The double affine Hecke algebra and generalizations of Macdonald polynomials

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Introduction

This thesis focuses on generalizations of Macdonald polynomials in the superspace and m-symmetric settings. In particular, our goal will be to present the results in a unified manner starting from Macdonald polynomial theory. It is thus natural to first ask ourselves, what are Macdonald polynomials?

To understand this question, we first need to discuss symmetric functions. Symmetric functions naturally appear in many areas of mathematics, such as representation theory, where Schur functions arise as characters of the polynomial representations of Gl_n [**13**], and algebraic geometry, where they are in correspondence with the Schubert classes of the Grassmannians [**31**]. They also appear in physics, where the Jack symmetric functions are eigenfunctions of the Calogero-Sutherland model Hamiltonian [**19**],[**28**],[**27**]. All these families of symmetric functions have beautiful properties that make them fascinating combinatorial objects to study.

In 1987, Ian Macdonald defined a family of polynomials [**20**], now called Macdonald polynomials, that contains the aforementioned families as special cases. Despite the complexity of these new symmetric functions, Macdonald, using simple yet ingenious techniques, showed that they still satisfy a wealth of combinatorial properties.

Before we proceed, we need to briefly discuss these properties which can be divided into two major groups:

Constructive properties:

(1) Triangularity: Macdonald polynomials are triangular when expanded into the monomials, the most natural basis of symmetric functions.
(2) Orthogonality: Macdonald polynomials are orthogonal with respect to a natural scalar product.
(3) Eigenfunctions: Macdonald polynomials are eigenfunctions of a family of commuting q-difference operators.

Combinatorial properties:

(1) Norm: Macdonald polynomials are orthogonal, but not orthonormal, with respect to the natural scalar product. There is a combinatorial way to calculate their norm squared.
(2) Principal evaluation: for a certain evaluation, there is an elegant combinatorial formula.
(3) Duality: There is a fundamental symmetry satisfied by the evaluation of a Macdonald polynomial.
(4) Pieri rules: there is a combinatorial way to see the multiplication of a Macdonald polynomial by generators of the ring of symmetric function.

Among these properties, it is important to highlight the Pieri rules, as they do not only provide a beautiful and simple combinatorial formula for multiplication, but also have numerous applications. For instance, they yield a straightforward and combinatorial way to calculate Macdonald polynomials. It is also possible to derive the principal evaluation and the norm formulas using the Pieri rules.

In 1995, Ivan Cherednik defined a non-symmetric version of Macdonald polynomials as eigenfunctions of certain operators in the double affine Hecke algebra [**10**]. In this larger and more algebraic context, he was able to prove certain conjectures of Macdonald. Moreover, the properties of the symmetric Macdonald polynomials were recovered naturally using the double affine Hecke algebra [**23**]. Although the proofs are sometimes more technical, the methods are applicable to the generalizations of Macdonald polynomials that we will consider in this thesis. As such, non-symmetric Macdonald polynomials and the double affine Hecke algebra will be our starting point.

The connection between symmetric functions and physics also motivated the introduction of Macdonald polynomials in superspace [**8**]. These polynomials depend on two families of variables, one commuting and the other anticommuting. Impressively, these polynomials still exhibit fascinating constructive and combinatorial properties. In [**8**], constructive properties were demonstrated while in [**15**] the norm and the evaluation were obtained. At the start of this thesis, there were conjectures for the self-duality [**30**], for the explicit form of the coefficients of the Macdonald operator [**29**], and most challenging, for the Pieri rules [**14**].

Our work first focused on proving these three missing properties by establishing a connection between the supersymmetric Macdonald polynomials and the bisymmetric Macdonald polynomials, which allowed to use non-symmetric Macdonald theory in a more systematic way. Generally speaking, the Pieri rules can be derived using the self-duality and the explicit coefficients of the Macdonald operator. We thus concentrated on proving these two properties in the bisymmetric case. Although the self-duality was approachable with this method, obtaining an explicit form for the operator turned out to be quite complex and technical.

However, the operator $e_r(Y_1,\ldots,Y_N)$ presented in [**29**] can be written as a sum of products of the simpler bisymmetric operators $e_r(Y_1,\ldots,Y_m)$ and $e_r(Y_{m+1},\ldots,Y_N)$ in the following way

$$e_r(Y_1,\ldots,Y_N)=\sum_{i=0}^{r} e_{r-i}(Y_1,\ldots,Y_m)e_i(Y_{m+1},\ldots,Y_N). \tag{0.1}$$

We were able to obtain the explicit formula for the coefficients of those simpler operators, which then lead to two sets of Pieri rules (corresponding to the multiplication by $e_r(x_1,\ldots,x_m)$ and $e_r(x_{m+1},\ldots,x_N)$). This solved the Pieri rule problem in the bisymmetric setting.

Although this allows to obtain, in principle, the entire operator $e_r(Y_1,\ldots,Y_N)$, extracting the desired coefficients from 0.1 turns out to be combinatorially quite difficult. Even though the Pieri rule problem still remains unresolved in superspace, we are confident that this provides the best approach to solve it.

In order to find the explicit expansion of the two families of bisymmetric operators, we used reproducing kernels depending on certain regions in $\mathbb{Z}^2$. We needed in fact to determine how exactly the action of the Hecke operators on these reproducing kernels modified the regions in $\mathbb{Z}^2$ (this technique was later used to solve a similar problem in another context [**11**]). Although this approach led us to a closed form expression for the Pieri coefficients, this expression happened not to be the simplest one to compute. For this reason, we are currently seeking a better formula (involving familiar combinatorial concepts such as leg and arm-lengths) for the Pieri rules presented in [**14**].

In recent years, motivated by a combinatorial open problem related to Macdonald positivity, a new class of Macdonald polynomials, the m-symmetric Macdonald polynomials, were introduced in [**18**]. These polynomials are non-symmetric in the first m variables and symmetric in the remaining ones. The m-symmetric Macdonald polynomials actually interpolate between the symmetric and nonsymmetric world since when $m = 0$ the m-symmetric Macdonald polynomials are the usual symmetric Macdonald polynomials while when m is equal to the total number of variables, the m-symmetric Macdonald polynomials become the nonsymmetric Macdonald polynomials. Surprisingly, the m-symmetric Macdonald polynomials still possess many of the combinatorial and constructive properties that we mentioned earlier, for instance, triangularity and eigenoperator [**18**]. In the second part of our work, using the tools found in the bisymmetric case, we proved the orthogonality, principal evaluation, norm, and symmetry for m symmetric Macdonald polynomials [**12**].

As mentioned earlier, we aim to present our results concerning m-symmetric Macdonald polynomials and Macdonald polynomials in superspace in a unified manner using the double affine Hecke algebra. For this purpose, we have utilized several references, most of which are from L. Lapointe and I. Macdonald.

Here is the structure of the document:

First chapter: In this chapter, we define combinatorial objects that will serve as indices for our polynomial families. Although these objects appear in different contexts, we will adopt the viewpoints found in [**20**] and [**18**].

Second chapter: Here, we present the relevant definitions and results about the double affine Hecke algebra and the non-symmetric Macdonald polynomials. This will serve as the starting point in building the subsequent chapters. These results are drawn from [**23**] and [**22**].

Third chapter: We focus on defining the symmetric Macdonald polynomials using the non-symmetric Macdonald polynomials and demonstrating the aforementioned properties. This chapter will guide us in proving these properties for a wider class of functions. The techniques used in this chapter are mostly from [**20**], [**23**], and [**22**], and the combinatorial aspects are taken from [**25**].

Fourth chapter: The first part of this chapter, which introduces the m-symmetric functions and the m-symmetric Macdonald polynomials, is taken from [**18**]. The second part presents our preprint [**12**] in which we studied and demonstrated the aforementioned properties.

Fifth chapter: Here, we introduce the symmetric functions in superspace and the Macdonald polynomials in superspace. As mentioned earlier, we consider instead in this thesis bisymmetric Macdonald polynomials (which for our purposes are equivalent to the Macdonald polynomials in superspace). The final part of this chapter is thus devoted to translating into the bisymmetric language the properties of the Macdonald polynomials in superspace obtained in [**8**].

Sixth chapter: This chapter constitutes the core of this thesis. It includes the results of the article [**11**] in which we derive the duality and the Pieri rules for the bisymmetric Macdonald polynomials. Finally, the proof of the Pieri rules associated to the elementary symmetric functions in the variables $x_1, \dots, x_m$, which was not fully included in our article, are provided at the end of the chapter.

CHAPTER 1

Compositions and partitions

In this chapter we will introduce the combinatorial objects that will be needed in Chapters 2 and 3.

1. Partitions

Partitions play a crucial role in the theory of symmetric functions. We will see for instance that the bases of the space of symmetric functions are naturally indexed by these fundamental combinatorial objects.

DEFINITION 1. *A **partition** is a sequence $\lambda = (\lambda_1, \ldots, \lambda_N, \cdots)$ of non-negative integers in decreasing order, i.e.*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq \cdots$$

*and containing only finitely many non-zero terms. Note that we do not distinguish between two sequences which differ only by a string of zeros. For instance, $(2,1)$ and $(2,1,0,0)$ are the same partition. The non-zero entries λ_i are called the **parts** of λ. The number of parts is the **length** of λ, denoted by $l(\lambda)$. The sum of the parts is the **weight** (or **size**) of λ, denoted by $|\lambda|$, i.e.*

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots$$

REMARK 2. *Sometimes it is convenient to use a notation which indicates the number of times each integer occurs as a part:*

$$\lambda = (1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots)$$

where m_i is the number of times that i appear in λ, i.e.

$$m_i = \#\{j \mid \lambda_j = i\}$$

*is called the **multiplicity** of i in λ.*

DEFINITION 3. *Given $n \in \mathbb{N}$, we say that λ is a **partition of n** if $|\lambda| = n$, and we denote the set of partitions of n by $\mathscr{P}_n$*

EXAMPLE 4. *The set of partitions of* 5 *is*

$$\mathscr{P}_5 = \{(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)\}.$$

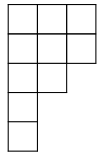
REMARK 5. *We can define the set of all partitions as*

$$\mathscr{P} = \bigcup_{n \geq 0} \mathscr{P}_n.$$

Up to this point, a partition has only been seen as a sequence. The following definition will associate a diagram to a partition, allowing us to enter the realm of combinatorics.

DEFINITION 6. *Let λ be a partition, we define the **diagram** of λ as the set of points $(i,j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$ for $1 \leq i \leq l(\lambda)$. When drawing these diagrams, we represent each point (i,j) by a square.*

Example 7. *The diagram of the partition* $(3,3,2,1,1)$ *is*



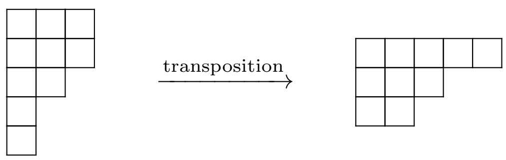
Remark 8. *When we write the symbol* λ *for a partition, it can either stand for a sequence or a diagram.*

Definition 9. *The* ***conjugate*** *of a partition, denoted by* λ', *is obtained by reflecting the diagram of the original partition* λ *along its main diagonal. Algebraically, this reflection is described by the following formula:*

$$\lambda'_i = \#\{j \mid \lambda_j \geq i\}.$$

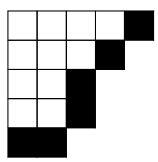
Remark 10. *We have that* $\lambda'_1 = l(\lambda)$, $\lambda_1 = l(\lambda')$ *and* $(\lambda')' = \lambda$.

Example 11. *If* $\lambda = (3,3,2,1,1)$ *then* $\lambda' = (5,3,2)$ *because*



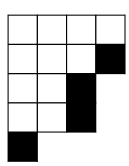
Definition 12. *Given two partitions* μ *and* λ, *we shall write* $\lambda \subset \mu$ *if the diagram of* μ *contains the diagram of* λ, *i.e.* $\lambda_i \leq \mu_i$ *for all* $i \geq 1$. *In this case, we define the* ***skew diagram*** *as the diagram obtained by the difference* $\mu - \lambda$ *which we will denote* μ/λ.

Example 13. *If* $\mu = (5,4,3,3,2)$ *and* $\lambda = (4,3,2,2)$ *we have that the diagram of* μ/λ *is the shaded part of the following diagram*



Definition 14. *We say that a skew diagram* μ/λ *is an* r-***vertical strip*** *(resp. horizontal strip) if the diagram* μ/λ *has* r *boxes and contains at most one box in each row (resp. each column).*

Example 15. *If* $\mu = (4,4,3,3,1)$ *and* $\lambda = (4,3,2,2)$ *we have that* μ/λ *is a* 4*-vertical strip with diagram*



It is crucial to define a partial order on partitions which will later enable us to compare symmetric functions.

Definition 16. *We can define an order on* $\mathscr{P}_n$. *Let* λ, μ *in* $\mathscr{P}_n$, *we define the* ***dominance order*** *as*

$$\lambda \geq \mu \Longleftrightarrow \sum_{j=1}^{k} \lambda_j \geq \sum_{j=1}^{k} \mu_j \text{ for all } k \geq 1$$

Example 17. *Taking* $\lambda = (4,3,2,2)$ *and* $\mu = (3,3,3,2)$ *we have that* $\lambda \geq \mu$.

Remark 18. *This order is not a total ordering. For example,* $\lambda = (3,1,1,1)$ *and* $\mu = (2,2,2)$ *are not comparable in this order.*

There are important combinatorial concepts associated to a partition will prove useful in the following chapters.

Definition 19. *For each partitions we define*

$$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$$

Example 20. *For example, if* $\lambda = (3,3,2,1,1)$ *we have that* $n(\lambda) = 14$. *We may see this number diagrammatically by taking the sum of the numbers in the following diagram:*

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

Definition 21. *For a box* $s = (i,j)$ *in a partition* λ *(i.e., in row* i *and column* j*), we introduce the* ***arm-lengths*** *and* ***leg-lengths*** *as*

$$a_\lambda(s) = \lambda_i - j \quad \text{and} \quad \ell_\lambda(s) = \lambda'_j - i$$

The leg-length thus corresponds to the number of cells in λ *strictly below* s *(and in the same column) while the arm-length corresponds to the number of cells in* λ *strictly to the right of* s *(and in the same row).*

Example 22. *The values of* $a(s)$ *and* $\ell(s)$ *in each cell of the diagram of* $\lambda = (3,3,2,1,1)$ *are*

$$\begin{array}{|c|c|c|}\hline 24 & 12 & 01 \\ \hline 23 & 11 & 00 \\ \hline 12 & 00 & \multicolumn{1}{c}{} \\ \cline{1-2} 01 & \multicolumn{2}{c}{} \\ \cline{1-1} 00 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

2. Compositions

Just as partitions index symmetric functions, compositions will index polynomials in general. Compositions lose the aesthetic appeal that diagrams provide for partitions, making them more intricate to compute. Hence, we will define the diagram of a composition using what has been defined for partitions in the last section.

Definition 23. *An element* $\eta = (\eta_1, \ldots, \eta_N)$ *of* $\mathbb{Z}^N_{\geq 0}$ *is called a (weak)* ***composition*** *with* N *parts (or entries).*

Definition 24. *An element* w *of the symmetric group* $\mathfrak{S}_N$ *acts on a vector* $(v_1, \ldots, v_N) \in \mathbb{Z}^N$ *as* $w(v_1, \ldots, v_N) = (v_{w^{-1}(1)}, \ldots, v_{w^{-1}(N)})$.

We let w_η be the unique minimal length permutation in $\mathfrak{S}_N$ such that

$$\eta = w_\eta \eta^+$$

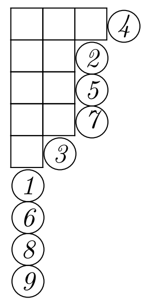
where η^+ is the partition obtained sorting out η.

Example 25. *If* $\eta = (1,3,0,2,1)$ *we have that* $w_\eta = (13542)$ *(in the cycle notation) and* $\eta^+ = (3,2,1,1,0)$.

It will prove convenient to represent a composition by a Young (or Ferrers) diagram.

Definition 26. *The **diagram** corresponding to η is the Young diagram of η^+ with an i-circle (a circle filled with an i) added to the right of the row of size η_i (if there are many rows of size η_i, the circles are ordered from top to bottom in increasing order).*

Example 27. *Given $\eta = (0, 2, 1, 3, 2, 0, 2, 0, 0)$, we have that the corresponding diagram is*



Definition 28. *The **Bruhat order** on compositions is defined as follows:*

$$\nu \prec \eta \quad \text{iff} \quad \nu^+ < \eta^+ \quad \text{or} \quad \nu^+ = \eta^+ \quad \text{and} \quad w_\eta < w_\nu,$$

where we recall that w_η is the unique permutation of minimal length such that $\eta = w_\eta \eta^+$. In the Bruhat order on the symmetric group $\mathfrak{S}_N$, $w_\eta < w_\nu$ iff w_η can be obtained as a proper subword of w_ν.

Example 29. *If $\eta = (1, 3, 0, 2, 1)$ and $\nu = (0, 1, 3, 2, 1)$ we have that $\eta^+ = (3, 2, 1, 1, 0) = \nu^+$, $w_\eta = s_4 s_2 s_3 s_1$ and $w_\nu = s_4 s_2 s_3 s_1 s_2 s_1$, which implies that $\nu \prec \eta$.*

CHAPTER 2

The non-symmetric Macdonald polynomials and the double affine Hecke algebra

In this chapter, we will introduce the theory of nonsymmetric Macdonald polynomials. We will extract certain results from the presentation [**22**] of the double affine Hecke algebras and the non-symmetric Macdonald polynomials for reduced root systems. Additionally, following [**23**], we will present more explicit results in the A_n case.

1. Double affine Hecke algebra

The non-symmetric Macdonald polynomials can be defined as the common eigenfunctions of the Cherednik operators [**10**], which are operators that belong to the double affine Hecke algebra and act on the ring $\mathbb{Q}(q,t)[x_1,\dots,x_N]$. In this section we will introduce all the necessary tools to define Cherednik operators.

Given a permutation $\sigma\in\mathfrak{S}_N$, the element K_σ acts on $f\in\mathbb{Q}(q,t)[x_1,\dots,x_N]$ in the following way:

$$K_\sigma f(x_1,\dots,x_N)=f(x_{\sigma(1)},\dots,x_{\sigma(N)})$$

In the case of an elementary permutation $\sigma=s_i=(i,i+1)$, we use $K_{i,i+1}$ for $K_{(i,i+1)}$.

Definition 30. *We define the generators T_i of the affine Hecke algebra as*

$$T_i=t+\frac{tx_i-x_{i+1}}{x_i-x_{i+1}}(K_{i,i+1}-1),\quad i=1,\dots,N-1,$$

and

$$T_0=t+\frac{qtx_N-x_1}{qx_N-x_1}(K_{1,N}\tau_1\tau_N^{-1}-1)\,,$$

where $\tau_i f(x_1,\dots,x_i,\dots,x_N)=f(x_1,\dots,qx_i,\dots,x_N)$ is the q-shift operator.

Sometimes it will prove convenient to write T_i as

$$T_i=\frac{x_{i+1}(t-1)}{x_{i+1}-x_i}+\frac{tx_i-x_{i+1}}{x_i-x_{i+1}}K_{i,i+1}.$$

Example 31. *If $\sigma=(321)$ then $\sigma=s_2s_1$ and*

$$\begin{aligned}T_{(321)}=T_2T_1\quad &=\left(\frac{x_3(t-1)}{x_3-x_2}+\frac{tx_2-x_3}{x_2-x_3}K_{2,3}\right)\left(\frac{x_2(t-1)}{x_2-x_1}+\frac{tx_1-x_2}{x_1-x_2}K_{1,2}\right)\\ &=\frac{x_3(t-1)}{x_3-x_2}\frac{x_2(t-1)}{x_2-x_1}+\frac{x_3(t-1)}{x_3-x_2}\frac{tx_1-x_2}{x_1-x_2}K_{1,2}+\frac{tx_2-x_3}{x_2-x_3}\frac{x_3(t-1)}{x_3-x_1}K_{2,3}\\ &\quad+\frac{tx_2-x_3}{x_2-x_3}\frac{tx_1-x_3}{x_1-x_3}K_{2,3}K_{1,2}.\end{aligned}$$

The T_i's satisfy the relations $(0 \le i \le N-1)$:

$$\begin{aligned}
&(T_i - t)(T_i + 1) = 0 \\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\
&T_i T_j = T_j T_i\,, \quad i - j \neq \pm 1 \mod N
\end{aligned}$$

where the indices are taken modulo N.

REMARK 32. *The quadratic relation $(T_j - t)(T_j + 1) = 0$ allows to compute the inverse of T_j explicitly*

$$\bar{T}_j := T_j^{-1} = t^{-1} - 1 + t^{-1} T_j.$$

To be more precise,

$$\bar{T}_j = t^{-1}\left(\frac{x_j(1-t)}{x_j - x_{j+1}} + \frac{t x_j - x_{j+1}}{x_j - x_{j+1}} K_{j,j+1}\right).$$

REMARK 33. *We extend the definition of the Hecke operator to any element $\sigma \in \mathcal{S}_N$ in the following way. If $\sigma = s_{i_1} \cdots s_{i_\ell}$ is a reduced decomposition then T_σ is given by*

$$T_\sigma = T_{i_1} \cdots T_{i_\ell}.$$

To define the Cherednik operators, we also need to introduce the operator ω defined as:

$$\omega = K_{N-1,N} \cdots K_{1,2}\, \tau_1.$$

We note that $\omega T_i = T_{i-1}\omega$ for $i = 2, \ldots, N-1$. We are now in position to define the Cherednik operators:

DEFINITION 34. *for $i \in 1, \ldots, N$, the* ***Cherednik operators*** *are*

$$Y_i = t^{-N+i} T_i \cdots T_{N-1} \omega \bar{T}_1 \cdots \bar{T}_{i-1}.$$

EXAMPLE 35. *For $N = 4$ we have*

- $Y_1 = t^{-3} T_1 T_2 T_3 \omega_4$,
- $Y_2 = t^{-2} T_2 T_3 \omega_4 \bar{T}_1$,
- $Y_3 = t^{-1} T_3 \omega_4 \bar{T}_1 \bar{T}_2$,
- $Y_4 = \omega_4 \bar{T}_1 \bar{T}_2 \bar{T}_3$.

The Cherednik operators obey the following relations:

$$\begin{aligned}
T_i\, Y_i &= Y_{i+1} T_i + (t-1) Y_i \\
T_i\, Y_{i+1} &= Y_i T_i - (t-1) Y_i \\
T_i Y_j &= Y_j T_i \quad \text{if } j \neq i, i+1.
\end{aligned} \tag{1.1}$$

It can be easily deduced from these relations that

$$(Y_i + Y_{i+1}) T_i = T_i (Y_i + Y_{i+1}) \qquad \text{and} \qquad (Y_i Y_{i+1}) T_i = T_i (Y_i Y_{i+1}). \tag{1.2}$$

2. Non-symmetric Macdonald polynomials

The Cherednik operators Y_i's commute among each others, $[Y_i, Y_j] = 0$, and can be simultaneously diagonalized. Their eigenfunctions are the (monic) non-symmetric Macdonald polynomials (labeled by compositions).

DEFINITION 36. *For $x = (x_1, \ldots, x_N)$, the non-symmetric Macdonald polynomial $E_\eta(x; q, t)$ is the unique polynomial with coefficients in $\mathbb{Q}(q,t)$ that is triangularly related to the monomials*

$$E_\eta(x; q, t) = x^\eta + \sum_{\nu \prec \eta} b_{\eta\nu}(q,t)\, x^\nu$$

where $\prec$ is the Bruhat order on compositions defined in 28.

The non-symmetric Macdonald polynomials are simultaneous eigenfunctions of the Cherednik operators.

PROPOSITION 37. *For all $i = 1, \ldots, N$,*

$$Y_i E_\eta = \bar{\eta}_i E_\eta, \qquad \textit{where} \qquad \bar{\eta}_i = q^{\eta_i} t^{1-r_\eta(i)}$$

with $r_\eta(i)$ standing for the row (starting from the top) in which the i-circle appears in the diagram of η.

EXAMPLE 38. *We have:*

(1) $Y_1 E_{(2,0,3)} = q^2 t^{1-2} E_{(2,0,3)}$
(2) $Y_2 E_{(2,0,3)} = q^0 t^{1-3} E_{(2,0,3)}$
(3) $Y_3 E_{(2,0,3)} = q^3 t^{1-1} E_{(2,0,3)}$

The Cherednik operators have a triangular action on monomials [**21**].

PROPOSITION 39. *The action of Y_i is given by,*

$$Y_i x^\eta = \bar{\eta}_i x^\eta + \text{ smaller terms}$$

where "smaller terms" means that the remaining monomials x^ν appearing in the expansion are such that $\nu \prec \eta$ in the Bruhat order.

3. Properties

The following properties will be fundamental in the proofs of many properties of the Macdonald polynomials and their variations. All the results in this section are facts that we will not prove and which were primarily extracted from [**21**] and [**23**].

3.1. Stability.

The first one expresses the stability of the polynomials E_η with respect to the number of variables (see e.g. [**23**, eq. (3.2)]):

$$E_\eta(x_1, \ldots, x_{N-1}, 0; q, t) = \begin{cases} E_{\eta_-}(x_1, \ldots, x_{N-1}; q, t) & \text{if } \eta_N = 0\,, \\ 0 & \text{if } \eta_N \neq 0\,. \end{cases}$$

where $\eta_- = (\eta_1, \ldots, \eta_{N-1})$.

3.2. Action of T_i.

The second one gives the action of the operators T_i on E_η:

$$T_i E_\eta = \begin{cases} \left(\frac{t-1}{1-\delta_{i,\eta}^{-1}}\right) E_\eta + t E_{s_i \eta} & \text{if } \eta_i < \eta_{i+1}\,, \\ t E_\eta & \text{if } \eta_i = \eta_{i+1}\,, \\ \left(\frac{t-1}{1-\delta_{i,\eta}^{-1}}\right) E_\eta + \frac{(1-t\delta_{i,\eta})(1-t^{-1}\delta_{i,\eta})}{(1-\delta_{i,\eta})^2} E_{s_i \eta} & \text{if } \eta_i > \eta_{i+1}\,, \end{cases}$$

where $\delta_{i,\eta} = \bar{\eta}_i / \bar{\eta}_{i+1}$ and $s_i \eta = (\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \eta_i, \eta_{i+2}, \ldots, \eta_N)$.

3.3. Recursivity.

The third property, together with the previous one, allows to construct the non-symmetric Macdonald polynomials recursively. Given $\Phi_q = t^{1-N} T_{N-1} \cdots T_1 x_1$, we have that [**5**]

$$\Phi_q E_\eta(x; q, t) = t^{r_\eta(1)-N} E_{\Phi\eta}(x; q, t)$$

where $\Phi\eta = (\eta_2, \eta_3, \ldots, \eta_{N-1}, \eta_1 + 1)$.

3.4. Symmetry. For the last property we need introduce the following specialization

Definition 40. *We define the evaluation $u_\emptyset$ on any function $f(x)$ as*

$$u_\emptyset\big(f(x_1,\ldots,x_N)\big) = f\left(1,t,\ldots,t^{N-1}\right) \tag{3.1}$$

and for any $f(x)$ and $g(x)$ Laurent polynomials in $x_1,\ldots,x_N$, we define

$$[f(x),g(x)] := u_\emptyset\left(f(Y^{-1})g(x)\right).$$

The symmetry says that

$$[f,g]=[g,f]$$

for any Laurent polynomials $f(x)$ and $g(x)$ in $x_1,\ldots,x_N$.

4. Symetrization operators

In this section we will introduce operators that will allow us to define the classical Macdonald polynomials and their variations. We will later see that the symmetric Macdonald polynomials are essentially symmetrized versions of the non-symmetric Macdonald polynomials.

Definition 41. *Let be $I\subset[N]$, we introduce the the* ***symmetrization*** *and* ***antisymmetrization*** *operators*

$$\mathcal{S}_I=\sum_{\sigma\in\mathfrak{S}_I}K_\sigma \qquad\text{and}\qquad \mathcal{A}_I=\sum_{\sigma\in\mathfrak{S}_I}(-1)^{\ell(\sigma)}K_\sigma \tag{4.1}$$

together with the t-symmetrization and t-antisymmetrization operators

$$\mathcal{S}^t_I=\sum_{\sigma\in\mathfrak{S}_I}T_\sigma \qquad\text{and}\qquad \mathcal{A}^t_I=\sum_{\sigma\in\mathfrak{S}_I}(-1)^{\ell(\sigma)}T_\sigma \tag{4.2}$$

where $\mathfrak{S}_I$ stands for the permutation group of the elements in I

For simplicity, when $I=[N]$ we denote the operator as

$$\mathcal{S}^t_N=\sum_{\sigma\in S_N}T_\sigma \tag{4.3}$$

There are some relations between the operators introduced in Definition 41. For this, we shall introduce the next notation.

Definition 42. *Let $[N]=\{1,\ldots,N\}$. For I a subset of $[N]$, recall that*

$$\Delta_I(x)=\prod_{\substack{i,j\in I\\ i<j}}(x_i-x_j)\,,\qquad \Delta^t_I(x)=\prod_{\substack{i,j\in I\\ i<j}}(tx_i-x_j)\qquad\text{and}\qquad A_I(x)=\prod_{\substack{i,j\in I\\ i<j}}\left(\frac{tx_i-x_j}{x_i-x_j}\right) \tag{4.4}$$

For simplicity, when $I=[m]=\{1,\ldots,m\}$, we will use the notation $\Delta_m(x)$, $\Delta^t_m(x)$ or $A_m(x)$ instead of $\Delta_{[m]}(x)$, $\Delta^t_{[m]}(x)$ or $A_{[m]}(x)$.

The following proposition will relate the t-symmetrization operators and the symmetrization operators defined in Definition 41,

Proposition 43. *We have the follow relations*

$$\mathcal{S}^t_I=\mathcal{S}_I\left(\prod_{\substack{i,j\in I\\ j<i}}\left(\frac{tx_i-x_j}{x_i-x_j}\right)\right)\qquad\textit{and}\qquad \mathcal{A}^t_I=t^{\binom{|I|}{2}}\frac{\Delta^t_I}{\Delta_I}\mathcal{A}_I$$

Proposition 44. *We have the following equalities*

$$T_i\,\mathcal{S}^t_{m+1,N}=\mathcal{S}^t_{m+1,N}\,T_i=t\,\mathcal{S}^t_{m+1,N}\qquad\text{for } i=m+1,\ldots,N-1$$

CHAPTER 3

Symmetric functions

Symmetric functions naturally appear in several areas of mathematics. In this chapter, we will provide their formal definition, offer some examples of bases, and then proceed to define symmetric Macdonald polynomials by symmetrizing non-symmetric Macdonald polynomials. For this, we will heavily rely on [**23**]. Additionally, we will demonstrate some fundamental properties of Macdonald polynomials. While there are elegant ways to prove these properties, as is done in [**20**], we will employ the theory of non-symmetric Macdonald polynomials presented in Chapter 2, utilizing techniques outlined in [**21**], [**23**], and [**25**]. The reason for taking this approach is that these techniques will later be applied to the variations of Macdonald polynomials that we will consider in Chapters 4 and 5.
This chapter thus has two main purposes. First, to introduce and motivate Macdonald polynomials and their properties, and second, to provide a guide on how to prove the analogous properties in wider contexts.

1. The ring of symmetric functions

In this section we will introduce the basic topics in symmetric function theory. Most of this chapter is taken from [**21**].

The space of **symmetric polynomials** in n variables is the space of polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$ that are invariant under the symmetric group action, i.e.

$$\Lambda_n = \mathbb{Q}[x_1, \ldots x_n]^{S_n} = \{f \in \mathbb{Q}[x_1, \ldots x_n] \mid \sigma \cdot f = f \quad \text{for all } \sigma \in S_n\}.$$

Example 45. *For example, we have*

(1) $f(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2 \in \Lambda_2$,
(2) $f(x_1, x_2, x_3) = x_1x_2 + x_1^2 + x_2x_3 + x_2^2 + x_1x_3 + x_3^2 \in \Lambda_3$.

This space has a graded structure given by

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$$

where Λ_n^k is the space of homogeneous symmetric polynomials of degree k in n variables. In this theory, it is often more convenient to work with infinitely many variables given that when the number of variables is large enough the symmetric polynomials essentially cease to depend on the number of variables. To do this, consider the homomorphism

$$\begin{aligned} \rho \colon \mathbb{Q}[x_1, \ldots x_{n+1}] &\to \mathbb{Q}[x_1, \ldots x_n] \\ f &\mapsto f|_{x_{n+1}=0}. \end{aligned}$$

Its restriction to the space Λ_n^k is bijective for all $n \geq k$, so we can take the inverse limit as a graded ring

$$\Lambda^k = \varprojlim \Lambda_n^k$$

where the elements in Λ^k are sequences $(f_n)_{n\geq 0}$ with $f_n \in \Lambda_n^k$ with the condition $f_{n+1}(x_1,\ldots,x_n,0) = f_n(x_1,\ldots,x_n)$ for all $n \geq k$, this means, the polynomials are essentially the same for $n \geq k$. We define the ring of **symmetric functions** as

$$\mathbf{\Lambda} = \bigoplus_{k\geq 0} \Lambda^k$$

Remark 46. *We have by* (30) *that if a polynomial $f(x_1,\ldots,x_N)$ is such that $T_i f(x_1,\ldots,x_N) = tf(x_1,\ldots,x_N)$, then $f(x_1,\ldots,x_N)$ is symmetric in the variables x_i and x_{i+1}. Moreover, for any polynomial $f(x_1,\ldots,x_N)$, we get from Proposition 44 that $\mathcal{S}_N^t f(x_1,\ldots,x_N)$ is symmetric. We thus conclude that symmetric polynomials and t-symmetric polynomials are the same.*

2. Bases of the ring of symmetric functions

We want to study certain bases of the ring of symmetric functions, putting a special emphasis on their combinatorial properties.

Monomial symmetric functions. It is an elementary fact that the monomials x^η, fol all compositions η, form a basis for the space $\mathbb{Q}[x_1,\ldots,x_n]$, where

$$x^\eta = x_1^{\eta_1}\cdots x_n^{\eta_n}$$

We can thus naturally define a family of monomial symmetric polynomials by acting with the symmetrization operator on x^η, i.e.

$$\mathcal{S}_N \cdot x^\eta = \sum_{\sigma\in S_N} \sigma\cdot x^\eta = \sum_{\sigma\in S_N} x^{\sigma(\eta)}.$$

Let $G_{N,\lambda} = \{\sigma \in S_N \mid \sigma\lambda = \lambda\}$ be the stabilizer subgroup of λ. Note that each monomial appears $|G_{N,\lambda}|$ times in the r.h.s. of the previous equation

$$\sum_{\sigma\in S_N} x^{\sigma(\eta)} = |G_{N,\lambda}| \sum_{\sigma\in S_N/G_{N,\lambda}} x^{\sigma(\eta)}.$$

In order to have monomials without coefficients, we define the monomial symmetric polynomial as follows.

Definition 47. *Given λ a partition, we define the* ***monomial symmetric function*** *as*

$$m_\lambda(x_1,\ldots,x_N) = \frac{1}{|G_{N,\lambda}|}\mathcal{S}_N\cdot x^\eta$$

where η is any composition such that $\eta^+ = \lambda$.

Example 48. *We have some examples*

- $m_{(1)}(x_1,x_2) = x_1+x_2,$
- $m_{(2,1,1)}(x_1,x_2,x_3) = x_1^2x_2x_3 + x_2^2x_1x_3 + x_3^2x_1x_3,$
- $m_{(2,2)}(x_1,x_2,x_3) = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2.$

Remark 49. *Note that the set of all the m_λ's, where λ runs over all partitions, is a natural basis of $\mathbf{\Lambda}$.*

Elementary symmetric functions.

Definition 50. *For $r \geq 0$, we define the* ***elementary symmetric function*** *as*

$$e_r = m_{(1^r)}$$

We can then extend this definition to a partition λ in the following way

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_n}$$

Example 51. *Here are some examples of elementary symmetric functions.*

- $e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$,
- $e_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$,
- $e_{(3)}(x_1, x_2, x_3) = x_1x_2x_3$,
- $e_{(2,1,1)}(x_1, x_2, x_3) = (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3)^2$.

The generating function for the elementary symmetric functions is given by

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t).$$

Proposition 52. *The elementary symmetric functions have a triangular decomposition into monomial symmetric functions*

$$e_{\lambda'} = m_\lambda + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu$$

where $a_{\lambda\mu}$ are non-negative integers, and $\mu < \lambda$ is the dominance order introduced in Definition 16.

Homogeneous symmetric functions.

Definition 53. *For $r \geq 0$, we define the* ***complete symmetric function*** *as*

$$h_r = \sum_{|\lambda| = r} m_\lambda$$

We can extend again this definition to a partition λ in the following way

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_n}.$$

Example 54. *Here are some examples of homogeneous symmetric functions*

- $h_1(x_1, x_2, x_3) = m_{(1)}(x_1, x_2, x_3) = x_1 + x_2 + x_3$,
- $h_2(x_1, x_2, x_3) = m_{(2)}(x_1, x_2, x_3) + m_{(1,1)}(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$,
- $h_3(x_1, x_2, x_3) = m_{(3)}(x_1, x_2, x_3) + m_{(2,1)}(x_1, x_2, x_3) + m_{(1,1,1)}(x_1, x_2, x_3)$,
- $h_{(2,1)}(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3)$.

The generating function for the complete symmetric function is given by

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} \frac{1}{1 - x_i t}.$$

Power sum symmetric functions.

Definition 55. *For $r \geq 0$, we define the* ***power sum symmetric function*** *as*

$$p_r = \sum_{i=1}^{N} x_i^r$$

As usual, we extend this definition to a partition λ:

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_n}.$$

EXAMPLE 56. *Here are some examples of power sum symmetric functions*

- $p_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$,
- $p_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$,
- $p_{(2,1)}(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3)$.

The generating function for the power symmetric function is given by

$$P(t) = \sum_{r\geq 0} p_r t^r = \prod_{i\geq 1} \frac{x_i}{1 - x_i t}.$$

3. Symmetric Macdonald polynomials

In this section, we will define Macdonald polynomials, demonstrate their stability as the number of variables increases, and provide a characterization using the Macdonald operator. The results of this chapter are taken mostly from [**23**].

DEFINITION 57. *We define the symmetric Macdonald polynomials as*

$$P_\lambda(x_1, \ldots, x_N; q, t) = \frac{1}{u_{\lambda,N}(t)} \mathcal{S}_N^t \, E_{\eta_{\lambda,N}}(x_1, \ldots, x_N; q, t)$$

where $\eta_{\lambda,N} = (\lambda_N, \ldots, \lambda_2, \lambda_1)$ *with the normalization constant* $u_{\lambda,N}(t)$ *given by*

$$u_{\lambda,N}(t) = \left(\prod_{i\geq 0} [n_\lambda(i)]_{t^{-1}}! \right) t^{N(N-1)/2}$$

where $n_\lambda(i)$ *is the number of entries in* $\lambda_1, \ldots, \lambda_N$ *that are equal to* i *(note that* i *can be equal to zero), and where*

$$[k]_q = \frac{(1-q)(1-q^2)\cdots(1-q^k)}{(1-q)^k}.$$

REMARK 58. *Observe that the normalization constant* $u_{\lambda,N}(t)$ *is chosen such that the coefficient of* m_λ *in* $P_\lambda(x; q, t)$ *is equal to 1.*

The first consequence we draw from the non-symmetric world is analogous to the stability in Section 3.1, stating that as the number of variables increases, Macdonald polynomials are essentially the same. As such, Macdonald polynomials only depend on the partition λ.

PROPOSITION 59. ***[Stability]*** *The symmetric Macdonald polynomial* P_λ *is stable with respect the number of variables, that is,*

$$P_\lambda(x_1, \ldots, x_{N-1}, 0; q, t) = \begin{cases} P_\lambda(x_1, \ldots, x_{N-1}; q, t) & \text{if } N > \ell(\lambda) \\ 0 & \text{otherwise} \end{cases}.$$

PROOF. From the definition of P_λ it suffices to prove that

$$[\mathcal{S}_N^t E_{\eta_{\lambda,N}}]_{x_N=0} = \begin{cases} \dfrac{u_{\bar\lambda,N-1}(t)}{u_{\lambda,N}(t)} \mathcal{S}_{N-1}^t E_{\overline{\eta}_{\lambda,N-1}} & if \quad \lambda_N = 0 \\ \\ 0 & if \quad \lambda_N \neq 0 \end{cases}$$

where $\bar\lambda = (\lambda_1, \ldots, \lambda_{N-1})$ and $\overline{\eta}_{\lambda,N-1} = (0, \lambda_{N-1}, \ldots, \lambda_1)$, note that from Property **??**

$$\mathcal{S}_N^t = \mathcal{S}_{N-1}^t(1 + T_{N-1} + T_{N-1}T_{N-2} + \cdots + T_{N-1}\cdots T_1)$$

so that

$$[\mathcal{S}_N^t E_{\eta_{\lambda,N}}]_{x_N=0} = [\mathcal{S}_{N-1}^t(1 + T_{N-1} + T_{N-1}T_{N-2} + \cdots + T_{N-1}\cdots T_1) E_{\overline{\eta}_{\lambda,N-1}}]_{x_N=0}$$

Now, if λ has $k = n_\lambda(0)$ zero entries, it is easy to see from Properties 3.1 and 3.2 that

$$[T_{N-1}\cdots T_i E_\eta]_{x_N=0} = \begin{cases} t^{N-i} E_{\overline{\eta}_{\lambda,N-1}} & if \quad i \le k \\ \\ 0 & if \quad i > k \end{cases}$$

which then implies that

$$[\mathcal{S}_N^t E_{\eta_{\lambda,N}}]_{x_N=0} = \begin{cases} t^{N-k}[k]_t \mathcal{S}_{N-1}^t E_{\overline{\eta}_{\lambda,N-1}} & if \quad i \le k \\ \\ 0 & if \quad i > k \end{cases}$$

with $t^{N-k}[k]_t = c_{\bar{\lambda}}/c_\lambda$. □

The characterization of Macdonald polynomials provided in Definition 57 is not very practical because we first need to construct non-symmetric Macdonald polynomials and then take a sum over the symmetric group. This process can be computationally challenging for large values of N. In this section, we will present another characterization as eigenfunctions of a certain operator. Motivated by Proposition 37, we will define the Macdonald operator.

Definition 60. *We define the **Macdonald operator** as*

$$E_N = D_N - \sum_{i=1}^{N} t^{1-i}$$

where

$$D_N = Y_1 + \cdots + Y_N.$$

This operator satisfies very strong properties.

Proposition 61. *The symmetric Macdonald polynomials are simultaneous eigenfunctions of the operator E_N. To be precise,*

$$E_N P_\lambda = c_\lambda P_\lambda$$

where c_λ is given by,

$$\bar{\eta}_1 + \cdots + \bar{\eta}_N - \sum_{i=1}^{N} t^{1-i}$$

with $\bar{\eta}_i = q^{\eta_i} t^{1-r_\eta(i)}$.

Proof. Since $P_\lambda = \mathcal{S}_N^t E_{\eta_{\lambda,N}}$ and $D_N \mathcal{S}_N^t = \mathcal{S}_N^t D_N$, we have

$$\begin{aligned} E_N P_\lambda &= \frac{1}{u_{\lambda,N}(t)} E_N \mathcal{S}_N^t E_{\eta_{\lambda,N}}, \\ &= \frac{1}{u_{\lambda,N}(t)} \left(D_N - \sum_{i=1}^{N} t^{1-i} \right) \mathcal{S}_N^t E_{\eta_{\lambda,N}}, \\ &= \frac{1}{u_{\lambda,N}(t)} \mathcal{S}_N^t \left(D_N - \sum_{i=1}^{N} t^{1-i} \right) E_{\eta_{\lambda,N}}, \\ &= \frac{1}{u_{\lambda,N}(t)} \mathcal{S}_N^t \left(\bar{\eta}_1 + \cdots + \bar{\eta}_N - \sum_{i=1}^{N} t^{1-i} \right) E_{\eta_{\lambda,N}}, \\ &= \left(\bar{\eta}_1 + \cdots + \bar{\eta}_N - \sum_{i=1}^{N} t^{1-i} \right) P_\lambda \\ &= c_\lambda P_\lambda, \end{aligned}$$

which completes the proof. □

We have our second characterization of the symmetric Macdonald polynomials. The proof of the next Lemma can be found in [**23**]:

PROPOSITION 62. ***[Triangularity]*** *The symmetric Macdonald polynomials form the unique basis of the space of symmetric polynomials such that*

(1) the decomposition over the monomials are triangular

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\mu,\lambda} m_\mu$$

where $<$ is the dominance order in partitions.

(2)

$$E_N P_\lambda = c_\lambda P_\lambda$$

This characterization is still not ideal since it depends on the number of variables (since the action of E_N depends on N). It will prove nevertheless crucial in the next section for proving the orthogonality.

4. Orthogonality

In the previous section, we observed that Macdonald polynomials can be characterized by being eigenfunctions of an operator as well as having a triangular expansion in terms of the symmetric monomials. In this section, we will explore another characterization of Macdonald polynomials wchich asserts that they are orthogonal with respect to an inner product. Together with the triangularity, this inner product uniquely determines these polynomials. Most of this chapter can be found in [**20**].

DEFINITION 63. *We define the following scalar product on the power-sum symmetric functions:*

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda(q,t)$$

where

$$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \qquad \text{and} \qquad z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!,$$

where m_i is the number of times that i appear in λ.

We will prove that the symmetrics Macdonald polynomials are orthogonal with respect to the above scalar product. For this, we have to define the following kernel

DEFINITION 64. *The symmetric kernel is*

$$K_0 = \prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} \quad \text{with} \quad (a;q)_\infty = \prod_{i=1}^{\infty}(1-aq^{i-1})$$

.

LEMMA 65. *We have the following relation*

$$K_0 = \sum_\lambda z_\lambda(q,t)^{-1} p_\lambda(x) p_\lambda(y).$$

PROOF. We calculate

$$\begin{aligned} log(K_0) &= \sum_{i,j}\sum_{r\geq 0}(log(1-q^r t x_i y_j) - log(1-q^r x_i y_j)) \\ &= \sum_{i,j}\sum_{r\geq 0}\sum_{n\geq 1}\frac{1}{n}(1-t^n)(q^r x_i y_j)^n \\ &= \sum_{i,j}\sum_{n\geq 1}\frac{1}{n}\frac{1-t^n}{1-q^n}(x_i y_j)^n \\ &= \sum_{n\geq 1}\frac{1}{n}\frac{1-t^n}{1-q^n}p_n(x)p_n(y), \end{aligned}$$

from which it follows that

$$K_0 = \prod_{n\geq 1} exp\left(\frac{1}{n}\frac{1-t^n}{1-q^n}p_n(x)p_n(y)\right) = \prod_{n\geq 1}\sum_{r_n\geq 0}\frac{1}{n^{r_n}r_n!}\frac{(1-t^n)^{r_n}}{(1-q^n)^{r_n}}p_n(x)^{r_n}p_n(y)^{r_n}$$

From the r.h.s. of the equation it is then clear that the coefficient of $p_\lambda(x)p_\lambda(y)$ is $z_\lambda(q,t)^{-1}$. □

The next lemma gives us a connection between the scalar product introduced in Definition 63 and the kernel K_0 defined above.

LEMMA 66. *Let $\{u_\lambda\}, \{v_\mu\}$ be bases of the space of symmetric functions. Then the following criteria is verified*

$$K_0 = \sum_\lambda u_\lambda(x)v_\lambda(y) \Longleftrightarrow \langle u_\lambda, v_\mu\rangle = \delta_{\lambda\mu}.$$

PROOF. Let $p^*_\lambda = z_\lambda(q,t)^{-1}p_\lambda$, so that $\langle p^*_\lambda, p_\mu\rangle = \delta_{\lambda\mu}$. Because u_λ and v_μ are symmetric functions, we can write

$$u_\lambda = \sum_\rho a_{\lambda\rho}p^*_\rho \quad \text{and} \quad v_\mu = \sum_\sigma a_{\mu\sigma}p_\sigma$$

We then have

$$\langle u_\lambda, v_\mu\rangle = \sum_\rho a_{\lambda\rho}b_{\mu\rho}$$

so the r.h.s. of the equivalence amounts to

$$\sum_\rho a_{\lambda\rho}b_{\mu\rho} = \delta_{\lambda\mu} \tag{a}$$

From Lemma 65, the l.h.s. of the equivalence gives

$$\sum_\lambda u_\lambda(x)u_\lambda(y) = \sum_\rho p^*_\rho(x)p_\rho(y)$$

from which we deduce that the l.h.s. of the equivalence takes the form

$$\sum_\lambda a_{\lambda\rho}b_{\lambda\sigma} = \delta_{\rho\sigma}. \tag{b}$$

Since (a) and (b) are equivalent, this concludes the proof. □

The following lemma is the key point in demonstrating the orthogonality of Macdonald polynomials, this can be found in [**20**]

LEMMA 67. *The operator defined in 79 is symmetric if you exchanges operators in x or y:*

$$E^{(x)}K_0 = E^{(y)}K_0$$

Theorem 68. ***[Cauchy formula]*** *We have the following relationship between the symmetric kernel and the symmetric Macdonald polynomials:*

$$K_0 = \sum_{\lambda} b_\lambda^{-1}(q,t) P_\lambda(x) P_\lambda(y).$$

Proof. First, note that we can write K_0 as

$$K_0 = \sum_{\lambda,\mu} d_{\lambda\mu}(q,t) P_\lambda(x) P_\mu(y)$$

this is because K_0 is symmetric in x and y, and P_λ is a basis of the space of symmetric functions. Furthermore,

$$E^x K_0 = \sum_{\lambda,\mu} d_{\lambda\mu}(q,t) E^x P_\lambda(x) P_\mu(y) = \sum_{\lambda,\mu} d_{\lambda\mu}(q,t) c_\lambda = d_{\lambda\mu}(q,t) c_\lambda(q,t) P_\lambda(x) P_\mu(y)$$

and, because $E^x K_0 = E^y K_0$, we have the relation

$$d_{\lambda\mu}(q,t) c_\lambda(q,t) = d_{\lambda\mu}(q,t) c_\mu(q,t)$$

But c_λ is uniquely determined by λ, so we conclude that $d_{\lambda\mu} = 0$ unless $\lambda = \mu$. Taking $d_{\lambda\lambda} = b_\lambda^{-1}$ proves the theorem. □

Theorem 69. ***[Orthogonality]*** *The Macdonald polynomials are orthogonal with respect to the scalar product in Definition 63, i.e.*

$$\langle P_\lambda, P_\mu \rangle = 0 \text{ if } \lambda \neq \mu$$

and

$$\langle P_\lambda, P_\lambda \rangle = b_\lambda(q,t)$$

for some coefficient $b_\lambda(q,t)$ that we will give explicitly in the next section.

Proof. Both equations are direct consequence of Lemmas 67 and 66. □

We are now in a position to state yet another characterization of Macdonald polynomials.

