = \mathrm{Res}_{z=\lambda_2^-} \frac{-1}{\chi_\kappa^-(z)} \int_{\mathbb{R}} e^{-zs} Q(s)ds = \frac{-1}{(\chi_\kappa^-)'(\lambda_2^-)} \int_{\mathbb{R}} e^{-\lambda_2 s} Q(s)ds > 0.$$

This contradicts to the assumption $y(t) = O(e^{\lambda_1 t}),\ t \to +\infty$, and shows that $a > 0$.

Remark 3. The proof of Lemma 19 shows that, for each $(h, c) \in \mathrm{Int}\ \mathcal{D}_{\mathfrak{L}}$, it holds that $y(t) = O(e^{\lambda_2 t})$, $t \to +\infty$, even if the sub-tangency conditions of the lemma are not assumed. Similarly, $\phi(t) = O(e^{\lambda t}),\ t \to -\infty$. In order to establish the same growth estimates for the derivatives $y'(t), \phi'(t)$, we can proceed as follows. For example, let us consider $y'(t)$ at $+\infty$. After integrating (3.31) on $(t, +\infty)$, we obtain

$$y'(t) = cy(t) + \int_t^{+\infty} (\alpha_\kappa + N(s))y(s) + (\beta_\kappa + M(s))y(s-ch)ds = O(e^{\lambda_2 t}),\ t \to +\infty.$$

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] M.J. Ablowitz and A. Zeppetella. Explicit solution of fisher's equation for a special wave speed. *Bull. Math. Biol.*, 41:835–840, 1979.

[2] M. Aguerrea, C. Gomez, and S. Trofimchuk. On uniqueness of semi-wavefronts (diekmann-kaper theory of a nonlinear convolution equation re-visited). *Math. Ann.*, 354:73–109, 2012.

[3] M. Aguerrea, S. Trofimchuk, and G. Valenzuela. Uniqueness of fast traveling fronts in a single species reaction-diffusion equation with delay. *Proc. R. Soc. A*, 464:2591–2608, 2008.

[4] S. Ai. Traveling wave fronts for generalized fisher equations with spatio-temporal delays. *J. Differential Equations*, 232:104–133, 2007.

[5] P. Ashwin, M. V. Bartuccelli, and S. A. Gourley T. J. Bridges. Travelling fronts for the kpp equation with spatio-temporal delay. *Z. Angew. Math. Phys.*, 53:103–122, 2002.

[6] R. Benguria and M. C. Depassier. Variational characterization of the speed of propagation of fronts for the nonlinear diffusion equation. *Comm. Math. Phys.*, 175:221–227, 1996.

[7] H. Berestycki, G. Nadin, B. Perthame, and L. Ryzhik. The non-local fisher-kpp equation: travelling waves and steady states. *Nonlinearity*, 22:2813–2844, 2009.

[8] A. Boumenir and V.M. Nguyen. Perron theorem in the monotone iteration method for traveling waves in delayed reaction-diffusion equations. *J. Differential Equations*, 244:1551–1570, 2008.

[9] M. Bramson. *Convergence of solutions of the Kolmogorov equation to traveling waves.* Mem. Amer. Math. Soc., 1983.

[10] A. Calamai, C. Marcelli, and F. Papalini. A general approach for front-propagation in functional reaction-diffusion equations. *J. Dyn. Diff. Eqns.*, 21:567–593, 2009.

[11] J. Carr and A. Chmaj. Uniqueness of travelling waves for nonlocal monostable equations, proc. amer. math. soc. *Proc. Amer. Math. Soc.*, 132:2433–2439, 2004.

[12] J. Coville, J. Dávila, and S.Martínez. Nonlocal anisotropic dispersal with monostable nonlinearity. *J. Differential Equations*, 244:3080–3118, 2008.

[13] J. L. Daleckii and M. G. Krein. *The Asymptotic Solution of Linear Differential Systems*, volume 43 of *Translations of Mathematical Monographs.* Amer. Math. Soc., Providence, R.I., 1974.

[14] O. Diekmann and H. G. Kaper. On the bounded solutions of a nonlinear convolution equation. *Nonlinear Anal.*, 2:721–737, 1978.