Proposition 70. *The symmetric Macdonald polynomials are the unique basis of the space of symmetric polynomials which satisfy:*

(1) the decomposition in monomials is triangular

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\mu,\lambda} m_\mu$$

(2)

$$\langle P_\lambda, P_\mu \rangle = 0 \text{ if } \lambda \neq \mu.$$

Proof. By the Theorem 69 and 62, we know that Macdonald polynomials satisfy (1) and (2). On the other hand, properties (1) and (2) determine uniquely these polynomials from the Gram-Schmidt process. □

This definition of Macdonald polynomials has the advantage of not depending explicitly on the variables $x_1, x_2, \ldots$. As such, it allows to compute Macdonald polynomials explicitly.

If we analyze what has been done in this section, we can see that a relation was established between the kernel and the scalar product. Then, we proved Lemma 67, which was fundamental as it led quite directly to the orthogonality. In general, this method will be the standard approach: search for a kernel on which the Macdonald operator acts symmetrically with respect to variables x and y. This in turn translates into an orthogonality relation with respect to the scalar product associated to the kernel.

5. Symmetries

In this section, we will explore remarkable symmetries satisfied by the Macdonald polynomials. These symmetries will provide us with useful tools to demonstrate more intricate properties.

THEOREM 71. ***[Symmetry]*** *The Macdonald polynomials P_λ satisfy the following symmetry*

$$P_\lambda(x_1,\ldots,x_N;q,t) = P_\lambda(x_1,\ldots,x_N;q^{-1},t^{-1})$$

PROOF. It is immediately from Definition 63 that

$$z_\lambda(q^{-1},t^{-1}) = (q^{-1}t)^{|\lambda|} z_\lambda(q,t),$$

We then have that

$$\langle f,g\rangle_{q^{-1},t^{-1}} = (q^{-1}t)^n \langle f,g\rangle_{q,t}.$$

Thus, $P_\lambda(x_1,\ldots,x_N;q^{-1},t^{-1})$ is $P_\lambda(x_1,\ldots,x_N;q,t)$, up to a constant. But by Theorem 69, the coefficient of m_λ in $P_\lambda(x_1,\ldots,x_N;q^{-1},t^{-1})$ and $P_\lambda(x_1,\ldots,x_N;q,t)$ is 1, which entails that

$$P_\lambda(x_1,\ldots,x_N;q,t) = P_\lambda(x_1,\ldots,x_N;q^{-1},t^{-1}),$$

as we wanted. □

DEFINITION 72. *We define the principal evaluation u_λ on any symmetric function $f(x)$ as*

$$u_\lambda\big(f(x_1,\ldots,x_N)\big) = f\left(q^{-\lambda_1}t^0,\ldots,q^{-\lambda_N}t^{N-1}\right). \tag{5.1}$$

REMARK 73. *Note that in the case $\lambda = (0^N)$ we have $u_\lambda = u_\emptyset$, where $u_\emptyset$ was defined in* (3.1)*.*

LEMMA 74. *If f is a symmetric function in N variables then*

$$f(Y^{-1})P_\lambda(x;q,t) = u_\lambda(f)P_\lambda(x;q,t). \tag{5.2}$$

PROOF. Let $\eta = \eta_{\lambda,N}$. We have that $Y_i^{-1}E_\eta = \bar{\eta}_i^{-1}E_\eta$, where we recall that $\bar{\eta}_i = q^{\eta_i}t^{1-r_\eta(i)}$. Using the fact that f is symmetric, we then have that

$$\begin{aligned} f(Y_1^{-1},\ldots,Y_N^{-1})P_\lambda(x;q,t) &= f(Y_1^{-1},\ldots,Y_N^{-1})d_\eta(q,t)\mathcal{S}_N^t E_\eta \\ &= d_\eta(q,t)\mathcal{S}_N^t f(Y_1^{-1},\ldots,Y_N^{-1})E_\eta \\ &= d_\eta(q,t)\mathcal{S}_N^t f(\bar{\eta}_1^{-1},\ldots,\bar{\eta}_N^{-1})E_\eta \\ &= f(\bar{\eta}_1^{-1},\ldots,\bar{\eta}_N^{-1})P_\lambda(x;q,t). \end{aligned}$$

It is easy to see that because f is symmetric $f(\bar{\eta}_1^{-1},\ldots,\bar{\eta}_N^{-1})$ is exactly the evaluation defined in (72), which completes the proof. □

EXAMPLE 75. *We have,*

(1) $u_\emptyset(P_{(3,1,1)}) = \dfrac{t^3(t^3-1)(qt^3-1)}{(t-1)(qt-1)}$,

(2) $u_{(3,2,1)}(P_{(3,1,1)}) = \dfrac{t^3(q^2t^2+qt+1)(q^3t^3-q^2t^2-q^2t+qt^2+qt-1)}{(qt-1)q^{12}}$.

Note that in the first example, we have a simple factorization. Later, we will prove that $u_\emptyset(P_\lambda)$ always has a beautiful factorization. However, this is not the case in general. As seen in the second example, the expression is not elegant at all. Although we lack an explicit expression for the evaluation u_λ, we will see that it obeys a very beautiful symmetry. Since $u_\emptyset(P_\lambda(x,q,t)) \neq 0$ we can define

DEFINITION 76. *Let $\tilde{P}_\lambda(x,q,t)$ be the normalization of the Macdonald polynomials given by*

$$\tilde{P}_\lambda(x,q,t) = \frac{P_\lambda(x;q,t)}{u_\emptyset(P_\lambda(x,q,t))}$$

Proposition 77. ***[Self-duality]*** *The following symmetry holds:*

$$u_\mu(\tilde{P}_\lambda) = u_\lambda(\tilde{P}_\mu)$$

Proof. For $f(x)$ and $g(x)$ Laurent polynomials in $x_1, \ldots, x_N$, we get from (3.4) that the pairing

$$[f(x), g(x)] := u_\emptyset\left(f(Y^{-1})g(x)\right)$$

is such that $[f, g] = [g, f]$. From Lemma 74, we thus get

$$\begin{aligned}
[P_\lambda(x, q, t), P_\mu(x, q, t)] &= u_\emptyset\big(P_\lambda(Y_i^{-1})P_\mu(x, q, t)\big) \\
&= u_\mu(P_\lambda(x, q, t))u_\emptyset\big(P_\mu(x, q, t)\big)
\end{aligned}$$

From the symmetry of the pairing $[\cdot, \cdot]$, it then follows that

$$u_\mu(P_\lambda(x, q, t))u_\emptyset\big(P_\mu(x, q, t)\big) = u_\lambda(P_\mu(x, q, t))u_\emptyset\big(P_\lambda(x, q, t)\big)$$

which proves the proposition. $\square$

Example 78.

$$u_{(4,4,3,2,2,1)}(\tilde{P}_{(6,6,2,1,1)}) = u_{(6,6,2,1,1)}(\tilde{P}_{(4,4,3,2,2,1)})$$

6. Operator and Pieri rules

We obtained earlier a characterization of the Macdonald polynomials as eigenfunctions of the Macdonald operator. In this section, we will provide an explicit formula for this operator. In fact, more generally, we will see that Macdonald polynomials are eigenfunctions of a commuting family of q-difference operators. Additionally, we will demonstrate the Pieri rules for Macdonald polynomials, which are impressive combinatorial rules for the product of an elementary symmetric function and a Macdonald polynomial.

Definition 79. *For $r = 1, \ldots, N$, we let the q-difference operator D_r^N be*

$$D_r^N = \sum_{|I|=r} A_I(x, t) \prod_{i \in I} \tau_i$$

where

$$A_I(x, t) = t^{r(r-1)/2} \prod_{i \in I, j \notin I} \frac{tx_i - x_j}{x_i - x_j}.$$

Example 80. *We have that*

$$D_2^3 = t\left(\frac{tx_1 - x_3}{x_1 - x_3}\right)\left(\frac{tx_2 - x_3}{x_2 - x_3}\right)\tau_1\tau_2 + t\left(\frac{tx_2 - x_1}{x_2 - x_1}\right)\left(\frac{tx_3 - x_1}{x_3 - x_1}\right)\tau_2\tau_3 + t\left(\frac{tx_1 - x_2}{x_1 - x_2}\right)\left(\frac{tx_3 - x_2}{x_3 - x_2}\right)\tau_1\tau_3.$$

The following proposition essentially tells us that over the space of symmetric functions, the operators D_r^N act like elementary functions of Cherednik operators.

Proposition 81. ***[Operator]*** *Let f be a symmetric function. Then*

$$e_r(Y)f = t^{r(1-N)} D_r^N f.$$

Proof. From Lemma 95, we have

$$e_r(Y)f = \frac{1}{[N-r]![r]!}\mathcal{S}_N^t Y_{N-r+1} \cdots Y_N f,$$

From Lemma 94, this then implies that

$$e_r(Y)f = \frac{t^{(r+1-2N)r/2}}{[N-r]![r]!}\mathcal{S}_N^t \tau_{N-r+1} \cdots \tau_N f,$$

from which we deduce that

$$e_r(Y)f = \frac{t^{(r+1-2N)r/2}}{[N-r]![r]!}\sum_{\sigma\in S_N}\sigma\left(\prod_{\substack{i,j\in[N-r+1,N]\\ i<j}} A_{j,i}\right)\tau_{N-r+1}\cdots\tau_N f. \tag{a}$$

Notice that by symmetry we just have to prove the proposition for $J=[N-r+1,N]$, so we will find the coefficient of $\tau_{N-r+1}\cdots\tau_N$ in (a). Moreover, the coefficient is given by

$$\frac{t^{(r+1-2N)r/2}}{[N-r]![r]!}\sum_{\substack{\sigma\in S_N\\ \sigma(I_r)=I_r}}\sigma\left(\prod_{\substack{i,j\in[N-r+1,N]\\ i<j}} A_{j,i}\right)\tau_{N-r+1}\cdots\tau_N f$$

which, from Lemma 93, says that

$$t^{(r+1-2N)r/2}A_{J\times L}\tau_{N-r+1}\cdots\tau_N f$$

is the coefficient of $\tau_{N-r+1}\cdots\tau_N$ in D_r^N, where $L=[N]\setminus J$. $\square$

Remark 82. *Previous Lemma said that D_r^N is essentially $e_r(Y)$ and since the operators Y_i commute with each others, it immediately follows that the Macdonald operators commute among themselves. Moreover, the Lemma 74 also works for Y_i, then taking $f=e_r$, we obtain that Macdonald polynomials are eigenfunctions of the $e_r(Y)$ operators, and because D_r^N is essentially $e_r(Y)$ we have that the Macdonald polynomials are eigenfuntions of the Macdonald operators D_r^N.*

Lemma 83. *Let θ be a $\{0,1\}$-tuple with exactly r entries equal to 1. We have that $u_\lambda(A_I(x,t))\neq 0$ iff $\lambda+\theta$ is a partition.*

Proof. We calculate

$$u_\lambda(A_I)=\prod_{i<j}\frac{q^{\lambda_i}t^{N-i+\theta_i}-q^{\lambda_j}t^{N-j+\theta_j}}{q^{\lambda_i}t^{N-i}-q^{\lambda_j}t^{N-j}}$$

If $u_\lambda(A_I)=0$ then there exist i and j such that $i<j$, $\lambda_i=\lambda_j$ and $i-j=\theta_i-\theta_j$. Since $|\theta_i-\theta_j|\leq 1$, it follows that $j=i+1$, $\theta_i=0$ and $\theta_j=1$, so that $\lambda_i=\lambda_{i+1}$ with $\lambda_i+\theta_i<\lambda_{i+1}+\theta_{i+1}$. This means that $\lambda+\theta$ is not a partition.
Conversely, if $\lambda+\theta$ is not a partition then there exists an $i\leq n-1$ such that $\lambda_i=\lambda_{i+1}$, $\theta_i=0$ and $\theta_j=1$, whence $u_\lambda(A_I)=0$. $\square$

The Macdonald polynomials are known to form a basis of the space of symmetric functions. Furthermore, considering $r\in\mathbb{N}$ and λ a partition, the expression $e_r(x)P_\lambda$ is also a symmetric function. Therefore, we can represent it as a linear combination of Macdonald polynomials,

$$e_r(x)P_\lambda(x)=\sum_\mu C_{\lambda,\mu}P_\mu$$

However, we have no information a priori about the partitions μ and the coefficients $C_{\lambda,\mu}$. The Pieri rules are an impressive property of Macdonald polynomials that provide an explicit combinatorial expressions for the partitions μ and the coefficients $C_{\lambda,\mu}$. It turns out that it is simpler to first obtain the Pieri rules for $\tilde{P}_\lambda$ defined in Definition 76.

Theorem 84. ***[Pieri Rule]** We have the following Pieri rules for the symmetric Macdonald polynomials*

$$e_r(x)\tilde{P}_\lambda=\sum_\mu \tilde{C}_{\lambda,\mu}\tilde{P}_\mu$$

where λ/μ is a vertical r-strip and the coefficients are given by

$$\tilde{C}_{\lambda,\mu}=u_\mu(A_I(x;t)).$$

Proof. For each partition ν we know from Lemma 74 that

$$e_r(Y)\tilde{P}_\nu = u_\nu(e_r)\tilde{P}_\nu$$

On the other hand, from Proposition 81 we get

$$e_r(Y)\tilde{P}_\nu = \sum_{|I|=r} A_I(x;t)\prod_{i\in I}\tau_i P_\nu$$

Joining the two previous equations leads to

$$u_\nu(e_r)\tilde{P}_\nu = \sum_{|I|=r} A_I(x;t)\prod_{i\in I}\tau_i \tilde{P}_\nu$$

Then, applying the evaluation u_μ on both sides of the equation gives

$$u_\nu(e_r)u_\mu(\tilde{P}_\nu) = \sum_{|I|=r} u_\mu(A_I(x;t))\prod_{i\in I}\tau_i u_\mu(\tilde{P}_\nu)$$

From $\prod_{i\in I}\tau_i u_\mu(\tilde{P}_\nu) = u_{\mu+I}(\tilde{P}_\nu)$, this simplifies to

$$u_\nu(e_r)u_\mu(\tilde{P}_\nu) = \sum_{|I|=r} u_\mu(A_I(x;t))u_{\mu+I}(\tilde{P}_\nu)$$

Applying the symmetry in Poroposition 77, we obtain

$$u_\nu(e_r)u_\nu(\tilde{P}_\mu) = \sum_{|I|=r} u_\mu(A_I(x;t))u_\nu(\tilde{P}_{\mu+I})$$

Because ν can be any partition, we can conclude that the above equations holds for any value of the variables:

$$e_r(x)\tilde{P}_\mu(x) = \sum_{|I|=r} u_\mu(A_I(x;t))\tilde{P}_{\mu+I}(x)$$

where, from Lemma 83, the coefficient is non-zero only if $\mu + I$ is a partition (in which case it is a vertical strip). $\square$

The previous proof is found in Macdonald's book [**21**] and essentially tells us that, having the operator $e_r(Y)$ explicitly as well as the self-duality, gives us the Pieri rules. This will be the canonical method we use in this document to derive Pieri rules.

Corollary 85. *We have the following Pieri rules for the symmetric Macdonald polynomials*

$$e_r(x)P_\lambda = \sum_\mu C_{\lambda,\mu}P_\mu$$

where λ/μ is a vertical r-strip and the coefficients are given by

$$C_{\lambda,\mu} = \frac{u_\mu(A_I(x;t))u_\emptyset(P_\mu)}{u_\emptyset(P_{\mu+I})}.$$

Proof. It is immediate from Definition 76 and Theorem 84. $\square$

7. Combinatorial formulas

So far, we have defined Macdonald polynomials, characterized them and explored their algebraic properties. However, we have not yet seen how these properties are connected to combinatorics. In this section, we will explore the combinatorial form of Pieri rules, from which we will deduce the principal evaluation and the norm squared. This section is primarily based on [**25**].

Definition 86. *For $k \in \mathbb{N}$, we define*

$$(a;q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a)$$

Theorem 87. ***[Combinatorial Pieri Rule]*** *The symmetric Macdonal polynomials satisfy the following Pieri rules:*

$$e_r(x)\tilde{P}_\lambda = \sum_\mu \tilde{C}_{\lambda,\mu}\tilde{P}_\mu$$

where λ/μ is a vertical r-strip and the coefficient is given by

$$\tilde{C}_{\lambda,\mu} = \frac{t^{n(\lambda)}\prod\limits_{s\in\lambda}(1-q^{a_\lambda(s)}t^{N-l'_\lambda(s)})}{t^{n(\mu)}\prod\limits_{s\in\mu}(1-q^{a_\mu(s)}t^{N-l'_\mu(s)})}\cdot\frac{\prod\limits_{s\in\mu\cap R_{\lambda/\mu}}(1-q^{a_\mu(s)}t^{l_\mu(s)+1})}{\prod\limits_{s\in\lambda\cap R_{\lambda/\mu}}(1-q^{a_\lambda(s)}t^{l_\lambda(s)+1})}\frac{\prod\limits_{s\in\mu\cap R_{\lambda/\mu}}(1-q^{a_\mu(s)+1}t^{l_\mu(s)})}{\prod\limits_{s\in\lambda\cap R_{\lambda/\mu}}(1-q^{a_\lambda(s)+1}t^{l_\lambda(s)})}$$

with $R_{\lambda/\mu}$ denoting the union of the rows intersecting the vertical strip λ/μ.

Proof. Let λ/μ be a vertical r-strip. We can then write $\lambda=\mu+I$, with $I\subset[N]$ and $|I|=r$. Using Theorem 84 and Definition 79, the coefficient of $\hat{P}_\lambda$ in the expansion is

$$\tilde{C}_{\lambda,\mu} = u_\mu(A_I(x;t)) = u_\mu\left(t^{r(r-1)/2}\prod_{i\in I,j\in J}\frac{1-tx_i/x_j}{1-x_i/x_j}\right)$$

where $J=I^c$. Applying u_μ, we obtain

$$\tilde{C}_{\lambda,\mu} = t^{r(r-1)/2}\prod_{\substack{1\le a<b\le N\\ a\in I,b\in J}}\frac{1-q^{\mu_a-\mu_b}t^{b-a+1}}{1-q^{\mu_a-\mu_b}t^{b-a}}\prod_{\substack{1\le a<b\le N\\ a\in J,b\in I}}\frac{1-q^{\mu_a-\mu_b}t^{b-a-1}}{1-q^{\mu_a-\mu_b}t^{b-a}}$$

For simplicity of notation, we will work with the conjugate partitions λ', μ' in Definition 86, noting that they satisfy the interlacing property

$$n\ge\lambda'_1\ge\mu'_1\ge\lambda'_2\ge\mu'_2\ge\cdots$$

The sets I and J can then be expressed as the following disjoint unions

$$\begin{aligned} I &= \textstyle\bigcup_{k\ge1} I_k & I_k &= (\mu'_k,\lambda'_k] \\ J &= \textstyle\bigcup_{k\ge1} J_k & J_k &= (\lambda'_k,\mu'_{k-1}] \end{aligned} \tag{7.1}$$

where $(a,b]=\{k\in\mathbb{Z}\mid a<k\le b\}$ and $\mu'_0=\lambda'_0=N$. Note that, $\mu_i=k-1,\lambda_i=k$ if $i\in I_k$ and $\mu_j=\lambda_j=k-1$ if $j\in J_k$. We then have

$$\tilde{C}_{\lambda,\mu} = t^{r(r-1)/2}\prod_{\substack{j\le i\\ a\in I_i\\ b\in J_j}}\frac{(1-q^{i-j}t^{b-a+1})}{(1-q^{i-j}t^{b-a})}\prod_{\substack{i<j\\ a\in J_j\\ b\in I_i}}\frac{(1-q^{i-j}t^{b-a+1})}{(1-q^{i-j}t^{b-a})}$$

Note that, if we fix i,j and a, we have, using $J_j=(\lambda'_j,\mu'_{j-1}]$, that

$$\prod_{b\in J_j}\frac{1-q^{i-j}t^{b-a+1}}{1-q^{i-j}t^{b-a}} = \frac{1-q^{i-j}t^{\mu'_{j-1}-a+1}}{1-q^{i-j}t^{\lambda'_j-a+1}},$$

From the definition of $I_i=(\mu'_i,\lambda'_i]$, we can then express this product as

$$\tilde{C}_{\lambda,\mu} = t^{r(r-1)/2}\prod_{j\le i}\frac{(q^{i-j}t^{\mu'_{j-1}-\lambda'_i+1};t)_{\lambda'_i-\mu'_i}}{(q^{i-j}t^{\lambda'_j-\lambda'_i+1};t)_{\lambda'_i-\mu'_i}}\prod_{i<j}\frac{(q^{i-j}t^{\mu'_i-\mu'_{j-1}+1};t)_{\mu'_{j-1}-\lambda'_j}}{(q^{i-j}t^{\lambda'_i-\mu'_{j-1}};t)_{\mu'_{j-1}-\lambda'_j}}$$

By separating the term $j=1$ in the numerator of the first product and making a change of variables, we obtain

$$\tilde{C}_{\lambda,\mu} = t^{r(r-1)/2} \prod_{1\le j} (q^{j-1}t^{N-\lambda'_j+1};t)_{\lambda'_j-\mu'_j} \frac{\prod_{j<i}(q^{j-i-1}t^{\mu'_j-\lambda'_j+1};t)_{\lambda'_j-\mu'_j}}{\prod_{j\le i}(q^{j-i}t^{\lambda'_i-\lambda'_j+1};t)_{\lambda'_j-\mu'_j}} \frac{\prod_{i<j}(q^{i-j}t^{\mu'_i-\mu'_j+1};t)_{\mu'_j-\lambda'_{j+1}}}{\prod_{i\le j}(q^{i-j}t^{\lambda'_i-\mu'_j};t)_{\mu'_j-\lambda'_{j+1}}},$$

Finally, using the combinatorial notation, we conclude that

$$\tilde{C}_{\lambda,\mu} = \frac{t^{n(\lambda)}\prod_{s\in\lambda}(1-q^{a_\lambda(s)}t^{N-l'_\lambda(s)})}{t^{n(\mu)}\prod_{s\in\mu}(1-q^{a_\mu(s)}t^{N-l'_\mu(s)})} \cdot \frac{\prod_{s\in\mu\cap R_{\lambda/\mu}}(1-q^{a_\mu(s)}t^{l_\mu(s)+1})}{\prod_{s\in\lambda\cap R_{\lambda/\mu}}(1-q^{a_\lambda(s)}t^{l_\lambda(s)+1})} \frac{\prod_{s\in\mu\cap R_{\lambda/\mu}}(1-q^{a_\mu(s)+1}t^{l_\mu(s)})}{\prod_{s\in\lambda\cap R_{\lambda/\mu}}(1-q^{a_\lambda(s)+1}t^{l_\lambda(s)})},$$

as wanted. □

THEOREM 88. ***[Principal Evaluation]*** *The principal evaluation of a Macdonald polynomial has the following combinatorial formula:*

$$u_\emptyset(P_\lambda(x;q,t)) = t^{n(\lambda)} \prod_{s\in\lambda} \frac{(1-q^{a(s)}t^{N-\ell'(s)})}{(1-q^{a(s)}t^{\ell(s)+1})}$$

PROOF. We have from Theorem 84 that

$$e_r(x)\tilde{P}_\mu = \sum_\nu \tilde{C}_{\mu,\nu}\tilde{P}_\nu, \tag{7.2}$$

Keeping in mind that we can rewrite $\tilde{P}_\lambda$ as

$$\tilde{P}_\lambda = \frac{m_\lambda(x)}{u_\emptyset(P_\lambda)} + \text{ lower terms}$$

we can take $\lambda = \mu + (1)^r$ and compare the coefficient of m_λ on each side of (7.2). This yields

$$\frac{1}{u_\emptyset(P_\mu)} = \tilde{C}_{\lambda\mu}\frac{1}{u_\emptyset(P_\lambda)}$$

and therefore

$$u_\emptyset(P_\lambda) = \tilde{C}_{\lambda\mu}u_\emptyset(P_\mu)$$

where the coefficient $\tilde{C}_{\lambda\mu}$ is given in Theorem 87. In this case $\mu\cap R_{\lambda/\mu} = \mu/R_{\lambda/\mu} = \lambda/R_{\lambda/\mu} = \emptyset$, from which we get the following recurrence formula

$$u_\emptyset(P_\lambda) = t^{n(\lambda/\mu)} \prod_{s\in\lambda/\mu} \frac{(1-q^{a_\lambda(s)}t^{N-i+1})}{(1-q^{a_\lambda(s)+1}t^{l_\lambda(s)})} u_\emptyset(P_\mu),$$

Since $P_\emptyset = 1$, we deduce the desired formula. □

EXAMPLE 89. *For $\lambda = (3,1,1)$ we have*

$$u_\emptyset(P_{(3,1,1)}) = t^3 \frac{(1-t^3)}{(1-q^2t^3)} \frac{(1-qt^3)}{(1-qt)} \frac{(1-q^2t^3)}{(1-t)} \frac{(1-t^2)}{(1-t^2)} \frac{(1-t)}{(1-t)} = t^3 \frac{(1-t^3)}{(1-t)} \frac{(1-qt^3)}{(1-qt)}$$

COROLLARY 90. *The symmetric Macdonald polynomials satisfy the following combinatorial Pieri rules*

$$e_r(x)P_\lambda = \sum_\mu C_{\lambda,\mu}P_\mu$$

where λ/μ is a vertical r-strip, and where the coefficient $\tilde{C}_{\lambda,\mu}$ is given by

$$\tilde{C}_{\lambda,\mu} = \prod_{s\in\lambda/R_{\lambda/\mu}} \frac{(1-q^{a_\lambda(s)}t^{l_\lambda(s)+1})}{(1-q^{a_\lambda(s)+1}t^{l_\lambda(s)})} \prod_{s\in\mu/R_{\lambda/\mu}} \frac{(1-q^{a_\mu(s)+1}t^{l_\mu(s)})}{(1-q^{a_\mu(s)}t^{l_\mu(s)+1})} = \prod_{s\in C_{\mu/\lambda}/R_{\mu/\lambda}} \frac{(1-q^{a_\mu(s)+1}t^{l_\mu(s)})}{(1-q^{a_\mu(s)}t^{l_\mu(s)+1})},$$

with $R_{\lambda/\mu}$ (resp. $C_{\lambda/\mu}$) denoting the union of the rows (resp. columns) intersecting the vertical strip λ/μ.

Proof. It is immediate from Definition 87 and Theorem 88. □

Example 91. *Taking $\lambda = (3,2)$ and $r = 2$, we have*

$$e_2 \cdot P_{(3,2)} = \psi_{\lambda(4,3)} P_{(4,3)} + \psi_{\lambda(4,2,1)} P_{(4,2,1)} + \psi_{\lambda(3,3,1)} P_{(3,3,1)} + \psi_{\lambda(3,2,1,1)} P_{(3,2,1,1)},$$

where, for instance, the coefficient of $P_{(4,2,1)}$ is given by

$$\psi_{\lambda(4,2,1)} = \left(\frac{1-qt^2}{1-q^2t}\right)\left(\frac{1-q^2}{1-qt}\right).$$

The next combinatorial result can be found in [**20**]

Theorem 92. ***[Norm]*** *The evaluation over the partition $\emptyset$ is given by the follow combinatorial formula*

$$\langle P_\lambda(x,q,t), P_\lambda(x,q,t)\rangle = \prod_{s\in\lambda} \frac{1-q^{a(s)+1}t^{\ell(s)}}{1-q^{a(s)}t^{\ell(s)+1}}$$

8. Double affine Hecke algebra relations for Symmetric case

In this section, we establish a few results involving the Hecke algebra and the Double affine Hecke algebra that we used in the previous sections.

Lemma 93. *Let $J \subseteq [N]$ and $L = [N] \setminus J$. We then have*

$$\sum_{\substack{\sigma([N-r+1,N])=J \\ \sigma\in\mathfrak{S}_N}} K_\sigma \left(\prod_{1\le i<j\le N} \frac{x_i - tx_j}{x_i - x_j}\right) = a_{r,N}(t) A_{J\times L}(x,x)$$

where $r = |J|$ and

$$a_{r,N}(t) = [r]_t![N-r]_t!$$

Proof. For convenience, we will let

$$\bar{A}_I(x) = \prod_{\substack{i,j\in I \\ i<j}} \left(\frac{x_i - tx_j}{x_i - x_j}\right)$$

We first prove the special case when $J = [r]$ and $L = [r+1, N]$. Let γ be the permutation $[r+1,\ldots,N,1,\ldots,r]$ (in one-line notation). In this case, we have

$$\begin{aligned}
&\sum_{\substack{\sigma([N-r+1,N])=[r] \\ \sigma\in\mathfrak{S}_N}} K_\sigma \bar{A}_N(x) \\
&\quad= \sum_{w\in\mathfrak{S}_r} \sum_{w'\in\mathfrak{S}_{r+1,N}} K_w K_{w'} K_\gamma \bar{A}_{N-r}(x) \bar{A}_{[N-r+1,N]}(x) A_{[N-r+1,N]\times[N-r]}(x,x) \\
&\quad= A_{[r]\times[r+1,N]}(x,x) \left(\sum_{w\in\mathfrak{S}_r} K_w \bar{A}_r(x)\right) \left(\sum_{w'\in\mathfrak{S}_{r+1,N}} K_{w'} \bar{A}_{[r+1,N]}(x)\right)
\end{aligned}$$

since w and w' leave $A_{[r]\times[r+1,N]}$ invariant. Using [**?**]

$$\mathcal{S}_N^t \cdot 1 = \sum_{\sigma\in\mathfrak{S}_N} K_\sigma \bar{A}_N(x) = [N]_t! \tag{8.1}$$

the formula is seen to hold in that case.

As for the general case, let δ be any permutation such that $\delta([r]) = J$ (and thus also such that $\delta([r+1,\ldots,N]) = L$). Applied on both sides of the special case that we just showed, we get

$$\sum_{\substack{\sigma([N-r+1,N])=[r]\\ \sigma\in\mathfrak{S}_N}} K_\delta K_\sigma \left(\prod_{1\le i<j\le N} \frac{x_i - tx_j}{x_i - x_j}\right) = a_{r,N}(t) K_\delta A_{[r]\times[r+1,N]}(x,x) = a_{r,N}(t) A_{J\times L}(x,x)$$

which amounts to

$$\sum_{\substack{\delta\sigma([N-r+1,N])=J\\ \delta\sigma\in\mathfrak{S}_N}} K_{\delta\sigma} \left(\prod_{1\le i<j\le N} \frac{x_i - tx_j}{x_i - x_j}\right) = a_{r,N}(t) A_{J\times L}(x,x)$$

The lemma then follows immediately. □

We now show that the product $Y_{N-r+1}\cdots Y_N$ of Cherednik operators can be simplified quite significantly in certain cases.

Lemma 94. *Let $r \le N$. For any symmetric function $f(x)$, we have that*

$$Y_{N-r+1}\cdots Y_N f(x) = t^{(r+1-2N)r/2} \tau_{N-r+1}\cdots\tau_N f(x)$$

Proof. We first show that

$$Y_{N-r+1}\cdots Y_N = t^{-r(r-1)/2} (\omega \bar{T}_1 \cdots \bar{T}_{N-r})^r \tag{8.2}$$

The result obviously holds by definition when $r = 1$. Assuming that it holds for $r-1$, we have that

$$\begin{aligned} Y_{N-r+1}\cdots Y_N &= Y_N \cdots Y_{N-r+1} \\ &= t^{-(r-1)(r-2)/2} (\omega\bar{T}_1\cdots\bar{T}_{N-r+1})^{r-1} \big(t^{-r+1} T_{N-r+1}\cdots T_{N-1}\omega\bar{T}_1\cdots\bar{T}_{N-r}\big) \end{aligned}$$

Making use of the relation $\bar{T}_{i-1}\omega = \omega\bar{T}_i$, we can move the term $\bar{T}_{N-r+1}$ of every product to the right to get

$$\begin{aligned} Y_{N-r+1}\cdots Y_N &= t^{-r(r-1)/2} (\omega\bar{T}_1\cdots\bar{T}_{N-r})^{r-1} \bar{T}_{N-1}\cdots\bar{T}_{N-r+1} T_{N-r+1}\cdots T_{N-1}\omega\bar{T}_1\cdots\bar{T}_{N-r} \\ &= t^{-r(r-1)/2} (\omega\bar{T}_1\cdots\bar{T}_{N-r})^r \end{aligned}$$

which proves (2.2) by induction.

Using $\bar{T}_{i-1}\omega = \omega\bar{T}_i$ again and again, we then get from (2.2) that

$$Y_{N-r+1}\cdots Y_N = t^{-r(r-1)/2}\omega^r (\bar{T}_r\cdots\bar{T}_{N-1})\cdots(\bar{T}_1\cdots\bar{T}_{N-r})$$

If $f(x)$ is a symmetric function, the rightmost $N-r$ terms in every product in the previous equation can be pushed to the right and made to act as $1/t$ on $f(x)$. This yields,

$$Y_{N-r+1}\cdots Y_N f(x) = t^{-r(N-r)-r(r-1)/2}\, \tau_{N-r+1}\cdots\tau_N f(x)$$

which proves the lemma. □

The next result shows that $e_r(Y_1,\ldots,Y_N)$ can be recovered from $\mathcal{S}_N^t$ acting on $Y_{N-r+1}\cdots Y_N$.

Lemma 95. *For $r \le N$, we have that if $f(x)$ is a symmetric function then*

$$e_r(Y_1,\ldots,Y_N) f(x) = \frac{1}{[N-r]_t! [r]_t!} \mathcal{S}_N^t Y_{N-r+1}\cdots Y_N f(x)$$

Proof. First, if $w \in \mathfrak{S}_r$ and $\sigma \in \mathfrak{S}_{r+1,N}$ then $(T_w T_\sigma) Y_{N-r+1} \cdots Y_N = Y_{N-r+1} \cdots Y_N (T_w T_\sigma)$ by (1.2). This yields

$$T_w T_\sigma Y_{N-r+1} \cdots Y_N f(x) = t^{\ell(w)+\ell(\sigma)} Y_{N-r+1} \cdots Y_N f(x)$$

given that $f(x)$ is symmetric. Hence, summing over all the elements of $\mathfrak{S}_r \times \mathfrak{S}_{r+1,N}$ in $\mathcal{S}_N^t = \sum_{\sigma \in \mathfrak{S}_N} T_\sigma$ gives a factor of $[N-r]_t![r]_t!$ from (2.1). We thus have left to prove that

$$e_r(Y_1, \ldots, Y_N) f(x) = \sum_{[\sigma^*] \in \mathfrak{S}_N / (\mathfrak{S}_r \times \mathfrak{S}_{r+1,N})} T_{\sigma^*} Y_{N-r+1} \cdots Y_N f(x)$$

where the sum is over all left-coset representatives σ^* of minimal length. Such minimal length representatives are of the form (in one-line notation) $\sigma^* = [i_1, \ldots, i_{N-r}, i_{N-r+1}, \ldots, i_N]$ with $i_1 < i_2 < \cdots < i_{N-r}$ and $i_{N-r+1} < i_{N-r+2} < \cdots < i_N$. A reduced decomposition of σ^* is then given by

$$(s_{i_N} \cdots s_{N-1}) \cdots (s_{i_{N-r+1}} s_{i_{N-r+1}+1} \ldots s_{N-r}) \tag{8.3}$$

We will now see that the factor $T_{i_{N-r+1}} T_{i_{N-r+1}+1} \ldots T_{N-r}$ of T_{σ^*} changes Y_{N-r+1} into $Y_{i_{N-r+1}}$ and leaves the rest of the terms invariant. First, we use the relation $T_i Y_{i+1} = t Y_i \bar{T}_i$ to obtain

$$T_{N-r} Y_{N-r+1} Y_{N-r+2} \cdots Y_N f(x) = t Y_{N-r} \bar{T}_{N-r} Y_{N-r+2} \cdots Y_N f(x) = Y_{N-r} Y_{N-r+2} \cdots Y_N f(x)$$

Proceeding in this way again and again, we then get that

$$T_{i_{N-r+1}} T_{i_{N-r+1}+1} \ldots T_{N-r} Y_{N-r+1} Y_{N-r+2} \cdots Y_N f(x) = Y_{i_{N-r+1}} Y_{N-r+2} \cdots Y_N f(x)$$

as wanted. By assumption, all of the remaining indices of the s_j's in (2.3) are larger than i_{N-r+1}. Hence $Y_{i_{N-r+1}}$ will not be affected by the remaining terms in T_{σ^*}. Following as we just did, it is then immediate that

$$T_{\sigma^*} Y_{N-r+1} \cdots Y_N f(x) = Y_{i_{N-r+1}} \cdots Y_{i_N} f(x)$$

Finally, summing over all σ^*, the lemma is then seen to hold. $\square$

CHAPTER 4

The m-Symmetric Macdonald polynomials

In [**18**], aiming to understand Macdonald positivity, a novel class of Macdonald polynomials was defined. The m-symmetric Macdonald polynomials are defined as the t-symmetrization of the last variables of a non-symmetric Macdonald polynomial, while leaving the first m variables non-symmetric. As such, they coincide with the symmetric Macdonald polynomials when $m = 0$, and as m becomes sufficiently large, they transition into non-symmetric Macdonald polynomials. These polynomials, remarkably, satisfy most of the properties of the Macdonald polynomials outlined in the preceding chapter. This not only implies that these properties hold in the non-symmetric case (for sufficiently large m) but also facilitates a seamless transition between the symmetric and non-symmetric realms without sacrificing their inherent elegance.

1. The ring of m-symmetric functions

We define the ring R_m of m-symmetric functions as the subring of $\mathbb{Q}(q,t)[[x_1, x_2, x_3, \dots]]$ made of formal power series that are symmetric in the variables $x_{m+1}, x_{m+2}, x_{m+3}, \dots$. In other words, we have

$$R_m \simeq \mathbb{Q}(q,t)[x_1, \dots, x_m] \otimes \mathbf{\Lambda}_m$$

where $\mathbf{\Lambda}_m$ is the ring of symmetric functions in the variables $x_{m+1}, x_{m+2}, x_{m+3}, \dots$.

Example 96. *(1) We have that $f(x_1, x_2, x_3x_4) = x_1x_2(x_3^2 + x_4^2)$ is 2-symmetric,*
(2) We have that $f(x_1, x_2, x_3x_4) = x_1x_2x_3x_4 + x_2x_3 + x_2x_4 + x_3x_4 + x_1^2$ is 1-symmetric,

Remark 97. *It is immediate that $R_0 = \mathbf{\Lambda}$ is the usual ring of symmetric functions and that*

$$R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots .$$

2. m-Partitions

We know that bases of the space of symmetric functions are indexed by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0)$ while bases of the ring of polynomials are indexed by composition. Bases of R_m are naturally indexed by m-partitions which are pairs $\Lambda = (\boldsymbol{a}; \lambda)$, where $\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{Z}_{\geq 0}^m$ is a composition with m parts, and where λ is a partition.

Definition 98. *Given a composition $\boldsymbol{a}$ and a partition λ, a m-**partition** is $\Lambda = \boldsymbol{a} \cup \lambda$ that denote the partition obtained by reordering the entries of the concatenation of $\boldsymbol{a}$ and λ.*

$$\Lambda = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_m; \lambda_{m+1}, \dots, \lambda_l)$$

*We call the entries of $\boldsymbol{a}$ and λ the non-symmetric and symmetric entries of Λ respectively. The non-zero entries Λ_i are called the **parts** of Λ. We define the length of Λ as $\ell(\Lambda) = m + \ell(\lambda)$. The **degree** of an m-partition Λ, denoted $|\Lambda|$, is the sum of the degrees of $\boldsymbol{a}$ and λ, that is,*

$$|\Lambda| = a_1 + \cdots + a_m + \lambda_1 + \lambda_2 + \cdots .$$

Observe that we use a different notation for the composition $\boldsymbol{a}$ with m parts (which corresponds to the non-symmetric entries of Λ) than for the composition η with N parts (which will typically index a non-symmetric Macdonald polynomial).

Example 99. *The* 2*-partitions of* 3 *are*

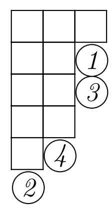
$$(0,0;2,1),(0,0;1,1,1),(1,0;2),(1,0;1,1),(0,1;2),(0,1;1,1),$$
$$(0,0;3),(2,0;1),(0,2;1),(1,1;1),(3,0;),(0,3;),(2,1;),(1,2;).$$

We will say that $\boldsymbol{a}$ is dominant if $a_1 \geq a_2 \geq \cdots \geq a_m$, and by extension, we will say that $\Lambda = (\boldsymbol{a};\lambda)$ is dominant if $\boldsymbol{a}$ is dominant. If $\boldsymbol{a}$ is not dominant, we let $\boldsymbol{a}^+$ be the dominant composition obtained by reordering the entries of $\boldsymbol{a}$.

There is a natural way to represent an m-partition by a Young diagram.

Definition 100. *The diagram corresponding to* Λ *is the Young diagram of* $\boldsymbol{a}\cup\lambda$ *with an* i*-circle added to the right of the row of size* a_i *for* $i = 1,\ldots,m$ *(if there are many rows of size* a_i*, the circles are ordered from top to bottom in increasing order).*

Example 101. *For instance, given* $\Lambda = (2,0,2,1;3,2)$*, we have*

$\Lambda \quad \longleftrightarrow$ 

Remark 102. *Observe that when* $m = 0$*, the diagram associated to* $\Lambda = (;\lambda)$ *coincides with the Young diagram associated to* λ*. Also note that if* η *is a composition with* m *parts, then the diagram of* η *coincides with the diagram of the* m*-partition* $\Lambda = (\boldsymbol{a};\emptyset)$*, where* $\boldsymbol{a} = \eta$*.*

Remark 103. *Since that two circle can not be in the same row, there is not a natural way to define the conjugate diagram in this context.*

Definition 104. *We let* $\Lambda^{(0)} = \boldsymbol{a}\cup\lambda$*, that is,* $\Lambda^{(0)}$ *is the partition obtained from the diagram of* Λ *by discarding all the circles. More generally, for* $i = 1,\ldots,m$*, we let* $\Lambda^{(i)} = (\boldsymbol{a}+1^i)\cup\lambda$*, where* $\boldsymbol{a}+1^i = (a_1+1,\ldots,a_i+1,a_{i+1},\ldots,a_m)$*. In other words,* $\Lambda^{(i)}$ *is the partition obtained from the diagram associated to* Λ *by changing all of the* j*-circles, for* $1 \leq j \leq i$*, into squares and discarding the remaining circles.*

Example 105. *Taking as above* $\Lambda = (2,0,2,1;3,2)$*, we have* $\Lambda^{(0)} = (3,2,2,2,1)$*,* $\Lambda^{(1)} = (3,3,2,2,1)$*,* $\Lambda^{(2)} = (3,3,2,2,1,1)$*,* $\Lambda^{(3)} = (3,3,3,2,1,1)$ *and* $\Lambda^{(4)} = (3,3,3,2,2,1)$*.*

Definition 106. *We define the dominance ordering on* m*-partitions to be such that*

$$\Lambda \geq \Omega \iff \Lambda^{(i)} \geq \Omega^{(i)} \qquad \text{for all } i = 0,\ldots,m \tag{2.1}$$

where the order on the r.h.s. is the usual dominance order on partitions defines in .

We will associate arm and leg-lengths to the cells of the diagram of an m-partition.

Definition 107. *Because of the circles, we will need two notions of arm-lengths as well as two notions of leg-lengths. The arm-length* $a(s)$ *is equal to the number of cells in* Λ *strictly to the right of* s *(and in the same row). Note that if there is a circle at the end of its row, then it adds one to the arm-length of* s*. The arm-length* $\tilde{a}(s)$ *is exactly as* $a(s)$ *except that the circle at the end of the row does not contribute to* $\tilde{a}(s)$*.*

The leg-length $\ell(s)$ *is equal to the number of cells in* Λ *strictly below* s *(and in the same column). If at the bottom of its column there are* k *circles whose fillings are smaller than the filling of the circle at the end of its row, then they add* k *to the value of the leg-length of* s*. If the row does not end with a circle then none of the circles at the bottom of its column contributes to the leg-length. The leg-length* $\tilde{\ell}(s)$ *is exactly as* $\ell(s)$ *except that the circles at the bottom of the column contribute to* $\tilde{\ell}(s)$ *when there is no circle at the end of the row of* s*.*

Example 108. *The values of $a(s)$ and $\ell(s)$ in each cell of the diagram of $\Lambda = (2,0,0,2;4,1,1)$ are*

$$\begin{array}{|c|c|c|c|}
\hline
34 & 22 & 10 & 00 \\ \hline
23 & 11 & 1 & \\ \cline{1-2}
24 & 10 & 4 & \\ \cline{1-2}
01 & & & \\ \cline{1-1}
00 & & & \\ \cline{1-1}
2 & & & \\
3 & & &
\end{array}$$

while those of $\tilde{a}(s)$ and $\tilde{\ell}(s)$ are

$$\begin{array}{|c|c|c|c|}
\hline
36 & 22 & 12 & 00 \\ \hline
13 & 01 & 1 & \\ \cline{1-2}
14 & 00 & 4 & \\ \cline{1-2}
03 & & & \\ \cline{1-1}
02 & & & \\ \cline{1-1}
2 & & & \\
3 & & &
\end{array}$$

3. Bases of the space of m-symmetric functions

We have the following bases, indexed by m-partitions, of the space of m-symmetric functions:

Monomial m-symmetric functions. Let the m-symmetric monomial function $m_\Lambda(x)$ be defined as

$$m_\Lambda(x) := x_1^{a_1} \cdots x_m^{a_m}\, m_\lambda(x_{m+1}, x_{m+2}, \dots) = x^{\boldsymbol{a}}\, m_\lambda(x_{m+1}, x_{m+2}, \dots)$$

where $m_\lambda(x_{m+1}, x_{m+2}, \dots)$ is the usual monomial symmetric function in the variables $x_{m+1}, x_{m+2}, \dots$

$$m_\lambda(x_{m+1}, x_{m+2}, \dots) = \sum_\alpha x_{m+1}^{\alpha_1} x_{m+2}^{\alpha_2} \cdots$$

with the sum being over all derrangements α of $(\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}, 0, 0, \dots)$. It is immediate that $\{m_\Lambda(x)\}_\Lambda$ is a basis of R_m.

Hall Litlewood m-symmetric functions. Let $H_{\boldsymbol{a}}(x_1, \dots, x_m; t) = E_{(a_1,\dots,a_m,0^{N-m})}(x_1, \dots, x_N; 0, t)$ be the non-symmetric Hall-Littlewood polynomial (the non-symmetric Macdonald polynomials only depend on the variables $x_1, \dots, x_m$ when $q = 0$ and the indexing compositions have length at most m). For simplicity, we will denote the non-symmetric Hall-Littlewood polynomial $H_{\boldsymbol{a}}(x;t)$ instead of $H_{\boldsymbol{a}}(x_1, \dots, x_m; t)$. We should note that the polynomial $H_{\boldsymbol{a}}(x;t)$ can be constructed recursively as follows. If $\boldsymbol{a}$ is dominant then $H_{\boldsymbol{a}}(x;t) = x^{\boldsymbol{a}}$. Otherwise, $T_i H_{\boldsymbol{a}}(x;t) = H_{s_i\boldsymbol{a}}(x;t)$ if $a_i > a_{i+1}$ (with $s_i\boldsymbol{a} = (a_1 \dots, a_{i+1}, a_i, \dots, a_m)$). Since $H_{\boldsymbol{a}}(x;1) = x^{\boldsymbol{a}}$, the following t-deformation of the m-symmetric power sum basis

$$p_\Lambda(x;t) = H_{\boldsymbol{a}}(x;t) p_\lambda(x) \tag{3.1}$$

also provides a basis of R_m.

Power-sum m-symmetric functions. Another basis of R_m is provided by the m-symmetric power sums.

$$p_\Lambda(x) := x_1^{a_1} \dots x_m^{a_m}\, p_\lambda(x) = x^{\boldsymbol{a}}\, p_\lambda(x)$$

It should be observed that the variables in p_λ, contrary to those of m_λ in $m_\Lambda(x)$, start at x_1 instead of x_{m+1}. In this expression, $p_\lambda(x)$ is the usual power-sum symmetric function

$$p_\lambda(x) = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(x)$$

where $p_r(x) = x_1^r + x_2^r + \cdots$.

4. m-Symmetric Macdonald polynomials

To construct Macdonald polynomials in N variables in the symmetric case, we use the t-symmetrization operator $\mathcal{S}_N^t$ on non-symmetric Macdonald polynomials. Similarly, the m-symmetric Macdonald polynomials in N variables can be obtained by applying the t-symmetrization operator $\mathcal{S}_{m+1,N}^t$ to non-symmetric Macdonald polynomials. It is worth noting that the findings presented in this section are once again taken from [**18**].

Definition 109. *The m-symmetric Macdonald polynomials in N variables are defined as*

$$P_\Lambda(x_1, \ldots, x_N; q, t) = \frac{1}{u_{\Lambda,N}(t)} \mathcal{S}_{m+1,N}^t E_{\eta_{\Lambda,N}}(x_1, \ldots, x_N; q, t) \tag{4.1}$$

with $\eta_{\Lambda,N} = (a_1, \ldots, a_m, \lambda_{N-m}, ..., \lambda_1)$, where we consider that $\lambda_i = 0$ if $i > \ell(\lambda)$ and the normalization constant $u_{\Lambda,N}(t)$ given by

$$u_{\Lambda,N}(t) = \left(\prod_{i \geq 0} [n_\lambda(i)]_{t^{-1}}! \right) t^{(N-m)(N-m-1)/2} \tag{4.2}$$

where $n_\lambda(i)$ is the number of entries in $\lambda_1, \ldots, \lambda_{N-m}$ that are equal to i (note that i can be equal to zero), and where

$$[k]_q = \frac{(1-q)(1-q^2)\cdots(1-q^k)}{(1-q)^k}$$

Observe that the normalization constant $u_{\Lambda,N}(t)$ is chosen such that the coefficient of m_Λ in $P_\Lambda(x; q, t)$ is equal to 1.

Remark 110. *If γ is any composition such that $\gamma_i = a_i$ for $i = 1, \ldots, m$ and such that the remaining entries rearrange to λ, then*

$$P_\Lambda(x_1, \ldots, x_N; q, t) = d_\gamma(q, t) \mathcal{S}_{m+1,N}^t E_\gamma(x_1, \ldots, x_N; q, t)$$

*for some non-zero coefficient $d_\gamma(q, t) \in \mathbb{Q}(q, t)$. This is an easy consequence of (**??**) and (**??**) (see for instance Lemma 55 in [**18**] for a more precise statement).*

In the case where $\Lambda = (\boldsymbol{a}; \emptyset)$, an m-symmetric Macdonald polynomial is simply a non-symmetric Macdonald polynomial:

$$P_{(\boldsymbol{a};\emptyset)}(x; q, t) = E_\eta(x, q, t) \tag{4.3}$$

where $\eta = (a_1, \ldots, a_m, 0^{N-m})$.

The m-symmetric Macdonald polynomials are stable with respect to the number of variables

Proposition 111. *Let N be the number of variables and suppose that $N > m$. Then*

$$P_\Lambda(x_1, \ldots, x_{N-1}, 0; q, t) = \begin{cases} P_\Lambda(x_1, \ldots, x_{N-1}; q, t) & \text{if } N > m + \ell(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

The m-symmetric Macdonald polynomials are the common eigenfunctions of a set of $m + 1$ commuting operators. First, they are eigenfunctions of the Cherednik operators Y_i, for $i = 1, \ldots, m$:

$$Y_i P_\Lambda(x_1, \ldots, x_N; q, t) = \varepsilon_\Lambda^{(i)}(q, t) P_\Lambda(x_1, \ldots, x_N; q, t) \qquad \text{with} \qquad \varepsilon_\Lambda^{(i)}(q, t) = q^{a_i} t^{1 - r_\Lambda(i)} \tag{4.4}$$

where we recall that $r_\Lambda(i)$ is the row in which the i-circle appears in the diagram associated to Λ. They are also eigenfunctions of the operator

$$D = Y_{m+1} + \cdots + Y_N - \sum_{i=m+1}^{N} t^{1-i}$$

which is such that

$$D\,P_\Lambda(x_1,\dots,x_N;q,t)=\varepsilon_\Lambda^D(q,t)P_\Lambda(x_1,\dots,x_N;q,t)\quad\text{with}\quad \varepsilon_\Lambda^D(q,t)=\sum_i{}' q^{\Lambda_i^{(0)}}t^{1-i}-\sum_{i=m+1}^{m+\ell(\lambda)}t^{1-i} \tag{4.5}$$

where the prime indicates that the sum is only over the rows of the diagram of Λ that do not end with a circle. We stress that the eigenvalues $\varepsilon_\Lambda^{(i)}(q,t)$ and $\varepsilon_\Lambda^D(q,t)$ do not depend on the number N of variables and uniquely determine the m-partition Λ.

Letting the number of variables to be infinite, the m-symmetric Macdonald polynomials then form a basis of R_m.

Proposition 112. *We have that*

$$P_\Lambda(x;q,t)=m_\Lambda+\sum_{\Omega<\Lambda}d_{\Lambda\Omega}(q,t)\,m_\Omega \tag{4.6}$$

Hence, the m-symmetric Macdonald polynomials form a basis of R_m.

Finally, let

$$c_\Lambda(q,t)=\prod_{s\in\Lambda}(1-q^{a(s)}t^{\ell(s)+1}) \tag{4.7}$$

where the product is over all the squares in the diagram of Λ (not including the circles), and where the arm and leg-lengths were defined in Definition 107. The integral form of the m-symmetric Macdonald polynomials is then defined as $J_\Lambda(x;q,t)=c_\Lambda(q,t)P_\Lambda(x;q,t)$.

5. Inclusion, evaluation and symmetry

Since an m-symmetric function is also an $(m+1)$-symmetric function, it is natural to consider the inclusion $i:R_m\to R_{m+1}, a\mapsto a$. The inclusion of an m-symmetric Macdonald polynomial turns out to have a simple formula which will prove fundamental in the next section. The proof, being quite long and technical, will be relegated to Appendix 7.1.

Theorem 113. *The inclusion $i:R_m\to R_{m+1}$ is such that*

$$i(P_\Lambda)=\sum_\Omega \psi_{\Omega/\Lambda}(q,t)P_\Omega$$

where the sum is over all $(m+1)$-partitions Ω whose diagram is obtained from that of Λ by adding an $(m+1)$-circle at the end of a symmetric row (a row that does not end with a circle). The coefficient $\psi_{\Omega/\Lambda}(q,t)$ is given explicitly as

$$\psi_{\Omega/\Lambda}(q,t)=\prod_{s\in\mathrm{col}_{\Omega/\Lambda}}\frac{1-q^{a_\Lambda(s)+1}t^{\tilde\ell_\Lambda(s)}}{1-q^{a_\Omega(s)+1}t^{\tilde\ell_\Omega(s)}}$$

where $\mathrm{col}_{\Omega/\Lambda}$ stands for the set of squares in the diagram of Ω that lie in the column of the $(m+1)$-circle and in a symmetric row (a row that does not end with a circle), and where the arm and leg-lengths were defined before Example 108 (with the indices specifying with respect to which m-partition they are computed).

Now, let $\mathrm{Inv}(\boldsymbol{a})$ be the number of inversions in $\boldsymbol{a}$:

$$\mathrm{Inv}(\boldsymbol{a})=\#\{1\le i<j\le m\,|\,a_i<a_j\}$$

and let

$$\mathrm{coInv}(\boldsymbol{a})=m(m-1)/2-\mathrm{Inv}(\boldsymbol{a})=\#\{1\le i<j\le m\,|\,a_i\ge a_j\}$$

As usual, for a partition λ, we let $n(\lambda)=\sum_i (i-1)\lambda_i$. In the case of an m-partition, we will define $n(\Lambda):=n(\Lambda^{(m)})$, where we recall that $\Lambda^{(m)}$ is the partition obtained from the diagram of Λ by converting all the circles into squares.

We now give the principal specialization $u_\emptyset\big(f(x_1,x_2,\dots,x_N)\big):=f(1,t,\dots,t^{N-1})$ of an m-symmetric Macdonald polynomial.

Proposition 114. *For $\Lambda=(\boldsymbol{a};\lambda)$, the principal specialization is given by*

$$u_\emptyset(P_\Lambda(x;q,t))=t^{n(\Lambda)-\mathrm{coInv}(\boldsymbol{a})}\frac{[N-m]_t!}{[N]_t!}\prod_{s\in\Lambda^\circ}\frac{(1-q^{a'(s)}t^{N-\ell'(s)})}{(1-q^{a(s)}t^{\ell(s)+1})}$$

where Λ° stands for the set of cells (including the circles) in the diagram of Λ, and where the co-arm and co-leg are given respectively by $a'(s)=j-1$ and $\ell'(s)=i-1$ for the cell $s=(i,j)$.

Proof. Define the operator

$$\Psi_N=(1-t)(1+T_{N-1}+T_{N-2}T_{N-1}+\cdots+T_m\cdots T_{N-1})\Phi_q$$

where Φ_q was introduced in (**??**). The action of Ψ_N on an m-Macdonald polynomial turns out to be quite simple [**18**]:

$$\Psi_N J_\Lambda(x;q,t)=t^{-\#\{2\le j\le m\,|\,a_j\le a_1\}}J_{\Lambda^\square}(x,q,t) \tag{5.1}$$

where $\Lambda^\square=\big(a_2,\dots,a_m;\lambda\cup(a_1+1)\big)$, and where the integral form of the m-Macdonald polynomials was introduced in Section 3. Note that the diagram of $\Lambda^\square$ can be obtained from that of Λ by transforming the 1-circle into a square (and then relabeling the remaining circles so that they go from 1 to $m-1$ instead of from 2 to m).

Using $(T_i f)(1,\dots,t^{N-1})=tf(1,\dots,t^{N-1})$ for all $i=1,\dots,N-1$, and for all $f(x_1,\dots,x_N)\in\mathbb{Q}[x_1,\dots,x_N]$, we easily deduce that

$$u_\emptyset(\Psi_N g)=(1-t)(1+t+\cdots+t^{N-m})u_\emptyset(g)=(1-t^{N-m+1})u_\emptyset(g)$$

for all $g(x_1,\dots,x_N)\in\mathbb{Q}[x_1,\dots,x_N]$. Applying $u_\emptyset$ on both sides of (5.1) and using $J_\Lambda(x;q,t)=c_\Lambda(q,t)P_\Lambda(x;q,t)$, we thus get that

$$u_\emptyset(P_\Lambda)=\frac{t^{-\#\{2\le j\le m\,|\,a_j\le a_1\}}}{(1-t^{N-m+1})}\frac{c_{\Lambda^\square}(q,t)}{c_\Lambda(q,t)}u_\emptyset(P_{\Lambda^\square}) \tag{5.2}$$

Since $\Lambda^\square$ is an $(m-1)$-partition while Λ is an m-partition, this recursion will allow us to prove the proposition by induction on m.

In the base case $m=0$, we have $\mathrm{coInv}(\boldsymbol{a})=0$ and $\Lambda=(;\lambda)=\Lambda^{(m)}=\Lambda^\circ$ can be identified with λ. Hence the proposition simply becomes

$$u_\emptyset(P_\lambda(x;q,t))=t^{n(\lambda)}\prod_{s\in\lambda}\frac{(1-q^{a'(s)}t^{N-\ell'(s)})}{(1-q^{a(s)}t^{\ell(s)+1})}$$

which is the well-known evaluation of a Macdonald polynomial [**20**].

We will now see that the general case holds. Let

$$c'_\Lambda(q,t)=\prod_{s\in\Lambda^\circ}(1-q^{a(s)}t^{\ell(s)+1})\qquad\text{and}\qquad d_\Lambda(q,t)=\prod_{s\in\Lambda^\circ}(1-q^{a'(s)}t^{N-\ell'(s)})$$

First observe that

$$\frac{c_\Lambda(q,t)}{c'_\Lambda(q,t)}=\frac{c_{\Lambda^\square}(q,t)}{(1-t)c'_{\Lambda^\square}(q,t)}$$

since the 1-circle contributes in $c'_\Lambda(q,t)/c_\Lambda(q,t)$ while not in $c'_{\Lambda^\square}(q,t)/c_{\Lambda^\square}(q,t)$. Using $d_\Lambda(q,t)=d_{\Lambda^\square}(q,t)$ and (5.2), the proposition will thus hold by induction if we can show that

$$t^{n(\Lambda)-\mathrm{coInv}(\boldsymbol{a})}\frac{[N-m]_t!}{[N]_t!}\frac{1}{(1-t)}=\frac{t^{-\#\{2\le j\le m\,|\,a_j\le a_1\}}}{(1-t^{N-m+1})}t^{n(\Lambda^\square)-\mathrm{coInv}(\boldsymbol{a}')}\frac{[N-m+1]_t!}{[N]_t!}$$

where $\boldsymbol{a}' = (a_2, \ldots, a_m)$. But this easily follows from $n(\Lambda) = n(\Lambda^\square)$,

$$\mathrm{coInv}(\boldsymbol{a}) = \mathrm{coInv}(\boldsymbol{a}') + \#\{2 \leq j \leq m \,|\, a_j \leq a_1\}$$

and

$$[N-m+1]_t! = \frac{(1-t^{N-m+1})}{(1-t)}[N-m]_t!$$

□

In the special case $m = N$, our evaluation formula for the m-symmetric Macdonald polynomials can be simplified. It provides a reformulation of the principal specialization of the non-symmetric Macdonald polynomials which can be found for instance in [**23**].

Corollary 115. *The non-symmetric Macdonald polynomials are such that*

$$E_\eta(1, t, \ldots, t^{N-1}; q, t) = t^{n(\eta^+)+\mathrm{Inv}(\eta)} \prod_{s \in \eta} \frac{(1 - q^{a(s)} t^{N-\ell'(s)})}{(1 - q^{a(s)} t^{\ell(s)+1})}$$

where we recall that η^+ is the partition obtained by reordering the entries of η.

Proof. From (4.3), we have in the case $m = N$ that $P_{(\boldsymbol{a};\emptyset)}(x; q, t) = E_\eta(x; q, t)$ with $\eta = (a_1, \ldots, a_N)$. Using Proposition 114, we thus get that

$$E_\eta(1, t, \ldots, t^{N-1}; q, t) = t^{n(\eta^+)+N(N-1)/2-\mathrm{coInv}(\eta)} \frac{1}{[N]_t!} \prod_{s \in \eta^\circ} \frac{(1 - q^{a'(s)} t^{N-\ell'(s)})}{(1 - q^{a(s)} t^{\ell(s)+1})} \tag{5.3}$$

where we have used the fact that $n(\Lambda) = n(\eta^+ + 1^N) = n(\eta^+) + N(N-1)/2$ since every row of η ends with a circle. It is straightforward to check that

$$\prod_{s \in \eta^\circ} (1 - q^{a'(s)} t^{N-\ell'(s)}) = \prod_{s \in \eta^\circ} (1 - q^{a(s)} t^{N-\ell'(s)})$$

Hence

$$\prod_{s \in \eta^\circ} \frac{(1 - q^{a'(s)} t^{N-\ell'(s)})}{(1 - q^{a(s)} t^{\ell(s)+1})} = \left[\prod_{s \in \eta} \frac{(1 - q^{a(s)} t^{N-\ell'(s)})}{(1 - q^{a(s)} t^{\ell(s)+1})}\right] \left[\prod_{s \in \circ} \frac{(1 - q^{a(s)} t^{N-\ell'(s)})}{(1 - q^{a(s)} t^{\ell(s)+1})}\right] \tag{5.4}$$

where $\circ$ stands for the cells of the diagram of η corresponding to circles. Note that when s is in the position of a circle, we have $a(s) = 0$ and $\ell(s) = 0$. Because there is a circle in every row of η, we thus obtain

$$\prod_{s \in \circ} \frac{(1 - q^{a(s)} t^{N-\ell'(s)})}{(1 - q^{a(s)} t^{\ell(s)+1})} = [N]_t$$

Using the previous result in (5.4), the corollary follows immediately from (5.3) and the relation $\mathrm{Inv}(\eta) = N(N-1)/2 - \mathrm{coInv}(\eta)$. □

We now introduce an evaluation depending on an m-partition. We will see that it satisfies a natural symmetry property. First, to the m-partition $\Lambda = (a_1, \ldots, a_m; \lambda)$, we associate the composition

$$\gamma_\Lambda = (a_1, \ldots, a_m, \lambda_1, \ldots, \lambda_\ell, 0^{N-m-\ell})$$

Let w be the minimal length permutation such that $w\gamma_\Lambda$ is weakly decreasing. If we forget about the extra zeroes, we thus have that $w\gamma_\Lambda = \Lambda^{(0)}$, where we recall that $\Lambda^{(0)}$ stands for the partition whose diagram is obtained from that of Λ by removing all the circles. The evaluation u_Λ is then defined on any m-symmetric function $f(x)$ as

$$u_\Lambda\big(f(x_1, \ldots, x_N)\big) = f\left(q^{-\Lambda^{(0)}_{w(1)}} t^{w(1)-1}, \ldots, q^{-\Lambda^{(0)}_{w(N)}} t^{w(N)-1}\right) \tag{5.5}$$

Observe that in the case $\Lambda = (0^m; \emptyset)$, u_Λ corresponds to the principal evaluation $u_\emptyset$.

LEMMA 116. *If f is an m-symmetric function in N variables then*

$$f(Y^{-1})P_\Lambda(x;q,t) = u_\Lambda(f)P_\Lambda(x;q,t) \tag{5.6}$$

PROOF. Let $\eta = \gamma_\Lambda$. We have that $Y_i^{-1}E_\eta = \bar{\eta}_i^{-1}E_\eta$, where we recall that $\bar{\eta}_i = q^{\eta_i}t^{1-r_\eta(i)}$. Using the fact that f is m-symmetric, we then have from Remark 110 that

$$\begin{aligned} f(Y_1^{-1},\dots,Y_N^{-1})P_\Lambda(x;q,t) &= f(Y_1^{-1},\dots,Y_N^{-1})d_\eta(q,t)\mathcal{S}^t_{m+1,,N}E_\eta \\ &= d_\eta(q,t)\mathcal{S}^t_{m+1,,N}f(Y_1^{-1},\dots,Y_N^{-1})E_\eta \\ &= d_\eta(q,t)\mathcal{S}^t_{m+1,,N}f(\bar{\eta}_1^{-1},\dots,\bar{\eta}_N^{-1})E_\eta \\ &= f(\bar{\eta}_1^{-1},\dots,\bar{\eta}_N^{-1})P_\Lambda(x;q,t) \end{aligned}$$

It thus only remains to show that the specialization $x_i = \bar{\eta}_i^{-1}$ corresponds to the evaluation defined in (5.5). Let w be the minimal length permutation such that $w\eta = \Lambda^{(0)}$. We have immediately that $\eta_i = \Lambda^{(0)}_{w(i)}$. Hence, from the definition of u_Λ, we only need to show that $r_\eta(i) = w(i)$. But this is a consequence of the minimality of w. Indeed, if $\eta_i = \eta_j$ and $i < j$ then the minimality of w ensures that $w(i) < w(j)$, which implies that the circles increase from top to bottom in equal rows. □

The next proposition extends a well-known property of the Macdonald polynomials [**20**]. Recall that $u_\emptyset(P_\Lambda(x,q,t))$ was given explicitly in Proposition 114.

PROPOSITION 117. *Let $\tilde{P}_\Lambda(x,q,t)$ be the normalization of the m-Macdonald polynomials given by*

$$\tilde{P}_\Lambda(x,q,t) = \frac{P_\Lambda(x;q,t)}{u_\emptyset(P_\Lambda(x,q,t))}$$

Then, the following symmetry holds:

$$u_\Omega(\tilde{P}_\Lambda) = u_\Lambda(\tilde{P}_\Omega)$$

PROOF. For $f(x)$ and $g(x)$ Laurent polynomials in $x_1,\dots,x_N$, it is known [**22**] that the pairing

$$[f(x),g(x)] := u_\emptyset\left(f(Y^{-1})g(x)\right)$$

is such that $[f,g]=[g,f]$. From Lemma 189, we thus get

$$\begin{aligned} \left[P_\Lambda(x_{(N)},q,t),P_\Omega(x_{(N)},q,t)\right] &= u_\emptyset\big(P_\Lambda(Y_i^{-1})P_\Omega(x_{(N)},q,t)\big) \\ &= u_\Omega(P_\Lambda(x_{(N)},q,t))u_\emptyset\big(P_\Omega(x_{(N)},q,t)\big) \end{aligned}$$

From the symmetry of the pairing $[\cdot,\cdot]$, it then follows that

$$u_\Omega(P_\Lambda(x_{(N)},q,t))u_\emptyset\big(P_\Omega(x_{(N)},q,t)\big) = u_\Lambda(P_\Omega(x_{(N)},q,t))u_\emptyset\big(P_\Lambda(x_{(N)},q,t)\big)$$

which proves the proposition. □

The final result of this section is concerned with the behavior of $P_\Lambda(x;q,t)$ when q and t are sent to q^{-1} and t^{-1}. For $\sigma \in S_N$ with a reduced decomposition $s_{i_1}\cdots s_{i_r}$, we let $K_\sigma = K_{i_1,i_1+1}\cdots K_{i_r,i_r+1}$, and $T_\sigma = T_{i_1}\cdots T_{i_r}$. We also let $\omega_m = [m,m-1,\dots,1]$ be the longest permutation in the symmetric group S_m (which we consider as the element $[m,\dots,1,m+1,\dots,N]$ of S_N), and denote the inverse of T_{ω_m} by $\bar{T}_{\omega_m}$.

PROPOSITION 118. *We have that*

$$q^{|\boldsymbol{a}|}t^{\mathrm{Inv}(\boldsymbol{a})}P_\Lambda(x;q^{-1},t^{-1}) = t^{\binom{m}{2}}\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}P_\Lambda(x;q,t)$$

or, equivalently, that

$$q^{|\boldsymbol{a}|}t^{\mathrm{Inv}(\boldsymbol{a})}P_\Lambda(x_mq^{-1},\dots,x_1q^{-1},x_{m+1},x_{m+2},\dots;q^{-1},t^{-1}) = t^{\binom{m}{2}}\bar{T}_{\omega_m}P_\Lambda(x;q,t)$$

Remark 119. *The proposition is an extension of a similar result on non-symmetric polynomials (see Lemma 2.3 a) in* [**23**]*) which states that*

$$t^{\mathrm{Inv}(\eta)}E_\eta(x_m,\dots,x_1;q^{-1},t^{-1})=t^{\binom{m}{2}}\bar T_{\omega_m}E_\eta(x_1,\dots,x_m;q,t) \tag{5.7}$$

This relation was also proven in a broader context in [**1**] *in connection with permuted-basement Macdonald polynomials* [**17**]. *When the number of variables N is equal to m and $\Lambda=(\eta;\emptyset)$, Proposition 118 becomes* (5.7) *(the q powers canceling from the homogeneity of E_η). But when the number of variables is larger than m and $\Lambda=(\eta;\emptyset)$, Proposition 118 is actually stronger than* (5.7) *since it says that for any non-symmetric Macdonald polynomial such that $\ell(\eta)\leq m$ we have*

$$q^{|\eta|}t^{\mathrm{Inv}(\eta)}E_\eta(x_mq^{-1},\dots,x_1q^{-1},x_{m+1},x_{m+2},\dots;q^{-1},t^{-1})=t^{\binom{m}{2}}\bar T_{\omega_m}E_\eta(x;q,t)$$

Proof of Proposition 118. It suffices to prove the result in N variables. Let $Y_i^\star$ be the Cherednik operator Y_i with parameters q^{-1} and t^{-1} instead of q and t, and similarly for the operator $D^\star$. We thus have from (4.4) and (4.5) that

$$Y_i^\star\,P_\Lambda(x;q^{-1},t^{-1})=\varepsilon_\Lambda^{(i)}(q^{-1},t^{-1})\,P_\Lambda(x;q^{-1},t^{-1})\qquad i=1,\dots,m$$

and

$$D^\star\,P_\Lambda(x;q^{-1},t^{-1})=\varepsilon_\Lambda^{D}(q^{-1},t^{-1})\,P_\Lambda(x;q^{-1},t^{-1})$$

The main part of the proof thus consists in proving that $\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}P_\Lambda(x;q,t)$ is an eigenfunction of $Y_1^\star,\dots,Y_m^\star$ and $D^\star$ with the right eigenvalues. In order to achieve this, we prove that for any $f\in R_m$ we have

$$Y_i^\star\left(\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}\right)f=\left(\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}\right)\bar Y_if\qquad i=1,\dots,m \tag{5.8}$$

and

$$D^\star\left(\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}\right)f=\left(\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}\right)\bar Df \tag{5.9}$$

where $\bar D=\bar Y_{m+1}+\cdots+\bar Y_N+\sum_{i=m+1}^N t^{i-1}$. It is then immediate that

$$\begin{aligned}Y_i^\star\left(\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}\right)P_\Lambda(x;q,t)&=\left(\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}\right)\bar Y_iP_\Lambda(x;q,t)\\&=\varepsilon_\Lambda^{(i)}(q^{-1},t^{-1})\left(\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}\right)P_\Lambda(x;q,t)\qquad i=1,\dots,m\end{aligned}$$

and

$$\begin{aligned}D^\star\left(\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}\right)P_\Lambda(x;q,t)&=\left(\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}\right)\bar DP_\Lambda(x;q,t)\\&=\varepsilon_\Lambda^{D}(q^{-1},t^{-1})\left(\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}\right)P_\Lambda(x;q,t)\end{aligned}$$

as wanted. The proof of (5.8) and (5.9), which is somewhat technical, is provided in Appendix 7.3.

We have thus proven that $\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}P_\Lambda(x;q,t)$ is equal to $P_\Lambda(x;q^{-1},t^{-1})$ up to a constant. Hence, we have left to prove that the proportionality constant corresponds to the powers of q and t in the statement of the proposition. In the proof of Lemma 2.3 a) in [**23**], it is shown that

$$K_{\omega_m}\bar T_{\omega_m}x^{\boldsymbol a}=t^{\mathrm{Inv}(\boldsymbol a)-\binom{m}{2}}x^{\boldsymbol a}+\text{smaller terms}$$

where the order is the order on compositions defined in Section **??**. We thus deduce straightforwardly that the action of $\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}$ on the dominant term $m_\Lambda=x^{\boldsymbol a}m_\lambda(x_{m+1},\dots,x_N)$ of $P_\Lambda(x;q^{-1},t^{-1})$ is such that

$$\tau_1\cdots\tau_mK_{\omega_m}\bar T_{\omega_m}m_\Lambda=q^{|\boldsymbol a|}t^{\mathrm{Inv}(\boldsymbol a)-\binom{m}{2}}m_\Lambda+\text{smaller terms}$$

where the smaller terms are of the form $x^{\boldsymbol b}m_\lambda(x_{m+1},\dots,x_N)$ with $\boldsymbol b$ smaller than $\boldsymbol a$. This concludes the proof of the proposition. $\square$

6. Orthogonality

Recall that $\mathrm{Inv}(\boldsymbol{a})$ is the number of inversions in $\boldsymbol{a}$, and let $|\boldsymbol{a}| = a_1 + \cdots + a_m$. The following scalar product in R_m is defined on the t-deformation of the m-symmetric power sums introduced in (3.1):

$$\langle p_\Lambda(x;t)\,,\, p_\Omega(x;t)\rangle_m = \delta_{\Lambda\Omega}\, q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} z_\lambda(q,t) \tag{6.1}$$

where

$$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

with $z_\lambda = \prod_{i\geq 1} i^{n_\lambda(i)} \cdot n_\lambda(i)!$ (recall that $n_\lambda(i)$ is the number of occurrences of i in λ). Observe that when $m=0$, this corresponds to the usual Macdonald polynomial scalar product [**20**].

The main goal of this section is to show that the m-symmetric Macdonald polynomials are orthogonal with respect to the scalar product (6.1) and to provide the value of the squared norm $\|P_\Lambda(x;q,t)\|^2$. But before proving the theorem, we need to establish a few results.

Definition 120. *We define $K_m(x,y)$ is a reproducing kernel for the scalar product* (6.1)*:*

$$K_m(x,y) = t^{-\binom{m}{2}} K_0(x,y) T^{(x)}_{\omega_m} \mathcal{NF}_m \tag{6.2}$$

where

$$\mathcal{NF}_m = \left[\frac{\prod_{i+j\leq m}(1-tq^{-1}x_iy_j)}{\prod_{i+j\leq m+1}(1-q^{-1}x_iy_j)}\right]$$

and the superscript (x) indicates that the Hecke algebra operator $T^{(x)}_{\omega_m}$ acts on the x variables. We will see later in Proposition 125 that

We first show that the eigenoperators $Y_1,\ldots,Y_m, D$ are symmetric with respect to $K_m(x,y)$ when the number of variables is finite. In order not to disrupt the flow of the presentation, the proof will be relegated to Appendix 7.2.

Proposition 121. *For $x_{(N)} = (x_1,\ldots,x_N)$ and $y_{(N)} = (y_1,\ldots,y_N)$, we have that*

$$Y_i^{(x)} K_m(x_{(N)}, y_{(N)}) = Y_i^{(y)} K_m(x_{(N)}, y_{(N)})$$

for $i = 1,\ldots,m$, and

$$D^{(x)} K_m(x_{(N)}, y_{(N)}) = D^{(y)} K_m(x_{(N)}, y_{(N)})$$

As already mentioned, the eigenvalues of the Y_i's and D do not depend on the number of variables N. Using Proposition 111, we can thus define the operators $\tilde{Y}_i$ and $\tilde{D}$ as their inverse limits. In other words, the operators $\tilde{Y}_i : R_m \to R_m$ (for $i = 1,\ldots,m$) and $\tilde{D} : R_m \to R_m$ are defined such that

$$\tilde{Y}_i P_\Lambda(x;q,t) = \varepsilon^{(i)}_\Lambda(q,t) P_\Lambda(x;q,t) \quad \text{and} \quad \tilde{D} P_\Lambda(x;q,t) = \varepsilon^{D}_\Lambda(q,t) P_\Lambda(x;q,t)$$

for all m-partitions Λ, where $\varepsilon^{(i)}_\Lambda(q,t)$ and $\varepsilon^{D}_\Lambda(q,t)$ are such as defined in (4.4) and (4.5) respectively. We will now see that the previous proposition also holds for $\tilde{Y}_i$ and $\tilde{D}$.

Proposition 122. *We have that*

$$\tilde{Y}_i^{(x)} K_m(x,y) = \tilde{Y}_i^{(y)} K_m(x,y)$$

for $i = 1,\ldots,m$, and

$$\tilde{D}^{(x)} K_m(x,y) = \tilde{D}^{(y)} K_m(x,y)$$

Moreover,

$$K_m(x,y) = \sum_\Lambda b_\Lambda(q,t) P_\Lambda(x;q,t) P_\Lambda(y;q,t) \tag{6.3}$$

for certain coefficients $b_\Lambda(q,t) \in \mathbb{Q}(q,t)$ that will be given explicitely in Corollary 130.

Proof. From (??) and the fact that $K_m(x,y)/K_0(x,y)$ only depends on $x_1,\dots,x_m$, we have that

$$K_m(x,y) = \sum_{\Lambda,\Omega} d_{\Lambda\Omega}(q,t) P_\Lambda(x;q,t) P_\Omega(y;q,t)$$

for some coefficients $d_{\Lambda\Omega}(q,t)$. From Proposition 111, we then get in N variables that

$$K_m(x_{(N)},y_{(N)}) = \sum_{\ell(\Lambda),\ell(\Omega)\le N} d_{\Lambda\Omega}(q,t) P_\Lambda(x_{(N)};q,t) P_\Omega(y_{(N)};q,t)$$

Therefore, if for instance the action of $\tilde D^{(x)}$ were different from that of $\tilde D^{(y)}$ then there would exist a coefficient $d_{\Lambda\Omega}(q,t)$ such that $\varepsilon^D_\Lambda(q,t) d_{\Lambda\Omega}(q,t) \neq \varepsilon^D_\Omega(q,t) d_{\Lambda\Omega}(q,t)$. But then, choosing N large enough, this would contradict the fact that $D^{(x)}$ and $D^{(y)}$ have the same action on $K_m(x_{(N)},y_{(N)})$. Hence, the first part of the proposition holds.

Now, this entails that for all Λ,Ω we have

$$\varepsilon^D_\Lambda(q,t) d_{\Lambda\Omega}(q,t) = \varepsilon^D_\Omega(q,t) d_{\Lambda\Omega}(q,t) \qquad \text{and} \qquad \varepsilon^{(i)}_\Lambda(q,t) d_{\Lambda\Omega}(q,t) = \varepsilon^{(i)}_\Omega(q,t) d_{\Lambda\Omega}(q,t), \quad i=1,\dots,m$$

Given that the eigenvalues as a whole uniquely determine Λ, we must have that $d_{\Lambda\Omega}(q,t)=0$ if $\Lambda\neq\Omega$. Letting $b_\Lambda(q,t) = d_{\Lambda\Lambda}(q,t)$, we get that (6.3) also holds. □

The following proposition will be instrumental in the proof that $K_m(x,y)$ is a reproducing kernel for the scalar product (6.1).

Proposition 123. *We have*

$$t^{-\binom{m}{2}} T^{(x)}_{\omega_m} \left[\frac{\prod_{i+j\le m}(1-tx_iy_j)}{\prod_{i+j\le m+1}(1-x_iy_j)} \right] = \sum_{\boldsymbol{a}} t^{-\mathrm{Inv}(\boldsymbol{a})} H_{\boldsymbol{a}}(x;t) H_{\boldsymbol{a}}(y;t)$$

where the sum is over all $\boldsymbol{a} \in \mathbb{Z}^m_{\ge 0}$.

Proof. Letting $y_i \mapsto qy_i$ in (6.3), we obtain

$$\begin{aligned} t^{-\binom{m}{2}} K_0(x,qy) T^{(x)}_{\omega_m} \left[\frac{\prod_{i+j\le m}(1-tx_iy_j)}{\prod_{i+j\le m+1}(1-x_iy_j)} \right] = K_m(x,qy) &= \sum_\Lambda b_\Lambda(q,t) P_\Lambda(x;q,t) P_\Lambda(qy;q,t) \\ &= \sum_\Lambda b_\Lambda(q,t) q^{|\Lambda|} P_\Lambda(x;q,t) P_\Lambda(y;q,t) \end{aligned}$$

by the homogeneity of $P_\Lambda(y;q,t)$. Using $P_\Lambda(x_1,\dots,x_m;q,t)=0$ if $\ell(\Lambda)>m$ by Proposition 111, we get when restricting to m variables that

$$t^{-\binom{m}{2}} K_0(\bar x,q\bar y) T^{(x)}_{\omega_m} \left[\frac{\prod_{i+j\le m}(1-tx_iy_j)}{\prod_{i+j\le m+1}(1-x_iy_j)} \right] = \sum_{\boldsymbol{a}} \bar b_{(\boldsymbol{a};\emptyset)}(q,t) P_{(\boldsymbol{a};\emptyset)}(\bar x;q,t) P_{(\boldsymbol{a};\emptyset)}(\bar y;q,t)$$

where $\bar x = (x_1,\dots,x_m)$ (and similarly for $\bar y$), and where $\bar b_{(\boldsymbol{a};\emptyset)}(q,t) = b_{(\boldsymbol{a};\emptyset)}(q,t) q^{|\boldsymbol{a}|}$. Letting $q=0$ and using $K_0(\bar x,\bar z)=1$ whenever $\bar z=(0,\dots,0)$, we then obtain

$$t^{-\binom{m}{2}} T^{(x)}_{\omega_m} \left[\frac{\prod_{i+j\le m}(1-tx_iy_j)}{\prod_{i+j\le m+1}(1-x_iy_j)} \right] = \sum_{\boldsymbol{a}} \bar b_{(\boldsymbol{a};\emptyset)}(0,t) H_{\boldsymbol{a}}(x;t) H_{\boldsymbol{a}}(y;t) \tag{6.4}$$

Note that we have used the fact that $P_{(\boldsymbol{a};\emptyset)}(\bar x;0,t) = H_{\boldsymbol{a}}(\bar x;t) = H_{\boldsymbol{a}}(x;t)$ given that $H_{\boldsymbol{a}}(x;t)$ only depends on $x_1,\dots,x_m$. We thus only need to show that $\bar b_{(\boldsymbol{a};\emptyset)}(0,t) = t^{-\mathrm{Inv}(\boldsymbol{a})}$, which will be achieved by using the specialization $y_i = t^{i-1}$ in the previous equation. First observe that setting $q=0$ in

Corollary 115 yields $H_{\boldsymbol{a}}(1,t,\dots,t^{m-1};t)=t^{n(\boldsymbol{a}^+)}t^{\mathrm{Inv}(\boldsymbol{a})}$ ($a(s)$ is always larger than zero given that every row in the diagram of $\eta=\boldsymbol{a}$ ends with a circle), and that

$$\frac{\prod_{i+j\leq m}(1-x_it^j)}{\prod_{i+j\leq m+1}(1-x_it^{j-1})}=\frac{1}{\prod_{i=1}^m(1-x_i)}$$

Specializing the variables y in (6.4) thus gives

$$t^{-\binom{m}{2}}T^{(x)}_{\omega_m}\left[\frac{1}{\prod_{i=1}^m(1-x_i)}\right]=\frac{1}{\prod_{i=1}^m(1-x_i)}=\sum_{\boldsymbol{a}}\bar{b}_{(\boldsymbol{a};\emptyset)}(0,t)t^{n(\boldsymbol{a}^+)}t^{\mathrm{Inv}(\boldsymbol{a})}H_{\boldsymbol{a}}(x;t)$$

since $\prod_{i=1}^m(1-x_i)$ is symmetric. Finally, we observe that $\prod_{i=1}^m(1-tx_i)^{-1}$ is the generating series of the complete symmetric functions $h_n(x_1,\dots,x_m)$ to deduce [**20**] that

$$\frac{1}{\prod_{i=1}^m(1-x_i)}=\sum_n h_n(x_1,\dots,x_m)=\sum_{\lambda\,;\,\ell(\lambda)\leq m}t^{n(\lambda)}P_\lambda(\bar{x};t)=\sum_{\lambda\,;\,\ell(\lambda)\leq m}t^{n(\lambda)}\sum_{\boldsymbol{a}\,:\,\boldsymbol{a}^+=\lambda}H_{\boldsymbol{a}}(x;t)$$

where $P_\lambda(\bar{x};t)$ is a Hall-Littlewood polynomial, and where the elementary relation $P_\lambda(\bar{x};t)=\sum_{\boldsymbol{a}\,:\,\boldsymbol{a}^+=\lambda}H_{\boldsymbol{a}}(x;t)$ can be found for instance in [**21**]. Comparing the previous two equations, we obtain immediately that $\bar{b}_{(\boldsymbol{a};\emptyset)}(0,t)=t^{-\mathrm{Inv}(\boldsymbol{a})}$, as wanted. $\square$

Corollary 124. *We have*

$$K_m(x,y)=\sum_{\Lambda}q^{-|\boldsymbol{a}|}t^{-\mathrm{Inv}(\boldsymbol{a})}z_\lambda(q,t)^{-1}p_\Lambda(x;t)p_\Lambda(y;t)$$

Proof. Letting $y_i\mapsto q^{-1}y_i$ in Proposition 123 yields

$$t^{-\binom{m}{2}}T^{(x)}_{\omega_m}\left[\frac{\prod_{i+j\leq m}(1-tq^{-1}x_iy_j)}{\prod_{i+j\leq m+1}(1-q^{-1}x_iy_j)}\right]=\sum_{\boldsymbol{a}}q^{-|\boldsymbol{a}|}t^{-\mathrm{Inv}(\boldsymbol{a})}H_{\boldsymbol{a}}(x;t)H_{\boldsymbol{a}}(y;t)$$

since $H_{\boldsymbol{a}}(q^{-1}y;t)=q^{-|\boldsymbol{a}|}H_{\boldsymbol{a}}(y;t)$. The corollary then follows from (**??**) and the definition of $p_\Lambda(x;t)$. $\square$

We immediately get that the function $K_m(x,y)$ is a reproducing kernel for the scalar product (6.1).

Proposition 125. *Let $\{f_\Lambda(x)\}_\Lambda$ and $\{g_\Lambda(x)\}_\Lambda$ be two bases of R_m. Then the following two statements are equivalent.*

(1) $K_m(x,y)=\sum_\Lambda f_\Lambda(x)g_\Lambda(y)$

(2) $\langle f_\Lambda(x)\,,\,g_\Omega(x)\rangle_m=\delta_{\Lambda\Omega}$ for all Λ,Ω.

Proof. Using the bases $\{p^*_\Lambda(x;t)\}_\Lambda$ and $\{p_\Lambda(x;t)\}_\Lambda$, where $p^*_\Lambda(x;t)=q^{|\boldsymbol{a}|}t^{\mathrm{Inv}(\boldsymbol{a})}z_\lambda(q,t)p_\Lambda(x;t)$, the proof is exactly as the proof of the similar statement in the usual Macdonald polynomial case [**20**]. $\square$

Before stating the main theorem of this section, we need to relate the inclusion and the restriction. First, it is straightforward to verify that the inclusion $i:R_m\to R_{m+1}$ is such that [**18**]

$$i(p_\Lambda(x;t))=p_{\Lambda^0}(x;t)$$

where $\Lambda^0=(\boldsymbol{a},0;\lambda)$. The restriction $r:R_{m+1}\to R_m$, which is defined as

$$r(f)=f(x_1,\dots,x_m,0,x_{m+2},x_{m+3},\dots)\big|_{(x_{m+2},x_{m+3},\dots)\mapsto(x_{m+1},x_{m+2},\dots)}$$

is on the other hand such that [**18**]

$$r(p_\Omega(x;t)) = \begin{cases} p_{\Omega_-}(x;t) & \text{if } b_{m+1}=0 \\ 0 & \text{otherwise} \end{cases} \tag{6.5}$$

where $\Omega_- = (\boldsymbol{b}_-;\mu)$ with $\boldsymbol{b}_- = (b_1,\dots,b_m)$.

The following proposition can easily be verified using the basis $\{p_\Lambda(x;t)\}_\Lambda$ of R_m.

Proposition 126. *We have*

$$\langle i(f), g\rangle_{m+1} = \langle f, r(g)\rangle_m \tag{6.6}$$

for all $f \in R_m$ and all $g \in R_{m+1}$.

We can now establish the orthogonality and the squared norm of the m-symmetric Macdonald polynomials.

Theorem 127. *We have*

$$\langle P_\Lambda(x;q,t)\, ,\, P_\Omega(x;q,t)\rangle_m = 0 \qquad \text{if } \Lambda \neq \Omega$$

and

$$\langle P_\Lambda(x;q,t)\, ,\, P_\Lambda(x;q,t)\rangle_m = q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} \prod_{s\in\Lambda} \frac{1-q^{\tilde a(s)+1}t^{\tilde\ell(s)}}{1-q^{a(s)}t^{\ell(s)+1}}$$

where the product is over the cells of Λ (excluding the circles), and where the arm and leg-lengths were defined before Example 108.

Example 128. *Using $\Lambda = (2,0,0,2;4,1,1)$ such as in Example 108, we get that $\|P_\Lambda(x;q,t)\|^2$ is given by*

$$\frac{(1-q)(1-q^2t^2)(1-q^3t^2)(1-q^4t^6)(1-qt)(1-q^2t^3)(1-q)(1-q^2t^4)(1-qt^3)(1-qt^2)}{(1-t)(1-qt)(1-q^2t^3)(1-q^3t^5)(1-qt^2)(1-q^2t^4)(1-qt)(1-q^2t^5)(1-t^2)(1-t)}$$

Proof. Proposition 125 and (6.3) immediately imply that the m-symmetric Macdonald polynomials are orthogonal, that is,

$$\langle P_\Lambda(x;q,t)\, ,\, P_\Omega(x;q,t)\rangle_m = 0 \qquad \text{if } \Lambda \neq \Omega$$

We thus only have to prove the formula for the squared norm of an m-Macdonald polynomial. Let $\Lambda = (a_1,\dots,a_{m-1},a_m;\lambda)$ and $\hat\Lambda = (a_1,\dots a_{m-1};\lambda\cup\{a_m\})$. Observe that $\hat\Lambda$ can be obtained from Λ by discarding the m-circle. The restriction of an m-Macdonald polynomial is given (in the integral form) by [**18**]

$$r(J_\Lambda(x,q,t)) = q^{a_m} t^{\#\{i\,|\,a_i<a_m\}} J_{\hat\Lambda}(x,q,t)$$

which amounts to

$$r(P_\Lambda(x,q,t)) = q^{a_m} t^{\#\{i\,|\,a_i<a_m\}} \varphi_{\Lambda/\hat\Lambda}(q,t) P_{\hat\Lambda}(x,q,t) \tag{6.7}$$

where

$$\varphi_{\Lambda/\hat\Lambda}(q,t) = \frac{c_{\hat\Lambda}(q,t)}{c_\Lambda(q,t)} = \prod_{s\in\Lambda} \frac{1-q^{a_{\hat\Lambda}(s)}t^{\ell_{\hat\Lambda}(s)+1}}{1-q^{a_\Lambda(s)}t^{\ell_\Lambda(s)+1}}$$

From the definition of the arm and leg-length, we see that $a_{\hat\Lambda}(s) = a_\Lambda(s)$ and $\ell_{\hat\Lambda}(s) = \ell_\Lambda(s)$ for all $s\in\Lambda$ except those in $\mathrm{row}_{\Lambda/\hat\Lambda}$, the row in which the m-circle of Λ lies. Hence

$$\varphi_{\Lambda/\hat\Lambda}(q,t) = \prod_{s\in\mathrm{row}_{\Lambda/\hat\Lambda}} \frac{1-q^{a_{\hat\Lambda}(s)}t^{\ell_{\hat\Lambda}(s)+1}}{1-q^{a_\Lambda(s)}t^{\ell_\Lambda(s)+1}} \tag{6.8}$$

Also observe that the formula for the inclusion in Theorem 113 gives

$$\psi^{-1}_{\Lambda/\hat{\Lambda}}(q,t) = \prod_{s\in \mathrm{col}_{\Lambda/\hat{\Lambda}}} \frac{1-q^{\tilde{a}_{\Lambda}(s)+1}t^{\tilde{\ell}_{\Lambda}(s)}}{1-q^{\tilde{a}_{\hat{\Lambda}}(s)+1}t^{\tilde{\ell}_{\hat{\Lambda}}(s)}} \tag{6.9}$$

where we observe that for $s \in \mathrm{col}_{\Lambda/\hat{\Lambda}}$ we have that $\tilde{a}(s) = a(s)$.

The proof will proceed by induction on m. In the case $m = 0$, as was already observed, the scalar product is the usual Macdonald polynomial scalar product. Using $\Lambda = \lambda$, $|\boldsymbol{a}| = 0$, $\mathrm{Inv}(\boldsymbol{a}) = 0$, $\tilde{a}(s) = a(s)$ y $\tilde{\ell}(s) = \ell(s)$, we thus have to show that the Macdonald polynomials are such that

$$\langle P_\lambda(x,q,t), P_\lambda(x,q,t)\rangle_0 = \prod_{s\in\lambda} \frac{1-q^{a(s)+1}t^{\ell(s)}}{1-q^{a(s)}t^{\ell(s)+1}}$$

But this is the well known formula for the norm squared of a Macdonald polynomial [**20**].

Supposing that the theorem holds for the $(m-1)$-symmetric Macdonald polynomials, we will see that it also holds for the m-symmetric Macdonald polynomials. Let Λ and $\hat{\Lambda}$ be as before. From the formula for the inclusion of an $(m-1)$-symmetric Macdonald polynomial, we have

$$i(P_{\hat{\Lambda}}(x,q,t)) = \sum_{\Omega} \psi_{\Omega/\hat{\Lambda}}(q,t) P_\Omega(x,q,t) \tag{6.10}$$

where Ω is obtained from $\hat{\Lambda}$ by adding an m-circle. Taking $\Omega = \Lambda$, we get from the orthogonality of the m-Macdonald polynomials that

$$\langle i(P_{\hat{\Lambda}}(x,q,t)), P_\Lambda(x,q,t)\rangle_m = \psi_{\Lambda/\hat{\Lambda}}(q,t)\langle P_\Lambda(x,q,t), P_\Lambda(x,q,t)\rangle_m$$

or equivalently, that

$$\langle P_\Lambda(x,q,t), P_\Lambda(x,q,t)\rangle_m = \psi^{-1}_{\Lambda/\hat{\Lambda}}(q,t)\langle i(P_{\hat{\Lambda}}(x,q,t)), P_\Lambda(x,q,t)\rangle_m$$

From Proposition 126, we then obtain

$$\langle P_\Lambda(x,q,t), P_\Lambda(x,q,t)\rangle_m = \psi^{-1}_{\Lambda/\hat{\Lambda}}(q,t)\langle P_{\hat{\Lambda}}(x,q,t), r(P_\Lambda(x,q,t))\rangle_{m-1}$$

which amounts, using (6.7), to

$$\langle P_\Lambda(x,q,t), P_\Lambda(x,q,t)\rangle_m = q^{a_{m+1}} t^{\{i\,|\,a_i<a_{m+1}\}} \psi^{-1}_{\Lambda/\hat{\Lambda}}(q,t)\varphi_{\Lambda/\hat{\Lambda}}(q,t)\langle P_{\hat{\Lambda}}(x,q,t), P_{\hat{\Lambda}}(x,q,t)\rangle_{m-1}$$

By induction, we thus get

$$\begin{aligned}\langle P_\Lambda(x,q,t), P_\Lambda(x,q,t)\rangle_m &= q^{a_m} t^{\{i\,|\,a_i<a_m\}} \psi^{-1}_{\Lambda/\hat{\Lambda}}(q,t)\varphi_{\Lambda/\hat{\Lambda}}(q,t) q^{|\hat{\boldsymbol{a}}|} t^{\mathrm{Inv}(\hat{\boldsymbol{a}})} \prod_{s\in\hat{\Lambda}} \frac{1-q^{\tilde{a}_{\hat{\Lambda}}(s)+1}t^{\tilde{\ell}_{\hat{\Lambda}}(s)}}{1-q^{a_{\hat{\Lambda}}(s)}t^{\ell_{\hat{\Lambda}}(s)+1}}\\ &= q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} \psi^{-1}_{\Lambda/\hat{\Lambda}}(q,t)\varphi_{\Lambda/\hat{\Lambda}}(q,t) \prod_{s\in\Lambda} \frac{1-q^{\tilde{a}_{\hat{\Lambda}}(s)+1}t^{\tilde{\ell}_{\hat{\Lambda}}(s)}}{1-q^{a_{\hat{\Lambda}}(s)}t^{\ell_{\hat{\Lambda}}(s)+1}}\end{aligned}$$

Now, $\tilde{a}_\Lambda(s) = \tilde{a}_{\hat{\Lambda}}(s)$ for all $s \in \Lambda$, $\tilde{\ell}_\Lambda(s) = \tilde{\ell}_{\hat{\Lambda}}(s)$ for all $s \in \Lambda \setminus \mathrm{col}_{\Lambda/\hat{\Lambda}}$ while $a_{\hat{\Lambda}}(s) = a_\Lambda(s)$ and $\ell_{\hat{\Lambda}}(s) = \ell_\Lambda(s)$ for all $s \in \Lambda \setminus \mathrm{row}_{\Lambda/\hat{\Lambda}}$. The squared norm of P_Λ is thus equal to

$$q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} \psi^{-1}_{\Lambda/\hat{\Lambda}}(q,t)\varphi_{\Lambda/\hat{\Lambda}}(q,t) \frac{\displaystyle\prod_{\Lambda\setminus \mathrm{col}_{\Lambda/\hat{\Lambda}}} (1-q^{\tilde{a}_\Lambda(s)+1}t^{\tilde{\ell}_\Lambda(s)})}{\displaystyle\prod_{\Lambda\setminus \mathrm{row}_{\Lambda/\hat{\Lambda}}} (1-q^{a_\Lambda(s)}t^{\ell_\Lambda(s)+1})} \frac{\displaystyle\prod_{s\in \mathrm{col}_{\Lambda/\hat{\Lambda}}} (1-q^{\tilde{a}_{\hat{\Lambda}}(s)+1}t^{\tilde{\ell}_{\hat{\Lambda}}(s)})}{\displaystyle\prod_{s\in \mathrm{row}_{\Lambda/\hat{\Lambda}}} (1-q^{a_{\hat{\Lambda}}(s)}t^{\ell_{\hat{\Lambda}}(s)+1})}$$

Finally, using (3.3) and (6.9), we obtain

$$\langle P_\Lambda(x,q,t), P_\Lambda(x,q,t)\rangle_m = q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} \frac{\prod_{s\in\Lambda}(1-q^{\tilde{a}_\Lambda(s)+1}t^{\tilde{\ell}_\Lambda(s)})}{\prod_{s\in\Lambda}(1-q^{a_\Lambda(s)}t^{\ell_\Lambda(s)+1})}$$

as wanted. □

Corollary 129. *The operators $\tilde{Y}_i$, for $i=1,\dots,m$, and $\tilde{D}$ defined before Proposition 122 are self-adjoint with respect to the scalar product $\langle\cdot,\cdot\rangle_m$, that is,*

$$\langle \tilde{Y}_i f, g\rangle_m = \langle f, \tilde{Y}_i g\rangle_m \quad \text{for } i=1,\dots,m \quad \text{and} \quad \langle \tilde{D} f, g\rangle_m = \langle f, \tilde{D} g\rangle_m$$

We can also immediately deduce from Proposition 125 the value of the coefficients $b_\Lambda(q,t)$ in (6.3).