[15] R.D. Driver. *Ordinary and delay differential equations.* Springer-Verlag, 1932.

[16] M. S. P. Eastham. *The Asymptotic Solution of Linear Differential Systems.* London Mathematical Society Monographs. Clarendon Press, Oxford, 1989.

[17] T. Erneux. *Applied delay differential equations.* Springer, 2009.

[18] J. Fang and J. Wu. Monotone travelling waves for delayed lotka-volterra competition systems. *Discrete Contin. Dynam. Systems*, 32:3043–3058, 2012.

[19] J. Fang and X.-Q. Zhao. Monotone wavefronts of the nonlocal fisher-kpp equation. *Nonlinearity*, 24:3043–3054, 2011.

[20] T. Faria, W. Huang, and J. Wu. Traveling waves for delayed reaction-diffusion equations with non-local response. *Proc. R. Soc. A*, 462:229–261, 2006.

[21] T. Faria and S. Trofimchuk. Non-monotone traveling waves in a single species reaction-diffusion equation with delay. *J. Differential Equations*, 228:357–376, 2006.

[22] T. Faria and S. Trofimchuk. Positive travelling fronts for reaction-diffusion systems with distributed delay. *Nonlinearity*, 23:2457–2481, 2010.

[23] R. A. Fisher. The wave of advance of advantageous gene. *Ann. Eugen.*, 7:355–369, 1937.

[24] G. Friesecke. Exponentially growing solutions for a delay-diffusion equation with negative feedback. *J. Differential Equations*, 98:1–18, 1992.

[25] B. Gilding and R. Kersner. *Travelling Waves in Nonlinear Diffusion-Convection Reaction.* Birkhäuser, 2004.

[26] A. Gomez and S. Trofimchuk. Monotone travelling wavefronts of the kpp-fisher delayed equation. *J. Differential equations*, 250:1767–1787, 2011.

[27] A. Gomez and S. Trofimchuk. Global continuation of monotone wavefronts. *prerpint arXiv:1210.6419*, 2012.

[28] K. Gopalsamy, X.-Z. He, and D.Q. Sun. Oscillations and convergence in a diffusive delay logistic equation. *Math. Nachr.*, 164:219–237, 1993.

[29] S. A. Gourley. Travelling front solutions of a nonlocal fisher equation. *J. Math. Biology*, 41:272–284, 2000.

[30] K.P. Hadeler. Transport, reaction, and delay in mathematical biology, and the inverse problem for traveling fronts. *J. Math. Sciences*, 149:1658–1678, 2008.

[31] J. Hale and S. M. Verduyn Lunel. *Introduction to functional differential equations.* Applied Mathematical Sciences. Springer- Verlag, 1993.

[32] J.K. Hale and X.-B. Lin. Heteroclinic orbits for retarded functional differential equations. *J. Differential Equations*, 65:175–202, 1985.

[33] K. Hasik and S. Trofimchuk. Slowly oscillating wavefronts of the kpp-fisher delayed equation. *preprint arXiv:1206.0484v1.*, 2012.

[34] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Springer-Verlag, 1981.

[35] G.E. Hutchingson. Circular causal systems in ecology. *Ann. N.Y. Acad. Sci.*, 50:221–246, 1948.

[36] K. Kobayashi. On the semilinear heat equation with time-lag. *Hiroshima Math. J.*, 7:459–472, 1977.

[37] A. Kolmogorov, I. Petrovskii, and N. Piskunov. Study of a diffusion equation that is related to the growth of a quality of matter, and its application to a biological problem. *Byul. Mosk. Gos. Univ. Ser. A Mat. Mekh.*, 1:1–26, 1937.

[38] M. K. Kwong and C. Ou. Existence and nonexistence of monotone traveling waves for the delayed fisher equation. *J. Differential Equations*, 249:728–745, 2010.

[39] K.-S. Lau. On the nonlinear diffusion equation of kolmogorov, petrovsky, and piscounov. *J. Differential Equations*, 59:44–70, 1985.

[40] X.. Liang and X.-Q Zhao. Spreading speeds and travelling waves for abstract monostable evolution systems. *J. Func. Anal.*, 259:857–903, 2010.

[41] S. Luckhaus. Global boundedness for a delay-differential equation. *Trans. Amer. Math. Soc.*, 294:767–774, 1986.

[42] S. Ma. Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem. *J. Differential Equations*, 171:294–314, 2001.

[43] S. Ma. Traveling waves for non-local delayed diffusion equations via auxiliary equations. *J. Differential Equations*, 237:259–277, 2007.

[44] J. Mallet-Paret. The fredholm alternative for functional differential equations of mixed type. *J. Dynam. Differential Equations*, 11:1–48, 1999.

[45] J. Mallet-Paret and G.R. Sell. Systems of delay differential equations, i: Floquet multipliers and discrete lyapunov functions. *J. Differential Equations*, 125:385–440, 1996.

[46] M. Mei, C.-K. Lin, C.-T. Lin, and J. So. Traveling wavefronts for time-delayed reaction-diffusion equation: (i) local nonlinearity. *J. Differential Equations*, 247:495–510, 2009.

[47] J.D. Murray. *Mathematical Biology: I. An introduction*, volume 17 of *Interdisciplinary Applied Mathematics*. Springer, 2002.

[48] C. Ou and J. Wu. Traveling wavefronts in a delayed food-limited population model. *SIAM J. Math. Anal.*, 39:103–125, 2007.

[49] S. pan. Asymptotic behavior of traveling fronts of the delayed fisher equation. *Nonlinear Analysis: Real World Applications*, 10:1173–1182, 2009.

[50] G. Raugel and K. Kirchgässner. Stability of fronts for a kpp-system. ii. the critical case. *J. Differential Equations*, 146:399–456, 1998.

[51] A. M. Samoilenko and N. A. Perestyuk. *Impulsive differential equations*. World Scientific Publishing, River Edge, NJ, 1995.

[52] K. Schaaf. Asymptotic behavior and traveling wave solutions for parabolic functional differential equations. *Trans. Amer. Math. Soc.*, 302:587–615, 1987.

[53] J. W.-H. So and X. Zou. Traveling waves for the diffusive nicholson's blowflies equation. *App. Math. Comp.*, 122.

[54] E. Trofimchuk, P. Alvarado, and S. Trofimchuk. On the geometry of wave solutions of a delayed reaction-diffusion equation. *J. Differential Equations*, 246:1422–11444, 2009.

[55] E. Trofimchuk, M. Pinto, and S. Trofimchuk. Pushed travelling fronts in monostable equations with monotone delayed reaction. *Discrete Contin. Dyn. Syst. A*, 2012.

[56] E. Trofimchuk, V. Tkachenko, and S. Trofimchuk. Slowly oscillating wave solutions of a single species reaction-diffusion equation with delay. *J. Differential Equations*, 245:2307–2332, 2008.

[57] E. Trofimchuk and S. Trofimchuk. Admissible wavefront speeds for a single species reaction-diffusion equation with delay. *Discrete Contin. Dyn. Syst. A*, 20:407–423, 2008.

[58] Z.-C. Wang, W.T. Li, and S. Ruan. Travelling wave fronts in reaction-diffusion systems with spatio-temporal delays. *J. Differential Equations*, 222:185–232, 2006.

[59] Z.-C. Wang, W.T. Li, and S. Ruan. Traveling fronts in monostable equations with nonlocal delayed effects. *J. Dynam. Differential Equations*, 20:573–607, 2008.

[60] D.V. Widder. *The Laplace Transform.* Princeton Mathematical Series, no. 6. Princeton University Press, 1941.

[61] J. Wu and X. Zou. Traveling wave fronts of reaction-diffusion systems with delay. *J. Dynam. Differential Equations*, 13:651–687, 2001.

[62] E. Yanagida. Irregular behavior of solutions for fisher's equation. *J. Dynam. Differential Equations*, 19:895–914, 2007.

[63] K. Yoshida. The hopf bifurcation and its stability for semilinear diffusion equations with time delay arising in ecology. *Hiroshima Math. J.*, 12:321–348, 1982.

[64] X. Zou. Delay induced traveling wave fronts in reaction diffusion equations of kpp-fisher type. *J. Comp. Appl Math.*, 146:309–321, 2002.