Corollary 130. *We have*

$$K_m(x,y) = \sum_\Lambda b_\Lambda(q,t) P_\Lambda(x;q,t) P_\Lambda(y;q,t) \tag{6.11}$$

where

$$b_\Lambda(q,t)^{-1} = \langle P_\Lambda(x;q,t)\,,\, P_\Lambda(x;q,t)\rangle_m = q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} \prod_{s\in\Lambda} \frac{1-q^{\tilde{a}(s)+1}t^{\tilde{\ell}(s)}}{1-q^{a(s)}t^{\ell(s)+1}}$$

The function $K_m(x,y)$ is not totally explicit due to the presence of the operator T_{ω_m}. Using Proposition 118, this defect in Corollary 130 can be corrected.

Proposition 131. *The following Cauchy-type identity holds*

$$K_0(x,\tilde{y}) \left[\prod_{1\le i<j\le m} \frac{1-tx_iy_j}{1-x_iy_j}\right] \left[\prod_{1\le i\le m} \frac{1}{1-x_iy_i}\right] = \sum_\Lambda a_\Lambda(q,t) P_\Lambda(x;q,t) P_\Lambda(y;q^{-1},t^{-1}) \tag{6.12}$$

where $\tilde{y}$ stands for the alphabet

$$\tilde{y} = (qy_1,\dots,qy_m,y_{m+1},y_{m+2},\cdots)$$

and where

$$a_\Lambda(q,t)^{-1} = q^{-|\boldsymbol{a}|} t^{-\mathrm{Inv}(\boldsymbol{a})} b_\Lambda(q,t)^{-1} = \prod_{s\in\Lambda} \frac{1-q^{\tilde{a}(s)+1}t^{\tilde{\ell}(s)}}{1-q^{a(s)}t^{\ell(s)+1}}$$

Proof. We get from (6.11) that $K_m(x,y)$ is symmetric in x and y. Therefore, using $T^{(y)}_{\omega_m}$ instead of $T^{(x)}_{\omega_m}$ in $K_m(x,y)$, we obtain from (6.11) that

$$K_0(x,y) \left[\frac{\prod_{i+j\le m}(1-tq^{-1}x_iy_j)}{\prod_{i+j\le m+1}(1-q^{-1}x_iy_j)}\right] = \sum_\Lambda b_\Lambda(q,t) P_\Lambda(x;q,t) \left[t^{\binom{m}{2}} \bar{T}^{(y)}_{\omega_m} P_\Lambda(y;q,t)\right]$$

Applying $(\tau_1\cdots\tau_m K_{\omega_m})^{(y)}$ on both sides of the equation and using Proposition 118 thus yields

$$K_0(x,\tilde{y}) K^{(y)}_{\omega_m} \left[\frac{\prod_{i+j\le m}(1-tx_iy_j)}{\prod_{i+j\le m+1}(1-x_iy_j)}\right] = \sum_\Lambda b_\Lambda(q,t) P_\Lambda(x;q,t) \left[q^{|\boldsymbol{a}|} t^{\mathrm{Inv}(\boldsymbol{a})} P_\Lambda(y;q^{-1},t^{-1})\right]$$

The proposition is then immediate after checking that

$$K^{(y)}_{\omega_m} \left[\frac{\prod_{i+j\le m}(1-tx_iy_j)}{\prod_{i+j\le m+1}(1-x_iy_j)}\right] = \left[\prod_{1\le i<j\le m} \frac{1-tx_iy_j}{1-x_iy_j}\right] \left[\prod_{1\le i\le m} \frac{1}{1-x_iy_i}\right]$$

□

Remark 132. *The previous proposition suggests that there is a natural sesquilinear scalar product $\langle \cdot,\cdot\rangle'$ in R_m such that*

$$\langle P_\Lambda(x;q,t), P_\Omega(x;q,t)\rangle' = \delta_{\Lambda\Omega}\, c_\Lambda(q,t)^{-1} = \delta_{\Lambda\Omega} \prod_{s\in\Lambda} \frac{1-q^{\tilde a(s)+1}t^{\tilde\ell(s)}}{1-q^{a(s)}t^{\ell(s)+1}} \tag{6.13}$$

and for which the l.h.s. of (6.12) *is a reproducing kernel. Indeed, defining $\langle\cdot,\cdot\rangle'$ as*

$$\langle f(x;q,t), g(x;q,t)\rangle' = t^{-\binom{m}{2}} \left\langle f(x;q,t), \overline{\tau_1\cdots\tau_m K_{\omega_m}\bar T_{\omega_m} g(x;q,t)}\right\rangle_m$$

where $\overline{h(x;q,t)} = h(x;q^{-1},t^{-1})$ for any $h(x;q,t)\in R_m$, we have from Proposition 118 that (6.13) *holds. Note that, for $i=1,\dots,m-1$, the adjoint of T_i with respect to this sesquilinear scalar product is $\bar T_i$ while, as was seen in* [**18**], *T_i is self-adjoint with respect to the scalar product $\langle\cdot,\cdot\rangle_m$.*

Working in $N=m$ variables, we obtain a Cauchy-type identity for the non-symmetric Macdonald polynomials. Note that since there is no restriction on m, the result holds for any number of variables.

Proposition 133. *For $\bar x=(x_1,\dots,x_m)$ and $\bar y=(x_1,\dots,x_m)$, we have*

$$K_0(\bar x, q\bar y)\left[\prod_{1\le i<j\le m}\frac{1-tx_iy_j}{1-x_iy_j}\right]\left[\prod_{1\le i\le m}\frac{1}{1-x_iy_i}\right] = \sum_{\eta\in\mathbb{Z}^m_{\ge 0}} a_\eta(q,t)E_\eta(\bar x;q,t)E_\eta(\bar y;q^{-1},t^{-1}) \tag{6.14}$$

where

$$a_\eta(q,t)^{-1} = \prod_{s\in\eta}\frac{1-q^{\tilde a(s)+1}t^{\tilde\ell(s)}}{1-q^{\tilde a(s)+1}t^{\ell(s)+1}}$$

Proof. From Proposition 111, we have that $P_{(\boldsymbol{a};\lambda)}(x_{(m)};q,t)=0$ if $\lambda\neq\emptyset$. Moreover, when $\lambda=\emptyset$, we get from (4.3) that $P_{(\boldsymbol{a};\emptyset)}(x_{(m)};q,t)=E_\eta(x_{(m)};q,t)$ for $\eta=\boldsymbol{a}=(a_1,\dots,a_m)$. The result is then immediate from Corollary 130 since the diagram of the composition η is equal to that of the m-partition $(\boldsymbol{a};\emptyset)$, and since $a(s)=\tilde a(s)+1$ given that every row of the diagram of η ends with a circle. □

Remark 134. *A different Cauchy-type identity for the non-symmetric Macdonald polynomials was provided in* [**24**]. *For $\bar x=(x_1,\dots,x_m)$ and $\bar y=(x_1,\dots,x_m)$, it reads in our language as*

$$K_0(\bar x,\bar y)\left[\prod_{1\le j<i\le m}\frac{1-x_iy_j}{1-tx_iy_j}\right]\left[\prod_{1\le i\le m}\frac{1}{1-tx_iy_i}\right] = \sum_{\eta\in\mathbb{Z}^m_{\ge0}} a_\eta(q,t)E_\eta(\bar x;q,t)E_\eta(\bar y;q^{-1},t^{-1}) \tag{6.15}$$

We will now see that (6.14) *and* (6.15) *are essentially equivalent by recovering* (6.14) *from* (6.15). *First observe that*

$$a_\eta(q^{-1},t^{-1}) = t^{-|\eta|}a_\eta(q,t)$$

and

$$E_\eta(t\bar x;q^{-1},t^{-1}) = t^{|\eta|}E_\eta(\bar x;q^{-1},t^{-1})$$

We also have using (**??**) *that*

$$K_0(\bar x,\bar y)\Big|_{(q,t)\mapsto(q^{-1},t^{-1})} = \sum_\lambda z_\lambda(q^{-1},t^{-1})^{-1}p_\lambda(\bar x)p_\lambda(\bar y) = \sum_\lambda z_\lambda(q,t)^{-1}p_\lambda(q\bar x/t)p_\lambda(\bar y) = K_0(q\bar x/t,\bar y)$$

Letting $(q,t)\mapsto(q^{-1},t^{-1})$ in (6.15) *followed by $x_i\mapsto tx_i$, for $i=1,\dots,m$, thus yields*

$$K_0(q\bar x,\bar y)\left[\prod_{1\le j<i\le m}\frac{1-tx_iy_j}{1-x_iy_j}\right]\left[\prod_{1\le i\le m}\frac{1}{1-x_iy_i}\right] = \sum_{\eta\in\mathbb{Z}^m_{\ge0}} a_\eta(q,t)E_\eta(\bar x;q^{-1},t^{-1})E_\eta(\bar y;q,t)$$

Interchanging $\bar x$ and $\bar y$ then leads to (6.14).

Finally, we get from Proposition 112 and Theorem 127 that the m-symmetric Macdonald polynomials also have and orthogonality/unitriangularity characterization akin to that of the usual Macdonald polynomials.

Proposition 135. *The m-symmetric Macdonald polynomials form the unique basis $\{P_\Lambda(x;q,t)\}_\Lambda$ of R_m such that*

(1) $\langle P_\Lambda(x;q,t)\,,\,P_\Omega(x;q,t)\rangle_m = 0$ if $\Lambda \neq \Omega$ (orthogonality)
(2) $P_\Lambda(x;q,t) = m_\Lambda + \sum_{\Omega<\Lambda} d_{\Lambda\Omega}(q,t)\, m_\Omega$ (unitriangularity)

for certain coefficients $d_{\Lambda\Omega}(q,t) \in \mathbb{Q}(q,t)$. We recall that the dominance order on m-partitions was defined in (2.1).

7. Appendix

7.1. Proof of Theorem 113.

Proof. We will prove the result in N variables. The case $N \to \infty$ will then be immediate. Recall that

$$P_\Lambda(x_1,\dots,x_N;q,t) = \frac{1}{u_{\Lambda,N}(t)} \mathcal{S}^t_{m+1,N} E_{\eta_{\Lambda,N}}(x_1,\dots,x_N;q,t)$$

First consider the case $\Lambda = (\boldsymbol{a}; b^{N-m})$ for any $b \geq 0$. The case $b=0$ was proven in [**18**] (the proof is essentially the same as the one we will provide in the case $b>0$) so we can assume that $b>0$. In that case, there are two possibilities for the $(m+1)$-circle. It can be added in the uppermost row of size b to give $\Lambda^b = (\boldsymbol{a}, b; b^{N-m-1})$ or in a row of size 0 to give $\Lambda^0 = (\boldsymbol{a}, 0; b^{N-m})$. But from Proposition 111, we have that $P_{\Lambda^0}(x;q,t) = 0$ in N variables given that $b>0$ by assumption. The only possibility is thus Λ^b. Observe that in this case, all the rows above that of the $(m+1)$-circle in Λ^b end with a circle and thus do not contribute to $\psi_{\Lambda^b/\Lambda}$. As such, we need to show that $i(P_\Lambda) = P_{\Lambda^b}$.

Recall from (**??**) that $\mathcal{S}^t_{m+1,N} = \mathcal{S}^t_{m+2,N}\mathcal{R}_{m+1,N}$. Using $\eta_{\Lambda,N} = (a_1,\dots,a_m,b^{N-m})$, we obtain from (**??**) that

$$\mathcal{R}_{m+1,N} E_{\eta_{\Lambda,N}}(x_1,\dots,x_N;q,t) = (1+t+t^2+\cdots+t^{N-m-1}) E_{\eta_{\Lambda,N}}(x_1,\dots,x_N;q,t)$$

Hence, in order to prove that $i(P_\Lambda) = P_{\Lambda^b}$, we need to show that

$$\frac{1}{u_{\Lambda,N}(t)} \mathcal{S}^t_{m+2,N}(1+t+t^2+\cdots+t^{N-m-1}) E_{\eta_{\Lambda,N}} = \frac{1}{u_{\Lambda^b,N}(t)} \mathcal{S}^t_{m+2,N} E_{\eta_{\Lambda^b,N}}$$

But this is easily seen to be the case given that $\eta_{\Lambda,N} = \eta_{\Lambda^b,N}$ and

$$\frac{(1+t+\cdots+t^{N-m-1})}{u_{\Lambda,N}(t)} = \frac{(1+t^{-1}+\cdots+t^{-(N-m-1)})}{u_{\Lambda,N}(t)} \frac{t^{(N-m)(N-m-1)/2}}{t^{(N-m-1)(N-m-2)/2}} = \frac{1}{u_{\Lambda^b,N}(t)}$$

We now consider the general case. We let $\eta_{\Lambda,N} = (a_1,\dots,a_m,\lambda_{N-m},\dots,\lambda_2,\lambda_1)$, where we consider that λ_{N-m} can be equal to 0. Observe that we can assume that $\lambda_{N-m} < \lambda_1$ since the case $\lambda_{N-m} = \lambda_1$ corresponds to the case $\Lambda = (\boldsymbol{a}; b^{N-m})$, which was already established.

We will proceed by induction on $N-m$. The theorem is readily seen to hold when $N = m+1$ since in this case Λ can only be of the form $\Lambda = (\boldsymbol{a}; b)$, which has been seen to hold. In the following, we will denote by $\lambda \setminus b$ the partition obtained by removing one row of length b from λ (similarly $\lambda \setminus \{b,c\}$ will stand for the partition obtained by removing the entries b and c from λ).

We will first show that the theorem holds for the $(m+1)$-partitions $\Omega = (\boldsymbol{a}, b; \lambda \setminus b)$ such that $b \neq \lambda_{N-m}$. For simplicity, we will let $s = \lambda_{N-m}$ and use the notation $\Lambda^+ = (\boldsymbol{a}, s; \lambda \setminus s)$ and $\Omega^+ = (\boldsymbol{a}, s, b; \lambda \setminus \{b, s\})$. By induction on $N - m$, we have that

$$\frac{1}{u_{\Lambda^+,N}(t)} \mathcal{S}^t_{m+2,N} E_{\eta_{\Lambda^+,N}} = \sum_{\Delta} \psi_{\Delta/\Lambda^+} \frac{1}{u_{\Delta,N}(t)} \mathcal{S}^t_{m+3,N} E_{\eta_{\Delta,N}} \tag{7.1}$$

Our goal is to show that

$$\frac{1}{u_{\Lambda,N}(t)} \mathcal{S}^t_{m+1,N} E_{\eta_{\Lambda,N}} \Big|_{P_\Omega} = \psi_{\Omega^+/\Lambda^+} \tag{7.2}$$

This will prove our claim since $\psi_{\Omega^+/\Lambda^+} = \psi_{\Omega/\Lambda}$ given that the only difference between Λ^+ and Λ is the extra $(m+1)$-circle in row s (which does no affect $\psi_{\Omega^+/\Lambda^+}$).

Using $\mathcal{R}_{m+1,N} = 1 + T_{m+1} \mathcal{R}_{m+2,N}$, we get

$$\mathcal{S}^t_{m+1,N} = \mathcal{S}^t_{m+2,N} \mathcal{R}_{m+1,N} = \mathcal{S}^t_{m+2,N} + \mathcal{S}^t_{m+2,N} T_{m+1} \mathcal{R}_{m+2,N} \tag{7.3}$$

Now, when acting on $E_{\eta_{\Lambda,N}}$, the operator $\mathcal{S}^t_{m+2,N}$ will produce a linear combination of E_ν's such that the first $m+1$ entries of ν are $a_1, \ldots, a_m, s$ with $s \neq b$. This implies that the term $\mathcal{S}^t_{m+2,N}$ in the r.h.s. of (7.3) will not contribute to the coefficient $\psi_{\Omega/\Lambda}$. Therefore, using

$$\begin{aligned} \mathcal{S}^t_{m+2,N} T_{m+1} \mathcal{R}_{m+2,N} &= \mathcal{L}_{m+2,N} \mathcal{S}^t_{m+3,N} T_{m+1} \mathcal{R}_{m+2,N} \\ &= \mathcal{L}_{m+2,N} T_{m+1} \mathcal{S}^t_{m+3,N} \mathcal{R}_{m+2,N} = \mathcal{L}_{m+2,N} T_{m+1} \mathcal{S}^t_{m+2,N} \end{aligned}$$

we obtain from (7.3) that

$$\frac{1}{u_{\Lambda,N}(t)} \mathcal{S}^t_{m+1,N} E_{\eta_{\Lambda,N}} \Big|_{P_\Omega} = \frac{1}{u_{\Lambda,N}(t)} \mathcal{L}_{m+2,N} T_{m+1} \mathcal{S}^t_{m+2,N} E_{\eta_{\Lambda,N}} \Big|_{P_\Omega}$$

Given that $\eta_{\Lambda,N} = \eta_{\Lambda^+,N}$, we then obtain

$$\frac{1}{u_{\Lambda,N}(t)} \mathcal{S}^t_{m+1,N} E_{\eta_{\Lambda,N}} \Big|_{P_\Omega} = \frac{u_{\Lambda^+,N}(t)}{u_{\Lambda,N}(t)} \mathcal{L}_{m+2,N} T_{m+1} \left(\frac{1}{u_{\Lambda^+,N}(t)} \mathcal{S}^t_{m+2,N} E_{\eta_{\Lambda^+,N}} \right) \Bigg|_{P_\Omega}$$

From (7.1), we then have

$$\frac{1}{u_{\Lambda,N}(t)} \mathcal{S}^t_{m+1,N} E_{\eta_{\Lambda,N}} \Big|_{P_\Omega} = \sum_{\Delta} \psi_{\Delta/\Lambda^+} \frac{1}{u_{\Delta,N}(t)} \frac{u_{\Lambda^+,N}(t)}{u_{\Lambda,N}(t)} \mathcal{L}_{m+2,N} T_{m+1} \mathcal{S}^t_{m+3,N} E_{\eta_{\Delta,N}} \Big|_{P_\Omega}$$

Now, we observe that the only way to generate P_Ω is to act with T_{m+1} on $\mathcal{S}^t_{m+3,N} E_{\eta_{\Omega^+,N}}$ (otherwise the resulting indexing composition will not have a b in the $(m+1)$-th position). From (**??**), we see that

$$T_{m+1} \mathcal{S}^t_{m+3,N} E_{\eta_{\Omega^+,N}} = \mathcal{S}^t_{m+3,N} T_{m+1} E_{\eta_{\Omega^+,N}} = t \mathcal{S}^t_{m+3,N} E_{\eta_{\Omega,N}} + A(q,t) \mathcal{S}^t_{m+3,N} E_{\eta_{\Omega^+,N}}$$

for some coefficient $A(q,t)$. This yields

$$\begin{aligned} \frac{1}{u_{\Lambda,N}(t)} \mathcal{S}^t_{m+1,N} E_{\eta_{\Lambda,N}} \Big|_{P_\Omega} &= \psi_{\Omega^+/\Lambda^+} \frac{u_{\Lambda^+,N}(t)}{u_{\Lambda,N}(t)} \frac{u_{\Omega,N}(t)}{u_{\Omega^+,N}(t)} \mathcal{L}_{m+2,N} \left(\frac{t}{u_{\Omega,N}(t)} \mathcal{S}^t_{m+3,N} E_{\eta_{\Omega,N}} \right) \Bigg|_{P_\Omega} \\ &= \psi_{\Omega^+/\Lambda^+} \frac{u_{\Lambda^+,N}(t)}{u_{\Lambda,N}(t)} \frac{u_{\Omega,N}(t)}{u_{\Omega^+,N}(t)} \left(\frac{t}{u_{\Omega,N}(t)} \mathcal{S}^t_{m+2,N} E_{\eta_{\Omega,N}} \right) \Bigg|_{P_\Omega} \\ &= t \psi_{\Omega^+/\Lambda^+} \frac{u_{\Lambda^+,N}(t)}{u_{\Lambda,N}(t)} \frac{u_{\Omega,N}(t)}{u_{\Omega^+,N}(t)} \end{aligned}$$

The result thus holds since it is not too difficult to show that

$$t \frac{u_{\Lambda^+,N}(t)}{u_{\Lambda,N}(t)} \frac{u_{\Omega,N}(t)}{u_{\Omega^+,N}(t)} = 1$$

given that

$$\frac{u_{\Lambda^+,N}(t)}{u_{\Lambda,N}(t)} = \frac{[n-1]_{1/t}!}{t^{N-m-1}[n]_{1/t}!} \qquad \text{and} \qquad \frac{u_{\Omega,N}(t)}{u_{\Omega^+,N}(t)} = \frac{t^{N-m-2}[n]_{1/t}!}{[n-1]_{1/t}!}$$

with n the number of occurrences of s in $\lambda_{N-m}, \ldots, \lambda_1$.

Finally, we have to consider the case where the $(m+1)$-partition $\Omega = (\boldsymbol{a}, b; \lambda \setminus b)$ is such that $b = \lambda_{N-m}$. This case turns out to be somewhat more complicated. This time, we use the relation

$$\mathcal{R}_{m+1,N} = \mathcal{R}_{m+1,N-1}(1 + T_{N-1}) - \mathcal{R}_{m+1,N-2}T_{N-1}$$

to get from (**??**) that

$$\begin{aligned}
\mathcal{S}^t_{m+1,N} &= \mathcal{S}^t_{m+2,N}\big(\mathcal{R}_{m+1,N-1}(1+T_{N-1}) - \mathcal{R}_{m+1,N-2}T_{N-1}\big) \\
&= \mathcal{L}'_{m+2,N}\mathcal{S}^t_{m+2,N-1}\mathcal{R}_{m+1,N-1}(1+T_{N-1}) - \mathcal{L}'_{m+2,N}\mathcal{L}'_{m+2,N-1}\mathcal{S}^t_{m+2,N-2}\mathcal{R}_{m+1,N-2}T_{N-1} \\
&= \mathcal{L}'_{m+2,N}\mathcal{S}^t_{m+1,N-1} + \mathcal{L}'_{m+2,N}\mathcal{S}^t_{m+1,N-1}T_{N-1} - \mathcal{L}'_{m+2,N}\mathcal{L}'_{m+2,N-1}\mathcal{S}^t_{m+1,N-2}T_{N-1} \qquad (7.4)
\end{aligned}$$

We first establish the result when $\lambda_1 = \lambda_2$. In this case, $T_{N-1}E_{\eta_{\Lambda,N}} = tE_{\eta_{\Lambda,N}}$, which implies from the previous equation that

$$\frac{1}{u_{\Lambda,N}(t)}\mathcal{S}^t_{m+1,N}E_{\eta_{\Lambda,N}} = \frac{(1+t)}{u_{\Lambda,N}(t)}\mathcal{L}'_{m+2,N}\mathcal{S}^t_{m+1,N-1}E_{\eta_{\Lambda,N}} - \frac{t}{u_{\Lambda,N}(t)}\mathcal{L}'_{m+2,N}\mathcal{L}'_{m+2,N-1}\mathcal{S}^t_{m+1,N-2}E_{\eta_{\Lambda,N}} \tag{7.5}$$

In order to use induction, we will need the relations

$$\Phi_q\mathcal{S}^t_{m+2,N} = \mathcal{S}^t_{m+1,N-1}\Phi_q \qquad \text{and} \qquad \Phi_q^2\mathcal{S}^t_{m+3,N} = \mathcal{S}^t_{m+1,N-2}\Phi_q^2 \tag{7.6}$$

where Φ_q, which was defined in (**??**), is such that $\Phi_q E_{\eta_{\hat\Lambda,N}} = t^{r-N}E_{\eta_{\Lambda,N}}$ with $\hat\Lambda = (\lambda_1 - 1, \boldsymbol{a}; \lambda \setminus \lambda_1)$ and r the row in the diagram of $\eta_{\hat\Lambda,N}$ corresponding to the entry $\lambda_1 - 1$ (the highest row of size $\lambda_1 - 1$ in the diagram of $\eta_{\hat\Lambda,N}$). The first term in the r.h.s. of (7.5) thus gives

$$\frac{(1+t)}{u_{\Lambda,N}(t)}\mathcal{L}'_{m+2,N}\mathcal{S}^t_{m+1,N-1}E_{\eta_{\Lambda,N}} = \frac{t^{-r+N}(1+t)}{u_{\Lambda,N}(t)}\mathcal{L}'_{m+2,N}\Phi_q\mathcal{S}^t_{m+2,N}E_{\eta_{\hat\Lambda,N}}$$

Hence, by induction on $N-m$ (using a similar expression as in (7.1)) we have that the terms $\mathcal{S}^t_{m+3,N}E_{\eta_{\hat\Delta,N}}$ that can appear in $\mathcal{S}^t_{m+2,N}E_{\eta_{\hat\Lambda,N}}$ are such that $\hat\Delta = (\lambda_1 - 1, \boldsymbol{a}, \lambda_i; \lambda \setminus \{\lambda_1, \lambda_i\})$. Letting $\Delta = (\boldsymbol{a}, \lambda_i; \lambda \setminus \lambda_i)$, we observe using $\mathcal{L}'_{m+2,N}\mathcal{S}^t_{m+2,N-1} = \mathcal{S}^t_{m+2,N}$ that

$$\mathcal{L}'_{m+2,N}\Phi_q\mathcal{S}^t_{m+3,N}E_{\eta_{\hat\Delta,N}} = \mathcal{L}'_{m+2,N}\mathcal{S}^t_{m+2,N-1}\Phi_q E_{\eta_{\hat\Delta,N}} = t^{r-N}\mathcal{S}^t_{m+2,N}E_{\eta_{\Delta,N}}$$

since the row in the diagram of $\eta_{\hat\Delta,N}$ corresponding to the entry $\lambda_1 - 1$ is the same as that of the entry $\lambda_1 - 1$ in the diagram of $\eta_{\hat\Lambda,N}$. Therefore, focusing on the term Ω, we have

$$\frac{(1+t)}{u_{\Lambda,N}(t)}\mathcal{L}'_{m+2,N}\mathcal{S}^t_{m+1,N-1}E_{\eta_{\Lambda,N}}\Big|_{P_\Omega} = (1+t)\psi_{\hat\Omega/\hat\Lambda}\frac{u_{\hat\Lambda,N}(t)}{u_{\Lambda,N}(t)}\frac{u_{\Omega,N}(t)}{u_{\hat\Omega,N}(t)}$$

where $\hat\Omega = (\lambda_1 - 1, \boldsymbol{a}, s; \lambda \setminus \{\lambda_1, s\})$. Doing a similar analysis for the second term in the rhs of (7.5), we obtain that

$$-\frac{t}{u_{\Lambda,N}(t)}\mathcal{L}'_{m+2,N}\mathcal{L}'_{m+2,N-1}\mathcal{S}^t_{m+1,N-1}E_{\eta_{\Lambda,N}}\Big|_{P_\Omega} = -t\psi_{\tilde\Omega/\tilde\Lambda}\frac{u_{\tilde\Lambda,N}(t)}{u_{\Lambda,N}(t)}\frac{u_{\Omega,N}(t)}{u_{\tilde\Omega,N}(t)}$$

with $\tilde\Lambda = (\lambda_2 - 1, \lambda_1 - 1, \boldsymbol{a}; \lambda \setminus \{\lambda_1, \lambda_2\})$ and $\tilde\Omega = (\lambda_2 - 1, \lambda_1 - 1, \boldsymbol{a}, s; \lambda \setminus \{\lambda_1, \lambda_2\})$. We thus have to prove that

$$(1+t)\psi_{\hat\Omega/\hat\Lambda}\frac{u_{\hat\Lambda,N}(t)}{u_{\Lambda,N}(t)}\frac{u_{\Omega,N}(t)}{u_{\hat\Omega,N}(t)} - t\psi_{\tilde\Omega/\tilde\Lambda}\frac{u_{\tilde\Lambda,N}(t)}{u_{\Lambda,N}(t)}\frac{u_{\Omega,N}(t)}{u_{\tilde\Omega,N}(t)} = \psi_{\Omega/\Lambda} \tag{7.7}$$

Since $s < \lambda_1 = \lambda_2$ by assumption, we have that

$$\frac{u_{\Omega,N}(t)}{u_{\Lambda,N}(t)} = \frac{[n-1]_{1/t}!}{t^{N-m-1}[n]_{1/t}!}, \quad \frac{u_{\hat{\Lambda},N}(t)}{u_{\hat{\Omega},N}(t)} = \frac{t^{N-m-2}[n]_{1/t}!}{[n-1]_{1/t}!}, \quad \text{and} \quad \frac{u_{\tilde{\Lambda},N}(t)}{u_{\tilde{\Omega},N}(t)} = \frac{t^{N-m-3}[n]_{1/t}!}{[n-1]_{1/t}!} \tag{7.8}$$

with n the number of occurrences of s in $\lambda_{N-m}, \dots, \lambda_1$. It thus remains to prove that

$$\frac{(1+t)}{t}\psi_{\hat{\Omega}/\hat{\Lambda}} - \frac{1}{t}\psi_{\tilde{\Omega}/\tilde{\Lambda}} = \psi_{\Omega/\Lambda} \tag{7.9}$$

Comparing $\psi_{\hat{\Omega}/\hat{\Lambda}}$, $\psi_{\tilde{\Omega}/\tilde{\Lambda}}$ and $\psi_{\Omega/\Lambda}$, we get that their factors are identical except in the rows corresponding to λ_1 and λ_2 in Λ (they are consecutive rows, with that of λ_2 just above that of λ_1). The squares in those rows contribute

$$\frac{1-q^{\lambda_1+1}t^{\ell-1}}{1-q^{\lambda_1+1}t^{\ell}}, \qquad 1 \qquad \text{and} \qquad \frac{(1-q^{\lambda_1+1}t^{\ell-1})}{(1-q^{\lambda_1+1}t^{\ell})}\frac{(1-q^{\lambda_1+1}t^{\ell-2})}{(1-q^{\lambda_1+1}t^{\ell-1})}$$

respectively to $\psi_{\hat{\Omega}/\hat{\Lambda}}$, $\psi_{\tilde{\Omega}/\tilde{\Lambda}}$ and $\psi_{\Omega/\Lambda}$, where ℓ is the leg-length of the square in the row of λ_2 in the diagram of Ω. We therefore get that (7.9) holds given that

$$\frac{(1+t)}{t}\frac{(1-q^{\lambda_1+1}t^{\ell-1})}{(1-q^{\lambda_1+1}t^{\ell})} - \frac{1}{t} = \frac{1-q^{\lambda_1+1}t^{\ell-2}}{1-q^{\lambda_1+1}t^{\ell}} = \frac{(1-q^{\lambda_1+1}t^{\ell-1})}{(1-q^{\lambda_1+1}t^{\ell})}\frac{(1-q^{\lambda_1+1}t^{\ell-2})}{(1-q^{\lambda_1+1}t^{\ell-1})}$$

Finally, we need to prove the result when $\lambda_1 > \lambda_2$. In this case,

$$T_{N-1}E_{\eta_{\Lambda,N}} = tE_{s_{N-1}(\eta_{\Lambda,N})} + \frac{(t-1)}{1-q^{\lambda_1-\lambda_2}t^r}E_{\eta_{\Lambda,N}}$$

where r is the difference between the row of λ_2 and that of λ_1 in the diagram associated to $\eta_{\Lambda,N}$. Using (7.4), we obtain this time

$$\begin{aligned}\frac{1}{u_{\Lambda,N}(t)}\mathcal{S}^t_{m+1,N}E_{\eta_{\Lambda,N}} &= \frac{(t-q^{\lambda_1-\lambda_2}t^r)}{u_{\Lambda,N}(t)(1-q^{\lambda_1-\lambda_2}t^r)}\mathcal{L}'_{m+2,N}\mathcal{S}^t_{m+1,N-1}E_{\eta_{\Lambda,N}} \\ &\quad - \frac{(t-1)}{u_{\Lambda,N}(t)(1-q^{\lambda_1-\lambda_2}t^r)}\mathcal{L}'_{m+2,N}\mathcal{L}'_{m+2,N-1}\mathcal{S}^t_{m+1,N-2}E_{\eta_{\Lambda,N}} \\ &\quad + \frac{t}{u_{\Lambda,N}(t)}\mathcal{L}'_{m+2,N}\mathcal{S}^t_{m+1,N-1}E_{s_{N-1}(\eta_{\Lambda,N})} \\ &\quad - \frac{t}{u_{\Lambda,N}(t)}\mathcal{L}'_{m+2,N}\mathcal{L}'_{m+2,N-1}\mathcal{S}^t_{m+1,N-2}E_{s_{N-1}(\eta_{\Lambda,N})}\end{aligned}$$

Using (7.6) and doing a similar analysis as in the $\lambda_1 = \lambda_2$ case, we obtain by induction that

$$\frac{(t-q^{\lambda_1-\lambda_2}t^r)}{u_{\Lambda,N}(t)(1-q^{\lambda_1-\lambda_2}t^r)}\mathcal{L}'_{m+2,N}\mathcal{S}^t_{m+1,N-1}E_{\eta_{\Lambda,N}}\Big|_{P_\Omega} = \frac{(t-q^{\lambda_1-\lambda_2}t^r)}{(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\hat{\Omega}/\hat{\Lambda}}\frac{u_{\hat{\Lambda},N}(t)}{u_{\Lambda,N}(t)}\frac{u_{\Omega,N}(t)}{u_{\hat{\Omega},N}(t)}$$

and that

$$-\frac{(t-1)}{u_{\Lambda,N}(t)(1-q^{\lambda_1-\lambda_2}t^r)}\mathcal{L}'_{m+2,N}\mathcal{L}'_{m+2,N-1}\mathcal{S}^t_{m+1,N-1}E_{\eta_{\Lambda,N}}\Big|_{P_\Omega} = -\frac{(t-1)}{(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\tilde{\Omega}/\tilde{\Lambda}}\frac{u_{\tilde{\Lambda},N}(t)}{u_{\Lambda,N}(t)}\frac{u_{\Omega,N}(t)}{u_{\tilde{\Omega},N}(t)}$$

Letting $\hat{\Gamma} = (\lambda_2 - 1, \boldsymbol{a}; \lambda \setminus \lambda_2)$, $\hat{\Gamma}^s = (\lambda_2 - 1, \boldsymbol{a}, s; \lambda \setminus \lambda_2)$, $\tilde{\Gamma} = (\lambda_1 - 1, \lambda_2 - 1, \boldsymbol{a}; \lambda \setminus \{\lambda_1, \lambda_2\})$, and $\tilde{\Gamma}^s = (\lambda_1 - 1, \lambda_2 - 1, \boldsymbol{a}, s; \lambda \setminus \{\lambda_1, \lambda_2\})$ as well as making use of the relation (see Lemma 55 in [**18**])

$$\mathcal{S}^t_{m+1,N}E_{s_{N-1}(\eta_{\Omega,N})} = \frac{(1-q^{\lambda_1-\lambda_2}t^{r+1})}{t(1-q^{\lambda_1-\lambda_2}t^r)}E_{\eta_{\Omega,N}}$$

we get similarly by induction that

$$\frac{t}{u_{\Lambda,N}(t)}\mathcal{L}'_{m+2,N}\mathcal{S}^t_{m+1,N-1}E_{s_{N-1}(\eta_{\Lambda,N})}\Big|_{P_\Omega} = \frac{(1-q^{\lambda_1-\lambda_2}t^{r+1})}{(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\hat{\Gamma}^s/\hat{\Gamma}}\frac{u_{\hat{\Gamma},N}(t)}{u_{\Lambda,N}(t)}\frac{u_{\Omega,N}(t)}{u_{\hat{\Gamma}^s,N}(t)}$$

and

$$-\frac{t}{u_{\Lambda,N}(t)}\mathcal{L}'_{m+2,N}\mathcal{L}'_{m+2,N-1}\mathcal{S}^t_{m+1,N-2}E_{s_{N-1}(\eta_{\Lambda,N})}\Big|_{P_\Omega} = -\frac{(1-q^{\lambda_1-\lambda_2}t^{r+1})}{(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\tilde\Gamma^s/\tilde\Gamma}\frac{u_{\tilde\Gamma,N}(t)}{u_{\Lambda,N}(t)}\frac{u_{\Omega,N}(t)}{u_{\tilde\Gamma^s,N}(t)}$$

We thus have to show that

$$\begin{aligned}\frac{(t-q^{\lambda_1-\lambda_2}t^r)}{(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\hat\Omega/\hat\Lambda}\frac{u_{\hat\Lambda,N}(t)}{u_{\Lambda,N}(t)}\frac{u_{\Omega,N}(t)}{u_{\hat\Omega,N}(t)} - \frac{(t-1)}{(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\tilde\Omega/\tilde\Lambda}\frac{u_{\tilde\Lambda,N}(t)}{u_{\Lambda,N}(t)}\frac{u_{\Omega,N}(t)}{u_{\tilde\Omega,N}(t)}\\ + \frac{(1-q^{\lambda_1-\lambda_2}t^{r+1})}{(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\hat\Gamma^s/\hat\Gamma}\frac{u_{\hat\Gamma,N}(t)}{u_{\Lambda,N}(t)}\frac{u_{\Omega,N}(t)}{u_{\hat\Gamma^s,N}(t)} - \frac{(1-q^{\lambda_1-\lambda_2}t^{r+1})}{(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\tilde\Gamma^s/\tilde\Gamma}\frac{u_{\tilde\Gamma,N}(t)}{u_{\Lambda,N}(t)}\frac{u_{\Omega,N}(t)}{u_{\tilde\Gamma^s,N}(t)} = \psi_{\Omega/\Lambda}\end{aligned}\tag{7.10}$$

Since $s \leq \lambda_2$, the only case where $\tilde\Omega$, $\hat\Gamma^s$ or $\tilde\Gamma^s$ may not exist (for the lack of an extra s in λ) is the case $\Lambda = (\boldsymbol{a}; \lambda_1, \lambda_2)$ in $N = m+2$ variables. Let us consider it first. Using (7.8) with $n = 1$, as well as $\psi_{\tilde\Omega/\tilde\Lambda} = \psi_{\tilde\Gamma^s/\tilde\Gamma} = \psi_{\hat\Gamma^s/\hat\Gamma} = 0$ in (3.9), we have to prove in this case that

$$\frac{(t-q^{\lambda_1-\lambda_2}t^r)}{t(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\hat\Omega/\hat\Lambda} = \psi_{\Omega/\Lambda}$$

But this is immediate given that $\psi_{\hat\Omega/\hat\Lambda} = 1$ and

$$\psi_{\Omega/\Lambda} = \frac{1-q^{\lambda_1-\lambda_2}t^{r-1}}{1-q^{\lambda_1-\lambda_2}t^r}$$

As previously mentioned, the remaining cases will involve all terms: $\hat\Omega$, $\tilde\Omega$, $\hat\Gamma^s$ and $\tilde\Gamma^s$. Using (7.8) together with

$$\frac{u_{\hat\Gamma,N}(t)}{u_{\hat\Gamma^s,N}(t)} = \frac{t^{N-m-2}[n]_{t^{-1}}!}{[n-1]_{t^{-1}}!} \quad\text{and}\quad \frac{u_{\tilde\Gamma,N}(t)}{u_{\tilde\Gamma^s,N}(t)} = \frac{t^{N-m-3}[n]_{1/t}!}{[n-1]_{\frac{1}{t}}!}$$

in (3.9), we have to prove this time that

$$\frac{(t-q^{\lambda_1-\lambda_2}t^r)}{t(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\hat\Omega/\hat\Lambda} - \frac{(t-1)}{t^2(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\tilde\Omega/\tilde\Lambda} + \frac{(1-q^{\lambda_1-\lambda_2}t^{r+1})}{t(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\hat\Gamma^s/\hat\Gamma} - \frac{(1-q^{\lambda_1-\lambda_2}t^{r+1})}{t^2(1-q^{\lambda_1-\lambda_2}t^r)}\psi_{\tilde\Gamma^s/\tilde\Gamma} = \psi_{\Omega/\Lambda}$$

Comparing $\psi_{\hat\Omega/\hat\Lambda}$, $\psi_{\tilde\Omega/\tilde\Lambda}$, $\psi_{\hat\Gamma^s/\hat\Gamma}$, $\psi_{\tilde\Gamma^s/\tilde\Gamma}$, and $\psi_{\Omega/\Lambda}$, we get that their factors are identical except in the rows corresponding to λ_1 and λ_2 in Ω/Λ. The squares in those rows contribute

$$\frac{1-q^{\lambda_2+1}t^{l-1}}{1-q^{\lambda_2+1}t^l}, \qquad 1, \qquad \frac{1-q^{\lambda_1+1}t^{r+l-1}}{1-q^{\lambda_1+1}t^{r+l}}, \qquad 1 \qquad\text{and}\qquad \frac{(1-q^{\lambda_1+1}t^{r+l-1})}{(1-q^{\lambda_1+1}t^{r+l})}\frac{(1-q^{\lambda_2+1}t^{l-1})}{(1-q^{\lambda_2+1}t^l)}$$

respectively to $\psi_{\hat\Omega/\hat\Lambda}$, $\psi_{\tilde\Omega/\tilde\Lambda}$, $\psi_{\hat\Gamma^s/\hat\Gamma}$, $\psi_{\tilde\Gamma^s/\tilde\Gamma}$, and $\psi_{\Omega/\Lambda}$, where l is the leg-length of the square in the row of λ_2 in the diagram of Ω, and where we recall that r is the difference between the row of λ_2 and that of λ_1 in the diagram associated to Λ (or, equivalently, in the diagram associated to $\eta_{\Lambda,N}$). The result then follows from the relation

$$\begin{aligned}\frac{(t-q^{\lambda_1-\lambda_2}t^r)}{t(1-q^{\lambda_1-\lambda_2}t^r)}\frac{(1-q^{\lambda_2+1}t^{l-1})}{(1-q^{\lambda_2+1}t^l)} - \frac{(t-1)}{t^2(1-q^{\lambda_1-\lambda_2}t^r)}\\ + \frac{(1-q^{\lambda_1-\lambda_2}t^{r+1})}{t(1-q^{\lambda_1-\lambda_2}t^r)}\frac{(1-q^{\lambda_1+1}t^{r+l-1})}{(1-q^{\lambda_1+1}t^{r+l})} - \frac{(1-q^{\lambda_1-\lambda_2}t^{r+1})}{t^2(1-q^{\lambda_1-\lambda_2}t^r)} = \frac{(1-q^{\lambda_1+1}t^{r+l-1})}{(1-q^{\lambda_1+1}t^{r+l})}\frac{(1-q^{\lambda_2+1}t^{l-1})}{(1-q^{\lambda_2+1}t^l)}\end{aligned}$$

which can straightforwardly be checked using

$$-\frac{(t-1)}{t^2(1-q^{\lambda_1-\lambda_2}t^r)} - \frac{(1-q^{\lambda_1-\lambda_2}t^{r+1})}{t^2(1-q^{\lambda_1-\lambda_2}t^r)} = -\frac{1}{t}$$

and then

$$\frac{(1-q^{\lambda_1-\lambda_2}t^{r+1})}{t(1-q^{\lambda_1-\lambda_2}t^r)}\frac{(1-q^{\lambda_1+1}t^{r+l-1})}{(1-q^{\lambda_1+1}t^{r+l})} - \frac{1}{t} = q^{\lambda_1-\lambda_2}t^{r-1}\frac{(1-t)(1-q^{\lambda_2+1}t^{l-1})}{(1-q^{\lambda_1-\lambda_2}t^r)(1-q^{\lambda_1+1}t^{r+l})}$$

followed by

$$\frac{(t-q^{\lambda_1-\lambda_2}t^r)}{t(1-q^{\lambda_1-\lambda_2}t^r)}\frac{(1-q^{\lambda_2+1}t^{l-1})}{(1-q^{\lambda_2+1}t^l)}+q^{\lambda_1-\lambda_2}t^{r-1}\frac{(1-t)(1-q^{\lambda_2+1}t^{l-1})}{(1-q^{\lambda_1-\lambda_2}t^r)(1-q^{\lambda_1+1}t^{r+l})}$$
$$=\frac{(1-q^{\lambda_1+1}t^{r+l-1})}{(1-q^{\lambda_1+1}t^{r+l})}\frac{(1-q^{\lambda_2+1}t^{l-1})}{(1-q^{\lambda_2+1}t^l)}$$

□

7.2. Proof of Proposition 121. Let $H_m(t)$ be the Hecke algebra generated by $T_1,\dots,T_{m-1}$. We define the linear antihomomorphism $\varphi_m : H_m(t) \to H_m(t)$ to be such that

$$\varphi_m(T_i) = T_{m-i} \qquad \text{for} \quad i = 1,\dots,m-1$$

For any permutation $\sigma \in S_m$, we thus have that $\varphi_m(T_\sigma) = T_{\tilde\sigma}$ for a certain permutation $\tilde\sigma \in S_m$ of the same length as σ. Hence

$$\varphi_m(T_{\omega_m}) = T_{\omega_m} \tag{7.11}$$

given that ω_m is the unique longest permutation in S_m. Moreover, it is easy to see that $\varphi_m \circ \varphi_m$ is the identity and that

$$\varphi_m(\bar T_i) = \bar T_{m-i} \tag{7.12}$$

Lemma 136. *Let $\omega'_{m-1} = [1, m, m-1, \dots, 2]$ be the longest permutation in the symmetric group on $\{2,\dots,m\}$. Then, for all $i \in \{1,\dots,m\}$, we have*

$$T_{\omega_m} = T_{i-1}\cdots T_1 \cdot T_{\omega'_{m-1}} \cdot \varphi_m(T_i \cdots T_{m-1}) \tag{7.13}$$

and

$$\varphi_m(\bar T_{\omega_m}) = \varphi_m(\bar T_1 \cdots \bar T_{i-1}) \cdot \varphi_m(\bar T_{\omega'_{m-1}}) \cdot \bar T_{m-1} \cdots \bar T_i \tag{7.14}$$

Proof. It suffices to prove (7.13) since (7.11) can be obtained from (7.13) by taking the inverse and then applying φ_m. Since ω_m is of length $m(m-1)/2$, (7.13) will hold if we can prove that

$$\omega_m = s_{i-1}\cdots s_1 \omega'_{m-1} s_1 \cdots s_{m-i}$$

for $i = 1,\dots,m$. The result is well known when $i = 1$. For an arbitrary i, we use the simple relation

$$s_\ell\, \omega_m\, s_{m-\ell} = \omega_m$$

successively (for $\ell = 1,\dots,i-1$) on the $i = 1$ case

$$\omega_m = \omega'_{m-1} s_1 \cdots s_{m-1}$$

□

For simplicity, we now define

$$\bar K_m(x,y) = t^{\binom{m}{2}} \bar T^{(x)}_{\omega_m} K_m(x,y) = K_0(x,y)\left[\frac{\prod_{i+j\le m}(1-tq^{-1}x_iy_j)}{\prod_{i+j\le m+1}(1-q^{-1}x_iy_j)}\right] \tag{7.15}$$

Lemma 137. *For $i = 1,\dots,m-1$, we have*

$$T_i^{(x)} \bar K_m(x_{(N)}, y_{(N)}) = T^{(y)}_{m-i} \bar K_m(x_{(N)}, y_{(N)}) \tag{7.16}$$

Hence

$$T_\sigma^{(x)} \bar K_m(x_{(N)}, y_{(N)}) = \varphi_m^{(y)}\left(T_\sigma^{(y)}\right) \bar K_m(x_{(N)}, y_{(N)}) \tag{7.17}$$

for any $\sigma \in S_m$.

Proof. We have that $\bar{K}_m(x_{(N)}, y_{(N)})$ is symmetric in x_i and x_{i+1} and in y_{m-i} and y_{m-i+1} except for the term

$$B := \frac{(1 - tq^{-1}x_i y_{m-i})}{(1 - q^{-1}x_i y_{m-i+1})(1 - q^{-1}x_{i+1}y_{m-i})(1 - q^{-1}x_i y_{m-i})}$$

We thus only have to prove that $T_i^{(x)}B = T_{m-i}^{(y)}B$, or equivalently, that $(T_i^{(x)} - t)B = (T_{m-i}^{(y)} - t)B$. The lemma thus holds after checking that

$$(T_i^{(x)} - t)B = (T_{m-i}^{(y)} - t)B = \frac{q^{-1}(tx_i - x_{i+1})(ty_{m-i} - y_{m-i+1})}{(1 - q^{-1}x_{i+1}y_{m-i})(1 - q^{-1}x_i y_{m-i})(1 - q^{-1}x_i y_{m-i+1})(1 - q^{-1}x_{i+1}y_{m-i+1})}$$

□

Proof of Proposition 121. Recall that our claim is that

$$Y_i^{(x)} K_m(x_{(N)}, y_{(N)}) = Y_i^{(y)} K_m(x_{(N)}, y_{(N)}) \tag{7.18}$$

for $i = 1, \ldots, m$, and that

$$D^{(x)} K_m(x_{(N)}, y_{(N)}) = D^{(y)} K_m(x_{(N)}, y_{(N)}) \tag{7.19}$$

For the remainder of the section, we will consider that $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$. We first prove (7.18). For simplicity, we write $Y_i = a \cdot b \cdot \omega \cdot c$, where $a = T_i \ldots T_{m-1}$, $b = T_m \ldots T_{N-1}$ and $c = \bar{T}_1 \ldots \bar{T}_{i-1}$ (note that if $i = 1$ then $c = 1$). We will also let $d = T_{\omega_m}$ and use a for $a^{(x)}$ and $\boldsymbol{a}$ for $a^{(y)}$ (and similarly for b, c, d and ω). Using this notation, (7.18) translates into

$$a\, b\, \omega\, c\, d\, \bar{K}_m(x, y) = \boldsymbol{a}\, \boldsymbol{b}\, \boldsymbol{\omega}\, \boldsymbol{c}\, d\, \bar{K}_m(x, y)$$

But, since $d\, \bar{K}_m(x, y) = \varphi^{(y)}(\boldsymbol{d})\bar{K}_m(x, y) = \boldsymbol{d}\, \bar{K}_m(x, y)$ from (7.11) and (7.17), this amounts to

$$a\, b\, \omega\, c\, d\, \bar{K}_m(x, y) = \boldsymbol{a}\, \boldsymbol{b}\, \boldsymbol{\omega}\, \boldsymbol{c}\, \boldsymbol{d}\, \bar{K}_m(x, y)$$

which we can rewrite as

$$\bar{K}_m(x, y) = \boldsymbol{a}\, \boldsymbol{b}\, \boldsymbol{\omega}\, \boldsymbol{c}\, \boldsymbol{d}\, \bar{d}\, \bar{c}\, \bar{\omega}\, \bar{b}\, \bar{a}\, \bar{K}_m(x, y)$$

where $\bar{a}$ stands for the inverse of a (and similarly for the other terms). From Lemma 137, this becomes

$$\bar{K}_m(x, y) = \boldsymbol{a}\, \boldsymbol{b}\, \boldsymbol{\omega}\, \boldsymbol{c}\, \boldsymbol{d} \varphi_m^{(y)}(\bar{\boldsymbol{a}})\, \bar{d}\, \bar{c}\, \bar{\omega}\, \bar{b}\, \bar{K}_m(x, y)$$

It was proven in [**8**] that $\bar{\omega}\, \bar{b}\, \bar{K}_m(x, y) = \bar{\boldsymbol{\omega}}\, \bar{\boldsymbol{b}}\, \bar{K}_m(x, y)$ (see the symmetry of $G(x, y)$ in (140) therein). Therefore, we have to prove, using Lemma 137, that

$$\begin{aligned}
\bar{K}_m(x, y) &= \boldsymbol{a}\, \boldsymbol{b}\, \boldsymbol{\omega}\, \boldsymbol{c}\, \boldsymbol{d}\, \varphi_m^{(y)}(\bar{\boldsymbol{a}})\, \bar{\boldsymbol{\omega}}\, \bar{\boldsymbol{b}}\, \bar{d}\, \bar{c}\, \bar{K}_m(x, y) \\
&= \boldsymbol{a}\, \boldsymbol{b}\, \boldsymbol{\omega}\, \boldsymbol{c}\, \boldsymbol{d}\, \varphi_m^{(y)}(\bar{\boldsymbol{a}})\, \bar{\boldsymbol{\omega}}\, \bar{\boldsymbol{b}}\, \varphi_m^{(y)}(\bar{\boldsymbol{c}})\, \bar{d}\, \bar{K}_m(x, y) \\
&= \boldsymbol{a}\, \boldsymbol{b}\, \boldsymbol{\omega}\, \boldsymbol{c}\, \boldsymbol{d}\, \varphi_m^{(y)}(\bar{\boldsymbol{a}})\, \bar{\boldsymbol{\omega}}\, \bar{\boldsymbol{b}}\, \varphi_m^{(y)}(\bar{\boldsymbol{c}})\, \bar{\boldsymbol{d}}\, \bar{K}_m(x, y)
\end{aligned}$$

since, as we have seen before, $\bar{d}\, \bar{K}_m(x, y) = \varphi^{(y)}(\bar{\boldsymbol{d}})\bar{K}_m(x, y) = \bar{\boldsymbol{d}}\, \bar{K}_m(x, y)$ from (7.11) and (7.17). The equality will thus follow if we can prove that

$$a\, b\, \omega\, c\, d\, \varphi_m(\bar{a})\, \bar{\omega}\, \bar{b}\, \varphi_m(\bar{c})\, \bar{d} = 1$$

where 1 stands for the identity operator. Now, we use Lemma 136 to get $d = \bar{c}\, e\, \varphi_m(a)$ and $\bar{d} = \varphi_m(\bar{d}) = \varphi_m(c)\varphi_m(\bar{e})\bar{a}$, where $e = T_{\omega'_{m-1}}$. This yields

$$\begin{aligned}
a\, b\, \omega\, c\, d\, \varphi_m(\bar{a})\, \bar{\omega}\, \bar{b}\, \varphi_m(\bar{c})\, \bar{d} \quad &= a\, b\, \omega\, c\, \big(\bar{c}\, e\, \varphi_m(a)\big)\, \varphi_m(\bar{a})\, \bar{\omega}\, \bar{b}\, \varphi_m(\bar{c})\, \big(\varphi_m(c)\, \varphi_m(\bar{e})\, \bar{a}\big) \\
&= a\, b\, \omega\, e\, \bar{\omega}\, \bar{b}\, \varphi_m(\bar{e})\, \bar{a}
\end{aligned} \tag{7.20}$$

Finally, we have that $\varphi_m(\bar{e}) = \varphi_m(\bar{T}_{\omega'_{m-1}}) = \bar{T}_{\omega_{m-1}} = \bar{e}'$ since φ_m changes the set $\{T_2, \ldots T_{m-1}\}$ to $\{T_1, \ldots, T_{m-2}\}$. Using $\bar{b}\, \bar{e}' = \bar{e}'\, \bar{b}$ and $\bar{\omega}\, \bar{e}' = \bar{e}\, \bar{\omega}$, we obtain that

$$a\, b\, \omega\, e\, \bar{\omega}\, \bar{b}\, \varphi_m(\bar{e})\, \bar{a} = a\, b\, \omega\, e\, \bar{\omega}\, \bar{e}'\, \bar{b}\, \bar{a} = a\, b\, \omega\, e\, \bar{e}\, \bar{\omega}\, \bar{b}\, \bar{a} = 1$$

as wanted.

For (7.19), we have to prove that

$$D^{(x)}T^{(x)}_{\omega_m}\bar{K}_m(x,y) = D^{(y)}T^{(y)}_{\omega_m}\bar{K}_m(x,y)$$

Since $T^{(x)}_{\omega_m}$ commutes with $D^{(x)}$ and $T^{(y)}_{\omega_m}\bar{K}_m(x,y) = T^{(x)}_{\omega_m}\bar{K}_m(x,y)$ from (7.11) and (7.17), we only need to show that

$$D^{(x)}\bar{K}_m(x,y) = D^{(y)}\bar{K}_m(x,y)$$

Now, $D^{(x)}$ commutes with $\mathcal{S}^{t(x)}_{m+1,N}$ and

$$\mathcal{S}^{t(x)}_{m+1,N}\bar{K}_m(x,y) \propto \bar{K}_m(x,y)$$

since $\bar{K}_m(x_{(N)}, y_{(N)})$ is symmetric in $x_{m+1},\dots,x_N$ (and similarly when x is replaced by y). It is thus equivalent to show that

$$\mathcal{S}^{t(x)}_{m+1,N}D^{(x)}\bar{K}_m(x,y) = \mathcal{S}^{t(y)}_{m+1,N}D^{(y)}\bar{K}_m(x,y)$$

From the definition of D, the result will thus follow if we can show that

$$\mathcal{S}^{t(x)}_{m+1,N}Y^{(x)}_i\bar{K}_m(x,y) = \mathcal{S}^{t(y)}_{m+1,N}Y^{(y)}_i\bar{K}_m(x,y)$$

for $i = m+1,\dots,N$. Using $\mathcal{S}^{t(x)}_{m+1,N}T^{(x)}_j = t\mathcal{S}^{t(x)}_{m+1,N}$ and $\bar{T}^{(x)}_j\bar{K}_m(x,y) = t^{-1}\bar{K}_m(x,y)$ for $j = m+1,\dots,N$, we obtain in those cases that (up to a power of t)

$$\mathcal{S}^{t(x)}_{m+1,N}Y^{(x)}_i\bar{K}_m(x,y) \propto \mathcal{S}^{t(x)}_{m+1,N}\omega^{(x)}\bar{T}^{(x)}_1\cdots\bar{T}^{(x)}_m\bar{K}_m(x,y)$$

Using the same relation with x replaced by y, we thus have left to prove that

$$\mathcal{S}^{t(x)}_{m+1,N}\omega^{(x)}\bar{T}^{(x)}_1\cdots\bar{T}^{(x)}_m\bar{K}_m(x,y) = \mathcal{S}^{t(y)}_{m+1,N}\omega^{(y)}\bar{T}^{(y)}_1\cdots\bar{T}^{(y)}_m\bar{K}_m(x,y)$$

But this was proven in [**8**] (see the symmetry of $L(x,y)$ in (154) therein). □

7.3. Missing piece in the proof of Proposition 118. We will prove the following lemma which was needed in the proof of Proposition 118.

LEMMA 138. *For any $f \in R_m$ in N variables, we have*

$$Y^\star_i\left(\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\right)f = \left(\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\right)\bar{Y}_i f \qquad i=1,\dots,m \tag{7.21}$$

and

$$D^\star\left(\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\right)f = \left(\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\right)\bar{D}f \tag{7.22}$$

where $\bar{D} = \bar{Y}_{m+1}+\cdots+\bar{Y}_N+\sum_{i=m+1}^N t^{i-1}$.

PROOF. We first prove (7.21) in the case $i=1$. Observe that $\bar{Y}_1$ and $\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}$ both preserve R_m. We can thus prove instead that

$$Y^\star_1\left(\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\right)f = K_{\sigma_{m+1}}\left(\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\right)\bar{Y}_1 f$$

where $\sigma_{m+1} = [1,\dots,m,N,N-1,\dots,m+1]$. Hence, using the expression for Y_1, we need to prove that

$$T^\star_1\cdots T^\star_{N-1}s_{N-1}\cdots s_1\tau_2\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}f = K_{\sigma_{m+1}}\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\tau_1^{-1}s_1\dots s_{N-1}\bar{T}_{N-1}\cdots\bar{T}_1 f$$

where we use $s_i = K_{i,i+1}$ for simplicity. Using $K_{\omega_m}\bar{T}_{\omega_m} = T^\star_{\omega_m}K_{\omega_m}$ and multiplying both sides by $\bar{T}^\star_{m-1}\cdots\bar{T}^\star_1$, this is equivalent to proving that

$$T^\star_m\cdots T^\star_{N-1}s_{N-1}\cdots s_1\tau_2\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}f = K_{\sigma_{m+1}}\tau_1\cdots\tau_m T^\star_{\omega_{m-1}}K_{\omega_m}\tau_1^{-1}s_1\dots s_{N-1}\bar{T}_{N-1}\cdots\bar{T}_1 f$$

where we used $T_{\omega_{m-1}} = \bar{T}_{m-1}\cdots\bar{T}_1 T_{\omega_m}$. Letting $f' = \bar{T}_{m-1}\cdots\bar{T}_1 f$, we thus have to prove that

$$T^\star_m\cdots T^\star_{N-1}s_{N-1}\cdots s_1\tau_2\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_{m-1}}f' = K_{\sigma_{m+1}}\tau_1\cdots\tau_m T^\star_{\omega_{m-1}}K_{\omega_m}\tau_1^{-1}s_1\dots s_{N-1}\bar{T}_{N-1}\cdots\bar{T}_m f'$$

for $f' \in R_m$. Owing to the relations $K_{\omega_m}\tau_1^{-1} = \tau_m^{-1}K_{\omega_m}$ and $s_{N-1}\cdots s_1\tau_2\cdots\tau_m = \tau_1\cdots\tau_{m-1}s_{N-1}\cdots s_1$, we can extract $\tau_1\cdots\tau_{m-1}$ to the left on both sides of the equation. We thus have left to prove that

$$T_m^\star\cdots T_{N-1}^\star s_{N-1}\cdots s_1 K_{\omega_m}\bar{T}_{\omega_{m-1}}f' = K_{\sigma_{m+1}}T_{\omega_{m-1}}^\star K_{\omega_m}s_1\ldots s_{N-1}\bar{T}_{N-1}\cdots\bar{T}_m f'$$

With the help of $s_{m-1}\cdots s_1 K_{\omega_m} = K_{\omega_m}s_1\cdots s_{m-1} = K_{\omega_{m-1}}$ and $K_{\omega_{m-1}}\bar{T}_{\omega_{m-1}} = T_{\omega_{m-1}}^\star K_{\omega_{m-1}}$, this now amounts to showing that

$$T_m^\star\cdots T_{N-1}^\star s_{N-1}\cdots s_m f' = K_{\sigma_{m+1}}s_m\ldots s_{N-1}\bar{T}_{N-1}\cdots\bar{T}_m f'$$

Letting $\sigma_m = [1,\ldots,m-1,N,N-1,\ldots,m]$ and using the fact that $f' \in R_m$, this is seen to hold since

$$\begin{aligned}T_m^\star\cdots T_{N-1}^\star s_{N-1}\cdots s_m f' = T_m^\star\cdots T_{N-1}^\star K_{\sigma_m}f' &= K_{\sigma_m}\bar{T}_{N-1}\cdots\bar{T}_m f' \\ &= K_{\sigma_{m+1}}s_m\ldots s_{N-1}\bar{T}_{N-1}\cdots\bar{T}_m f'\end{aligned}$$

We now consider the general case in (7.21). From the relation

$$Y_i = \bar{T}_{i-1}\cdots\bar{T}_1 Y_1\bar{T}_1\cdots\bar{T}_{i-1}$$

we have to show that, for $1 \le i \le m$, we have

$$\big(\bar{T}_{i-1}^\star\cdots\bar{T}_1^\star Y_1^\star\bar{T}_1^\star\cdots\bar{T}_{i-1}^\star\big)\tau_1\ldots\tau_m K_{\omega_m}\bar{T}_{\omega_m} = \tau_1\ldots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\big(T_{i-1}\cdots T_1 Y_1 T_1\cdots T_{i-1}\big)$$

But this is indeed the case since using the Y_1 case and $T_i^\star K_{\omega_m}\bar{T}_{\omega_m} = K_{\omega_m}\bar{T}_{\omega_m}T_i$ for all $i = 1,\ldots,m$, we get that

$$\begin{aligned}\big(\bar{T}_{i-1}^\star\cdots\bar{T}_1^\star Y_1^\star\bar{T}_1^\star\cdots\bar{T}_{i-1}^\star\big)\tau_1\ldots\tau_m K_{\omega_m}\bar{T}_{\omega_m} &= \bar{T}_{i-1}^\star\cdots\bar{T}_1^\star Y^\star\tau_1\ldots\tau_m K_{\omega_m}\bar{T}_{\omega_m}T_1\cdots T_{i-1} \\ &= \bar{T}_{i-1}^\star\cdots\bar{T}_1^\star\tau_1\ldots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\bar{Y}_1 T_1\cdots T_{i-1} \\ &= \tau_1\ldots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\big(T_{i-1}\cdots T_1\bar{Y}_1 T_1\cdots T_{i-1}\big)\end{aligned}$$

Finally, proceeding as in the proof of (7.19), in order to prove (7.22) we only have to show that

$$(\mathcal{S}_{m+1,N}^t)^\star Y_{m+1}^\star\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}f = \tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\mathcal{S}_{m+1,N}^t\bar{Y}_{m+1}f$$

on any $f \in R_m$. This is equivalent to proving that

$$K_{\sigma_{m+1}}(\mathcal{S}_{m+1,N}^t)^\star Y_{m+1}^\star\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}f = \tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\mathcal{S}_{m+1,N}^t\bar{Y}_{m+1}f$$

since $\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\mathcal{S}_{m+1,N}^t\bar{Y}_{m+1}$ preserves R_m. Using $(\mathcal{S}_{m+1,N}^t)^\star T_i^\star = t^{-1}(\mathcal{S}_{m+1,N}^t)^\star$ and $\bar{T}_i f = t^{-1}f$ for $i = m+1,\ldots,N$, this thus amounts to showing that

$$\begin{aligned}&\mathcal{S}_{m+1,N}^t K_{\sigma_{m+1}}s_{N-1}\cdots s_1\tau_1^{-1}\bar{T}_1^\star\cdots\bar{T}_m^\star\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}f \\ &\qquad = \tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}\mathcal{S}_{m+1,N}^t T_m\cdots T_1\tau_1^{-1}s_1\cdots s_m f\end{aligned} \tag{7.23}$$

where we have used the relation $K_{\sigma_{m+1}}(\mathcal{S}_{m+1,N}^t)^\star = \mathcal{S}_{m+1,N}^t K_{\sigma_{m+1}}$. We will now see that (7.23) holds. We first use $K_{\sigma_{m+1}}s_{N-1}\cdots s_{m+1} = K_{\sigma_{m+2}}$ to obtain

$$\begin{aligned}&\mathcal{S}_{m+1,N}^t K_{\sigma_{m+1}}s_{N-1}\cdots s_1\tau_1^{-1}\bar{T}_1^\star\cdots\bar{T}_m^\star\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}f \\ &\qquad = \mathcal{S}_{m+1,N}^t K_{\sigma_{m+2}}s_m\cdots s_1\tau_1^{-1}\bar{T}_1^\star\cdots\bar{T}_m^\star\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}f \\ &\qquad = \mathcal{S}_{m+1,N}^t s_m\cdots s_1\tau_1^{-1}\bar{T}_1^\star\cdots\bar{T}_m^\star\tau_1\cdots\tau_m K_{\omega_m}\bar{T}_{\omega_m}f\end{aligned}$$

since $K_{\sigma_{m+2}}f = f$. We then observe that

$$\bar{T}_1^\star\cdots\bar{T}_m^\star T_{\omega_m}^\star K_{\omega_{m+1}} = K_{\omega_{m+1}}T_m\cdots T_1\bar{T}_{\omega_m'} = K_{\omega_{m+1}}\bar{T}_{\omega_m}T_m\cdots T_1$$

since $s_m \cdots s_1 \omega'_m = \omega_m s_m \cdots s_1$, where $\omega'_m = [1, m, m-1, \ldots, 2, m+1, \ldots, N]$. But then (7.23) holds given that

$$\begin{aligned}
&\mathcal{S}^t_{m+1,N} s_m \cdots s_1 \tau_1^{-1} \bar{T}_1^\star \cdots \bar{T}_m^\star \tau_1 \cdots \tau_m K_{\omega_m} \bar{T}_{\omega_m} f \\
&\qquad = \mathcal{S}^t_{m+1,N} s_m \cdots s_1 \tau_1^{-1} \bar{T}_1^\star \cdots \bar{T}_m^\star T^\star_{\omega_m} \tau_1 \cdots \tau_m K_{\omega_m} f \\
&\qquad = \mathcal{S}^t_{m+1,N} s_m \cdots s_1 \tau_1^{-1} \bar{T}_1^\star \cdots \bar{T}_m^\star T^\star_{\omega_m} \tau_1 \cdots \tau_m K_{\omega_{m+1}} s_1 \cdots s_m f \\
&\qquad = \mathcal{S}^t_{m+1,N} K_{\omega_m} \tau_{m+1}^{-1} \bar{T}_{\omega_m} T_m \cdots T_1 \tau_2 \cdots \tau_{m+1} s_1 \cdots s_m f \\
&\qquad = \mathcal{S}^t_{m+1,N} K_{\omega_m} \tau_1 \cdots \tau_m \bar{T}_{\omega_m} T_m \cdots T_1 \tau_1^{-1} s_1 \cdots s_m f \\
&\qquad = \tau_1 \cdots \tau_m K_{\omega_m} \bar{T}_{\omega_m} \mathcal{S}^t_{m+1,N} T_m \cdots T_1 \tau_1^{-1} s_1 \cdots s_m f
\end{aligned}$$

$\square$

CHAPTER 5

Symmetric functions in superspace and bisymmetric functions

1. Ring of symmetric functions ins superspace

In this section we will introduce the basics notions concerning symmetric functions in superspace. Most of this section is taken from [**8**].

Consider the ring $\mathbb{Q}[x;\theta] = \mathbb{Q}[x_1, x_2, \dots, x_N, \theta_1, \theta_2, \dots, \theta_N]$, where the variables obey the relations

$$x_i x_j = x_j x_i, \quad \theta_i x_j = x_j \theta_i, \quad \theta_i \theta_j = -\theta_j \theta_i$$

We can define an action of S_N on $\mathbb{Q}[x;\theta]$ as follows, given a permutation $\sigma \in S_N$ and $f \in \mathbb{Q}[x;\theta]$ the action of σ on f is

$$\mathcal{K}_\sigma f = K_\sigma^{(x)} K_\sigma^{(\theta)} f = f(x_{\sigma(1)}, \dots, x_{\sigma(N)}, \theta_{\sigma(1)}, \dots, \theta_{\sigma(N)})$$

The space of **symmetric polynomials in superspace** in N variables is the space of polynomials in $\mathbb{Q}[x;\theta]$ that are invariant under this action, i.e.

$$\Lambda_N = \mathbb{Q}[x;\theta]^{S_N} = \{f \in \mathbb{Q}[x;\theta] \mid \mathcal{K}_\sigma \cdot f = f \quad \text{for all } \sigma \in S_N\}$$

Example 139. *Here are two examples of symmetric functions in superspace when $N = 2$:*

(1) $f(x_1, x_2, \theta_1, \theta_2) = \theta_1 x_2^2 + \theta_2 x_1^2$,
(2) $f(x_1, x_2, \theta_1, \theta_2) = \theta_1\theta_2(x_1^4 - x_2^4)$,

The space of symmetric polynomials in superspace has a doubly graded structure given by

$$\Lambda_N = \bigoplus_{n,m \geq 0} \Lambda_N(n|m)$$

where $\Lambda_N(n|m)$ is the space of homogeneous symmetric polynomials in superspace of degree n in the x variables and degree m in the θ variables. Since $\theta_i\theta_i = 0$, the degree m in the θ variables implies that each term of the symmetric polynomial in superspace has exactly m distinct θ_i's.

As we will now see, as a vector space, the ring of symmetric functions in superspace is equivalent to the ring of bisymmetric functions. This relationship will prove crucial as it will allow us to disregard the θ variables to work with a single family of variables. The theory will thus be reformulated in a more tractable way, enabling us to use all the machinery introduced in the previous chapters to establish properties of the Macdonald polynoials in superspace.

Definition 140. *We will be concerned with the ring of bisymmetric functions*

$$\mathscr{R}_{m,N} = \mathbb{Q}[x_1, \dots, x_N]^{\mathfrak{S}_m \times \mathfrak{S}_{m+1,N}}.$$

Bases of $\mathscr{R}_{m,N}$ are naturally indexed by pairs of partitions (λ, μ), where λ (resp. μ) is a partition whose length is at most m (resp. $N - m$). We will adopt the language of symmetric functions in superspace [**8**] *and consider the bijection*

$$(\lambda, \mu) \longleftrightarrow (\Lambda^a; \Lambda^s) := (\lambda + \delta_m; \mu)$$

where $\delta_m = (m-1, m-2, \ldots, 0)$. The superpartition $\Lambda = (\Lambda^a, \Lambda^s)$ thus consists of a partition Λ^a with m non-repeated entries (one of them possibly equal to zero) and a usual partition Λ^s whose length is not larger than $N-m$.

Remark 141. *Since the variables θ's satisfy anticommutating relations, any polynomial in superspace $F(x;\theta)$ can be written as*

$$F(x;\theta) = \sum_{I\subseteq\{1,\ldots,N\};|I|=m} \theta_I \Delta_I(x) f_I(x) \tag{1.1}$$

where, for $I = \{i_1, \ldots, i_m\}$ with $i_1 < i_2 < \cdots < i_m$, we have $\theta_I = \theta_{i_1}\cdots\theta_{i_m}$. Observe that by symmetry a polynomial in superspace is completely determined by its coefficient $f_{\{1,\ldots,m\}}(x)$, and moreover, that $f_{\{1,\ldots,m\}}(x)$ needs to be bisymmetric.

Example 142. *(1) If $F(x_1, x_2, \theta_1, \theta_2) = \theta_1\theta_2(x_1^4 - x_2^4) = \theta_1\theta_2(x_1 - x_2)(x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3)$ then $f_{1,2}(x_1, x_2) = x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3$.*

For our purposes, working with symmetric functions in superspace is equivalent to working with bisymmetric functions.

2. Superpartitions

We want to study different bases for the space of symmetric functions in superspace. As we shall emphasize their combinatorial properties, we first need the analog of a partition in this context.

Definition 143. *A **superpartition** is a pair of partitions*

$$\Lambda = (\Lambda^a; \Lambda^s) = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_l)$$

with the conditions

$$\Lambda_1 > \Lambda_2 > \cdots > \Lambda_m \geq 0 \quad \text{and} \quad \Lambda_{m+1} \geq \Lambda_{m+2} \geq \cdots \geq \Lambda_l \geq 0$$

*Note that Λ^a has distinct parts. The length of Λ is $\ell(\Lambda) = \ell(\Lambda^a) + \ell(\Lambda^s)$. Sometimes it is convenient to consider Λ with exactly N parts. In this case we add $N - \ell(\Lambda)$ entries equal to zero to Λ^s. The number m (the number of part of Λ^a), is called the **fermionic degree** of Λ and the **bosonic degree** of Λ is*

$$n = |\Lambda^a| + |\Lambda^s| = \Lambda_1 + \Lambda_2 + \cdots \Lambda_l.$$

*A superpartition has **degree** $(n|m)$ if it has bosonic degree n and fermionic degree m.*

Definition 144. *Given $n, m \in \mathbb{N}$, the set of superpartitions of degree $(n|m)$ is denoted by $SPar(n|m)$.*

Example 145. *The set of superpartitions of $(4|2)$ is*

$$SPar(4|2) = \{(1,0;3), (1,0;2,1), (1,0;1,1,1), (2,0;2), (2,0;1,1), (2,1;1), (3,0;1), (3,1;), (4,0;)\}$$

Remark 146. *We can define the set of all superpartitions as*

$$SPar = \bigcup_{n,m\geq 0} SPar(n|m)$$

Definition 147. *A superpartition $\Lambda \vdash (n|m)$ of length l is described by a pair of partitions $(\Lambda^*, \Lambda^\circledast)$, which satisfy the following conditions:*

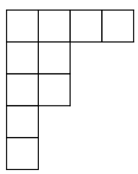
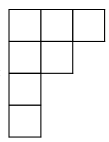
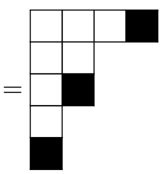
1. *$\Lambda^* \subset \Lambda^\circledast$;*
2. *the degree of Λ is n;*
3. *the length of $\Lambda^\circledast$ os l;*
4. *the skew diagram $\Lambda^\circledast/\Lambda^*$ os both a horizontal and a vertical m-strip.*

Remark 148. *There is a correspondence* $(\Lambda^a, \Lambda^s) \longrightarrow (\Lambda^*, \Lambda^\circledast)$ *between Definition 143 and Definition 147; Given* (Λ^a, Λ^s) *define* $\Lambda^* = (\Lambda^a, \Lambda^s)^+$ *the partition obtained when you sort the composition given by the concatenation of* Λ^a *and* Λ^s*, and* $\Lambda^\circledast = (\Lambda^a + (1^m), \Lambda^s)^+$*. It is not difficult to see that they satisfy the conditions in Definition 147. On the other hand, given* $(\Lambda^*, \Lambda^\circledast)$*, let* Λ^a *be the entries in* Λ^* *that correspond to a row of the the vertical* m*-strip* $\Lambda^\circledast/\Lambda^*$ *and let* Λ^s *be its complement.*

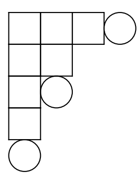
Example 149. *To* $\Lambda^* = (3, 2, 1, 1, 0)$ *and* $\Lambda^\circledast = (4, 2, 2, 1, 1)$ *correspond the superpartition* $(3, 1, 0; 2, 1)$*.*

Definition 150. *Given a superpartition* Λ *the* ***superdiagram*** *or the diagram of* Λ *is the diagram of* $\Lambda^\circledast$ *where the boxes corresponding to* $\Lambda^\circledast/\Lambda^*$ *are drawn as circles.*

Example 151. *Given the superpartition* $(3, 1, 0; 2, 1)$*, we know that* $\Lambda^* = (3, 2, 1, 1, 0)$ *and* $\Lambda^\circledast = (4, 2, 2, 1, 1)$*. Its corresponding diagrams are*

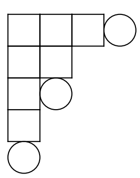
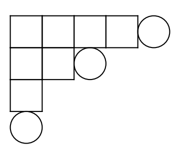
$\Lambda^\circledast =$  $\Lambda^* =$  *then* $\Lambda^\circledast/\Lambda^* =$ 

Thus, the diagram of the superpartition $(3, 1, 0; 2, 1)$ *is*



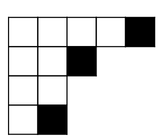
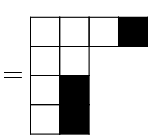
Definition 152. *The* ***conjugate*** *of a superpartition, denoted by* Λ'*, is obtained by reflecting the diagram of the original partition* λ *along its main diagonal.*

Example 153. *If* $\Lambda = (3, 1, 0; 2, 1)$ *then* $\Lambda' = (4, 2, 0; 1)$ *because*

 $\xrightarrow{reflecting}$ 

Definition 154. *We say that a skew diagram* Ω/Λ *is a* ***vertical*** r***-strip*** *if the diagrams of* $\Omega^\circledast/\Lambda^\circledast$ *and* Ω^*/Λ^* *are vertical* r*-strips.*

Example 155. *If* $\Omega = (4, 2, 0; 2, 2)$ *and* $\Lambda = (3, 1, 0; 2, 1)$ *we have that* Ω/Λ *is a vertical* 3*-strip with diagram*

$\Omega^\circledast/\Lambda^\circledast =$  *and* $\Omega^*/\Lambda^* =$ 

Definition 156. *We can define an order in* $SPar(n|m)$*. Let* Λ*,* Ω *in* $SPar(n|m)$*, we define the* ***dominance order*** *as*

$$\Lambda \geq \Omega \Longleftrightarrow \Lambda^* \geq \Omega^* \text{ and } \Lambda^\circledast \geq \Omega^\circledast$$

where the symbol $\geq$ *on the right side is the usual dominance order on partitions.*

Example 157. *Taking* $\Lambda = (3,0;4,1)$ *and* $\Omega = (2,1;3,2)$ *we have that*

$$\Lambda^* = (3,0;4,1) > \Omega^* = (2,1;3,2)$$
$$\Lambda^{\circledast} = (4,4;1,1) > \Omega^{\circledast} = (3,3;2,2)$$

which implies that $\Lambda > \Omega$.

Remark 158. *Again, this order is not a total ordering.*

3. Bases of the space of symmetric functions in superspace

We want to study natural bases for the space of symmetric functions in superspace and their bisymmetric counterparts.

Monomial symmetric functions in superspace.

Definition 159. *Given* λ *a partition, we define the* ***monomial symmetric function in superspace*** *as*

$$m_\Lambda(x_1,\dots,x_n) = \frac{1}{|G_{N,\Lambda}|}\mathcal{S}_N \cdot \theta_1\cdots\theta_m x^\eta$$

where η *is a composition such that* $\eta^+ = \Lambda$ *and* $G_{N,\Lambda} = \{\sigma \in S_N \mid K_\sigma\Lambda = \Lambda\}$ *and* $\mathcal{S}_N$ *correspond to the simetrization operator respect the action of symmetric group over* $\mathbb{Q}[x;\theta]$.

Example 160. *We have*

$$\begin{aligned} m_{(2,0;1)} &= \theta_1\theta_2x_1^2x_3 + \theta_2\theta_1x_2^2x_3 + \theta_3\theta_2x_3^2x_1 + \theta_1\theta_3x_1^2x_2 + \theta_2\theta_3x_2^2x_1 + \theta_3\theta_1x_3^2x_2 \\ &= \theta_1\theta_2(x_1^2x_3 - x_2^2x_3) + \theta_2\theta_3(x_2^2x_1 - x_3^2x_2) + \theta_1\theta_3(x_1^2x_2 - x_3^2x_2). \end{aligned}$$

Remark 161. *Note that* $\{m_\Lambda\}_\Lambda$, *where* Λ *runs over all superpartitions, is a natural basis of the space of symmetric functions in superspace.*

Remark 162. *Note that in this case, the bisymmetric part of this polynomials is just* x^η, *i.e.*

$$m_\Lambda(\theta_1,\dots,\theta_m;x_1,\dots,x_n) = \sum_{\sigma\in\mathcal{S}_N}\mathcal{K}_\sigma\theta_1\cdots\theta_m\Delta_m(x)x^\eta$$

where η *is a composition such that* $\eta^+ = \Lambda^*$.

Elementary symmetric functions in superspace.

Definition 163. *For* $r \geq 0$, *we define the* ***elementary symmetric functions in superspace*** e_r *and* $\tilde{e}_r$ *as*

$$\tilde{e}_r = m_{(0;1^r)} \text{ and } e_r = m_{(;1^r)}$$

We can extend this definition to a superpartition Λ *in the following way*

$$e_\Lambda = e_{\Lambda^a}e_{\Lambda^s} = \tilde{e}_{\Lambda_1^a}\cdots\tilde{e}_{\Lambda_m^a}e_{\Lambda_{m+1}^s}\cdots e_{\Lambda_N^s}$$

Example 164. *We have for instance*

- $\tilde{e}_1 = \theta_1(x_2+x_3) + \theta_2(x_1+x_3) + \theta_3(x_1+x_2)$
- $\tilde{e}_2 = \theta_1x_2x_3 + x_1\theta_2x_3 + x_1x_2\theta_3$
- $e_{(2,1;2)} = (\theta_1x_2x_3 + x_1\theta_2x_3 + x_1x_2\theta_3)(\theta_1(x_2+x_3) + \theta_2(x_1+x_3) + \theta_3(x_1+x_2))(x_1x_2x_3 + x_1x_2x_3 + x_1x_2x_3)$

Complete symmetric functions in superspace.

Definition 165. *For $r \geq 0$, we define the **complete symmetric functions in superspace** h_r and $\tilde{h}_r$ as*

$$\tilde{h}_r = \sum_{\Lambda \vdash (n|1)} (1+\Lambda_1) m_\Lambda \text{ and } h_r = \sum_{\lambda \vdash n} m_\Lambda$$

We can extend this definition to a superpartition Λ in the following way

$$h_\Lambda = h_{\Lambda^a} h_{\Lambda^s} = \tilde{h}_{\Lambda_1^a} \cdots \tilde{h}_{\Lambda_m^a} h_{\Lambda_{m+1}^s} \cdots h_{\Lambda_N^s}$$

Example 166. *We have*

- $\tilde{h}_1 = (1+0) m_{(0;1)} + (1+1) m_{(1;0)}$,
- $\tilde{h}_2 = (1+0) m_{(0;2)} + (1+0) m_{(0;1,1)} + (1+1) m_{(1;1)} + (1+2) m_{(2;0)}$,
- $h_{(1;2)} = (m_{(0;1)} + 2m_{(1;0)})(m_{(2)} + m_{(1,1)})$.

Power-sum symmetric functions in superspace.

Definition 167. *For $r \geq 0$, we define the **power-sum symmetric functions in superspace** p_r and $\tilde{p}_r$ as*

$$\tilde{p}_r = m_{(r;0)} = \sum \theta_i x_i^r \text{ and } p_r = m_{(;r)} = \sum x_i^r$$

We can extend this definition to a superpartition Λ in the following way

$$p_\Lambda = p_{\Lambda^a} p_{\Lambda^s} = \tilde{p}_{\Lambda_1^a} \cdots \tilde{p}_{\Lambda_m^a} p_{\Lambda_{m+1}^s} \cdots p_{\Lambda_N^s}$$

Example 168. *For example*

- $\tilde{p}_1 = \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3$
- $\tilde{p}_2 = \theta_1 x_1^2 + \theta_2 x_2^2 + \theta_3 x_3^2$
- $p_{(2,1;2)} = (\theta_1 x_1^2 + \theta_2 x_2^2 + \theta_3 x_3^2)(\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3)(x_1^2 + x_2^2 + x_3^2)$

Remark 169. *Note that in this case we can write*

$$p_\Lambda(\theta_1, \ldots, \theta_m; x_1, \ldots, x_n) = \sum_{\sigma \in \mathcal{S}_N} \mathcal{K}_\sigma \theta_1 \cdots \theta_m \Delta_m(x) \mathcal{A}_m x^a p_\lambda(x_1, \ldots, x_N)$$

We then have $\mathfrak{p}_\Lambda = \mathcal{A}_m x^a p_\lambda(x_1, \ldots, x_N)$, where a is a composition such that $a^+ = \Lambda^a$. Also, notice that $\mathfrak{p}_\Lambda$ corresponds to the antisymmetrization of the m-symmetric p_Λ.

4. Bisymmetric Macdonald polynomials

By Remark 140, the Macdonald polynomials in superspace are essentially equivalent to their bisymmetric counterpart. In the following chapters we will work with bisymmetric polynomials, for this reason up now, we will use the same notation for bisymmetric polynomials and polynomials in the super space. Most of this chapter is taken from [**8**].

Definition 170. *The bisymmetric Macdonald polynomial indexed by the superpartition Λ is defined as*

$$\mathcal{P}_\Lambda(x_1, \ldots, x_N; q, t) = \frac{c_\Lambda(t)}{\Delta_m^t(x)} \mathcal{A}_m \mathcal{S}_{m,N}^t E_{\eta_\Lambda}(x_1, \ldots, x_N; q, t)$$

with the normalization constant $c_\Lambda(t)$ such that

$$\frac{1}{c_\Lambda(t)} = \left(\prod_{i \geq 0} [n_{\Lambda^s}(i)]_{t^{-1}}! \right) t^{(N-m)(N-m-1)/2}$$

where $n_{\Lambda^s}(i)$ is the number of entries in Λ^s that are equal to i, and where

$$[k]_q = \frac{(1-q)(1-q^2)\cdots(1-q^k)}{(1-q)^k}$$

We observe that the normalization constant $c_\Lambda(t)$ is chosen such that the coefficient of

$$x_1^{\lambda_1}\cdots x_m^{\lambda_m} x_{m+1}^{\mu_1}\cdots x_N^{\mu_{N-m}}$$

in $\mathcal{P}_\Lambda(x;q,t)$ is equal to 1, where $(\lambda,\mu)\longleftrightarrow\Lambda$.

REMARK 171. *Note that*

$$\mathcal{P}_\Lambda(x_1,\dots,x_N;q,t) = \frac{1}{\Delta_m^t(x)}\,\mathcal{A}_m P_{(\Lambda_m^a,\dots,\Lambda_1^a,\Lambda^s)}(x_1,\dots,x_N;q,t)$$

where $P_{(\Lambda_m^a,\dots,\Lambda_1^a,\Lambda^s)}(x_1,\dots,x_N;q,t)$ is the m-symmetric Macdonald polynomial defined in the previous section.

The bisymmetric Macdonald polynomials are stable.

PROPOSITION 172. ***[Stability]*** *The symmetric Macdonald polynomials $\mathcal{P}_\lambda$ are stable with respect the number of variables, that means,*

$$\mathcal{P}_\Lambda(x_1,\dots,x_{N-1},0;q,t) = \begin{cases} \mathcal{P}_\Lambda(x_1,\dots,x_{N-1};q,t) & \text{if } N>\ell(\Lambda)\\ 0 & \text{otherwise}\end{cases}$$

PROOF. The proof is similar to that of Proposition 172. □

In this case, we have that two operators are needed to characterize the bisymmetric Macdonald polynomials.

DEFINITION 173. *We define the the **bisymmetric Macdonald operators** as*

$$E_{1,N} = Y_1+\cdots+Y_m \quad \text{and} \quad E_{2,N} = Y_{m+1}+\cdots+Y_N - \sum_{i=1}^N t^{1-i}$$

As expected, they have the bisymmetric Macdonald polynomials as eigenfunctions.

PROPOSITION 174. *The bisymmetric Macdonald polynomials are simultaneously eigenfunctions of the operator E_N, to be precise,*

$$E_{1,N}\mathcal{P}_\Lambda = c_\Lambda\mathcal{P}_\Lambda \qquad E_{2,N}\mathcal{P}_\Lambda = d_\Lambda\mathcal{P}_\Lambda$$

where $c_\Lambda = \bar\eta_1+\cdots+\bar\eta_m$ and $d_\Lambda = \bar\eta_{m+1}+\cdots+\bar\eta_N - \sum_{i=1}^N t^{1-i}$, with $\bar\eta_i = q^{\eta_i}t^{1-r_\eta(i)}$.

PROOF. The proof is quite similar to the proof of Proposition 61. □

Note that $E_{1,N}$ and $E_{2,N}$ do not depend on the numbers of variables N. We can then consider

$$E_1 = \varprojlim E_{1,N} \qquad E_2 = \varprojlim E_{2,N}$$

We name E_1 and E_2 the **Macdonald Operators the superspace**.

We have our first characterization of the symmetric Macdonald polynomials. The proof of this result can be found in [**8**], see Prop 10.

PROPOSITION 175. ***[Triangularity]*** *The Symmetric Macdonald polynomials are the unique symmetric polynomials indexed by partitions which satisfy*

(1) *the descomposition over the monomials are triangular*

$$\mathcal{P}_\Lambda = m_\Lambda + \sum_{\Omega<\Lambda} c_{\Omega,\Lambda} m_\mu$$

where $<$ is the dominance order in partitions.

(2)

$$E_1\mathcal{P}_\Lambda = c_\Lambda \mathcal{P}_\Lambda \qquad E_2\mathcal{P}_\Lambda = d_\Lambda \mathcal{P}_\Lambda$$

5. Orthogonality

Motivated by the section 4 of the chapter III, we will define a scalar product and emulating the same methods as in that section we will show that the bisymmetric Macdonald polynomials are othogonal with respect to that scalar product. Because many of this ideas are similar to section 4, we will skips some proofs. Most of this chapter can be found in [**7**].

Definition 176. *We define the following scalar product over bisymmetric power polynomials*

$$\langle\langle p_\Lambda, p_\Omega \rangle\rangle = \delta_{\Lambda\Omega} z_\Lambda(q,t)$$

We will prove that the bisymmetric Macdonald polynomials are orthogonal with respect to this scalar product. But we first have to define the following kernel:

Definition 177. *The symmetric kernel is*

$$\mathcal{K} = \mathcal{F}_m K_0$$

where K_0 was introduced in Definition 64 and where

$$\mathcal{F}_m = \frac{\Delta_m^t(x)}{\prod\limits_{1\le i,j\le m}(1-x_iy_j)}.$$

The following proposition can be found in [**8**].

Lemma 178. *The following identity holds:*

$$\mathcal{A}_m^{(y)} \mathcal{NF}_m(x,y) = (-1)^{\binom{m}{2}} \Delta_m(y)\mathcal{F}_m(x)$$

where $\mathcal{NF}_m(x,y)$ is defined in 120.

This Lemma give us the follow immediate consequence

Proposition 179. *We have the next relation between K_m defined in 120 and $\mathcal{F}_m$*

$$\mathcal{A}_m^{(y)} K_m = t^{\binom{m}{2}} \Delta_m(y)\mathcal{K}.$$

Proof. Note that by 120 and Lemma 178, we have

$$\begin{aligned}\mathcal{A}_m^{(y)} K_m &= (-t)^{\binom{m}{2}} \Delta_m(y) T_{\omega_m}^{(x)} \mathcal{F}_m K_0 \\ &= t^{\binom{m}{2}} \Delta_m(y)\mathcal{F}_m K_0 \\ &= t^{\binom{m}{2}} \Delta_m(y)\mathcal{K}.\end{aligned}$$

□

We again have a relation between our kernel and power functions.

Lemma 180.

$$\mathcal{K} = \sum_\lambda z_\lambda(q,t)^{-1} p_\lambda(x) p_\lambda(y)$$

The next lemma gives us a connection between the scalar product in Definition 176 and the kernel K_0 defined above.

LEMMA 181. *Let u_Λ, v_Ω symmetric functions, then the following criteria of orthogonality is verified*

$$\mathcal{K}_N = \sum_\Lambda u_\Lambda(x) v_\Lambda(y) \Longleftrightarrow \langle\langle u_\Lambda, v_\Omega \rangle\rangle = \delta_{\Lambda\Omega}$$

The following lemma is the key point in demonstrating the orthogonality of Macdonald polynomials,

LEMMA 182. *When acting on the kernel, the operators in Definition 173 are symmetric in x and y:*

$$E_1^{(x)} \Delta_m^t(y) \mathcal{K}_N = E_1^{(y)} \Delta_m^t(y) \mathcal{K}_N$$
$$E_2^{(x)} \Delta_m^t(y) \mathcal{K}_N = E_2^{(y)} \Delta_m^t(y) \mathcal{K}_N$$

PROOF. By Proposition 179 we have

$$\mathcal{A}_m^{(y)} K_m = t^{\binom{m}{2}} \Delta_m(y) \mathcal{K}$$

This yields

$$\begin{aligned}
(Y_{m+1}^{(x)} + \ldots + Y_N^{(x)}) \Delta_m^t(y) \mathcal{K} &= (Y_{m+1}^{(x)} + \ldots + Y_N^{(x)}) \frac{\Delta_m^t(y)}{t^{\binom{m}{2}} \Delta_m(y)} \mathcal{A}_m^{(y)} K_m \\
&= (Y_{m+1}^{(x)} + \ldots + Y_N^{(x)}) (A_m^t)^{(y)} K_m \\
&= (A_m^t)^{(y)} (Y_{m+1}^{(x)} + \ldots + Y_N^{(x)}) K_m \\
&= (A_m^t)^{(y)} (Y_{m+1}^{(y)} + \ldots + Y_N^{(y)}) K_m \\
&= (Y_{m+1}^{(y)} + \ldots + Y_N^{(y)}) \Delta_m^t(y) \mathcal{K}.
\end{aligned}$$

The proof for $Y_1 + \ldots + Y_m$ is exactly the same. □

THEOREM 183. ***[Cauchy formula]***

$$\mathcal{K}_N = \sum_\Lambda b_\Lambda^{-1}(q,t) \mathcal{P}_\Lambda(x) \mathcal{P}_\Lambda(y)$$

THEOREM 184. ***[Orthogonality]*** *The bisymmetric Macdonald polynomials are orthogonal with respect to the scalar product introduced in 176, i.e.*

$$\langle \mathcal{P}_\Lambda, \mathcal{P}_\Omega \rangle = 0 \ \textit{if}\ \Lambda \neq \Omega$$

and

$$\langle \mathcal{P}_\Lambda, \mathcal{P}_\Lambda \rangle = b_\Lambda(q,t)$$

where we will give $b_\Lambda(q,t)$ explicitly in the next section.

PROOF. Both equations are direct consequences of Lemmas 182 and 181. □

We are now in a position to state the second characterization of the bisymmetric Macdonald polynomials.

PROPOSITION 185. *The bisymmetric Macdonald polynomials are the unique family of bisymmetric polynomials which satisfy*

(1) the decomposition over the monomials is triangular

$$\mathcal{P}_\Lambda = m_\Lambda + \sum_{\Omega < \Lambda} c_{\Omega,\Lambda} m_\Omega$$

where $<$ is the dominance order in superpartitions.

(2)
$$\langle \mathcal{P}_\Lambda, \mathcal{P}_\Omega \rangle = 0 \text{ if } \Lambda \neq \Omega.$$

This provides us with a characterization of the bisymmetric Macdonald polynomials which does not depend on the number of variables.

THEOREM 186. ***[Symmetry]*** *The bisymmetric Macdonald polynomials $\mathcal{P}_\Lambda$ satisfy the following symmetry*
$$q^{\binom{m}{2}-|\Lambda^a|}\mathcal{P}_\Lambda(qx_1,\ldots,qx_m,x_{m+1},\ldots,x_N;q,t) = \mathcal{P}_\lambda(x_1,\ldots,x_N;q^{-1},t^{-1})$$

PROOF. Using Theorem 118 with $a = (\Lambda^a_m,\ldots,\Lambda^a_1)$, we have
$$q^{|\Lambda^a|}t^{\binom{m}{2}}P_\Lambda(x;q^{-1},t^{-1}) = t^{\binom{m}{2}}\tau_1\ldots\tau_m K_{\omega_m}\overline{T}_{\omega_m}P_\Lambda(x;q,t),$$
Taking $\mathcal{A}_m$ and dividing by $\Delta^{1/t}_m(x)$ we then obtain
$$q^{|\Lambda^a|}\mathcal{P}_\Lambda(x;q^{-1},t^{-1}) = q^{\binom{m}{2}}\tau_1\ldots\tau_m\frac{1}{\Delta^{1/t}_m(x)}\mathcal{A}_m K_{\omega_m}\overline{T}_{\omega_m}P_\Lambda(x;q,t).$$
But, $\mathcal{A}_m K_{\omega_m} = (-1)^{\binom{m}{2}}\mathcal{A}_m$, $\mathcal{A}_m\overline{T}_{\omega_m} = (-1)^{\binom{m}{2}}\mathcal{A}_m$ and $K_{\omega_m}\Delta^{1/t}(x) = (-t)^{\binom{m}{2}}\Delta^t_m(x)$, which gives
$$\mathcal{P}_\Lambda(x;q^{-1},t^{-1}) = q^{\binom{m}{2}-|\Lambda^a|}\tau_1\ldots\tau_m\frac{1}{\Delta^{1/t}_m(x)}\mathcal{A}_m P_\Lambda(x;q,t),$$
as we wanted. ∎

CHAPTER 6

Self-duality and Pieri rules for the bisymmetric Macdonald polynomials

In this chapter, we will focus on demonstrating symmetry, explicitly finding the Macdonald operator as a combination of q-difference operators, and proving Pieri rules for bisymmetric Macdonald polynomials. Particularly, the task of explicitly finding the expansion of the operator $e_r(Y_1,\dots,Y_m)$ and $e_r(Y_{m+1},\dots,Y_N)$ naturally led us to express the kernel $\mathcal{NF}_m$ defined in Chapter 4 as a product over areas in $\mathbb{Z}\times\mathbb{Z}$ that, under the action of Hecke operators, are modified by adding points. Although this method was quite technical, it allows us to find the desired expansion and opens the possibility of tackling similar problems using these tools.

1. Evaluations and symmetry

The element w of the symmetric group $\mathfrak{S}_N$ acts on a vector $(v_1,\dots,v_N)\in\mathbb{Z}^N$ as $w(v_1,\dots,v_N)=(v_{w^{-1}(1)},\dots,v_{w^{-1}(N)})$.

Let w be the minimal length permutation such that $w\Lambda=\Lambda^*$. The positive evaluation u_Λ^+ is defined on any $f(x)\in\mathscr{R}_{m,N}$ as

$$u_\Lambda^+(f(x_1,\dots,x_N))=f(q^{\Lambda^{\circledast}_{w(1)}}t^{1-w(1)},\dots,q^{\Lambda^{\circledast}_{w(N)}}t^{1-w(N)}) \tag{1.1}$$

while the negative evaluation u_Λ^- is defined as

$$u_\Lambda^-(f(x_1,\dots,x_N))=f(q^{-\Lambda^{*}_{w(1)}}t^{w(1)-1},\dots,q^{-\Lambda^{*}_{w(N)}}t^{w(N)-1}) \tag{1.2}$$

Remark 187. *In the case of $\Lambda_0=(\delta_m;\emptyset)$, where $\delta_m=(m-1,m-2,\dots,0)$, the negative evaluation corresponds to an evaluation considered in* [**7, 15**]. *To be more precise, if the symmetric function in superspace is $F(x,\theta)$ as given in* (141)*, then $u_{\Lambda_0}^-(f_{\{1,\dots,m\}}(x))=\varepsilon^m_{t^{N-m},q,t}\big(F(x,\theta)\big)$ in the language of* [**15**].

It turns out that we can use other permutations than the one of minimal length when taking the evaluations. We use the notation $\Lambda+(1^m)$ for the vector $(\Lambda_1+1,\dots,\Lambda_m+1,\Lambda_{m+1},\dots,\Lambda_N)$.

Lemma 188. *Let σ be any permutation such that $\sigma\Lambda=\Lambda^*$ and $\sigma(\Lambda+(1^m))=\Lambda^{\circledast}$. Then, when computing $u_\Lambda^+(f)$ and $u_\Lambda^-(f)$ for a bisymmetric function f, the permutation σ can be used in* (1.1) *and* (1.2) *instead of the minimal permutation w. That is, when computing $u_\Lambda^+(f)$ and $u_\Lambda^-(f)$ for a bisymmetric function f, we have in this case that*

$$u_\Lambda^+(f(x_1,\dots,x_N))=f(q^{\Lambda^{\circledast}_{\sigma(1)}}t^{1-\sigma(1)},\dots,q^{\Lambda^{\circledast}_{\sigma(N)}}t^{1-\sigma(N)})$$

and

$$u_\Lambda^-(f(x_1,\dots,x_N))=f(q^{-\Lambda^{*}_{\sigma(1)}}t^{\sigma(1)-1},\dots,q^{-\Lambda^{*}_{\sigma(N)}}t^{\sigma(N)-1})$$

Proof. We first prove that if w is the minimal permutation such that $w\Lambda=\Lambda^*$ then $w(\Lambda+(1^m))=\Lambda^{\circledast}$. Suppose that w is such a minimal permutation and suppose that $\Lambda_i=\Lambda_j$ with $i\le m$ and $j>m$. Then, by minimality, $w^{-1}(i)<w^{-1}(j)$ which means that Λ_i+1 occurs to the left of Λ_j in $w(\Lambda+(1^m))$. We then deduce immediately that $w(\Lambda+(1^m))=\Lambda^{\circledast}$.

Now, let σ be such that $\sigma\Lambda = \Lambda^*$ and $\sigma(\Lambda + (1^m)) = \Lambda^{\circledast}$. As we have just seen, the minimal permutation w is also such that $w\Lambda = \Lambda^*$ and $w(\Lambda + (1^m)) = \Lambda^{\circledast}$. Hence, $w^{-1}\sigma$ acts as the identity on Λ and $\Lambda + (1^m)$, which means in particular that $w^{-1}\sigma \in \mathfrak{S}_m \times \mathfrak{S}_{m+1,N}$. Since f is bisymmetric, we thus have $u_\Lambda^+(K_{w^{-1}\sigma} f) = u_\Lambda^+(f)$. Hence

$$u_\Lambda^+(f(x_{w^{-1}\sigma(1)}, \dots, x_{w^{-1}\sigma(N)})) = f(q^{\Lambda^{\circledast}_{\sigma(1)}} t^{1-\sigma(1)}, \dots, q^{\Lambda^{\circledast}_{\sigma(N)}} t^{1-\sigma(N)})$$

which is equivalent to (1.1) after performing the transformation $i \mapsto \sigma^{-1}w(i)$. The proof in the case of u_Λ^- is identical. □

We will say that (Λ, σ) **generates a superevaluation** whenever Λ is a superpartition such that

(1) $\sigma\Lambda = \Lambda^*$
(2) $\sigma(\Lambda + (1^m)) = \Lambda^{\circledast}$

Lemma 189. *If f is a bisymmetric function then*

$$f(Y^{-1})\Delta_m^t(x)\mathcal{P}_\Lambda(x;q,t) = u_\Lambda^-(f)\Delta_m^t(x)\mathcal{P}_\Lambda(x;q,t) \tag{1.3}$$

Moreover, if $g(x_{m+1}, \dots, x_N)$ is symmetric in the variables $x_{m+1}, \dots, x_N$ then

$$g(Y_{m+1}, \dots, Y_N)\Delta_m^t(x)\mathcal{P}_\Lambda(x;q,t) = u_\Lambda^+(g)\Delta_m^t(x)\mathcal{P}_\Lambda(x;q,t) \tag{1.4}$$

while if $g(x_1, \dots, x_m)$ is symmetric in the variables $x_1, \dots, x_m$ and of homogeneous degree d then

$$g(Y_1, \dots, Y_m)\Delta_m^t(x)\mathcal{P}_\Lambda(x;q,t) = q^{-d}u_\Lambda^+(g)\Delta_m^t(x)\mathcal{P}_\Lambda(x;q,t) \tag{1.5}$$

Proof. We first prove (1.3). We have that $Y_i^{-1}E_\eta = \bar{\eta}_i^{-1}E_\eta$, where we recall that $\bar{\eta}_i = q^{\eta_i}t^{1-r_\eta(i)}$. Using the fact that f is bisymmetric, we then have

$$\begin{aligned} f(Y_1^{-1}, \dots, Y_N^{-1})\Delta_m^t(x)\mathcal{P}_\Lambda(x;q,t) &= f(Y_1^{-1}, \dots, Y_N^{-1})c_\Lambda(t)\mathcal{A}_{1,m}^t\mathcal{S}_{m+1,,N}^t E_\eta \\ &= c_\Lambda(t)\mathcal{A}_{1,m}^t\mathcal{S}_{m+1,,N}^t f(Y_1^{-1}, \dots, Y_N^{-1})E_\eta \\ &= c_\Lambda(t)\mathcal{A}_{1,m}^t\mathcal{S}_{m+1,,N}^t f(\bar{\eta}_1^{-1}, \dots, \bar{\eta}_N^{-1})E_\eta \\ &= f(\bar{\eta}_1^{-1}, \dots, \bar{\eta}_N^{-1})\Delta_m^t(x)\mathcal{P}_\Lambda(x;q,t) \end{aligned}$$

It thus only remains to show that the specialization $x_i = \bar{\eta}_i^{-1}$ corresponds to the negative evaluation. We have that $r_\eta(i) = w(i)$, where w is the minimal permutation such that $w\eta = \Lambda^*$. Therefore, $x_i = \bar{\eta}_i^{-1} = q^{-\eta_i}t^{w(i)-1}$. By definition, we also have that $\Lambda_i^* = \eta_{w^{-1}(i)}$ or equivalently, that $\Lambda_{w(i)}^* = \eta_i$. Hence, the specialization $x_i = \bar{\eta}_i^{-1}$ amounts to $x_i = q^{-\Lambda^*_{w(i)}}t^{w(i)-1}$, as wanted.

As for the proof of (1.4) and (1.5), observe that $w^{-1}\Lambda^{\circledast} = \Lambda + (1^m)$ and $w^{-1}\Lambda^* = \Lambda$ imply that $\Lambda^{\circledast}_{w(i)} = \Lambda^*_{w(i)}$ for $i \in \{m+1, \dots, N\}$ while $\Lambda^{\circledast}_{w(i)} = \Lambda^*_{w(i)} + 1$ for $i \in \{1, \dots, m\}$. Proceeding as in the proof of (1.3), we get straightforwardly that the specialization is at $x_i = \bar{\eta}_i$ instead of $x_i = \bar{\eta}_i^{-1}$. We can then immediately deduce that (1.4) and (1.5) hold from our previous observation. □

1.1. The double affine Hecke algebra and a symmetric pairing. We will introduce in this subsection a symmetric pairing associated to the evaluation $u_{\Lambda_0}^-$ that generalizes the symmetric pairing in the double affine Hecke algebra. We first explain the symmetric pairing in the double affine Hecke algebra.

The double affine Hecke algebra has a natural basis (over $\mathbb{Q}(q,t)$) given by the elements of the form (see for instance (4.7.5) in [**22**])

$$x^\eta T_w Y^{-\gamma} \tag{1.6}$$

for all $\eta, \gamma \in \mathbb{Z}^N$ and all permutations $w \in \mathfrak{S}_N$. The map φ defined by [**22**]

$$\varphi\left(x^\eta T_w Y^{-\gamma}\right) = x^\gamma T_{w^{-1}} Y^{-\eta} \tag{1.7}$$

is an anti-automorphism. Notice that we have in particular that

$$\varphi(x^\eta) = Y^{-\eta} \qquad \text{and} \qquad \varphi(Y^{-\gamma}) = x^\gamma$$

The evaluation map Θ is then defined as

$$\Theta(a) = u_\emptyset^-(a \cdot 1) \tag{1.8}$$

where $u_\emptyset^-(f(x_1, \ldots, x_N) = f(1, t^1, \ldots, t^{N-1})$ for any Laurent polynomials $f(x)$. For instance, using $f(Y^{-1}) \cdot 1 = u_\emptyset^-(f)$ and $T_w \cdot 1 = t^{\ell(w)}$, we have

$$\begin{aligned}\Theta\big(g(x)T_w f(Y^{-1})\big) &= u_\emptyset^-\big(g(x)T_w f(Y^{-1}) \cdot 1\big) \\ &= u_\emptyset^-(f)u_\emptyset^-\big(g(x)T_w \cdot 1\big) \\ &= t^{\ell(w)}u_\emptyset^-(f)u_\emptyset^-\big(g\big)\end{aligned}$$

Using the fact that $\ell(w) = \ell(w^{-1})$, we see that

$$\Theta\big(g(x)T_w f(Y^{-1})\big) = \Theta\big(f(x)T_{w^{-1}}g(Y^{-1})\big) = \Theta \circ \varphi\big(g(x)T_w f(Y^{-1})\big)$$

Since the basis (1.6) is given by elements of the form $g(x)T_w f(Y^{-1})$, we thus have established that

$$\Theta = \Theta \circ \varphi \tag{1.9}$$

For all Laurent polynomials $f(x), g(x)$, let the pairing $[f, g]$ be defined as

$$[f, g] = \Theta\big(f(Y^{-1})g(x)\big) \tag{1.10}$$

Using the previous relation and the fact that φ is an anti-automorphism, we immediately get the symmetry of the pairing

$$[f, g] = \Theta\big(f(Y^{-1})g(x)\big) = \Theta \circ \varphi\big(f(Y^{-1})g(x)\big) = \Theta\big(g(Y^{-1})f(x)\big) = [g, f]$$

Now, consider the evaluation $u_{\Lambda_0}^-(f)$ given explicitly as

$$u_{\Lambda_0}^-(f(x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_N)) = f(q^{1-m}, q^{2-m}t, \ldots, q^0 t^{m-1}, t^m, \ldots, t^{N-1})$$

on any Laurent polynomial f, where Λ_0 is such as defined in Remark 187.

Our goal is to define a pairing associated to the evaluation $u_{\Lambda_0}^-$. We first need to extend the map Θ. Let Θ_m be such that

$$\Theta_m(a) = u_{\Lambda_0}^-\big(a \cdot E_{\delta_m}(x; q, t)\big) \tag{1.11}$$

where $E_{\delta_m}(x; q, t)$ is the non-symmetric Macdonald polynomial indexed by the composition

$$\delta_m = (m-1, m-2, \ldots, 1, 0, 0, \ldots, 0)$$

Observe that $f(Y^{-1}) \cdot E_{\delta_m}(x; q, t) = u_{\Lambda_0}^-(f)E_{\delta_m}(x; q, t)$.

LEMMA 190. *Let φ be the anti-automorphism defined in* (1.7). *We have that*

$$\Theta_m = \Theta_m \circ \varphi \tag{1.12}$$

PROOF. We first show that

$$\Theta_m(T_w) = \Theta_m \circ \varphi(T_w) = \Theta_m(T_{w^{-1}}) \tag{1.13}$$

for any permutation $w \in \mathfrak{S}_N$. Let $F_w(x) = T_w E_{\delta_m}(x; q, t) \cdot 1$. As this is a polynomial in $x_1, \ldots, x_N$, we can consider the quantity

$$\Theta\big(F_w(Y^{-1})E_{\delta_m}(x; q, t)\big) = u_{\Lambda_0}^-(F_w)u_\emptyset^-(E_{\delta_m}) = \Theta_m(T_w)u_\emptyset^-(E_{\delta_m}) \tag{1.14}$$

since $u_{\Lambda_0}^-(F_w) = u_{\Lambda_0}^-(T_w \cdot E_{\delta_m}(x; q, t)) = \Theta_m(T_w)$. From $\Theta = \Theta \circ \varphi$, we also have

$$\Theta\big(F_w(Y^{-1})E_{\delta_m}(x; q, t)\big) = \Theta\big(E_{\delta_m}(Y^{-1}; q, t)F_w(x)\big) = \Theta\big(E_{\delta_m}(Y^{-1}; q, t)T_w E_{\delta_m}(x)\big)$$

Using again $\Theta = \Theta \circ \varphi$, we can then replace w by w^{-1} in the term on the right to get

$$\Theta\big(F_w(Y^{-1})E_{\delta_m}(x; q, t)\big) = \Theta\big(E_{\delta_m}(Y^{-1}; q, t)T_{w^{-1}}E_{\delta_m}(x)\big) = \Theta\big(E_{\delta_m}(Y^{-1}; q, t)F_{w^{-1}}(x)\big)$$

Using $\Theta = \Theta \circ \varphi$ one last time, we can transform the term on the right to obtain

$$\Theta\big(F_w(Y^{-1})E_{\delta_m}(x;q,t)\big) = \Theta\big(F_{w^{-1}}(Y^{-1})E_{\delta_m}(x;q,t)\big) = \Theta_m(T_{w^{-1}})u^-_{\emptyset}(E_{\delta_m}) \tag{1.15}$$

Comparing (1.14) and (1.15), we can thus conclude that (1.13) holds given that $u^-_{\Lambda_0}(E_{\delta_m})$ is not equal to 0 (it can be deduced easily from the fact that $E_\eta(x;q,1) = x^\eta$ for any η).

We can now prove that $\Theta_m = \Theta_m \circ \varphi$ holds in general. Recall that any element of the affine Hecke algebra can be written in the form $f(x)T_w g(Y^{-1})$, where $f(x), g(x)$ are Laurent polynomials. On the one hand, we have

$$\Theta_m\big(f(x)T_w g(Y^{-1})\big) = u^-_{\Lambda_0}(g)\Theta_m\big(f(x)T_w\big) = u^-_{\Lambda_0}(g)u^-_{\Lambda_0}(f)\Theta_m(T_w)$$

while on the other hand, we have

$$\Theta_m \circ \varphi\big(f(x)T_w g(Y^{-1})\big) = \Theta_m\big(g(x)T_{w^{-1}}f(Y^{-1})\big) = u^-_{\Lambda_0}(f)u^-_{\Lambda_0}(g)\Theta_m(T_{w^{-1}})$$

Since we have previously established that $\Theta_m(T_{w^{-1}}) = \Theta_m(T_w)$, we conclude from the previous two equations that $\Theta_m = \Theta_m \circ \varphi$. $\square$

We now define our new pairing. For any Laurent polynomials f, g symmetric in the variables $x_1, \ldots, x_m$, let

$$[f,g]_m = u^-_{\Lambda_0}\big(f(Y^{-1})g(x)\Delta^t_m(x)\big) \tag{1.16}$$

This new pairing is again symmetric.

Lemma 191. *If f and g are two Laurent polynomials that are symmetric in the variables $x_1, \ldots, x_m$, then*

$$[f,g]_m = [g,f]_m$$

Proof. Let $\mathcal{A}^t_m$ be the t-antisymmetrizer in the first m variables

$$\mathcal{A}^t_m = \sum_{\sigma \in \mathfrak{S}_m} (-1/t)^{\ell(\sigma)} T_\sigma \tag{1.17}$$

We have that $T_i \mathcal{A}^t_m = -\mathcal{A}^t_m$ for any $i = 1, \ldots, m-1$. Hence, for every polynomial $f(x)$, we get that $\mathcal{A}^t_m f(x) = \Delta^t_m(x) g(x)$, where $g(x)$ is a polynomial symmetric in $x_1, \ldots, x_m$. In particular, by degree consideration, we have that

$$\Delta^t_m(x) = c_m(q,t)\mathcal{A}^t_m E_{\delta_m}(x;q,t)$$

for some non-zero constant $c_m(q,t)$ (at $t=1$, the r.h.s. produces the usual Vandermonde determinant, so $c_m(q,1) = 1 \neq 0$). It is also immediate that $\mathcal{A}^t_m\mathcal{A}^t_m = d_m(t)\mathcal{A}^t_m$ where $d_m(t) = \sum_{\sigma \in \mathfrak{S}_m} (1/t)^{\ell(\sigma)}$ is a non-zero constant. With these relations in hand, we can relate $[f,g]_m$ to Θ_m. Indeed, we have

$$\begin{aligned}
\Theta_m(\mathcal{A}^t_m f(Y^{-1})g(x)\mathcal{A}^t_m) &= d_m(t)\,\Theta_m(f(Y^{-1})g(x)\mathcal{A}^t_m) \\
&= d_m(t)u^-_{\Lambda_0}\big(f(Y^{-1})g(x)\mathcal{A}^t_m \cdot E_{\delta_m}(x;q,t)\big) \\
&= d_m(t)c_m(q,t)u^-_{\Lambda_0}\big(f(Y^{-1})g(x) \cdot \Delta^t_m(x)\big) \\
&= d_m(t)c_m(q,t)[f,g]_m
\end{aligned} \tag{1.18}$$

where, in the fist equality, we used the fact that $\mathcal{A}^t_m$ commutes with $f(Y^{-1})$ and $g(x)$ because they are both symmetric in the first m variables. We now use $\Theta_m = \Theta_m \circ \varphi$ and $\varphi(\mathcal{A}^t_m) = \mathcal{A}^t_m$ to interchange f and g:

$$\Theta_m\big(\mathcal{A}^t_m f(Y^{-1})g(x)\mathcal{A}^t_m\big) = \Theta_m \circ \varphi\big(\mathcal{A}^t_m f(Y^{-1})g(x)\mathcal{A}^t_m\big) = \Theta_m\big(\mathcal{A}^t_m g(Y^{-1})f(x)\mathcal{A}^t_m\big)$$

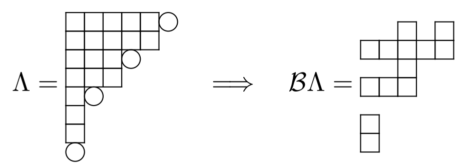
The symmetry $[f,g]_m = [g,f]_m$ then immediately follows from (1.18). $\square$

1.2. Symmetry of the bisymmetric Macdonald polynomials. We can now extend a well-known result on Macdonald polynomials to the bisymmetric case. But first, we need to give the explicit expressions for $u^-_{\Lambda_0}(\mathcal{P}_\Lambda(x,q,t))$ and $u^+_{\Lambda_0}(\mathcal{P}_\Lambda(x,q,t))$ that were obtained in [**15**].

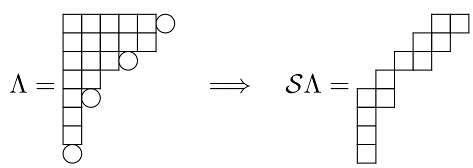
For a box $s=(i,j)$ in a partition λ (i.e., in row i and column j), we introduce the usual arm-lengths and leg-lengths:

$$a_\lambda(s)=\lambda_i-j \quad \text{and} \quad l_\lambda(s)=\lambda'_j-i \tag{1.19}$$

where we recall that λ' stands for the conjugate of the partition λ. Let $\mathcal{B}(\Lambda)$ denote the set of boxes in the diagram of Λ that do not appear at the same time in a row containing a circle and in a column containing a circle.



For Λ a superpartition of fermionic degree m, let $\mathcal{S}\Lambda$ be the skew diagrams $\Lambda^{\circledast}/\delta_{m+1}$.



Finally, for a partition λ, let $n(\lambda)=\sum_i(i-1)\lambda_i$. In the case of a skew partition λ/μ, $n(\lambda/\mu)$ stands for $n(\lambda)-n(\mu)$.

The following theorem was proved in [**15**] for Macdonald polynomials in superspace (in fact, only (1.20) was proved therein. But using (1.24), one can immediately deduce (1.21)). From Remark 187, it also applies to bisymmetric Macdonald polynomials.

Theorem 192. *Let Λ be of fermionic degree m. Then the evaluation formulas for the bisymmetric Macdonald polynomials read*

$$u^-_{\Lambda_0}(\mathcal{P}_\Lambda)=\frac{t^{n(\mathcal{S}\Lambda)+n((\Lambda')^a/\delta_m)}}{q^{(m-1)|\Lambda^a/\delta_m|-n(\Lambda^a/\delta_m)}}\frac{\prod_{(i,j)\in\mathcal{S}\Lambda}(1-q^{j-1}t^{N-(i-1)})}{\prod_{s\in\mathcal{B}\Lambda}(1-q^{a_{\Lambda^{\circledast}}(s)}t^{l_{\Lambda^*}(s)+1})} \tag{1.20}$$

and

$$u^+_{\Lambda_0}(\mathcal{P}_\Lambda)=\frac{q^{m|\Lambda^a/\delta_m|-n(\Lambda^a/\delta_m)}}{t^{n(\mathcal{S}\Lambda)+n((\Lambda')^a/\delta_m)}}\frac{\prod_{(i,j)\in\mathcal{S}\Lambda}(1-q^{1-j}t^{i-(N+1)})}{\prod_{s\in\mathcal{B}\Lambda}(1-q^{-a_{\Lambda^{\circledast}}(s)}t^{-l_{\Lambda^*}(s)-1})} \tag{1.21}$$

We can now state the two symmetries satisfied by the bisymmetric Macdonald polynomials.

Theorem 193. *Let $\tilde{\mathcal{P}}^-_\Lambda(x,q,t)$ and $\tilde{\mathcal{P}}^+_\Lambda(x,q,t)$ be the two normalizations of the bisymmetric Macdonald polynomials:*

$$\tilde{\mathcal{P}}^-_\Lambda(x,q,t)=\frac{\mathcal{P}_\Lambda(x;q,t)}{u^-_{\Lambda_0}(\mathcal{P}_\Lambda(x,q,t))} \qquad \text{and} \qquad \tilde{\mathcal{P}}^+_\Lambda(x,q,t)=\frac{\mathcal{P}_\Lambda(x;q,t)}{u^+_{\Lambda_0}(\mathcal{P}_\Lambda(x,q,t))}$$

where we recall that Λ_0 was defined in Remark 187. Then, the following two symmetries hold:

$$u^-_\Omega(\tilde{\mathcal{P}}^-_\Lambda)=u^-_\Lambda(\tilde{\mathcal{P}}^-_\Omega) \qquad \text{and} \qquad u^+_\Omega(\tilde{\mathcal{P}}^+_\Lambda)=u^+_\Lambda(\tilde{\mathcal{P}}^+_\Omega)$$

Proof. We first prove the symmetry involving the negative evaluation. From the definition of the pairing $[\cdot,\cdot]$ and from Lemma 189, we get that

$$\begin{aligned}[\mathcal{P}_\Lambda(x,q,t),\mathcal{P}_\Omega(x,q,t)]_m &= u^-_{\Lambda_0}\big(\mathcal{P}_\Lambda(Y_i^{-1})\Delta^t_m(x)\mathcal{P}_\Omega(x,q,t)\big)\\ &= u^-_\Omega(\mathcal{P}_\Lambda(x,q,t))u^-_{\Lambda_0}\big(\Delta^t_m(x)\mathcal{P}_\Omega(x,q,t)\big)\end{aligned} \tag{1.22}$$

Using Lemma 191, we then have

$$u_\Omega^-(\mathcal{P}_\Lambda(x,q,t))u_{\Lambda_0}^-\big(\Delta_m^t(x)\mathcal{P}_\Omega(x,q,t)\big) = u_\Lambda^-(\mathcal{P}_\Omega(x,q,t))u_{\Lambda_0}^-\big(\Delta_m^t(x)\mathcal{P}_\Lambda(x,q,t)\big)$$

and the first symmetry follows immediately.

We will now deduce the symmetry involving the positive evaluation from the negative one. As we will see, it essentially follows from Equation (4.6) in [**30**] which, when rewritten in our language, says that

$$\Delta_m(qx_1,\ldots,qx_m)\mathcal{P}_\Lambda(qx_1,\ldots,qx_m,x_{m+1},\ldots,x_N;q,t) = q^{|\Lambda^a|}\Delta_m(x)\mathcal{P}_\Lambda(x;1/q,1/t)$$

Simplifying the previous equation as

$$q^{m(m-1)/2-|\Lambda^a|}\mathcal{P}_\Lambda(qx_1,\ldots,qx_m,x_{m+1},\ldots,x_N;q,t) = \mathcal{P}_\Lambda(x;1/q,1/t) \tag{1.23}$$

we deduce that

$$\begin{aligned}
&\big[u_\Omega^-(\mathcal{P}_\Lambda(x;q,t))\big]_{(q,t)\mapsto(1/q,1/t)}\\
&\quad= \mathcal{P}_\Lambda(q^{\Omega_1}t^{1-w(1)},\ldots,q^{\Omega_m}t^{1-w(m)},q^{\Omega_{m+1}}t^{1-w(m+1)},\ldots,q^{\Omega_N}t^{1-w(N)};1/q,1/t)\\
&\quad= q^{m(m-1)/2-|\Lambda^a|}\mathcal{P}_\Lambda(q^{\Omega_1+1}t^{1-w(1)},\ldots,q^{\Omega_m+1}t^{1-w(m)},q^{\Omega_{m+1}}t^{1-w(m+1)},\ldots,q^{\Omega_N}t^{1-w(N)};q,t)\\
&\quad= q^{m(m-1)/2-|\Lambda^a|}u_\Omega^+(\mathcal{P}_\Lambda)
\end{aligned} \tag{1.24}$$

It is then immediate that

$$\begin{aligned}
\big[u_\Omega^-(\mathcal{P}_\Lambda(x;q,t))\big]_{(q,t)\mapsto(1/q,1/t)} &= \left[\frac{u_\Omega^-(\mathcal{P}_\Lambda(x;q,t))}{u_{\Lambda_0}^-(\mathcal{P}_\Lambda(x,q,t))}\right]_{(q,t)\mapsto(1/q,1/t)}\\
&= \frac{u_\Omega^+(\mathcal{P}_\Lambda(x;q,t))}{u_{\Lambda_0}^+(\mathcal{P}_\Lambda(x,q,t))}\\
&= u_\Omega^+(\tilde{\mathcal{P}}_\Lambda^+)
\end{aligned}$$

from which we conclude that the second symmetry also holds. □

2. Double affine Hecke algebra relations

In this section, we establish a few results involving the Hecke algebra and the Double affine Hecke algebra. They will be needed in the next section. We start with a generalization of a known result in symmetric function theory.

LEMMA 194. *Let $J \subseteq [N]$ and $L = [N]\setminus J$. We then have*

$$\sum_{\substack{\sigma([N-r+1,N])=J\\ \sigma\in\mathfrak{S}_N}} K_\sigma\left(\prod_{1\le i<j\le N}\frac{x_i-tx_j}{x_i-x_j}\right) = a_{r,N}(t)A_{J\times L}(x,x)$$

where $r=|J|$ and

$$a_{r,N}(t) = [r]_t![N-r]_t!$$

PROOF. For convenience, we will let

$$\bar{A}_I(x) = \prod_{\substack{i,j\in I\\ i<j}}\left(\frac{x_i-tx_j}{x_i-x_j}\right)$$

We first prove the special case when $J=[r]$ and $L=[r+1,N]$. Let γ be the permutation $[r+1,\dots,N,1,\dots,r]$ (in one-line notation). In this case, we have

$$\begin{aligned}
&\sum_{\substack{\sigma([N-r+1,N])=[r]\\ \sigma\in\mathfrak{S}_N}} K_\sigma \bar{A}_N(x)\\
&\qquad = \sum_{w\in\mathfrak{S}_r}\sum_{w'\in\mathfrak{S}_{r+1,N}} K_w K_{w'} K_\gamma \bar{A}_{N-r}(x)\bar{A}_{[N-r+1,N]}(x) A_{[N-r+1,N]\times[N-r]}(x,x)\\
&\qquad = A_{[r]\times[r+1,N]}(x,x)\left(\sum_{w\in\mathfrak{S}_r} K_w\bar{A}_r(x)\right)\left(\sum_{w'\in\mathfrak{S}_{r+1,N}} K_{w'}\bar{A}_{[r+1,N]}(x)\right)
\end{aligned}$$

since w and w' leave $A_{[r]\times[r+1,N]}$ invariant. Using [?]

$$\mathcal{S}_N^t\cdot 1=\sum_{\sigma\in\mathfrak{S}_N} K_\sigma\bar{A}_N(x)=[N]_t! \tag{2.1}$$

the formula is seen to hold in that case.

As for the general case, let δ be any permutation such that $\delta([r])=J$ (and thus also such that $\delta([r+1,\dots,N])=L$). Applied on both sides of the special case that we just showed, we get

$$\sum_{\substack{\sigma([N-r+1,N])=[r]\\ \sigma\in\mathfrak{S}_N}} K_\delta K_\sigma\left(\prod_{1\le i<j\le N}\frac{x_i-tx_j}{x_i-x_j}\right)=a_{r,N}(t)K_\delta A_{[r]\times[r+1,N]}(x,x)=a_{r,N}(t)A_{J\times L}(x,x)$$

which amounts to

$$\sum_{\substack{\delta\sigma([N-r+1,N])=J\\ \delta\sigma\in\mathfrak{S}_N}} K_{\delta\sigma}\left(\prod_{1\le i<j\le N}\frac{x_i-tx_j}{x_i-x_j}\right)=a_{r,N}(t)A_{J\times L}(x,x)$$

The lemma then follows immediately. $\square$

We now show that the product $Y_{N-r+1}\cdots Y_N$ of Cherednik operators can be simplified quite significantly in certain cases.

LEMMA 195. *Let $r\le N-m$. For any bisymmetric function $f(x)$, we have that*

$$Y_{N-r+1}\cdots Y_N f(x)=t^{(2m+r+1-2N)r/2}\omega^r(\bar{T}_r\cdots\bar{T}_{m+r-1})\cdots(\bar{T}_1\cdots\bar{T}_m)f(x)$$

PROOF. We first show that

$$Y_{N-r+1}\cdots Y_N=t^{-r(r-1)/2}(\omega\bar{T}_1\cdots\bar{T}_{N-r})^r \tag{2.2}$$

The result obviously holds by definition when $r=1$. Assuming that it holds for $r-1$, we have that

$$\begin{aligned}
Y_{N-r+1}\cdots Y_N &= Y_N\cdots Y_{N-r+1}\\
&= t^{-(r-1)(r-2)/2}(\omega\bar{T}_1\cdots\bar{T}_{N-r+1})^{r-1}\big(t^{-r+1}T_{N-r+1}\cdots T_{N-1}\omega\bar{T}_1\cdots\bar{T}_{N-r}\big)
\end{aligned}$$

Making use of the relation $\bar{T}_{i-1}\omega=\omega\bar{T}_i$, we can move the term $\bar{T}_{N-r+1}$ of every product to the right to get

$$\begin{aligned}
Y_{N-r+1}\cdots Y_N &= t^{-r(r-1)/2}(\omega\bar{T}_1\cdots\bar{T}_{N-r})^{r-1}\bar{T}_{N-1}\cdots\bar{T}_{N-r+1}T_{N-r+1}\cdots T_{N-1}\omega\bar{T}_1\cdots\bar{T}_{N-r}\\
&= t^{-r(r-1)/2}(\omega\bar{T}_1\cdots\bar{T}_{N-r})^r
\end{aligned}$$

which proves (2.2) by induction.

Using $\bar{T}_{i-1}\omega=\omega\bar{T}_i$ again and again, we then get from (2.2) that

$$Y_{N-r+1}\cdots Y_N=t^{-r(r-1)/2}\omega^r(\bar{T}_r\cdots\bar{T}_{N-1})\cdots(\bar{T}_1\cdots\bar{T}_{N-r})$$

If $f(x)$ is a bisymmetric function, the rightmost $N-r-m$ terms in every product in the previous equation can be pushed to the right and made to act as $1/t$ on $f(x)$. This yields,

$$Y_{N-r+1}\cdots Y_N f(x) = t^{-r(N-r-m)-r(r-1)/2}\,\omega^r(\bar{T}_r\cdots\bar{T}_{m+r-1})\cdots(\bar{T}_1\cdots\bar{T}_m)f(x)$$

which proves the lemma. □

Lemma 196. *Let $r\le m$. For any bisymmetric function $f(x)$, we have that*

$$\mathcal{A}_m^t Y_1\cdots Y_r\Delta_m^t f(x) = (-1)^{r(m-r)}t^{r(r+1-2N)/2}\mathcal{A}_m^t(T_m\cdots T_{N-1})\cdots(T_{m-(r-1)}\cdots T_{N-r})\omega^r$$

Proof. We first note that for the same argument used in Lemma 195, we have that

$$Y_1\cdots Y_r = t^{r(r+1-2N)/2}(T_r\cdots T_{N-1})\cdots(T_1\cdots T_{N-r})\omega^r$$

Because $\mathcal{A}_m^t T_i = -\mathcal{A}_m^t$ for $i\in[m]$ the leftmost $N-m$ terms in every product in the previous equation can be pushed to the left and made to act as -1 over $\mathcal{A}_m^t$. This yields,

$$\mathcal{A}_m^t Y_1\cdots Y_r = (-1)^{r(m-r)}t^{r(r+1-2N)/2}\,(T_m\cdots T_{N-1})\cdots(T_{m-r+1}\cdots T_{N-r})\omega^r$$

which proves the lemma. □

The next result shows that $e_r(Y_1,\ldots,Y_N)$ can be recovered from $\mathcal{S}_N^t$ acting on $Y_{N-r+1}\cdots Y_N$.

Lemma 197. *For $r\le N$, we have that if $f(x)$ is a symmetric function then*

$$e_r(Y_1,\ldots,Y_N)f(x) = \frac{1}{[N-r]_t![r]_t!}\mathcal{S}_N^t Y_{N-r+1}\cdots Y_N f(x)$$

Proof. First, if $w\in\mathfrak{S}_r$ and $\sigma\in\mathfrak{S}_{r+1,N}$ then $(T_wT_\sigma)Y_{N-r+1}\cdots Y_N = Y_{N-r+1}\cdots Y_N(T_wT_\sigma)$ by (1.2). This yields

$$T_wT_\sigma Y_{N-r+1}\cdots Y_N f(x) = t^{\ell(w)+\ell(\sigma)}Y_{N-r+1}\cdots Y_N f(x)$$

given that $f(x)$ is symmetric. Hence, summing over all the elements of $\mathfrak{S}_r\times\mathfrak{S}_{r+1,N}$ in $\mathcal{S}_N^t = \sum_{\sigma\in\mathfrak{S}_N}T_\sigma$ gives a factor of $[N-r]_t![r]_t!$ from (2.1). We thus have left to prove that

$$e_r(Y_1,\ldots,Y_N)f(x) = \sum_{[\sigma^*]\in\mathfrak{S}_N/(\mathfrak{S}_r\times\mathfrak{S}_{r+1,N})} T_{\sigma^*}Y_{N-r+1}\cdots Y_N f(x)$$

where the sum is over all left-coset representatives σ^* of minimal length. Such minimal length representatives are of the form (in one-line notation) $\sigma^*=[i_1,\ldots,i_{N-r},i_{N-r+1},\ldots,i_N]$ with $i_1< i_2<\cdots<i_{N-r}$ and $i_{N-r+1}<i_{N-r+2}<\cdots<i_N$. A reduced decomposition of σ^* is then given by

$$(s_{i_N}\cdots s_{N-1})\cdots(s_{i_{N-r+1}}s_{i_{N-r+1}+1}\ldots s_{N-r}) \tag{2.3}$$

We will now see that the factor $T_{i_{N-r+1}}T_{i_{N-r+1}+1}\ldots T_{N-r}$ of T_{σ^*} changes Y_{N-r+1} into $Y_{i_{N-r+1}}$ and leaves the rest of the terms invariant. First, we use the relation $T_iY_{i+1}=tY_i\bar{T}_i$ to obtain

$$T_{N-r}Y_{N-r+1}Y_{N-r+2}\cdots Y_N f(x) = tY_{N-r}\bar{T}_{N-r}Y_{N-r+2}\cdots Y_N f(x) = Y_{N-r}Y_{N-r+2}\cdots Y_N f(x)$$

Proceeding in this way again and again, we then get that

$$T_{i_{N-r+1}}T_{i_{N-r+1}+1}\ldots T_{N-r}Y_{N-r+1}Y_{N-r+2}\cdots Y_N f(x) = Y_{i_{N-r+1}}Y_{N-r+2}\cdots Y_N f(x)$$

as wanted. By assumption, all of the remaining indices of the s_j's in (2.3) are larger than i_{N-r+1}. Hence $Y_{i_{N-r+1}}$ will not be affected by the remaining terms in T_{σ^*}. Following as we just did, it is then immediate that

$$T_{\sigma^*}Y_{N-r+1}\cdots Y_N f(x) = Y_{i_{N-r+1}}\cdots Y_{i_N}f(x)$$

Finally, summing over all σ^*, the lemma is then seen to hold. □

Lemma 198. *For $r\le N$, we have that if $f(x)$ is a symmetric function then*

$$e_r(Y_1,\ldots,Y_m)\Delta_m^t(x)f(x) = \frac{1}{[m-r]_t![r]_t!}\mathcal{A}_m^t Y_1\cdots Y_r\Delta_m^t f(x)$$

PROOF. First, if $w \in \mathfrak{S}_r$ and $\sigma \in \mathfrak{S}_{r+1,N}$ then $(T_w T_\sigma) Y_{N-r+1} \cdots Y_N = Y_{N-r+1} \cdots Y_N (T_w T_\sigma)$ by (1.2). This yields

$$T_w T_\sigma Y_{N-r+1} \cdots Y_N \Delta_m^t f(x) = (-t)^{\ell(w)+\ell(\sigma)} Y_1 \cdots Y_m \Delta_m^t f(x)$$

given that $f(x)$ is symmetric. Hence, summing over all the elements of $\mathfrak{S}_r \times \mathfrak{S}_{r+1,m}$ in $\mathcal{A}_N^t = \sum_{\sigma \in \mathfrak{S}_N} T_\sigma$ gives a factor of $[m-r]_t![r]_t!$ from (**??**). We thus have left to prove that

$$e_r(Y_1, \ldots, Y_m) \Delta_m^t f(x) = \sum_{[\sigma^*] \in \mathfrak{S}_m / (\mathfrak{S}_r \times \mathfrak{S}_{r+1,m})} T_{\sigma^*} Y_1 \cdots Y_m \Delta_m^t f(x)$$

where the sum is over all left-coset representatives σ^* of minimal length. This follow in the same way that Lemma **??**. $\square$

3. The action of $e_r(Y_{m+1}, \ldots, Y_N)$ on bisymmetric functions

In this section, we will obtain the explicit action of the operator $e_r(Y_{m+1}, \ldots, Y_N)$ on a bisymmetric function. This will then be used in the next section to deduce the Pieri rules $e_r(x_{m+1}, \ldots, x_N)$ for the bisymmetric Macdonald polynomials.

We need a notation similar to 42 for subsets of $[N] \times [N]$. If $\mathcal{A} \subseteq [N] \times [N]$, we let

$$R_{\mathcal{A}}(x,y) = \prod_{(i,j) \in \mathcal{A}} (1 - x_i y_j), \quad \Delta_{\mathcal{A}}(x,y) = \prod_{(i,j) \in \mathcal{A}} (x_i - y_j), \quad \text{and} \quad A_{\mathcal{A}}(x,y) = \prod_{(i,j) \in \mathcal{A}} \left(\frac{t x_i - y_j}{x_i - y_j} \right)$$

With this notation in hand, we define

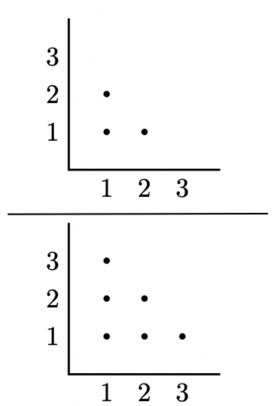
$$\mathcal{F}_m(x,y) = \frac{\Delta_m^t(x)}{R_{[m] \times [m]}(x,y)}$$

and

$$\mathcal{NF}_m(x,y) = \frac{R_{\mathcal{B}}(x,ty)}{R_{\mathcal{B}'}(x,y)}$$

where $\mathcal{B}$ is the set of integer points in the triangle with vertices $(1,1), (1, m-1)$ and $(m-1, 1)$, while $\mathcal{B}'$ is the set of integer points in the triangle with vertices $(1,1), (1,m)$ and $(m,1)$.

EXAMPLE 199. *The product $\mathcal{NF}_3(x,y)$ can be seen at the quotient of the factors stemming from the two following regions*



Hence $\mathcal{NF}_3(x,y)$ is equal to

$$\frac{(1 - t x_1 y_1)(1 - t x_1 y_2)(1 - t x_2 y_1)}{(1 - x_1 y_1)(1 - x_1 y_2)(1 - x_1 y_3)(1 - x_2 y_1)(1 - x_2 y_2)(1 - x_3 y_1)}$$

Let $\mathcal{A}_m^{(y)}$ stand for the antisymmetrizer $\mathcal{A}_m$ defined in (a) but acting on the y variables instead of the x variables. The next lemma was proven in [**8**] in another form.

Lemma 200. *We have*

$$\mathcal{A}_m^{(y)} R_{\mathcal{C}}(x,ty) R_{\mathcal{C}'}(x,y) = (-1)^{\binom{m}{2}} \Delta_m^t(x) \Delta_m(y)$$

where $\mathcal{C}$ is the set of integer points in the triangle with vertices $(1,1),(1,m-1)$ and $(m-1,1)$ while $\mathcal{C}'$ is the set of integer points in in the triangle with vertices $(2,m),(m,m)$ and $(m,2)$.

Proof. Equation (129) in [**8**] rewritten in our language (and with the t-power corrected) says that

$$\frac{1}{\Delta_m(y)} \mathcal{A}_m^{(y)} \left(\prod_{i+j\leq m} (1-tx_iy_j) \prod_{\substack{i+j>m+1\\ i,j\leq m}} (1-x_iy_j) \right) = (-1)^{\binom{m}{2}} \Delta_m^t(x)$$

The lemma then immediately follows. $\square$

Corollary 201. *The following identity holds:*

$$\mathcal{A}_m^{(y)} \mathcal{NF}_m(x,y) = (-1)^{\binom{m}{2}} \Delta_m(y) \mathcal{F}_m(x)$$

Proof. The identity can be deduced from Lemma 200 after completing the square in the triangle $\mathcal{B}'$ corresponding to the denominator of $\mathcal{NF}_m(x,y)$. $\square$

We now establish a few elementary relations that will be needed later on.

Lemma 202. *For all $i \in \{1,\dots,N-1\}$ and all $j \in \{1,\dots,N\}$, we have*

(1) $\bar{T}_i \dfrac{1}{(1-x_iy_j)} = \dfrac{(1-tx_iy_j)}{t(1-x_iy_j)(1-x_{i+1}y_j)}$

(2) $T_i \dfrac{1}{(1-x_{i+1}y_j)} = \dfrac{t(1-t^{-1}x_{i+1}y_j)}{(1-x_{i+1}y_j)(1-x_iy_j)}$

Proof. We only prove the first relation as the second one can be proven in the same fashion. Since $(1-x_iy_j)(1-x_{i+1}y_j)$ is symmetric in x_i and x_{i+1}, it commutes with $\bar{T}_i$. The first relation is thus equivalent to $\bar{T}_i(1-x_{i+1}y_j) = t^{-1}(1-tx_iy_j)$, which can easily be verified. $\square$

The following lemma concerns the function

$$\mathcal{NF}_m^k(x,y) = \frac{R_{\mathcal{B}_k}(x,ty)}{R_{\mathcal{B}'_k}(x,y)} \tag{3.1}$$

where $\mathcal{B}_k$ is the set of integer points in the trapezoid with vertices $(1,1),(1,m),(k,m)$ and $(m+k-1,1)$ while $\mathcal{B}'_k$ is the set of integer points in the trapezoid with vertices $(1,1),(1,m),(k+1,m)$ and $(m+k,1)$. That is, $\mathcal{B}'_k$ is the union of $\mathcal{B}_k$ with the line segment from $(k+1,m)$ to $(m+k,1)$ of slope -1. Note that $\mathcal{NF}_m^0(x,y) = \mathcal{NF}_m(x,y)$.

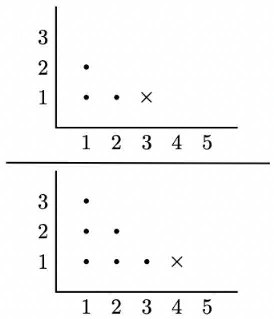
Lemma 203. *For $k\geq 1$, we have*

$$\bar{T}_k \cdots \bar{T}_{k+m-1} \mathcal{NF}_m^{k-1}(x,y) = t^{-m} \mathcal{NF}_m^k(x,y)$$

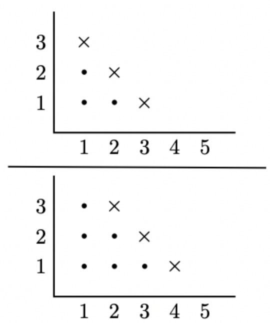
Consequently, for $r\geq 1$,

$$(\bar{T}_r \cdots \bar{T}_{r+m-1}) \cdots (\bar{T}_1 \cdots \bar{T}_m) \mathcal{NF}_m(x,y) = t^{-rm} \mathcal{NF}_m^r(x,y) \tag{3.2}$$

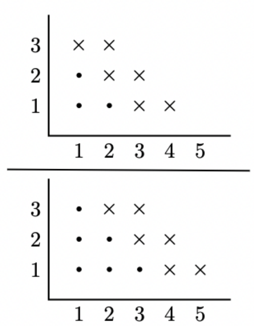
Proof. The lemma follows from Lemma 202. Diagrammatically, acting with $\bar{T}_m$ on $\mathcal{NF}_m$ amounts to adding a dot to the triangles associated to the numerator and the denominator in the diagram of $\mathcal{NF}_m$. For instance, if $m=3$ the diagram associated to $\bar{T}_3\mathcal{NF}_3$ is:



The product $\bar{T}_1 \cdots \bar{T}_m$ adds a diagonal above the triangles associated to $\mathcal{NF}_m$. For example, $\bar{T}_1\bar{T}_2\bar{T}_3\mathcal{NF}_3$ is



Finally, as r increases, extra diagonals are added. We get for instance, in the case $r = 2$ and $m = 3$, that acting with $(\bar{T}_2\bar{T}_3\bar{T}_4)(\bar{T}_1\bar{T}_2\bar{T}_3)$ on $\mathcal{NF}_3$ adds the following two diagonals:



$\square$

For the next lemma, we need to introduce some notation. Given a permutation $\sigma \in \mathfrak{S}_N$, we let

$$\mathfrak{A}_\sigma = \{i \in [1,m] \mid \sigma^{-1}(i) \in [1,m]\} \quad \text{and} \quad \mathfrak{B}_\sigma = \{i \in [m+1,N] \mid \sigma^{-1}(i) \in [m+1,N]\}$$

Their respective complements are

$$\mathfrak{A}^c_\sigma = [1,m] \setminus \mathfrak{A}_\sigma, \quad \text{and} \quad \mathfrak{B}^c_\sigma = [m+1,N] \setminus \mathfrak{B}_\sigma$$

Remark 204. *It is important to realize that if $w \in \mathfrak{S}_m \times \mathfrak{S}_{m+1,N}$ then $\mathfrak{A}_\sigma = \mathfrak{A}_{\sigma w}$ (and similarly for $\mathfrak{A}^c_\sigma$, $\mathfrak{B}_\sigma$ and $\mathfrak{B}^c_\sigma$).*

For $A \times B \subseteq [N] \times [N]$, let also $(A \times B)_< = \{(i,j) \in A \times B \,|\, i < j\}$. Finally, we define

$$\widetilde{\Delta}_{\mathcal{A}}(x,y) = \prod_{\substack{(i,j)\in\mathcal{A}\\ i\neq j}} (x_i - y_j) \quad \text{and} \quad \widetilde{A}_{\mathcal{A}}(x,y) = \prod_{\substack{(i,j)\in\mathcal{A}\\ i\neq j}} \left(\frac{tx_i - y_j}{x_i - y_j}\right)$$

LEMMA 205. *For $\sigma \in \mathfrak{S}_N$, let*

$$\Phi(\sigma) = (-1)^{D_\sigma} \left(\prod_{j\in\mathfrak{B}_\sigma^c} (t-1)x_j\right) \frac{\Delta_{\mathfrak{A}_\sigma^c}(x)\Delta_{\mathfrak{B}_\sigma^c}(x)}{\Delta_{\mathfrak{A}_\sigma^c\times\mathfrak{B}_\sigma^c}(x,x)} \tag{3.3}$$

where

$$D_\sigma = \frac{s(s-1)}{2} + \#(\mathfrak{A}_\sigma^c \times \mathfrak{A}_\sigma)_< + Z_\sigma$$

with $s = \#\mathfrak{A}_{\sigma^c}$ and $Z_\sigma = \#\{(i,j) \in [m]\times[m] \,|\, \sigma(i) > \sigma(j)\}$. We then have the following equality:

$$\Phi(\sigma) = \left(\prod_{j\in\mathfrak{B}_\sigma^c} (t-1)x_j\right) \cdot \frac{\widetilde{\Delta}_{\mathfrak{B}_\sigma^c\times\sigma([m])}(x,x)}{\Delta_{\mathfrak{A}_\sigma^c\times\sigma([m])}(x,x)} \frac{\Delta_m(x)}{K_\sigma(\Delta_m(x))} \tag{3.4}$$

Moreover, if $w \in \mathfrak{S}_m \times \mathfrak{S}_{m+1,N}$ then

$$\Phi(\sigma w) = (-1)^{Z_w}\Phi(\sigma) \tag{3.5}$$

PROOF. From Remark 204, (3.5) is easily seen to hold given that $K_w(\Delta_m(x)) = (-1)^{Z_w}\Delta_{\{w(1),\dots,w(m)\}}(x)$. We now prove (3.4). After simplifying the terms $\prod_{j\in\mathfrak{B}_\sigma^c}(t-1)x_j$ in $\Phi(\sigma)$ and in the r.h.s. of (3.4), we have left to prove that

$$(-1)^{D_\sigma} \frac{\Delta_{\mathfrak{A}_\sigma^c}(x)\Delta_{\mathfrak{B}_\sigma^c}(x)}{\Delta_{\mathfrak{A}_\sigma^c\times\mathfrak{B}_\sigma^c}(x,x)} = \frac{\widetilde{\Delta}_{\mathfrak{B}_\sigma^c\times\sigma([m])}(x,x)}{\Delta_{\mathfrak{A}_\sigma^c\times\sigma([m])}(x,x)} \frac{\Delta_m(x)}{K_\sigma(\Delta_m(x))} \tag{3.6}$$

As all the remaining products are of the form $(x_i - x_j)$, it will prove convenient to simply work with sets, taking special care of the signs that may appear. On the r.h.s. of (3.6), we have in the numerator

$$\big(\mathfrak{B}_\sigma^c \times \sigma([m])\big) \cup ([m]\times[m])_<$$

We observe that $\sigma([m]) = \mathfrak{A}_\sigma \cup \mathfrak{B}_\sigma^c$ since $\sigma^{-1}(i) \in [m] \iff i \in \sigma([m])$. Hence, the numerator on the r.h.s. of (3.6) is equal to

$$(\mathfrak{B}_\sigma^c\times\mathfrak{A}_\sigma)\cup(\mathfrak{B}_\sigma^c\times\mathfrak{B}_\sigma^c)_<\cup(\mathfrak{B}_\sigma^c\times\mathfrak{B}_\sigma^c)_>\cup(\mathfrak{A}_\sigma\times\mathfrak{A}_\sigma)_<\cup(\mathfrak{A}_\sigma\times\mathfrak{A}_\sigma^c)_<\cup(\mathfrak{A}_\sigma^c\times\mathfrak{A}_\sigma)_<\cup(\mathfrak{A}_\sigma^c\times\mathfrak{A}_\sigma^c)_< \tag{3.7}$$

Now, the denominator on the r.h.s. is equal, up to a sign $(-1)^{Z_\sigma}$, to

$$\big(\mathfrak{A}_\sigma^c \times \sigma([m])\big) \cup \big(\sigma([m]) \times \sigma([m])\big)_<$$

which is in turn equivalent to

$$(\mathfrak{A}_\sigma^c\times\mathfrak{A}_\sigma)\cup(\mathfrak{A}_\sigma^c\times\mathfrak{B}_\sigma^c)\cup(\mathfrak{A}_\sigma\times\mathfrak{A}_\sigma)_<\cup(\mathfrak{A}_\sigma\times\mathfrak{B}_\sigma^c)_<\cup(\mathfrak{B}_\sigma^c\times\mathfrak{A}_\sigma)_<\cup(\mathfrak{B}_\sigma^c\times\mathfrak{B}_\sigma^c)_< \tag{3.8}$$

It is immediate that $A\times B = (A\times B)_> \cup (A\times B)_<$ if A and B are disjoint. Moreover, $(A\times B)_> = (B\times A)_<$ (which accounts for an extra sign $(-1)^{\#(B\times A)_<}$). Hence, comparing (3.7) and (3.8), we have that $(\mathfrak{A}_\sigma^c\times\mathfrak{A}_\sigma^c)_<\cup(\mathfrak{B}_\sigma^c\times\mathfrak{B}_\sigma^c)_<$ is left on the numerator while $(\mathfrak{A}_\sigma^c\times\mathfrak{B}_\sigma^c)_< = \mathfrak{A}_\sigma^c\times\mathfrak{B}_\sigma^c$ is left on the denominator, with the extra sign being

$$(-1)^{\#(\mathfrak{B}_\sigma^c\times\mathfrak{B}_\sigma^c)_<+\#(\mathfrak{A}_\sigma\times\mathfrak{B}_\sigma^c)_<+\#(\mathfrak{A}_\sigma\times\mathfrak{A}_\sigma^c)_<}$$

Taking into account the sign $(-1)^{Z_\sigma}$ obtained earlier, we obtain

$$D_\sigma = \#(\mathfrak{B}_\sigma^c\times\mathfrak{B}_\sigma^c)_< + \#(\mathfrak{A}_\sigma\times\mathfrak{B}_\sigma^c)_< + \#(\mathfrak{A}_\sigma\times\mathfrak{A}_\sigma^c)_< + Z_\sigma$$

Observer that $\#(\mathfrak{B}^c_\sigma \times \mathfrak{B}^c_\sigma)_< = \dfrac{s(s-1)}{2}$ since $s = \#\mathfrak{A}^c_\sigma = \#\mathfrak{B}^c_\sigma$. The elements of $\mathfrak{A}_\sigma$ being all smaller than those of $\mathfrak{B}^c_\sigma$, we get

$$\#(\mathfrak{A}_\sigma \times \mathfrak{B}^c_\sigma)_< = \#\mathfrak{A}_\sigma \cdot \#\mathfrak{B}^c_\sigma = (m-s)s$$

Finally, given that the sets $\mathfrak{A}_\sigma$ are disjoint $\mathfrak{A}^c_\sigma$, we have

$$\#(\mathfrak{A}_\sigma \times \mathfrak{A}^c_\sigma)_< + \#(\mathfrak{A}^c_\sigma \times \mathfrak{A}_\sigma)_< = \#\mathfrak{A}^c_\sigma \cdot \#\mathfrak{A}_\sigma = s(m-s)$$

We thus have as wanted that

$$(-1)^{D_\sigma} = (-1)^{s(s-1)/2 + \#(\mathfrak{A}^c_\sigma \times \mathfrak{A}_\sigma)_< + Z_\sigma}$$

□

Before proving the main result of this section, we obtain a criteria to show the equivalence of two operators.

Lemma 206. *Let $\mathcal{O}$ and $\mathcal{O}'$ be any operators acting on bisymmetric functions. If for all symmetric functions $g(x)$ we have*

$$\mathcal{O}\left(\frac{g(x)}{R_{[m]\times[m]}(x,y)}\right) = \mathcal{O}'\left(\frac{g(x)}{R_{[m]\times[m]}(x,y)}\right)$$

then

$$\mathcal{O}f(x) = \mathcal{O}'f(x)$$

for all bisymmetric functions $f(x)$.

Proof. A basis of the space of bisymmetric functions is provided by products of Schur functions $\{s_\lambda(x_1,\ldots,x_m)s_\mu(x_1,\ldots,x_N)\,|\,\}_{\lambda,\mu}$ where λ and μ are partitions of length not larger than m and N respectively. It is well-known that [**20**]

$$\frac{1}{R_{[m]\times[m]}(x,y)} = \sum_{\lambda\,;\,\ell(\lambda)\le m} s_\lambda(x_1,\ldots,x_m)s_\lambda(y_1,\ldots,y_m)$$

Hence, by hypothesis,

$$0 = (\mathcal{O}-\mathcal{O}')\left(\frac{s_\mu(x)}{R_{[m]\times[m]}(x,y)}\right) = \sum_{\lambda\,;\,\ell(\lambda)\le m} (\mathcal{O}-\mathcal{O}')\big(s_\lambda(x_1,\ldots,x_m)s_\mu(x)\big)s_\lambda(y_1,\ldots,y_m)$$

Taking the coefficient of $s_\lambda(y_1,\ldots,y_m)$ in the expansion tells us that the action of $\mathcal{O}-\mathcal{O}'$ on the basis element $s_\lambda(x_1,\ldots,x_m)s_\mu(x)$ is null. We thus conclude that $\mathcal{O}$ and $\mathcal{O}'$ have the same action on the basis element $s_\lambda(x_1,\ldots,x_m)s_\mu(x)$, and thus on any bisymmetric function. □

Proposition 207. *Let $f(x)$ be any bisymmetric function. Then*

$$e_r(Y_{m+1},\ldots,Y_N)\Delta^t_m(x)f(x) = \sum_{\substack{J\subset[m+1,N]\\|J|=r}} \sum_{\substack{[\sigma]\in\mathfrak{S}_N/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})\\ \sigma([m])\cap L=\emptyset}} C_{J,\sigma}(x)\tau_J K_\sigma f(x)$$

where the coefficient $C_{J,\sigma}(x)$ is given by

$$C_{J,\sigma}(x) = t^{r(r+1-2N)/2} A_m(x) A_{J\times L}(x,x)\tau_J\big(\widetilde{A}_{J\times\sigma([m])}(x,x)\Phi(\sigma)K_\sigma(\Delta_m(x))\big).$$

with $L = [m+1,N]\setminus J$. We stress that $C_{J,\sigma}(x) = C_{J,\sigma w}(x)$ if $w\in\mathfrak{S}_m\times\mathfrak{S}_{m+1,N}$. As such, it makes sense to consider $[\sigma]\in\mathfrak{S}_N/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})$.

PROOF. From Lemma 3.4 and the relation $K_{\sigma w}(\Delta_m(x)) = (-1)^{Z_w} K_\sigma(\Delta_{\{w(1),\dots,w(m)\}}(x))$, we have immediately that $C_{J,\sigma}(x) = C_{J,\sigma w}(x)$ if $w \in \mathfrak{S}_m \times \mathfrak{S}_{m+1,N}$.

We now prove the central claim in the theorem. From Lemma 206, it suffices to show that

$$e_r(Y_{m+1},\dots,Y_N)\Delta_m^t(x)\frac{g(x)}{R_{[m]\times[m]}(x,y)}$$
$$= \sum_{\substack{J\subset[m+1,N]\\ |J|=r}} \sum_{\substack{[\sigma]\in\mathfrak{S}_N/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})\\ \sigma([m])\cap L=\emptyset}} C_{J,\sigma}\tau_J K_\sigma\left(\frac{g(x)}{R_{[m]\times[m]}(x,y)}\right)$$

for every symmetric function $g(x)$. Hence, the proposition will follow if we can prove that

$$e_r(Y_{m+1},\dots,Y_N)\mathcal{F}_m(x,y)g(x)$$
$$= \sum_{\substack{J\subset[m+1,N]\\ |J|=r}} \sum_{\substack{[\sigma]\in\mathfrak{S}_N/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})\\ \sigma([m])\cap L=\emptyset}} C_{J,\sigma}(x)\tau_J K_\sigma\left(\frac{g(x)}{R_{[m]\times[m]}(x,y)}\right) \tag{3.9}$$

for every symmetric function $g(x)$. The rest of the proof will be devoted to showing that (3.9) holds. Let $F(x,y) := e_r(Y_{m+1},\dots,Y_N)\mathcal{F}_m(x,y)g(x)$. Since $\mathcal{S}^t_{m+1,N}$ commutes with $F(x,y)$, we have by (2.1) that

$$\mathcal{S}^t_{m+1,N}\mathcal{O}(x,y) = [N-m]_t!\, F(x,y)$$

or, equivalently, that

$$F(x,y) = \frac{1}{[N-m]_t!}\mathcal{S}^t_{m+1,N} e_r(Y_{m+1},\dots,Y_N)\mathcal{F}_m(x,y)g(x) \tag{3.10}$$

Since $\mathcal{F}_m(x,y)g(x)$ is symmetric in $x_{m+1},\dots,x_N$, we can use Lemma **??** to rewrite $F(x,y)$ as

$$F(x,y) = \frac{1}{[N-m]_t![N-r-m]_t![r]_t!}\mathcal{S}^t_{m+1,N}\sum_{\sigma\in\mathfrak{S}_{m+1,N}} T_\sigma Y_{N-r+1}\cdots Y_N\mathcal{F}_m(x,y)g(x)$$

The relation $\mathcal{S}^t_{m+1,N}T_\sigma = t^{\ell(\sigma)}\mathcal{S}^t_{m+1,N}$ can then be used to get

$$F(x,y) = \frac{1}{[N-r-m]_t![r]_t!}\mathcal{S}^t_{m+1,N}Y_{N-r+1}\cdots Y_N\mathcal{F}_m(x,y)g(x)$$

It then follows by Lemma 195 that

$$F(x,y) = \frac{t^{(2m+r+1-2N)r/2}}{[N-r-m]_t![r]_t!}\mathcal{S}^t_{m+1,N}\omega^r(\overline{T}_r\cdots\overline{T}_{m+r-1})\cdots(\overline{T}_1\cdots\overline{T}_m)\mathcal{F}_m(x,y)g(x)$$

From Corollary **??**, we can use

$$\mathcal{F}_m(x,y) = \frac{(-1)^{\binom{m}{2}}}{\Delta_m(y)}\mathcal{A}_m^{(y)}\mathcal{N}\mathcal{F}_m(x,y)$$

to deduce that

$$F(x,y) = \frac{(-1)^{\binom{m}{2}}}{\Delta_m(y)}\frac{t^{(2m+r+1-2N)r/2}}{[N-r-m]_t![r]_t!}\mathcal{S}^t_{m+1,N}\omega^r\mathcal{A}_m^{(y)}(\overline{T}_r\cdots\overline{T}_{m+r-1})\cdots(\overline{T}_1\cdots\overline{T}_m)\mathcal{N}\mathcal{F}_m(x,y)g(x)$$

Since $g(x)$ commutes with all $\bar{T}_i$'s we can use Lemma **??** to get

$$F(x,y) = \frac{p(t)}{(-1)^{\binom{m}{2}}\Delta_m(y)}\mathcal{S}^t_{m+1,N}\omega^r\mathcal{A}_m^{(y)}\mathcal{N}\mathcal{F}^r_m(x,y)g(x)$$

where, for simplicity,, we have set

$$p(t) = \frac{t^{r(r+1-2N)/2}}{[N-r-m]_t![r]_t!}$$

Now, we will multiply and divide the quantity $\mathcal{NF}^r_m(x,y)$ by $R_{\mathcal{M}}(x,y)$, where $\mathcal{M}$ is the triangle with vertices $\{(r+2,m),(m+r,m),(m+r,2)\}$. This way, the denominator $R_{\mathcal{B}'_r}$ becomes a rectangle and we have

$$F(x,y)=\frac{p(t)}{(-1)^{\binom{m}{2}}\Delta_m(y)}\mathcal{S}^t_{m+1,N}\omega^r\mathcal{A}^{(y)}_m\frac{R_{\mathcal{B}_r}(x,ty)R_{\mathcal{M}}(x,y)}{R_{[m+r]\times[m]}(x,y)}g(x)$$

Observe that the rectangle $[r]\times[m]\subseteq\mathcal{B}_r$ is such that $R_{[r]\times[m]}(x,ty)$ commutes with $\mathcal{A}^{(y)}_m$. It is also obvious that $R_{[m+r]\times[m]}$ commutes with $\mathcal{A}^{(y)}_m$. Hence

$$F(x,y)=\frac{p(t)}{(-1)^{\binom{m}{2}}\Delta_m(y)}\mathcal{S}^t_{m+1,N}\omega^r\frac{R_{[r]\times[m]}(x,ty)}{R_{[m+r]\times[m]}(x,y)}\mathcal{A}^{(y)}_m R_{\mathcal{B}_r\setminus([r]\times[m])}(x,ty)R_{\mathcal{M}}(x,y)g(x)$$

But $\mathcal{B}_r\setminus([r]\times[m])$ is the triangle with vertices $(r+1,m-1),(r+1,1)$, and $(m+r-1,1)$. We can thus use Proposition **??** to get

$$F(x,y)=p(t)\mathcal{S}^t_{m+1,N}\omega^r\frac{R_{[r]\times[m]}(x,ty)}{R_{[m+r]\times[m]}(x,y)}\Delta^t_{\{r+1,\ldots,m+r\}}(x)g(x)$$

It will prove convenient to multiply and divide by $R_{[m+r+1,N]\times[m]}(x,y)$ so that $R_{[N]\times[m]}(x,y)$ appears in the denominator. This yields

$$F(x,y)=p(t)\mathcal{S}^t_{m+1,N}\omega^r\frac{R_{[r]\times[m]}(x,ty)R_{[m+r+1,N]\times[m]}(x,y)}{R_{[N]\times[m]}(x,y)}\Delta^t_{\{r+1,\ldots,m+r\}}(x)g(x)$$

Applying ω^r (which amounts to the permutation that maps $j\mapsto j-r$ modulo N followed by $\tau_{N-r+1,N}=\tau_{N-r+1}\tau_{N-r+2}\cdots\tau_N$) we obtain that

$$F(x,y)=p(t)\mathcal{S}^t_{m+1,N}\tau_{N-r+1,N}\frac{R_{[N-r+1,N]\times[m]}(x,ty)R_{[m+1,N-r]\times[m]}(x,y)}{R_{[N]\times[m]}(x,y)}\Delta^t_m(x)g(x)\qquad(3.11)$$

We should stress at this point that $F(x,y)/\Delta^t_m(x)$ is both symmetric in $x_1,\ldots,x_m$ and in $x_{m+1},\ldots,x_N$. This is because applying $\mathcal{S}^t_{m+1,N}$ ensures that the result is symmetric in $x_{m+1},\ldots,x_N$ while the symmetry in $x_1,\ldots,x_m$ is straightforward given that $\Delta^t_m(x)$ commutes with $\mathcal{S}^t_{m+1,N}\tau_{N-r+1,N}$.

Now, we need to use the expansion

$$\mathcal{S}^t_{m+1,N}=\left(\sum_{\sigma\in\mathfrak{S}_{m+1,N}}K_\sigma\right)\left(\prod_{m+1\le i<j\le N}\frac{x_i-tx_j}{x_i-x_j}\right)\qquad(3.12)$$

in (3.11). From the symmetry of $F(x,y)$, it will suffice to focus on the term $\tau_{N-r+1,N}$ as the remaining terms τ_J for $J\subseteq[m+1,N]$ and $|J|=r$ will be obtained by symmetry (only those terms can occur since $\mathcal{S}^t_{m+1,N}$ only contains K_σ's such that $\sigma\in\mathfrak{S}_{m+1,N}$). For simplicity, we will let $J_0=[N-r+1,N]$ and $L_0=[m+1,N-r]$. When we only focus on the term $\tau_{J_0}=\tau_{N-r+1,N}$, we need to sum over the σ's in (3.12) such that $\sigma(J_0)=J_0$. Observe that those permutations leave the expression to the right of $\mathcal{S}^t_{m+1,N}$ invariant in (3.11). Using Lemma 194 (in the case $J=[N-r+1,N]$ and with $[1,N]$ replaced by $[m+1,N-r]$) to obtain

$$\sum_{\substack{\sigma\in\mathfrak{S}_{m+1,N}\\ \sigma(J_0)=J_0}}K_\sigma\left(\prod_{m+1\le i<j\le N}\frac{x_i-tx_j}{x_i-x_j}\right)=[N-m-r]_t![r]_t!A_{J_0\times L_0}(x,x)$$

we thus conclude that the term in τ_{J_0} in $F(x,y)$ is given by

$$t^{r(r+1-2N)/2}A_{J_0\times L_0}(x,x)\tau_{J_0}\frac{R_{J_0\times[m]}(x,ty)R_{L_0\times[m]}(x,y)}{R_{[N]\times[m]}(x,y)}\Delta^t_m(x)g(x)$$

By symmetry, we thus get that

$$F(x,y) = t^{r(r+1-2N)/2} \sum_{J\subseteq[m+1,N]\,;\,|J|=r} A_{J\times L}(x,x)\tau_J \frac{R_{J\times[m]}(x,ty)R_{L\times[m]}(x,y)}{R_{[N]\times[m]}(x,y)} \Delta_m^t(x)g(x)$$

where $L = [m+1,N] \setminus J$.

Now, we want to expand $F(x,y)$ as

$$F(x,y) = \sum_{\substack{J\subseteq[m+1,N]\\ |J|=r}} \sum_{[\sigma]\in\mathfrak{S}_N/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})} C_{J,\sigma}(x)\tau_J K_\sigma\left(\frac{g(x)}{R_{[m]\times[m]}(x,y)}\right)$$

for some coefficients $C_{J,\sigma}(x)$. Since $g(x)$ is an arbitrary symmetric functions, the terms in $\tau_J g(x)$ need to be equal on both sides. We thus have that

$$t^{r(r+1-2N)/2} A_{J\times L}(q^{-1}x,x)\frac{R_{J\times[m]}(x,ty)R_{L\times[m]}(x,y)}{R_{[N]\times[m]}(x,y)}\Delta_m^t(x) = \sum_{\sigma\in\mathfrak{S}_{m,N}} \left(\tau_J^{-1}C_{J,\sigma}(x)\right) K_\sigma\left(\frac{1}{R_{[m]\times[m]}(x,y)}\right)$$

The coefficient $\tau_J^{-1}C_{J,w}(x)$ can be obtained by multiplying by $K_w\big(R_{[m]\times[m]}(x,y)\big)$ and then taking the specialization $y_i = x_{w(i)}^{-1}$ for $i = 1,\ldots,m$ (this way, all the terms such that $\sigma \neq w$ cancel on the r.h.s.). Hence

$$\tau_J^{-1}C_{J,w}(x) = t^{r(r+1-2N)/2} A_{J\times L}(q^{-1}x,x)\frac{R_{J\times[m]}(x,ty)R_{L\times[m]}(x,y)}{R_{[N]\times[m]}(x,y)}\Delta_m^t(x)K_w\big(R_{[m]\times[m]}(x,y)\big)\Big|_{y_i=x_{w(i)}^{-1}}$$

or, equivalently,

$$\tau_J^{-1}C_{J,w}(x) = t^{r(r+1-2N)/2} A_{J\times L}(q^{-1}x,x)\frac{R_{J\times[m]}(x,ty)R_{L\times[m]}(x,y)}{R_{([N]\setminus w([m]))\times[m]}(x,y)}\Delta_m^t(x)\Big|_{y_i=x_{w(i)}^{-1}} \tag{3.13}$$

When considering $y_i = x_{w(i)}^{-1}$, the following holds:

$$R(x_i,ay_j) = \begin{cases} -\left(\dfrac{ax_i - x_{\mu(j)}}{x_{\mu(j)}}\right) & \text{if} \quad i \neq w(j) \\ -(a-1) & \text{if} \quad i = w(j) \end{cases}$$

The extra sign that the specialization generates on the r.h.s. of (3.13) is then

$$\#(J\times[m]) + \#(L\times[m]) + \#\big((([N]\setminus w([m]))\times[m]\big) = \#J\cdot m + \#L\cdot m + (N-m)\cdot m$$

which is equal to $2(N-m)m$. The extra sign, being even, can thus be ignored.

We now split the set $J\times[m]$ as the disjoint union of G_1 and G_2, where

$$G_1 = \{(i,j)\in J\times[m] \mid i\neq w(j)\} \quad \text{and} \quad G_2 = \{(i,j)\in J\times[m] \mid i = w(j)\}$$

Hence, after multiplying and dividing the r.h.s. of (3.13) by $R_{G_1}(x,y)$, we obtain

$$\tau_J^{-1}C_{J,w}(x) = t^{r(r+1-2N)/2} A_{J\times L}(q^{-1}x,x)\frac{R_{G_1}(x,ty)}{R_{G_1}(x,y)}R_{G_2}(x,ty)\frac{R_{G_1}(x,y)R_{L\times[m]}(x,y)}{R_{([N]\setminus w([m]))\times[m]}(x,y)}\Delta_m^t(x)\Big|_{y_i=x_{w(i)}^{-1}}$$

It is easy to check that

$$\frac{R_{G_1}(x,ty)}{R_{G_1}(x,y)}\Big|_{y_i=x_{w(i)}^{-1}} = \widetilde{A}_{J\times w([m])}(x,x), \quad R_{G_2}(x,ty)\Big|_{y_i=x_{w(i)}^{-1}} = (t-1)^{\#(J\cap w([m]))}$$

as well as

$$R_{G_1}(x,y)\Big|_{y_i=x_{w(i)}^{-1}} = \frac{\widetilde{\Delta}_{J\times w([m])}(x,x)}{(x_{w(1)}\cdots x_{w(m)})^{\#J}} \prod_{i\in J\cap w([m])} x_i\,, \quad R_{L\times[m]}(x,y)\Big|_{y_i=x_{w(i)}^{-1}} = \frac{\widetilde{\Delta}_{L\times w([m])}(x,x)}{(x_{w(1)}\cdots x_{w(m)})^{\#L}}$$

and

$$R_{([N]\setminus w([m]))\times[m]}(x,y)\Big|_{y_i=x_{w(i)}^{-1}} = \frac{\Delta_{([N]\setminus w([m]))\times w([m])}(x,x)}{(x_{w(1)}\cdots x_{w(m)})^{N-\#L-\#J}}$$

Hence, using $J\cap w([m])=\mathfrak{B}_w^c$, we obtain

$$\tau_J^{-1}C_{J,w}(x) = t^{r(r+1-2N)/2}A_{J\times L}(q^{-1}x,x)\widetilde{A}_{J\times w([m])}\left(\prod_{i\in\mathfrak{B}_w^c} x_i(t-1)\right)\frac{\widetilde{\Delta}_{J\times w([m])}\Delta_{L\times w([m])}}{\Delta_{([N]\setminus w([m]))\times w([m])}}\Delta_m^t(x)$$

where the dependency is always in the variables (x,x) when not specified. We deduce immediately that $C_{J,w}=0$ whenever $L\cap w([m])\neq\emptyset$ since $\Delta_{L\times w([m])}(x,x)=0$ in that case. Finally, using $[N]=\mathfrak{A}_w\cup\mathfrak{A}_w^c\cup\mathfrak{B}_w\cup\mathfrak{B}_w^c$ and $w([m])=\mathfrak{A}_w\cup\mathfrak{B}_w^c$, we get that

$$\tau_J^{-1}C_{J,w}(x) = t^{r(r+1-2N)/2}A_{J\times L}(q^{-1}x,x)\widetilde{A}_{J\times w([m])}\left(\prod_{i\in B_\mu^c} x_i(t-1)\right)\frac{\widetilde{\Delta}_{\mathfrak{B}_w^c\times w([m])}}{\Delta_{\mathfrak{A}_w^c\times w([m])}}\Delta_m^t$$

when $L\cap w([m])=\emptyset$. From Lemma 205, this implies that

$$C_{J,w}(x) = t^{r(r+1-2N)/2}A_m(x)A_{J\times L}(x,x)\tau_J\big(\widetilde{A}_{J\times w([m])}(x,x)\Phi(w)K_w(\Delta_m)\big).$$

when $L\cap w([m])=\emptyset$. This proves (3.9) and the proposition thus holds. □

4. Pieri rules

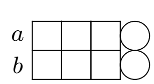
Before proving the Pieri rules for the bisymmetric Macdonald polynomials, we first need to establish a crucial lemma.

We will say that a composition $(\Lambda_1,\dots,\Lambda_N)$ is biordered if $\Lambda_1\geq\Lambda_2\geq\cdots\geq\Lambda_m$ and $\Lambda_{m+1}\geq\Lambda_2\geq\cdots\geq\Lambda_N$. Note that if Λ is not biordered then there exists a permutation $\sigma\in\mathfrak{S}_m\times\mathfrak{S}_{m+1,N}$ such that $\sigma\Lambda$ is biordered. For $J\subseteq[N]$, we will also let $\tau_J\Lambda=\Lambda+\varepsilon^J$, where $\varepsilon_i^J=1$ if $i\in J$ and 0 otherwise.

Lemma 208. *Suppose that $\sigma\in\mathfrak{S}_N$ and $J\subseteq[m+1,N]$ are such that $\sigma([m])\cap L=\emptyset$, where we recall that $L=[m+1,N]\setminus J$. Let (Λ,w) generate a superevaluation, and suppose that the composition $\Omega=\sigma^{-1}\tau_J(\Lambda+(1^m))-(1^m)$ is biordered. The following holds:*

- *(1) If $(\Omega,w\sigma)$ does not generate a superevaluation then $u_\Lambda^+(C_{J,\sigma})=0$, where $C_{J,\sigma}(x)$ is such as defined in Proposition 207.*
- *(2) Suppose that $(\Omega,w\sigma)$ generates a superevaluation. If $\delta\in\mathfrak{S}_N$ is also such that $(\Omega,w\delta)$ generates a superevaluation then $\sigma(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})=\delta(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})$ in $\mathfrak{S}_N/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})$.*
- *(3) If $I\subseteq[m+1,N]$ is such that $\Omega=\sigma^{-1}\tau_I(\Lambda+(1^m))-(1^m)$, then $I=J$.*

Proof. We first show that (*1*) holds. Suppose first that Ω is not a superpartition. Given that Ω is biordered, this can only happen if $\Omega_a=\Omega_{a+1}$ for a given $a\in[m-1]$, which can be visualized as



(4.1)

with $b = a+1$. Now, $\Omega = \sigma^{-1}\tau_J(\Lambda + (1^m)) - (1^m)$ translates in coordinates to

$$\big(\Omega + (1)^m\big)_a = \big(\Lambda + (1^m)\big)_{\sigma(a)} + \varepsilon^J_{\sigma(a)} \tag{4.2}$$

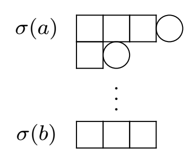
where $\varepsilon^J_i = 1$ if $i \in J$ and 0 otherwise. Hence there are two possible cases: (i) $\sigma(a), \sigma(b) \in [m+1, N]$ or (ii) $\sigma(a) \in [m], \sigma(b) \in [m+1, N]$ (the case $\sigma(a) \in [m+1, N], \sigma(b) \in [m]$ is equivalent).

Consider first the case (i). We have that $\sigma(a), \sigma(b) \in J$ since $\sigma([m]) \cap L = \emptyset$ by hypothesis. We thus deduce from (6.9) that $\Omega_a = \Lambda_{\sigma(a)}$ and $\Omega_b = \Lambda_{\sigma(b)}$, which implies that $\Lambda_{\sigma(a)} = \Lambda_{\sigma(b)}$. This in turn implies that the permutation w can be chosen such that $w\sigma(b) = w\sigma(a) + 1$, in which case we will have $\Lambda^{\circledast}_{w\sigma(a)} = \Lambda_{\sigma(a)} + 1$ and $\Lambda^{\circledast}_{w\sigma(b)} = \Lambda_{\sigma(b)} + 1$. Hence the term $\widetilde{A}_{J\times\sigma([m])}(x,x)$ in $C_{J,\sigma}(x)$ contains a factor $A_{\sigma(b),\sigma(a)}(x)$ such that

$$u^+_\Lambda(\tau_J A_{\sigma(b),\sigma(a)}(x)) = u^+_\Lambda\left(\frac{qtx_{\sigma(b)} - qx_{\sigma(a)}}{qx_{\sigma(b)} - qx_{\sigma(a)}}\right) = \frac{q^{\Lambda_{\sigma(b)}+2}t^{2-w\sigma(b)} - q^{\Lambda_{\sigma(a)}+2}t^{1-w\sigma(a)}}{q^{\Lambda_{\sigma(b)}+2}t^{1-w\sigma(b)} - q^{\Lambda_{\sigma(a)}+2}t^{1-w\sigma(a)}} = 0$$

and thus $C_{J,\sigma}(x)$ vanishes in that case.

The case (ii) is almost identical. We have that $\sigma(a) \in [m]$ and $\sigma(b) \in J$ since $\sigma([m]) \cap L = \emptyset$ by hypothesis.



We thus deduce from (6.9) that $\Omega_a = \Lambda_{\sigma(a)}$ and $\Omega_b = \Lambda_{\sigma(b)}$, which implies that $\Lambda_{\sigma(a)} = \Lambda_{\sigma(b)}$. This in turn implies that the permutation w can be chosen such that $w\sigma(b) = w\sigma(a) + 1$, in which case we will have $\Lambda^{\circledast}_{w\sigma(a)} = \Lambda_{\sigma(a)} + 1$ and $\Lambda^{\circledast}_{w\sigma(b)} = \Lambda_{\sigma(b)}$. Hence the term $\widetilde{A}_{J\times\sigma([m])}(x,x)$ in $C_{J,\sigma}(x)$ contains a factor $A_{\sigma(b),\sigma(a)}(x)$ such that

$$u^+_\Lambda(\tau_J A_{\sigma(b),\sigma(a)}(x)) = u^+_\Lambda\left(\frac{qtx_{\sigma(b)} - x_{\sigma(a)}}{qx_{\sigma(b)} - x_{\sigma(a)}}\right) = \frac{q^{\Lambda_{\sigma(b)}+1}t^{2-w\sigma(b)} - q^{\Lambda_{\sigma(a)}+1}t^{1-w\sigma(a)}}{q^{\Lambda_{\sigma(b)}+1}t^{1-w\sigma(b)} - q^{\Lambda_{\sigma(a)}+1}t^{1-w\sigma(a)}} = 0$$

and thus $C_{J,\sigma}(x)$ also vanishes in that case.

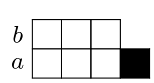
We now have to show that $C_{J,\sigma}(x) = 0$ when any of the two following cases occurs:

(1) $w\sigma(\Omega + (1^m)) \neq \Omega^{\circledast}$
(2) $w\sigma(\Omega) \neq \Omega^*$

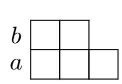
In the case (1), we have

$$\Omega^{\circledast} \neq w\sigma(\Omega + (1^m)) = w\tau_J(\Lambda + (1^m)) = \tau_{w(J)}\Lambda^{\circledast}$$

This can only happen if $\tau_{w(J)}\Lambda^{\circledast}$ is not a partition, that is, if we have the following situation:



where $1 + b = a$, $\Lambda^{\circledast}_a = \Lambda^{\circledast}_b$, $a \in w(J)$ and $b \notin w(J)$. Note that b cannot belong to $w([m])$ since otherwise the diagram of Λ^* would be of the form



(4.3)

and thus not a partition. We therefore conclude that $b \in w(L)$. Hence there exist $j \in J, l \in L$ such that $a = w(j)$ and $b = w(l)$, which implies that the term $A_{J\times L}(x, x)$ in $C_{J,\sigma}(x)$ contains a factor $A_{j,l}(x)$ such that

$$u^+_\Lambda(A_{j,l}(x)) = u^+_\Lambda\left(\frac{tx_j - x_l}{x_j - x_l}\right) = \frac{q^{\Lambda^\circledast_a} t^{2-a} - q^{\Lambda^\circledast_b} t^{1-b}}{q^{\Lambda^\circledast_a} t^{1-a} - q^{\Lambda^\circledast_b} t^{1-b}} = 0$$

as wanted.

We finally need to consider case (2) which amounts to

$$\Omega^* \neq w\sigma(\Omega + (1^m) - (1^m)) = w(\tau_J(\Lambda + (1^m)) - \mu(1^m)) = \tau_{w(J)}\Lambda^\circledast - w\sigma(1^m) \tag{4.4}$$

From Case (1) we know that $C_{J,\sigma}(x) = 0$ if $\tau_{w(J)}\Lambda^\circledast$ is not a partition from which we can suppose that $\tau_{w(J)}\Lambda^\circledast$ is a partition. Hence (4.4) will hold in the two following situations:

$$\begin{array}{r|c|c|c|}\cline{2-4} b & & & X \\ \cline{2-4} a & & & \\ \cline{2-4}\end{array} \qquad \text{and} \qquad \begin{array}{r|c|c|c|c|}\cline{2-5} b & & & & X \\ \cline{2-5} a & & & & \blacksquare \\ \cline{2-5}\end{array} \tag{4.5}$$

where X stands for a removed cell (the diagrams of $\Lambda^\circledast$ are those without X's and black square). We first show that the case to the left cannot occur. Indeed, we have in that case that

a) $b \notin w(J)$
b) $b \in w\sigma([m])$
c) $b \notin w([m])$ (otherwise Λ would not be a superpartition)

From *b)* and *c)* we deduce that $b \notin w(\sigma([m]) \cap [m])$. Therefore $b \in w(J)$ since $\sigma([m]) \cap L = \emptyset$ by hypothesis. But this contradicts *a)*.

Finally, we consider the case to the right in (4.5). We have $1 + b = a$, $\Lambda^\circledast_a + 1 = \Lambda^\circledast_b$, $b \in w\sigma([m])$ and $a \in w(J)$. Therefore, there exist $j \in J, s \in \sigma([m])$ such that $a = w(j)$ and $b = w(s)$. The term $\widetilde{A}_{J\times\sigma([m])}(x, x)$ in $C_{J,\sigma}(x)$ thus contains a factor $A_{j,s}(x)$ such that

$$u^+_\Lambda(\tau_J A_{j,s}(x)) = u^+_\Lambda\left(\frac{qtx_j - x_s}{qx_j - x_s}\right) = \frac{q^{\Lambda^\circledast_a+1} t^{2-a} - q^{\Lambda^\circledast_b} t^{1-b}}{q^{\Lambda^\circledast_a} t^{1-a} - q^{\Lambda^\circledast_b} t^{1-b}} = 0$$

which completes the proof of part (*1*) of the Lemma.

Part (*2*) and (*3*) of the lemma are much simpler to prove. We start with (2). Since both $(\Omega, w\delta)$ and $(\Omega, w\sigma)$ generate a superevaluation, we have that

$$\Omega^* = w\delta\Omega = w\sigma\Omega \qquad \text{and} \qquad \Omega^\circledast = w\delta(\Omega + (1^m)) = w\sigma(\Omega + (1^m))$$

Hence, $\sigma^{-1}\delta\Omega = \Omega$ and $\sigma^{-1}\delta(\Omega + (1^m)) = (\Omega + (1^m))$. We thus conclude that $\sigma^{-1}\delta \in \mathfrak{S}_m \times \mathfrak{S}_{m+1,N}$ or equivalently, that $\sigma(\mathfrak{S}_m \times \mathfrak{S}_{m+1,N}) = \delta(\mathfrak{S}_m \times \mathfrak{S}_{m+1,N})$, as wanted.

As for (*3*), we have

$$\Omega = \sigma^{-1}\tau_J(\Lambda + (1^m)) - (1^m) = \sigma^{-1}\tau_I(\Lambda + (1^m)) - (1^m) \iff \tau_I^{-1}\tau_J(\Lambda + (1^m)) = \Lambda + (1^m))$$

which implies that $I = J$. □

We can now state our main theorem. It is important to note that a more explicit characterization of the indexing superpartitions appearing in the Pieri rules will be provided in Corollary 215. Also recall that the evaluation $u_{\Lambda_0^+}(\mathcal{P}_\Lambda)$ was given in (1.21).

Theorem 209. *For $r \in \{1, \dots, N - m\}$, the bisymmetric Macdonald polynomial $\mathcal{P}_\Lambda(x; q, t)$ obeys the following Pieri rules*

$$e_r(x_{m+1}, \dots, x_N)\mathcal{P}_\Lambda(x; q, t) = \sum_\Omega \left(\frac{u^+_\Lambda(C_{J,\sigma})}{u^+_\Lambda(\Delta^t_m)} \frac{u_{\Lambda_0^+}(\mathcal{P}_\Lambda)}{u_{\Lambda_0^+}(\mathcal{P}_\Omega)}\right) \mathcal{P}_\Omega(x, q, t)$$

where the coefficients $C_{J,\sigma}(x)$ were obtained explicitly in Proposition 207 and where the sum is over all superpartitions Ω such that there exists a $\sigma \in \mathfrak{S}_N$ and a $J \subseteq [m+1,N]$ of size r such that

- $\sigma(\Omega + (1^m)) = \tau_J(\Lambda + (1^m))$
- $\sigma([m]) \cap L = \emptyset$*, where* $L = [m+1,\dots,N] \setminus J$
- $(\Omega, w\sigma)$ *is a superevaluation, where w is such that (Λ, w) generates a superevaluation*

Proof. We know from Theorem 207 that

$$e_r(Y_{m+1},\dots,Y_N)\Delta^t_m(x)\tilde{\mathcal{P}}^+_\Psi(x;q,t) = \sum_{\substack{J\subset[m+1,N]\\|J|=r}} \sum_{\substack{[\sigma]\in\mathfrak{S}_N/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})\\ \sigma([m])\cap L=\emptyset}} C_{J,\sigma}(x)\tau_J K_\sigma \tilde{\mathcal{P}}^+_\Psi(x;q,t)$$

where we recall that $\tilde{\mathcal{P}}^+_\Psi(x;q,t)$ was defined in Theorem 193. Using $e_r^{(m+1)}$ to denote $e_r(x_{m+1},\dots,x_N)$, we obtain from Lemma 189 that

$$u^+_\Psi(e_r^{(m+1)})\Delta^t_m(x)\tilde{\mathcal{P}}^+_\Psi(x;q,t) = \sum_{\substack{J\subset[m+1,N]\\|J|=r}} \sum_{\substack{[\sigma]\in\mathfrak{S}_N/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})\\ \sigma([m])\cap L=\emptyset}} C_{J,\sigma}(x)\tau_J K_\sigma \tilde{\mathcal{P}}^+_\Psi(x;q,t)$$

Let Λ be a superpartition such that (Λ, w) is a superevaluation. Applying u^+_Λ on both sides of the equation (and dropping the dependencies in x in the evaluations for simplicity) leads to

$$u^+_\Psi(e_r^{(m+1)})u^+_\Lambda(\Delta^t_m)u^+_\Lambda(\tilde{\mathcal{P}}^+_\Psi) = \sum_{\substack{J\subset[m+1,N]\\|J|=r}} \sum_{\substack{[\sigma]\in\mathfrak{S}_N/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})\\ \sigma([m])\cap L=\emptyset}} u^+_\Lambda(C_{J,\sigma})u^+_\Lambda(\tau_J K_\sigma \tilde{\mathcal{P}}^+_\Psi) \tag{4.6}$$

Now, in $u^+_\Lambda(\tau_J K_\sigma \tilde{\mathcal{P}}_\Psi)$, the evaluation amounts to the following substitution

$$x_i = q^{(\Lambda+(1^m))_{\sigma(i)}+\varepsilon^J_{\sigma(i)}}t^{1-w\sigma(i)}$$

where again $\varepsilon^J_i = 1$ if $i \in J$ and 0 otherwise. Comparing with (6.9), we have that the substitution is

$$x_i = q^{(\Omega+(1^m))_i}t^{1-w\sigma(i)}$$

where $\Omega = \sigma^{-1}\tau_J(\Lambda + (1^m)) - (1^m)$. Choosing σ in $[\sigma]$ such that Ω is biordered, we deduce from Lemma 208 that for $u^+_\Lambda(C_{J,\sigma})$ not to vanish, we need $(\Omega, w\sigma)$ to generate a superevaluation (and in particular for Ω to be a superpartition). We also get from Lemma 208 *2)* and *3)* that the superpartition Ω can arise in at most one way in the sums in the r.h.s. of (4.6). As such, we obtain that

$$u^+_\Psi(e_r^{(m+1)})u^+_\Lambda(\Delta^t_m)u^+_\Lambda(\tilde{\mathcal{P}}^+_\Psi) = \sum_\Omega u^+_\Lambda(C_{J,\sigma})u^+_\Omega(\tilde{\mathcal{P}}^+_\Psi)$$

where the sum is over all superpartitions Ω such that there exists a $\sigma \in \mathfrak{S}_N$ and a $J \subseteq [m+1,N]$ of size r such that $\sigma(\Omega + (1^m)) = \tau_J(\Lambda + (1^m))$, such that $\sigma([m]) \cap L = \emptyset$, and such that $(\Omega, w\sigma)$ is a superevaluation.

The symmetry established in Theorem 193 then implies that

$$u^+_\Psi\big(e_r^{(m+1)}u^+_\Lambda(\Delta^t_m)\tilde{\mathcal{P}}^+_\Lambda\big) = u^+_\Psi\left(\sum_\Omega u^+_\Lambda(C_{J,\sigma})\tilde{\mathcal{P}}^+_\Omega\right)$$

Now, the previous equation holds for every superpartition Ψ. Therefore,

$$e_r(x_{m+1},\dots,x_N)u^+_\Lambda(\Delta^t_m)\tilde{\mathcal{P}}^+_\Lambda(x;q,t) = \sum_\Omega u^+_\Lambda(C_{J,\sigma})\tilde{\mathcal{P}}^+_\Lambda(x,q,t)$$

from which we finally obtain that

$$e_r(x_{m+1},\dots,x_N)\mathcal{P}_\Lambda(x;q,t) = \sum_\Omega \left(\frac{u^+_\Lambda(C_{J,\sigma})}{u^+_\Lambda(\Delta^t_m)}\frac{u_{\Lambda_0^+}(\mathcal{P}_\Lambda)}{u_{\Lambda_0^+}(\mathcal{P}_\Omega)}\right)\mathcal{P}_\Omega(x,q,t)$$

□

5. Pieri rules and vertical strips

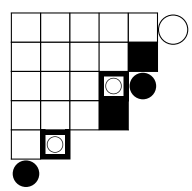
In this section, we will give explicitly which superpartition Ω appear in the Pieri rules of Theorem 209. They will turn out to be certain vertical strips.

For partitions λ and μ, we say that μ/λ is a vertical r-strip if $|\mu| - |\lambda| = r$ and $\mu_i - \lambda_i \in \{0, 1\}$ for all i, where we consider that $\mu_i = 0$ (resp. $\lambda_i = 0$) if i is larger than the length of μ (resp. λ).

Given the superpartitions Λ and Ω, we say that Ω/Λ is a vertical r-strip if both $\Omega^{\circledast}/\Lambda^{\circledast}$ and Ω^*/Λ^* are vertical r-strips. When describing the vertical strip Ω/Λ with Ferrers' diagram, we will use the following notation:

- the squares of Λ will be denoted by $\square$
- the squares of Ω/Λ that do not lie over a circle of Λ will be denoted by $\blacksquare$
- the squares of Ω/Λ that lie over a circle of Λ will be denoted by $\boxed{\circ}$
- the circles of Λ that are still circles in Ω will be denoted by $\bigcirc$
- the circles of Ω that were not circles in Λ will be denoted by $\bullet$

For instance, if $\Lambda = (5, 3, 1; 4, 3)$ and $\Omega = (5, 4, 0; 5, 4, 2)$ the cells of the vertical 4-strip Ω/Λ are represented as:

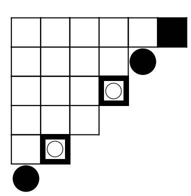


A row in the diagram of Ω/Λ that contains a $\blacksquare$ will be called a $\blacksquare$-row (and similarly for $\bigcirc$, $\bullet$ and $\boxed{\circ}$). A row that both contains a $\bullet$ and a $\boxed{\circ}$ will be called a $\boxed{\circ}\bullet$-row. For instance, in our previous example, the set of $\blacksquare$-rows is $\{2, 4\}$, the set of $\bullet$-rows is $\{3, 6\}$, the set of $\boxed{\circ}$-rows is $\{3, 5\}$ while the set of $\boxed{\circ}\bullet$-rows is $\{3\}$.

Definition 210. *We will say that Ω/Λ is a vertical r-strip of type I if*

(1) Ω/Λ is a vertical r-strip
(2) there are no $\boxed{\circ}\bullet$-rows in the diagram of Ω/Λ

For instance, if $\Lambda = (3, 1; 5, 4, 3)$ and $\Omega = (4, 0; 6, 4, 3, 2)$ then Ω/Λ is a vertical 3-strip of type I.



(5.1)

We first show that the Ω's that can appear in the Pieri rules of Theorem 209 are such that Ω/Λ is a vertical r-strip of type I.

Lemma 211. *Let σ and J be such as in Theorem 209, that is, such that*

(1) $\sigma(\Omega + (1^m)) = \tau_J(\Lambda + (1^m))$
(2) $\sigma([m]) \cap L = \emptyset$ con $L = [m + 1, N] - J$
(3) $(\Omega, w\sigma)$ is a superevaluation if (Λ, w) is a superevaluation.
(4) $J \subseteq [m + 1, N]$ with $|J| = r$.

Then Ω/Λ is a vertical r-strip of type I.

Proof. Applying w on both sides of *(1)* gives $w\sigma(\Omega+(1^m))=\tau_{w(J)}w\big(\Lambda+(1^m)\big)$. From *(3)* we then get that $\Omega^{\circledast}=\tau_{w(J)}\Lambda^{\circledast}$, which immediately implies that $\Omega^{\circledast}/\Lambda^{\circledast}$ is a vertical r-strip.

Subtracting (1^m) on both sides of *(1)* gives $\sigma(\Omega+(1^m))-(1^m)=\tau_J(\Lambda+(1^m))-(1^m)=\tau_J\Lambda$. Applying again w on both sides of the equation then yields $w\sigma(\Omega)+w\sigma(1^m)-w(1^m)=\tau_{w(J)}(w\Lambda)$, which from *(3)* amounts to $\Omega^*+w\sigma(1^m)-w(1^m)=\tau_{w(J)}\Lambda^*$, or equivalently, to

$$\Omega^*=\tau_{w(J)}\Lambda^*+w(1^m)-w\sigma(1^m)$$

Note that by the action of the symmetric group on vectors, $w(1^m)$ adds a 1 in the positions $w([m])$ (and similarly for $w\sigma(1^m)$). From *(4)*, we have that $w(J)\cap w([m])=\emptyset$, which gives $\Omega_i^*-\Lambda_i^*\leq 1$. Moreover, from *(2)*, we have that $\sigma([m])\subseteq[m]\cup J$, which implies that $w\sigma([m])\subseteq w([m])\cup wJ$. Hence $0\leq\Omega_i^*-\Lambda_i^*\leq 1$ and we have that Ω^*/Λ^* is a vertical r-strip as well.

Finally, suppose that row i in Ω/Λ is a $\boxed{\bigcirc}\bullet$-row. We have in this case that $i\in\Omega^{\circledast}/\Lambda^{\circledast}$ as well as $i\in w([m])$ since Λ has a circle in row i. But, as we have seen, $\Omega^{\circledast}=\tau_{w(J)}\Lambda^{\circledast}$. We thus have that $i\in w(J)\cap w([m])$, which contradicts (*4*). $\square$

Remark 212. *Observe that in a vertical r-strip, the rows of Ω^*/Λ^* correspond to the $\boxed{\bigcirc}$-rows together with the $\blacksquare$-rows. Similarly, the rows of $\Omega^{\circledast}/\Lambda^{\circledast}$ correspond in a vertical strip to the $\bullet$-rows together with the $\blacksquare$-rows. By this observation, if Ω/Λ is a vertical r-strip, then the number of $\boxed{\bigcirc}$-rows is equal to the number of $\bullet$-rows.*

We now show that all Ω's such that Ω/Λ is a vertical r-strip of type I do in fact appear in the Pieri rules of Theorem 209.

Lemma 213. *Given Ω/Λ a vertical r-strip of type I, let $\tilde\sigma$ be any permutation that interchanges the $\boxed{\bigcirc}$-rows and the $\bullet$-rows while leaving the remaining rows invariant (such a permutation can be defined by Remark 212). Let also $\tilde J$ be the set of $\boxed{\bigcirc}$-rows and $\blacksquare$-rows. If*

$$\sigma=w^{-1}\tilde\sigma w\quad\text{and}\quad J=w^{-1}\tilde\sigma(\tilde J)$$

then there exists a permutation $s\in\mathfrak{S}_m\times\mathfrak{S}_{m+1,N}$ such that $\sigma'=\sigma s$ obeys the following relations:

(1) $\sigma'(\Omega+(1^m))=\tau_J(\Lambda+(1^m))$
(2) $\sigma'([m])\cap L=\emptyset$ *con* $L=[m+1,N]-J$
(3) $(\Omega,w\sigma')$ *is a superevaluation if* (Λ,w) *is a superevaluation.*
(4) $J\subseteq[m+1,N]$.

As such, the superpartition Ω satisfies the conditions of Theorem **??** *(with $C_{J,\sigma'}(x)=C_{J,\sigma}(x)$).*

Proof. We first show that $(\Omega,w\sigma')$ is a superevaluation. By definition, we have to show that $w\sigma'\Omega=\Omega^*$ and that $w\sigma'\big(\Omega+(1^m)\big)=\Omega^{\circledast}$ for a certain $s\in\mathfrak{S}_m\times\mathfrak{S}_{m+1,N}$. We will show, equivalently, that $(w\sigma')^{-1}\Omega^*=\Omega$ and that $(w\sigma')^{-1}\Omega^{\circledast}=\Omega+(1^m)$. Observe that

$$w\sigma'=w\sigma s=ww^{-1}\tilde\sigma ws=\tilde\sigma ws$$

It thus suffices to show that $s^{-1}w^{-1}\tilde\sigma^{-1}\Omega^*=\Omega$ and $s^{-1}w^{-1}\tilde\sigma^{-1}\Omega^{\circledast}=\Omega+(1^m)$. From the definition of $\tilde\sigma$, it is immediate that $\tilde\sigma^{-1}$ also interchanges the $\bullet$-rows and the $\boxed{\bigcirc}$-rows. Hence $\tilde\sigma^{-1}\Omega^{\circledast}/\tilde\sigma^{-1}\Omega^*=\Lambda^{\circledast}/\Lambda^*$. Since by definition w^{-1} sends $\Lambda^{\circledast}/\Lambda^*$ to $[m]$, we have that $w^{-1}\tilde\sigma^{-1}$ sends the rows in the diagram of Ω ending with a circle to $[m]$, that is, $w^{-1}\tilde\sigma^{-1}\Omega^*=\boldsymbol{v}$ and $w^{-1}\tilde\sigma^{-1}\Omega^{\circledast}=\boldsymbol{v}+(1^m)$ for a certain $v\in\mathbb{Z}^N_{\geq 0}$. Using any $s^{-1}\in\mathfrak{S}_m\times\mathfrak{S}_{m+1,N}$ such that $s^{-1}\boldsymbol{v}=\Omega$, we obtain that $s^{-1}w^{-1}\tilde\sigma^{-1}\Omega^*=\Omega$ and $s^{-1}w^{-1}\tilde\sigma^{-1}\Omega^{\circledast}=\Omega+(1^m)$ as wanted. We will take this as the definition of s in the rest of the proof.

Observe that *1)* is equivalent to

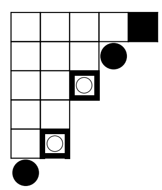
$$w\sigma'(\Omega+(1^m))=\tau_{w(J)}w(\Lambda+(1^m))=\tau_{\tilde\sigma(\tilde J)}\Lambda^{\circledast}$$

by definition of w and $\tilde{J}$. Since we have shown that *3)* holds, we only have left to show that $\Omega^{\circledast} = \tau_{\tilde{\sigma}(\tilde{J})}(\Lambda^{\circledast})$. But by definition of $\tilde{J}$ and $\tilde{\sigma}$, the set $\tilde{\sigma}(\tilde{J})$ corresponds to the ●-rows and ■-rows in the diagram of Ω/Λ, that is, to the rows of $\Omega^{\circledast}/\Lambda^{\circledast}$. We have thus shown that $\Omega^{\circledast} = \tau_{\tilde{\sigma}(\tilde{J})}(\Lambda^{\circledast})$.

As for *2)*, let $x \in \sigma'([m]) \cap L = \sigma([m]) \cap L$. Therefore, $w(x) \in w\sigma([m]) \cap w(L) = \tilde{\sigma}w([m]) \cap w(L)$. Now, $w([m])$ corresponds to the $\boxed{\circ}$-rows and the ○-rows in the diagram of Ω/Λ, which implies that $\tilde{\sigma}w([m])$ corresponds to the ○-rows and ●-rows in that diagram. Since $w([m]) \cap w(L) = \emptyset$, $w(L)$ cannot correspond to any $\boxed{\circ}$-row or any ○-row. Therefore, $w(x) \in \tilde{\sigma}w([m]) \cap w(L)$ needs to correspond to a ●-row. But this is impossible because $w(J) \cap w(L) = \emptyset$ and the ●-rows belong to $w(J) = \tilde{\sigma}(\tilde{J})$.

Finally, we have to show *4)*. By definition of a vertical strip of type I, the ●, ■, ○ and $\boxed{\circ}$ rows are all distinct. Now, $\tilde{\sigma}\tilde{J}$ corresponds to the ●-rows and the ■-rows, while $w([m])$ corresponds to the ○-rows and the $\boxed{\circ}$-rows. Hence, $\tilde{\sigma}\tilde{J} \subseteq w([m+1, N])$, which implies that $J = w^{-1}\tilde{\sigma}\tilde{J} \subseteq [m+1, N]$. □

EXAMPLE 214. *Consider the following vertical strip of type I:*



(5.2)

We have in this case that $\tilde{J} = \{1, 3, 5\}$. *Taking* $\tilde{\sigma} = [1, 3, 2, 4, 6, 5]$, *and* $w = [3, 5, 1, 2, 4, 6]$ *(in one-line notation), we get that* $J = \{3, 4, 6\}$ *and* $\sigma = [4, 6, 3, 1, 5, 2]$.

Using Lemma 211 and Lemma 213, we can rewrite Theorem 209 in a more precise fashion.

COROLLARY 215. *For* $r \in \{1, \dots, N-m\}$, *the bisymmetric Macdonald polynomial* $\mathcal{P}_\Lambda(x; q, t)$ *obeys the following Pieri rules*

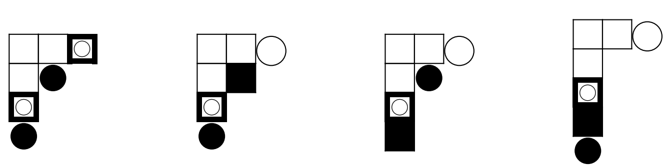
$$e_r(x_{m+1}, \dots, x_N)\mathcal{P}_\Lambda(x; q, t) = \sum_{\Omega} \left(\frac{u_\Lambda^+(C_{J,\sigma})}{u_\Lambda^+(\Delta_m^t)} \frac{u_{\Lambda_0^+}(\mathcal{P}_\Lambda)}{u_{\Lambda_0^+}(\mathcal{P}_\Omega)} \right) \mathcal{P}_\Omega(x, q, t)$$

where the sum is over all superpartitions Ω *such that* Ω/Λ *is a vertical* r*-strip of type I. Note that* $C_{J,\sigma}(x)$ *was defined in Proposition 207, where* σ *and* J *can be obtained in the following manner from the diagram of* Ω/Λ*: let* $\tilde{\sigma}$ *be any permutation that interchanges the* $\boxed{\circ}$*-rows and the* ●*-rows while leaving the remaining rows invariant, and let* $\tilde{J}$ *be the set of* $\boxed{\circ}$*-rows and* ■*-rows. Then*

$$\sigma = w^{-1}\tilde{\sigma}w \quad \text{and} \quad J = w^{-1}\tilde{\sigma}(\tilde{J})$$

where w *is such that* (Λ, w) *is a superevaluation.*

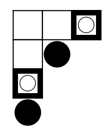
EXAMPLE 216. *The superpartitions that appear in the expansion of the multiplication of* $e_2(x_3, x_4, \dots, x_N)$ *and* $\mathcal{P}_{(2,0;1)}(x_1, \dots, x_N; q, t)$ *in terms of bisymmetric Macdonald polynomials are:*



To be more precise, we have that

$$
\begin{aligned}
e_2(x_3,x_4,\ldots,x_N)\mathcal{P}_{(2,0;1)} &= \frac{q(1-t)}{1-qt}\mathcal{P}_{(1,0;3,1)} + \frac{(1-q)(1-qt^2)}{(1-qt)^2}\mathcal{P}_{(2,0;2,1)} \\
&\quad - \frac{(t+1)(1-t)(1-q)(1-q^2t^4)}{(1-q^2t^3)(1-qt)(1-qt^2)}\mathcal{P}_{(2,1;1,1)} + \frac{(1-qt)(1-t^3)}{(1-t)(1-qt^3)}\mathcal{P}_{(2,0;1,1,1)}
\end{aligned}
$$

We now give more details on how the coefficient of $\mathcal{P}_{(1,0;3,1)}$ for instance was obtained. The diagram Ω/Λ is in this case



Choosing instance $\tilde{\sigma} = (12)(34)$, $\widetilde{J} = \{1,3\}$ and $w = (23)$, we then obtain from Corollary 215 that

$$\sigma = (23)(12)(34)(23) = [3,4,1,2] \quad \text{and} \quad J = (23)(12)(34)\{1,3\} = \{3,4\}$$

Lemma 205 gives

$$\Phi(\sigma) = -(t-1)^2 x_3x_4 \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)}$$

while Theorem 209 with $m=2$ and $r=2$ yields

$$C_{J,\sigma} = t^{3-2N}\frac{tx_1-x_2}{x_1-x_2}\tau_3\tau_4\left(\frac{tx_3-x_4}{x_3-x_4}\frac{tx_4-x_3}{x_4-x_3}\Phi(\sigma)K_{13}K_{24}(x_1-x_2)\right)$$

Taking

$$\frac{u^+_\Lambda(C_{J,\sigma})}{u^+_\Lambda(\Delta^t_m)}\frac{u_{\Lambda^+_0}(\mathcal{P}_\Lambda)}{u_{\Lambda^+_0}(\mathcal{P}_\Omega)}$$

we finally get the desired coefficient.

6. The $\mathbf{e}_r(x_1,\ldots,x_m)$*case*

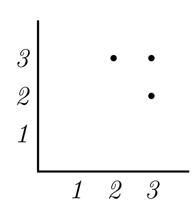
In Theorem 209 and Corollary 215, we obtained Pieri rules for the action of $e_r(x_{m+1},\ldots,x_N)$ on bisymmetric Macdonald polynomials. In this section, we will present Pieri rules for the the action of $e_r(x_1,\ldots,x_m)$. Although the proof in the $e_r(x_1,\ldots,x_m)$ case is quite similar to that in the $e_r(x_{m+1},\ldots,x_N)$ case, it was more challenging to explicitly find the coefficients of the operator $e_r(Y_1,\ldots,Y_m)$ viewed as a sum over the set τ_I, as we couldn't use Lemma 201 universally. To address this, we defined the operator S^r as $e_r(Y_1,\ldots,Y_m)\mathcal{A}^t_m S^r$, found the coefficients for the expansion of S^r, and antisymmetrized. First, we will define an analogue of $\mathcal{NF}_m$ in this context:

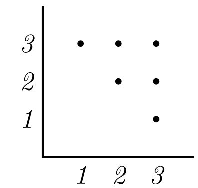
In the notation of section 5, we define

$$\mathcal{NG}_m(x,y) = \frac{R_{\mathcal{B}}(x,t^{-1}y)}{R_{\mathcal{B}'}(x,y)}$$

where $\mathcal{B}$ is the set of integer points in the triangle with vertices $(2,1),(m,m-1)$ and $(m,1)$, while $\mathcal{B}'$ is the set of integer points in the triangle with vertices $(1,1),(m,m)$ and $(m,1)$.

Example 217. *The product $NG_3(x,y)$ can be seen at the quotient of the factors stemming from the two following regions*





Hence $\mathcal{NG}_3(x,y)$ is equal to

$$\frac{(1-t^{-1}x_3y_2)(1-t^{-1}x_2y_3)(1-t^{-1}x_3y_3)}{(1-x_3y_1)(1-x_2y_2)(1-x_3y_2)(1-x_1y_3)(1-x_2y_3)(1-x_3y_3)}$$

The following corollary is an analogue of Corollary 201

Corollary 218. *We have the following equality*

$$\mathcal{A}^y_m\mathcal{NG}_m = (-t)^{-\binom{m}{2}}\Delta_m(y)\mathcal{F}_m$$

Proof. The identity can be deduced from Lemma 200 after completing the square in the triangle $\mathcal{B}'$ corresponding to the denominator of $\mathcal{NG}_m(x,y)$. □

The following lemma concerns the function

$$\mathcal{NG}^r_{m,N} = \frac{R_{\mathcal{P}_{r,N}}(x_i,t^{-1}y_j)R_{\mathcal{Q}_{r,N}}(x_i,qt^{-1}y_j)}{R_{\mathcal{P}'_{r,N}}(x_i,y_j)R_{\mathcal{Q}'_{r,N}}(x_i,qy_j)} \tag{6.1}$$

where $\mathcal{P}_{N,r}$ is the set of integer points in the trapezoid with vertices $\{(1,m),(1,m-r+1),(m-r,2),(m-r,m)\}$ and $\mathcal{Q}_{N,r}$ is the set of integer points in the trapezoid with vertices $\{(m-r+2,m),(m+1,m-r+1),(N,m),(N,m-r+1)\}$. That is, $\mathcal{P}'_{r,N}$ is the union of $\mathcal{P}_{N,r}$ with the line segment from $(1,m-r)$ to $(m-r,1)$ of slope -1 and $\mathcal{Q}'_{N,r}$ is the union of $\mathcal{Q}_{N,r}$ with the line segment from $(m-r+1,m)$ to $(m,m-r+1)$ of slope -1.

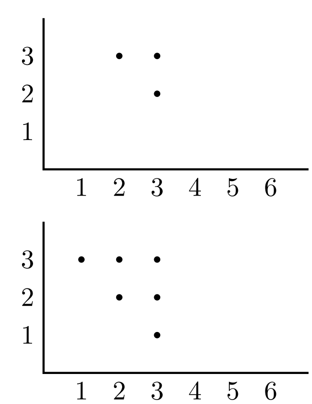
Lemma 219. *For $k\geq 1$, we have*

$$T_{m-r+1+k}\cdots T_{N-r+k}\mathcal{NG}^{k-1}_m(x,y) = t^{N-m}\mathcal{NG}^k_m(x,y)$$

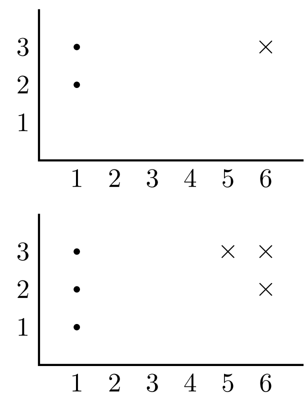
Consequently, for $r\geq 1$,

$$(T_m\cdots T_{N-1})\cdots(T_{m-r+1}\cdots T_{N-r})\omega^r_N\mathcal{NG}_m(x,y) = t^{r(N-m)}\mathcal{NG}^r_m(x,y) \tag{6.2}$$

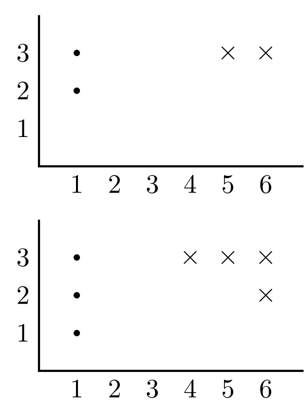
Proof. The lemma follows from Lemma 202. Diagrammatically, acting with $T_m\omega^r_N$ on $\mathcal{NG}_m$ amounts to adding a dot to the triangles associated to the numerator and the denominator in the diagram of $\mathcal{NG}_m$. For instance, if $N=6$, $m=3$ and $r=2$ the diagram associated to $\mathcal{NG}_3$ is:



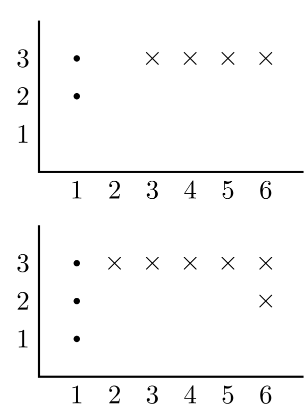
and when we apply ω_6^2 we obtain



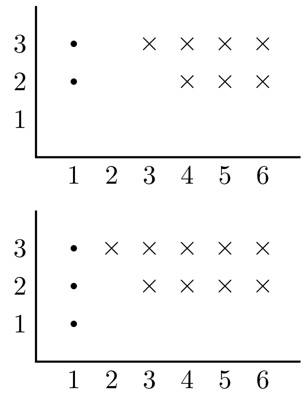
where $\times$ correspond to change $x_i \to q x_i$ in the respective factors. After that, we have to apply the Hecke operators; $T_4 \omega_6^2 \mathcal{N}\mathcal{G}_3$ is:



The product $\bar{T}_{m-r+1} \cdots \bar{T}_{N-r}$ complete the horizontal line in the m row of $\mathcal{N}\mathcal{F}_m$. For example, $T_2 T_3 T_4 \omega_6^2 \mathcal{N}\mathcal{G}_3$ is



Finally, as r increases, extra rows are added. We get for instance, in the case $N = 6$, $r = 2$ and $m = 3$, that acting with $(T_3 T_4 T_5)(T_2 T_3 T_4)$ on $\omega_6^2 \mathcal{N}\mathcal{G}_3$ adds the following two diagonals:



∎

Definition 220. *We define*

$$S_\sigma^r = (T_m \cdots T_{N-1}) \cdots (T_{m-r+1} \cdots T_{N-r})\omega_N^r$$

This operator has good propeties

Lemma 221. *S^r over a bisymmetric funciotn is t-symmetric in $x_{m+1}, \dots, x_N$, t-antisymmetric in $x_{m-r+1}, \dots, x_m$ and doesn't have variables $x_1, \dots, x_{m-r}$.*

Proof. Let f bisymmetric function, if $i \in [m+1, N-1]$ we have that

$$\begin{aligned} T_i S^r \Delta^t f(x) &= T_i T_\sigma \omega_N^r \Delta_m^t(x) f(x) \\ &= T_\sigma T_{i-r} \omega_N^r \Delta_m^t(x) f(x) \\ &= T_\sigma \omega_N^r T_i \Delta_m^t(x) f(x) \\ &= T_\sigma \omega_N^r \Delta_m^t(x) f(x) \\ &= S^r \Delta_m^t(x) f(x) \end{aligned}$$

Moreover, the word σ can be rewrite as

$$\sigma = (s_m \cdots s_{m-r+1}) \cdots (s_{N-1} \cdots s_{N-r})$$

So, if $i \in [1, r-1]$ we have

$$\begin{aligned} T_{m-i} S^r \Delta_m^t(x) f(x) &= T_{m-i} T_\sigma \omega_N^r \Delta_m^t(x) f(x) \\ &= T_\sigma T_{N-i} \omega_N^r \Delta_m^t(x) f(x) \\ &= T_\sigma \omega_N^r T_{N-i+r} \Delta_m^t(x) f(x) \\ &= -T_\sigma \omega_N^r \Delta_m^t(x) f(x) \\ &= -S^r \Delta_m^t(x) f(x) \end{aligned}$$

because $N - i + r \in [r-1]$ seeing as element module N. ∎

Lemma 222. *Let $O = \sum_{\sigma, I} C_{I,\sigma} \tau_I \sigma$ an operator t-antisymmetric in $x_1, \dots, x_m$ and t-symmetric in $x_{m+1}, \dots, x_N$ over a super symmetric function. We have the follow symmetries relations for its coefficients*

(1) If $1 \leq i < m$ then

$$\frac{C_{s_i(I), s_i\sigma}}{(tx_i - x_{i+1})} = K_{i(i+1)} \left(\frac{C_{I,\sigma}}{(tx_i - x_{i+1})} \right)$$

(2) and if $m < i \leq N-1$

$$C_{s_i(I), s_i\sigma} = C_{I,\sigma}$$

Proof. For simplicity of notation, write σ_1, $\mu = K_{i,i+1}\mu K_{i,i+1}$, $J = K_{i,i+1}I$ and

$$T_j = \frac{x_{j+1}(t-1)}{x_j - x_{j+1}} + \frac{tx_j - x_{j+1}}{x_j - x_{j+1}}K_{j,j+1}$$

Then, if $1 \leq i < m$ then we know that

$$T_i O\Delta_m^t(x)f(x) = -O\Delta_m^t(x)f(x)$$

Taking the coefficient of $\tau_I\sigma$ we have the relation

$$\frac{x_{i+1}(t-1)}{x_i - x_{i+1}}C_{I,\sigma} + \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}K_{i,i+1}(C_{J,\mu}) = -C_{I,\sigma}$$

which implies

$$\frac{C_{s_i(I),s_i\sigma}}{(tx_i - x_{i+1})} = K_{i(i+1)}\left(\frac{C_{I,\sigma}}{(tx_i - x_{i+1})}\right).$$

Case $m < i < N$ follows in the same way. □

Proposition 223. *Let $f(x)$ be any bisymmetric function. For N, r, m we have*

$$S^r\Delta_m^t(x)f(x) = \sum_{\substack{J\subset[m-r+1,N]\\|J|=r}}\ \sum_{\substack{[\sigma]\in\mathfrak{S}_{[m-r+1,N]}/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})\\ \sigma(J)\subset[m-r+1,m]}} C_{J,\sigma}(x)\tau_J K_\sigma f(x)$$

where the coefficient $C_{J,\sigma}(x)$ is given by

$$C_{J,\sigma} = (-1)^{\binom{m}{2}-r(m-r)+\#\mathfrak{B}^c_\sigma}A_{[m-r]}A_{[m-r+1,m]}A_{J,[m+1,N]}\Phi(\sigma)\tau_J A_{J,[m-r]}\sigma\Delta_m.$$

Proof. The main idea of this proof is the same that used in Proposition 207. By Lemma 221 and 222 is sufficient to prove the claim for a particular J and σ, let $J = [m-r+1, m-\tilde{r}]\cup[m+1, m+\tilde{r}]$ with $\tilde{r} \in \{0, \ldots, r\}$ and σ the permutation that send the set $[m-\tilde{r}+1, m]$ into $[m+1, \ldots, m+\tilde{r}]$ via the permutation $a \longrightarrow a+\tilde{r}$. Let $G(x,y) = S^r\mathcal{F}_m(x,y)g(x)$.

By Corollary 218, we have

$$G(x,y) = \frac{(-t)^{\binom{m}{2}}}{\Delta_m(y)}\mathcal{A}_m^{(y)}T_{\sigma_{m,N}}\omega_N^r\mathcal{N}\mathcal{G}_m(x,y)g(x)$$

and because $g(x)$ commute with T_i's and ω_N^r, we can use the Lemma 219, obtaining

$$\begin{aligned} G(x,y) &= \frac{(-t)^{\binom{m}{2}}t^{r(N-m)}}{\Delta_m(y)}\mathcal{A}_m^{(y)}\mathcal{N}\mathcal{G}_{m,N}^r(x,y)g(x) \\ &= \frac{(-t)^{\binom{m}{2}}t^{r(N-m)}}{\Delta_m(y)}\mathcal{A}_m^{(y)}\frac{R_{\mathcal{P}_{r,N}}(x_i,t^{-1}y_j)R_{\mathcal{Q}_{r,N}}(x_i,qt^{-1}y_j)}{R_{\mathcal{P}'_{r,N}}(x_i,y_j)R_{\mathcal{Q}'_{r,N}}(x_i,qy_j)}g(x) \end{aligned} \tag{6.3}$$

where $\mathcal{Q}_{r,N}, \mathcal{P}_{r,N}, \mathcal{Q}'_{r,N}$ and $\mathcal{P}'_{r,N}$ are the regions defined in 6.1. The problem now is $\mathcal{A}_m^{(y)}$ this expression, because when we apply $\mathcal{A}_m^{(y)}$ this expression does not factorize making heavy to control it. We know that we will follow the same idea gives in the proof of 207, then afterwards we will multiply $G(x,y)$ by $\tau_J K_\mu\big(R_{[m]\times[m]}(x,y)\big)$ and will take the specialization $y_i = \tau_J x^{-1}_{\mu(i)}$ for $i = 1, \ldots, m$, doing this, we can notice that the permutations that send elements in $[m-r]$ to $[m-r+1, m]$ are zero over $\sigma_y G(x,y)$. We wont give a proof of this claim, but it easy to see geometrically. Then we rewrite the regions $\mathcal{Q}_{r,N}, \mathcal{P}_{r,N}, \mathcal{Q}'_{r,N}$ and $\mathcal{P}'_{r,N}$ as

$$G(x,y) = \frac{t^{\binom{m}{2}}t^{r(N-m)}}{\Delta_m(y)}\mathcal{A}_m^{(y)}\frac{R_{\mathcal{P}_1}(x_i,t^{-1}y_j)R_{\mathcal{P}_2}(x_i,t^{-1}y_j)R_{\mathcal{Q}_1}(x_i,qt^{-1}y_j)R_{\mathcal{Q}_2}(x_i,qt^{-1}y_j)}{R_{\mathcal{P}'_1}(x_i,y_j)R_{\mathcal{P}_2}(x_i,y_j)R_{\mathcal{Q}'_1}(x_i,qy_j)R_{\mathcal{Q}_2}(x_i,qy_j)}g(x)$$

where the regions are $\mathcal{P}_1$ is the triangle $\{(2, m-r), (m-r, m-r), (m-r, 2)\}$, $\mathcal{Q}_1$ is the triangle $\{(m-r+2, m), (m, m), (m, m-r+2)\}$, $\mathcal{P}_2$ is the rectangle $[m-r]\times[m-r+1, m]$ y $\mathcal{Q}_2$ is the rectangle $[m+1,N]\times[m-r+1,m]$, while $\mathcal{P}_1'$ is the union of $\mathcal{P}_1$ with the line segment from $(1,m-r)$ to $(m-r,1)$ of slope -1 and $\mathcal{Q}_1'$ is the union of $\mathcal{Q}_1$ with the line segment from $(m-r+1,m)$ to $(m,m-r+1)$ of slope -1.

Now, we going to multiply the numerator and denominator by the triangles $R_{\mathcal{M}}(x,y)$ and $R_{\mathcal{N}}(x,qy)$ where $\mathcal{M}$ is the triangle $\{(1,1),(1,m-r-1),(m-r-1,1)\}$, $\mathcal{N}$ is the triangle $\{(m-r+1,m-1),(m-1,m-r+1),(m-r+1,m-r+1)\}$. Because afterwards $\mathcal{A}_m^{(y)}$ does not change the elements in $[m-r]$ with the elements in $[m-r+1,m]$, we can use the Lemma 200 (with $t\to t^{-1}$ and $x_i\to qx_i$) over the triangles $\mathcal{P}_1$ and $\mathcal{M}$, and the triangles $\mathcal{Q}_1$ and $\mathcal{N}$, which gives us

$$\begin{aligned}
G(x,y) &= p\cdot\frac{R_{\mathcal{P}_2}(x_i,t^{-1}y_j)R_{\mathcal{Q}_2}(x_i,qt^{-1}y_j)}{R_{\mathcal{P}_2}(x_i,y_j)R_{\mathcal{M}}(x_i,y_j)R_{\mathcal{Q}_2}(x_i,qy_j)R_{\mathcal{N}}(x_i,qy_j)}g(x),\\
&= p\cdot\frac{R_{\mathcal{P}_2}(x_i,t^{-1}y_j)R_{\mathcal{Q}_2}(x_i,qt^{-1}y_j)}{R_{[m-r]\times[m]}(x_i,y_j)R_{[m-r+1,m]\times[m-r+1,N]}(x_i,qy_j)}g(x),
\end{aligned}$$

where

$$\begin{aligned}
p &= (-1)^{\binom{m}{2}+\binom{m-r}{2}+\binom{r}{2}}\frac{q^{\binom{r}{2}}t^{\binom{m}{2}}t^{r(N-m)}}{t^{\binom{m-r}{2}}t^{\binom{r}{2}}}\frac{\Delta^t_{[m-r]}(x)\Delta_{[m-r]}(y)\Delta^t_{[m-r+1,m]}(x)\Delta_{[m-r+1,m]}(y)}{\Delta_m(y)},\\
&= (-1)^{\binom{m}{2}+\binom{m-r}{2}+\binom{r}{2}}q^{\binom{r}{2}}t^{\binom{m}{2}-\binom{m-r}{2}-\binom{r}{2}}\frac{\Delta^t_{[m-r]}(x)\Delta^t_{[m-r+1,m]}(x)}{\Delta_{[m-r]\times[m-r+1,m]}(y)}.
\end{aligned}$$

If we take the expansion of $G(x,y)$ we have

$$G(x,y)=\sum_{\substack{J\subseteq[m-r+1,N]\\|J|=r}}\;\sum_{[\sigma]\in\mathfrak{S}_N/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})}C_{J,\sigma}(x)\tau_J K_\sigma\left(\frac{g(x)}{R_{[m]\times[m]}(x_i,y_j)}\right)$$

for some coefficients $C_{J,\sigma}(x)$.

The coefficient $C_{J,\mu}(x)$ that we want, can be obtained multiplying by $\tau_J K_\mu\big(R_{[m]\times[m]}(x,y)\big)$ and then taking the specialization $y_i=\tau_J x^{-1}_{\mu(i)}$ for $i=1,\dots,m$. Hence,

$$C_{J,\mu}(x)=p\cdot\frac{R_{\mathcal{P}_2}(x_i,t^{-1}y_j)R_{\mathcal{Q}_2}(x_i,qt^{-1}y_j)}{R_{[m-r]\times[m]}(x_i,y_j)R_{[m-r+1,m]\times[m-r+1,N]}(x_i,qy_j)}\tau_J K_\mu\big(R_{[m]\times[m]}(x,y)\big)\Big|_{y_i=\tau_J x^{-1}_{\mu(i)}}$$

from this equation we obtain that $J\subset\mu([m-r+1,m])$ (and the $J=\mu([m-r+1,m])$), all other options becomes 0. Then the expression is

$$C_{J,\mu}(x)=p\cdot\frac{R_{\mathcal{P}_2}(x_i,t^{-1}y_j)R_{\mathcal{Q}_2}(x_i,qt^{-1}y_j)R_{\mathcal{E}}(x_i,qy_j)}{R_{\mathcal{D}}(x_i,qy_j)}\Bigg|_{y_i=\tau_J x^{-1}_{\mu(i)}}$$

where $\mathcal{D}=([m-\tilde{r}+1,m]\cup[m+\tilde{r}+1,N])\times[m-r+1,m]$ y $\mathcal{E}=([m-r+1,m-\tilde{r}]\cup[m+1,m+\tilde{r}])\times[m-r]$. Let $G_2=\{(i,j)\in\mathcal{Q}_2\mid i\neq\tau_J\sigma(j)\}$ thus we have

$$C_{J,\mu}(x)=p\cdot\frac{R_{\mathcal{P}_2}(x_i,t^{-1}y_j)R_{\mathcal{E}}(x_i,qy_j)R_{\mathcal{Q}_2}(x_i,qt^{-1}y_j)R_{G_2}(x_i,qy_j)}{R_{\mathcal{D}}(x_i,qy_j)R_{G_2}(x_i,qy_j)}\Bigg|_{y_i=\tau_J x^{-1}_{\mu(i)}}\tag{6.4}$$

it is easy to see

$$\frac{R_{\mathcal{Q}_2}(x,qt^{-1}y)}{R_{G_2}(x_i,qy_j)}\Bigg|_{y_i=\tau_J x^{-1}_{w(i)}}=\frac{(t-1)^{\tilde{r}}}{t^{r(N-m)}}\widetilde{A}_{J\times[m+1,N]}\tag{6.5}$$

and

$$\left.\frac{R_{\mathcal{E}}(x_i,qy)}{\Delta_{[m-r]\times[m-r+1,m]}(y)}\right|_{y_i=\tau_J x_{w(i)}^{-1}} = (x_J)^{m-r}q^{(m-r)r}, \quad R_{\mathcal{P}_2}(x,t^{-1}y)\Big|_{y_i=x_{w(i)}^{-1}} = \frac{\tau_J(A_{J\times[m-r]}\Delta_{J\times[m-r]})}{(-t)^{r(m-r)}(x_J)^{m-r}q^{(m-r)r}}$$

which imply

$$\left.\frac{R_{\mathcal{E}}(x_i,qy)R_{\mathcal{P}_2}(x,t^{-1}y)}{\Delta_{[m-r]\times[m-r+1,m]}(y)}\right|_{y_i=\tau_J x_{w(i)}^{-1}} = t^{-r(m-r)}\tau_J(A_{J\times[m-r]}\Delta_{J\times[m-r]}) = t^{-r(m-r)}\tau_J(A_{J\times[m-r]}\Delta_{J\times[m-r]}) \tag{6.6}$$

then we just have to know how much is

$$X := \left.\frac{R_{G_2}(x_i,qy_j)}{R_D(x_i,qy_j)}\right|_{y_i=\tau_J x_{\mu(i)}^{-1}}$$

but G_2 and $\mathcal{D}$ are the same in the rectangle $[m+\tilde{r}+1,N]\times[m-r+1,m]$, this gives us

$$\left.\frac{R_{G_2}(x_i,qy_j)}{R_D(x_i,qy_j)}\right|_{y_i=\tau_J x_{\mu(i)}^{-1}} = \left.\frac{R_{\widetilde{G}_2}(x_i,qy_j)}{R_{\widetilde{D}}(x_i,qy_j)}\right|_{y_i=\tau_J x_{\mu(i)}^{-1}}$$

where $\widetilde{G}_2 = \{(i,j)\in[m+1,m+\tilde{r}]\times[m-r+1,m] \mid i\neq\tau_J\sigma(j)\}$ y $\widetilde{\mathcal{D}} = [m-\tilde{r}+1,m]\times[m-r+1,m]$.

$$X = \left.\frac{R_{G_2}(x_i,qy_j)}{R_{\widetilde{\mathcal{D}}}(x_i,qy_j)}\right|_{y_i=\tau_J x_{\mu(i)}^{-1}} = (-1)^{\tilde{r}}x_{[m+1,m+\tilde{r}]}\frac{\widetilde{\Delta}_{[m+1,m+\tilde{r}]\times\sigma[m-r+1,m]}}{\Delta_{[m-\tilde{r}+1,m]\times\sigma[m-r+1,m]}}$$

but for the equation 3.4 from Lemma 205, for $\mathfrak{B}_\sigma^c = [m+1,m+\tilde{r}]$ and $\mathfrak{A}_\sigma^c = [m-\tilde{r}+1,m]$ we know that

$$\begin{aligned}
\Phi(\sigma) &= (t-1)^{\tilde{r}}x_{[m+1,m+\tilde{r}]}\cdot\frac{\widetilde{\Delta}_{[m+1,m+\tilde{r}]\times\sigma([m])}}{\Delta_{[m-\tilde{r}+1,m]\times\sigma([m])}}\frac{\Delta_m(x)}{K_\sigma(\Delta_m(x))}\\
&= (t-1)^{\tilde{r}}x_{[m+1,m+\tilde{r}]}\cdot\frac{\widetilde{\Delta}_{[m+1,m+\tilde{r}]\times[m-r]}\widetilde{\Delta}_{[m+1,m+\tilde{r}]\times\sigma([m-r+1,m])}}{\Delta_{[m-\tilde{r}+1,m]\times[m-r]}\Delta_{[m-\tilde{r}+1,m]\times\sigma([m-r+1,m])}}\frac{\Delta_{[m-r]\times[m-r+1,m]}\Delta_{[m-r+1,m]}}{\Delta_{[m-r]\times\sigma([m-r+1,m])}\Delta_{\sigma([m-r+1,m])}}\\
&= (t-1)^{\tilde{r}}\cdot\frac{\widetilde{\Delta}_{[m+1,m+\tilde{r}]\times[m-r]}}{\Delta_{[m-\tilde{r}+1,m]\times[m-r]}}\frac{\Delta_{[m-r]\times[m-r+1,m]}\Delta_{[m-r+1,m]}}{\Delta_{[m-r]\times\sigma([m-r+1,m])}\Delta_{\sigma([m-r+1,m])}}X
\end{aligned}$$

but

$$\frac{\widetilde{\Delta}_{[m+1,m+\tilde{r}]\times[m-r]}}{\Delta_{[m-\tilde{r}+1,m]\times[m-r]}}\frac{\Delta_{[m-r]\times[m-r+1,m]}}{\Delta_{[m-r]\times\sigma([m-r+1,m])}} = \frac{\Delta_{[m-r]\times[m-r+1,m+\tilde{r}]}}{\Delta_{[m-r]\times[m-\tilde{r}+1,m]}\Delta_{[m-r]\times[m-r+1,m+\tilde{r}]}} = 1$$

therefore

$$X(t-1)^{\tilde{r}} = (-1)^{\tilde{r}}\Phi(\sigma)\frac{K_\sigma(\Delta_{[m-r+1,m]})}{\Delta_{[m-r+1,m]}} \tag{6.7}$$

Finally using 6.5, 6.6, 6.7 and the fact

$$(-1)^{r(m-r)}q^{\binom{r}{2}}\Delta_{[m-r]}K_\sigma(\Delta_{[m-r+1,m]})\tau_J(\Delta_{J\times[m-r]}) = \tau_J(K_\sigma\Delta_m)$$

and

$$\binom{m}{2} - \binom{m-r}{2} - \binom{r}{2} - r(m-r) = 0,$$

in 6.4, we obtain

$$C_{J,\sigma} = (-1)^{r(m-r)+\tilde{r}}A_{[m-r]}A_{[m-r+1,m]}A_{J,[m+1,N]}\Phi(\sigma)\tau_J A_{J\times[m-r]}K_\sigma\Delta_m(x).$$

Finally, by Lemmas 221 and 222 we conclude that

$$C_{J,\sigma} = (-1)^{r(m-r)+\tilde{r}} A_{[m-r]} A_{[m-r+1,m]} A_{J,[m+1,N]} \Phi(\sigma) \tau_J A_{J,[m-r]} K_\sigma \Delta_m.$$

□

The analogous to the Proposition 207 is the following.

PROPOSITION 224. *Let $f(x)$ be any bisymmetric function. For any $1 \le r \le m$, we have that*

$$e_r(Y_1,\ldots,Y_m)\Delta_m^t(x)f(x) = \sum_{\substack{J\subseteq[N]\\|J|=r}} \sum_{\substack{[\sigma]\in\mathfrak{S}_N/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})\\ \sigma([m])\cap L=\emptyset,\,\sigma(J)\subseteq[m]}} D_{J,\sigma}(x)\tau_J K_\sigma f(x)$$

where the coefficient $D_{J,\sigma}(x)$ is given by

$$D_{J,\sigma}(x) = (-1)^{\#\mathfrak{B}^c_\sigma} t^{(r+1-2N)r/2} A_m(x) A_{J\times[m+1,N]}(x,x)\Phi(\sigma)\tau_J\big(A_{J\times\mathfrak{A}_\sigma - J}(x,x)K_\sigma\Delta_m(x)\big)$$

PROOF. Notice that by Lemma 198, we have:

$$e_r(Y_1,\ldots,Y_m)\Delta_m^t(x)f(x) = \frac{1}{[m-r]t![r]t!}\mathcal{A}_m^t Y_1\ldots Y_r\Delta_m^t(x)f(x)$$

and by Lemma 196, we have:

$$\begin{aligned}\mathcal{A}_m^t Y_1\ldots Y_r &= (-1)^{r(m-r)}t^{r(r+1-2N)/2}\mathcal{A}_m^t(T_m\cdots T_{N-1})\cdots(T_{m-(r-1)}\cdots T_{N-r})\omega^r\\ &= (-1)^{r(m-r)}t^{r(r+1-2N)/2}\mathcal{A}_m^t S^r,\end{aligned}$$

From these two equations, we have:

$$e_r(Y_1,\ldots,Y_m)\Delta_m^t(x)f(x) = \frac{(-1)^{r(m-r)}t^{r(r+1-2N)/2}}{[m-r]t![r]t!}\mathcal{A}_m^t S^r\Delta_m^t(x)f(x). \tag{a}$$

However, from Theorem 223, we can see that:

$$S^r\Delta^t(x)f(x) = (-1)^{r(m-r)}\mathcal{B}_{[m-r]}\mathcal{B}_{[m-r+1,m]}\sum_{\tilde{r}\in[r]}(-1)^{\tilde{r}}A_{J,[m+1,N]}\Phi(\sigma)\tau_J A_{J,[m-r]}K\sigma\Delta_m(x)$$

where $\mathcal{B}_\mathcal{K} = A_\mathcal{K}(x)\mathcal{A}_\mathcal{K}$. Using the relation $\mathcal{A}_m^t\mathcal{A}_m^t f(x) = [m]t^{-1}!\mathcal{A}_m^t f(x)$ and applying $\mathcal{A}_m^t$ to both sides of the previous equation, we obtain:

$$\mathcal{A}_m^t S^r\Delta_m^t(x)f(x) = (-1)^{r(m-r)}[m-r]t![r]t!\mathcal{A}_m^t\sum_{\tilde{r}\in[r]}(-1)^{\tilde{r}}A_{J,[m+1,N]}\Phi(\sigma)\tau_J A_{J,[m-r]}K_\sigma\Delta_m(x)$$

Using this in equation a, we get:

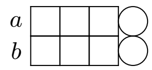
$$e_r(Y_1,\ldots,Y_m)\Delta_m^t(x)f(x) = t^{r(r+1-2N)/2}A_m(x)\mathcal{A}_m\sum_{\tilde{r}\in[r]}(-1)^{\tilde{r}}A_{J,[m+1,N]}\Phi(\sigma)\tau_J A_{J,[m-r]}K_\sigma\Delta_m(x)$$

□

LEMMA 225. *Suppose that $\sigma\in\mathfrak{S}_N$ and $J\subseteq[N]$ are such that $\sigma([m])\cap L=\emptyset$ and $\sigma(J)\subset[m]$, where we recall that $L=[N]\setminus J$. Let (Λ,w) generate a superevaluation, and suppose that the composition $\Omega=\sigma^{-1}\tau_J(\Lambda+(1^m))-(1^m)$ is biordered. The following holds:*

1. *If $(\Omega,w\sigma)$ does not generate a superevaluation then $u_\Lambda^+(D_{J,\sigma})=0$, where $D_{J,\sigma}(x)$ is such as defined in Proposition 224.*
2. *Suppose that $(\Omega,w\sigma)$ generates a superevaluation. If $\delta\in\mathfrak{S}_N$ is also such that $(\Omega,w\delta)$ generates a superevaluation then $\sigma(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})=\delta(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})$ in $\mathfrak{S}_N/(\mathfrak{S}_m\times\mathfrak{S}_{m+1,N})$.*
3. *If $I\subseteq[N]$ is such that $\Omega=\sigma^{-1}\tau_I(\Lambda+(1^m))-(1^m)$, then $I=J$.*

Proof. We first show that (*1*) holds. This is similar to the proof of Lemma 208. Suppose first that Ω is not a superpartition. Given that Ω is biordered, this can only happen if $\Omega_a = \Omega_{a+1}$ for a given $a \in [m-1]$, which can be visualized as

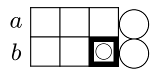


(6.8)

with $b = a+1$. Now, $\Omega = \sigma^{-1}\tau_J(\Lambda + (1^m)) - (1^m)$ translates in coordinates to

$$\big(\Omega + (1)^m\big)_a = \big(\Lambda + (1^m)\big)_{\sigma(a)} + \varepsilon^J_{\sigma(a)} \tag{6.9}$$

where $\varepsilon^J_i = 1$ if $i \in J$ and 0 otherwise. Hence there are three possible cases: (i) $\sigma(a), \sigma(b) \in [m+1, N]$, (ii) $\sigma(a) \in [m], \sigma(b) \in [m+1, N]$ or (iii) $\sigma(a), \sigma(b) \in [m]$. The cases (*i*) and (*ii*) follow in the same way that Lemma **??** using the term $\widetilde{A}_{J\times[m+1,N]}$, and for the case $\sigma(a), \sigma(b) \in [m]$ we have the situation

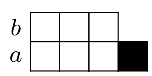


(6.10)

We have that $\sigma(b) \in J$ since $\sigma([m]) \cap L = \emptyset$ by hypothesis. We thus deduce from (6.9) that $\Omega_a = \Lambda_{\sigma(a)}$ and $\Omega_b = \Lambda_{\sigma(b)}$, which implies that $\Lambda_{\sigma(a)} = \Lambda_{\sigma(b)}$. This in turn implies that the permutation w can be chosen such that $w\sigma(b) = w\sigma(a) + 1$, in which case we will have $\Lambda^{\circledast}_{w\sigma(a)} = \Lambda_{\sigma(a)} + 1$ and $\Lambda^{\circledast}_{w\sigma(b)} = \Lambda_{\sigma(b)} + 1$. Hence the term $\widetilde{A}_{J\times\mathfrak{A}-J}(x,x)$ in $D_{J,\sigma}(x)$ contains a factor $A_{\sigma(b),\sigma(a)}(x)$ such that

$$u^+_\Lambda(\tau_J A_{\sigma(b),\sigma(a)}(x)) = u^+_\Lambda\left(\frac{qtx_{\sigma(b)} - qx_{\sigma(a)}}{qx_{\sigma(b)} - qx_{\sigma(a)}}\right) = \frac{q^{\Lambda_{\sigma(b)}+2}t^{2-w\sigma(b)} - q^{\Lambda_{\sigma(a)}+2}t^{1-w\sigma(a)}}{q^{\Lambda_{\sigma(b)}+2}t^{1-w\sigma(b)} - q^{\Lambda_{\sigma(a)}+2}t^{1-w\sigma(a)}} = 0$$

and thus $D_{J,\sigma}(x)$ vanishes in that case. The rest of the proof of (*1*) is similar to Lemma 208, considering that the case



does not happen because $\sigma(J) \subset [m]$.

Part (*2*) and (*3*) follows in the same way that Lemma 208. $\square$

Using a version of Lemma 208 (where $C_{J,\sigma}(x)$ is replaced by $D_{J,\sigma}$) and (1.5), which can be used since $e_r(x_1, \dots, x_m)$ is symmetric in the variables $x_1, \dots, x_m$ and of homogeneous degree r, we get the desired Pieri rules.

Theorem 226. *For any $1 \le r \le m$, the bisymmetric Macdonald polynomial $\mathcal{P}_\Lambda(x; q, t)$ is such that*

$$e_r(x_1, \dots, x_m)\mathcal{P}_\Lambda(x; q, t) = q^r \sum_\Omega \left(\frac{u^+_\Lambda(D_{J,\sigma})}{u^+_\Lambda(\Delta^t_m)} \frac{u_{\Lambda_0^+}(\mathcal{P}_\Lambda)}{u_{\Lambda_0^+}(\mathcal{P}_\Omega)}\right)\mathcal{P}_\Omega(x, q, t)$$

where the coefficients $D_{J,\sigma}(x)$ were obtained explicitly in Proposition 224 and where the sum is over all superpartitions Ω such that there exists a $\sigma \in \mathfrak{S}_N$ and a $J \subseteq [N]$ of size r such that

(*1*) $\sigma(\Omega + (1^m)) = \tau_J(\Lambda + (1^m))$
(*2*) $\sigma([m]) \cap L = \emptyset$ *con* $L = [N] - J$
(*3*) $(\Omega, w\sigma)$ *is a superevaluation if* (Λ, w) *is a superevaluation.*
(*4*) $J \subseteq [N]$ *with* $|J| = r$ *and* $\sigma(J) \subseteq [m]$.

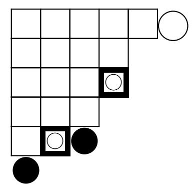
The Ω's that appear in the Pieri rules of Theorem 226 are also special vertical r-strips.

Proof. It is the same proof of Theorem 209. $\square$

DEFINITION 227. *We will say that Ω/Λ is a vertical r-strip of type II if*

(1) *Ω/Λ is a vertical r-strip*
(2) *there are no $\blacksquare$-rows in the diagram of Ω/Λ*

For instance, if $\Lambda = (5, 3, 1; 4, 3)$ and $\Omega = (5, 2, 0; 4, 4, 3)$ then Ω/Λ is a vertical 2-strip of type II.



(6.11)

As was the case in Corollary ??, we can rewrite Theorem ?? in a more precise way using vertical r-strips of type II.

LEMMA 228. *Let σ and J be such as in Theorem 226, that is, such that*

(1) $\sigma(\Omega + (1^m)) = \tau_J(\Lambda + (1^m))$
(2) $\sigma([m]) \cap L = \emptyset$ *con* $L = [N] - J$
(3) $(\Omega, w\sigma)$ *is a superevaluation if* (Λ, w) *is a superevaluation.*
(4) $J \subseteq [N]$ *with* $|J| = r$ *and* $\sigma(J) \subseteq [m]$.

Then Ω/Λ is a vertical r-strip of type II.

PROOF. The proof of Ω/Λ is a vertical r-strip is the same in Lemma 211, so we have to prove that there are no $\blacksquare$-rows in the diagram of Ω/Λ. Suppose that the row i in Ω/Λ is a $\blacksquare$-rows, then exist $i \in wJ$ with $i \notin w(\sigma^{-1}[m])$ since Λ does not have a circle in row i, which contradicts (*4*). $\square$

We now show that all Ω's such that Ω/Λ is a vertical r-strip of type II do in fact appear in the Pieri rules of Theorem 226.

LEMMA 229. *Given Ω/Λ a vertical r-strip of type II, let $\tilde{\sigma}$ be any permutation that interchanges the $\boxed{\circ}$-rows and the $\bullet$-rows while leaving the remaining rows invariant (such a permutation can be defined by Remark 212). Let also $\tilde{J}$ be the set of $\boxed{\circ}$-rows and $\blacksquare$-rows. If*

$$\sigma = w^{-1}\tilde{\sigma}w \quad \text{and} \quad J = w^{-1}\tilde{\sigma}(\tilde{J})$$

then there exists a permutation $s \in \mathfrak{S}_m \times \mathfrak{S}_{m+1,N}$ such that $\sigma' = \sigma s$ obeys the following relations:

(1) $\sigma'(\Omega + (1^m)) = \tau_J(\Lambda + (1^m))$
(2) $\sigma'([m]) \cap L = \emptyset$ *con* $L = [N] - J$
(3) $(\Omega, w\sigma')$ *is a superevaluation if* (Λ, w) *is a superevaluation.*
(4) $J \subseteq [N]$ *with* $|J| = r$ *and* $\sigma(J) \subseteq [m]$.

As such, the superpartition Ω satisfies the conditions of Theorem 226 (with $D_{J,\sigma'}(x) = D_{J,\sigma}(x)$).

PROOF. By Lemma 229 we have Ω/Λ is a r-strip imply (*1*), (*2*) and (*3*). Thus, we only have to show *4)*.

Let $J \subset [N]$ with $|J| = r$, note that $w\sigma J = \tilde{J}$ that correspond to $\boxed{\circ}$-rows and $\blacksquare$-rows, but by definition of a vertical strip of type II, there is not $\blacksquare$-rows in Ω/Λ, so $w\sigma J$ are $\boxed{\circ}$-rows that live in $w([m])$. Thus $\sigma J \subset [m]$. $\square$

Corollary 230. *For $r \in \{1, \dots, N-m\}$, the bisymmetric Macdonald polynomial $\mathcal{P}_\Lambda(x; q, t)$ obeys the following Pieri rules*

$$e_r(x_1, \dots, x_m)\mathcal{P}_\Lambda(x; q, t) = q^r \sum_{\Omega} \left(\frac{u_\Lambda^+(D_{J,\sigma})}{u_\Lambda^+(\Delta_m^t)} \frac{u_{\Lambda_0^+}(\mathcal{P}_\Lambda)}{u_{\Lambda_0^+}(\mathcal{P}_\Omega)} \right) \mathcal{P}_\Omega(x, q, t)$$

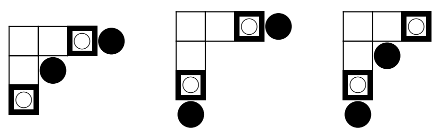
where the sum is over all superpartitions Ω such that Ω/Λ is a vertical r-strip of type II. Note that $D_{J,\sigma}(x)$ was defined in Proposition 224, where σ and J can be obtained in the following manner from the diagram of Ω/Λ: let $\tilde{\sigma}$ be any permutation that interchanges the $\boxed{\bigcirc}$-rows and the $\bullet$-rows, while leaving the remaining rows invariant (including the $\boxed{\bigcirc}\bullet$ rows), and let $\tilde{J}$ be the set of $\boxed{\bigcirc}$-rows. Then

$$\sigma = w^{-1}\tilde{\sigma}w \quad \text{and} \quad J = w^{-1}\tilde{\sigma}(\tilde{J})$$

where w is such that (Λ, w) is a superevaluation.

Proof. It is a immediate consequence of Theorem 226, Lemma 228 and Lemma 229. □

Example 231. *The superpartitions that appear in the expansion of the multiplication of $e_2(x_1, x_2)$ and $P_{(2,0;1)}(x; q, t)$ are given by:*



To be more precise, we have that

$$e_2(x_1, x_2)P_{(2,0;1)} = \mathcal{P}_{(3,1;1)} - \frac{q(1+t)(1-t)}{1-qt^2}\mathcal{P}_{(3,0;1,1)} + \frac{q^3(1-t)(1-q^2t^3)(1-qt)}{(1-qt)(1-q^3t^3)(1-q^2t)}\mathcal{P}_{(1,0;3,1)}$$

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